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Controllability Results for Evolution Inclusions with Non-Local Conditions

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Abstract. In this paper we prove controllability results for mild solutions defined on a compact real interval for first order differential evolution inclusions in Banach spaces with non-local conditions. By using suitable fixed point theorems we study the case when the multi-valued map has convex as well as non-convex values.

Keywords: *Control problems, evolution differential inclusions, set-valued operators*

AMS subject classification: 34A60, 34K35, 35G25, 47H04

1. Introduction

In this paper we shall establish sufficient conditions for the controllability of semilinear evolution inclusions in a Banach space with non-local conditions.

More precisely, in Section 3 we consider non-local evolution inclusion

$$y' - A(t,y)y \in F(t,y) + (Bu)(t) \quad (t \in [0,b]) \tag{1}$$

$$y(0) + \sum_{k=1}^{p} c_k y(t_k) = y_0 \tag{2}$$

where $F : [0,b] \times E \to \mathcal{P}(E)$ is a multi-valued map, $y_0 \in E$, $A(t,y)$ is a continuous operator on E for each $(t,y) \in [0,b] \times E$, $0 < t_1 < t_2 < \ldots < t_p < b, p \in \mathbb{N}$, $c_k \neq 0$, $\mathcal{P}(E)$ is the family of all subsets of E and E is a real separable Banach space with norm $|\cdot|$. The control function $u(\cdot)$ is given in $L^2([0,b],U)$, a Banach space of admissible control functions with U as a Banach space, and Θ is a bounded linear operator from U to E.

The non-local condition (2) was used recently by Byszewski in [6, 8] when he proved the existence and uniqueness of mild and classical solutions of non-local Cauchy problems. The constants c_k in the non-local condition (2) can satisfy the inequalities $|c_k| > 1$. As remarked by Byszewski [8], if all $c_k \neq 0$, the results can be applied to kinematics to

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determine the evolution $t \to y(t)$ of the location of a physical object for which we do not know the positions $y(0)$ and $y(t_k)$ for all k, but we know that the non-local condition (2) holds. Consequently, to describe some physical phenomena, the non-local condition can be more useful than the standard initial condition $y(0) = y_0$. From (2) it is clear that, when $c_k = 0$ for all k, we have the classical initial condition.

Existence and controllability results were proved by Benchohra and Ntouyas in [3] for equation (1) with non-local conditions of the form $y(0) + f(y) = y_0$, where $f \in C\big(C([0,b],E),E\big)$ under the assumption that f was bounded and the multi-valued map F has convex values. Here, we consider the non-local condition (2) and we prove controllability results in the cases when the multi-valued map F has convex or non-convex values. In the first case a fixed point theorem for condensing maps due to Martelli [20] is used. In the later we shall present two results. In the first one we rely on a fixed point theorem for contraction multi-valued maps, due to Covitz and Nadler [11], and for the second one on Schaefer's fixed point theorem combined with a selection theorem due to Bressan and Colombo [7] for lower semicontinuous multi-valued operators with non-empty closed and decomposable values. For recent controllability results in the convex case we refer to the papers by Benchohra and Ntouyas [3 - 5] and the references cited therein. Other results for the particular case $B \equiv 0$ can be found in the paper [2].

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multi-valued analysis which are used throughout this paper. We denote by

- $\mathcal{P}(E)$ the set of all subsets of a Banach space $(E, |\cdot|)$
- $C([0,b],E)$ the Banach space of continuous functions $y : [0,b] \to E$ normed by $\|y\|_\infty = \sup_{t\in[a,b]} |y(t)|$
- $B(E)$ the Banach space of bounded linear operators $N : E \to E$ with norm $\|N\|_{B(E)} = \sup_{|y|=1} |N(y)|$
- $L^1([0,b],E)$ the linear space of equivalence classes of all measurable functions $y : [0,b] \to E$ which are normed by $\|y\|_{L^1} = \int_0^b |y(t)|\,dt$

where a measurable function $y : [0,b] \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable (for properties of the Bochner integral see Yosida [21]).

Let $(X, |\cdot|)$ be a Banach space. A multi-valued map $G : X \to \mathcal{P}(X)$ is called

- *convex-valued* if $G(x)$ is convex for all $x \in X$
- *closed-valued* if $G(x)$ is closed for all $x \in X$
- *bounded* on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for any bounded set B of X, that is $\sup_{x\in B} \big\{ \sup_{y\in G(x)} |y| \big\} < \infty$
- *upper semicontinuous* on X if, for each $x_1 \in X$, $G(x_1)$ is a non-empty, closed subset of X and if, for each open set B of X containing $G(x_1)$, there exists an open neighbourhood A of x_1 such that $G(A) \subseteq B$

- *completely continuous* if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$.

If the multi-valued map G is completely continuous with non-empty compact values, then G is upper semicontinuous if and only if G has a closed graph (i.e. $x_n \to x_*$ and $y_n \to y_*$ with $y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. Further, we set

$$\begin{aligned} P(X) &= \{Y \in \mathcal{P}(X) : Y \neq \emptyset\} \\ P_{cl}(X) &= \{Y \in P(X) : Y \ \text{ closed}\} \\ P_b(X) &= \{Y \in \mathcal{P}(X) : Y \ \text{ bounded}\} \\ P_c(X) &= \{Y \in \mathcal{P}(X) : Y \ \text{ convex}\} \\ P_{cp}(X) &= \{Y \in \mathcal{P}(X) : Y \ \text{ compact}\}. \end{aligned}$$

A multi-valued map $G : [0,b] \to P_{cl}(X)$ is said to be measurable if for each $x \in X$ the function

$$t \mapsto d(x, G(t)) = \inf \{d(x,z) : z \in G(t)\}$$

is measurable on $[0,b]$. An upper semi-continuous map $G : X \to \mathcal{P}(X)$ is said to be condensing if for any subset $B \subseteq X$ with $\alpha(B) \neq 0$ we have $\alpha(G(B)) < \alpha(B)$, where α denotes the Kuratowski measure of non-compacteness. For properties of the Kuratowski measure, we refer to Banaś and Goebel [1] (comp. also [14]).

We remark that a completely continuous multi-valued map is the easiest example of a condensing map. For more details on multi-valued maps we refer to the books of Deimling [12], Górniewicz [14], and Hu and Papageorgiou [16].

3. The convex case

In this section we assume that F is a bounded, closed, convex multi-valued map. Let us list the basic hypotheses:

(H1) $A : [0,b] \times E \to B(E)$ is a continuous function so that for all $r > 0$ there exists $r_1 = r_1(r) > 0$ such that $|v| \leq r_1$ implies $\|A(t,v)\|_{B(E)} \leq r$ for all $t \in [0,b]$ and all $v \in E$.

Remark 3.1. From hypothesis (H1), for any fixed $u \in C([0,b], E)$ we are able to claim the existence of a unique continuous function $U_u : [0,b] \times [0,b] \to B(E)$ such that

$$U_u(t,s) = I + \int_s^t A_u(w) U_u(w,s)\, dw \tag{3}$$

(evolution operator of A), where I stands for the identity operator on E and $A_u(t) = A(t, u(t))$ (see, e.g., [19]).

From (3) one has

$$\left.\begin{aligned} U_u(t,t) &= I \\ U_u(t,s)U_u(s,r) &= U_u(t,r) \end{aligned}\right\} \qquad (t,s,r \in [0,b]).$$

Moreover,

$$\frac{\partial U_u(t,s)}{\partial t} = A_u(t)U_u(t,s) \qquad \text{for a.a. } t \in [0,b] \text{ and all } s \in [0,b].$$

Now, we continue with the presentation of the other hypotheses:

(H2) $F : [0,b] \times E \to P_{b,cl,c}(E), (t,y) \mapsto F(t,y)$ is measurable with respect to t for each $y \in E$, upper semicontinuous with respect to y for each $t \in [0,b]$, and for each fixed $y \in C([0,b],E)$ the set

$$S_{F,y} = \Big\{ g \in L^1([0,b],E) : g(t) \in F(t,y(t)) \text{ for a.e. } t \in [0,b] \Big\}$$

is non-empty.

(H3) There exists the operator Θ on E defined by

$$\Theta = \left(I + \sum_{k=1}^{p} c_k U_y(t_k,0) \right)^{-1}.$$

(H4) $\|F(t,y)\|_{\mathcal{P}} := \sup\{|v| : v \in F(t,y)\} \le p(t)\psi(|y|)$ for a.a. $t \in [0,b]$ and all $y \in E$, where $p \in L^1([0,b],\mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \to (0,\infty)$ is continuous and increasing with

$$M \int_0^b p(s)\,ds < \int_c^\infty \frac{du}{\psi(u)}$$

where

$$M = \sup_{(t,s)\in[0,b]\times[0,b]} \|U_y(t,s)\|_{B(E)}$$

and

$$c = M\|\Theta\|_{B(E)}|y_0| + M^2\|\Theta\|_{B(E)} \sum_{k=1}^{p} |c_k| \int_0^{t_k} p(t)\psi(|y|)\,dt + bMM_1\widehat{M}$$

with

$$\widehat{M} = M_2 \Bigg[|x_1| + \sum_{k=1}^{p} |c_k|\,|y(t_k)| + M\|\Theta\|_{B(E)}|y_0| + \sum_{k=1}^{p} M^2\|\Theta\|_{B(E)} \int_0^{t_k} p(s)\psi(|y|)\,ds + M\int_0^b p(s)\psi(|y|)\,ds \Bigg].$$

(H5) The linear operator $W : L^2([0,b],U) \to E$ defined by

$$Wu = \int_0^b U(b,s)Bu_y(s)\,ds$$

has an inverse W^{-1} which takes values in $L^2([0,b],U)\setminus\ker W$ and there exist constants $M_1>0$ and $M_2>0$ such that $\|B\|\le M_1$ and $\|W^{-1}\|\le M_2$.

(H6) For each bounded $D\subset C([0,b],E)$ and $t\in[0,b]$ the set

$$\left\{\begin{array}{l|l} U(t,0)\Theta y_0 & \\ \quad -\displaystyle\sum_{k=1}^{p} c_k U(t,0)\Theta\int_0^{t_k} U(t_k,s)g(s)\,ds & g\in S_{F,D} \\ \quad +\displaystyle\int_0^t U(t,s)g(s)\,ds+\int_0^t U(t,s)(Bu)(s)\,ds & \end{array}\right\}$$

is relatively compact in E, where $S_{F,D}=\cup_{y\in D}S_{F,y}$

Remark 3.2.

(i) If $\dim E<\infty$ then, for each $y\in C([0,b],E)$, $S_{F,y}\neq\emptyset$ (see Lasota and Opial [18]).

(ii) For construction of W^{-1} see [9].

(iii) The operator B in hypothesis (H3) exists if $\sum_{k=1}^{p}|c_k|<\frac{1}{M}$.

(iv) If $U_y(t,s)$ for $(t,s)\in[0,b]\times[0,b]$ is completely continuous, then hypothesis (H6) is satisfied. Also, if $\dim E<\infty$, then hypothesis (H6) is satisfied.

(v) From hypothesis (H1), if $u\in C([0,b],E)$, then $A_u\in C([0,b],B(E))$ and

$$\|u_n-u^*\|_\infty\to 0 \quad\Longrightarrow\quad \|A_{u_n}-A_{u^*}\|_\infty:=\max_{t\in[0,b]}\|A_{u_n}(t)-A_{u^*}(t)\|_{B(E)}\to 0$$

as $n\to\infty$.

Definition 3.1. A function $y\in C([0,b],E)$ is called a *mild solution* of problem (1) - (2) if there exists a function $v\in L^1([0,b],E)$ such that $v(t)\in F(t,y(t))$ a.e. on $[0,b]$ and

$$\begin{aligned} y(t)=U_y(t,0)\Theta y_0 & \\ &-\sum_{k=1}^{p}c_kU_y(t,0)\Theta\int_0^{t_k}U_y(t_k,s)v(s)\,ds \\ &+\int_0^t U_y(t,s)v(s)\,ds+\int_0^t U_y(t,s)(Bu)(s)\,ds. \end{aligned}\tag{4}$$

Definition 3.2. Non-local problem (1) - (2) is said to be *non-locally controllable* on the interval $[0,b]$, if for every $x_1\in E$ there exists a control $u\in L^2([0,b],U)$ such that the mild solution $t\to y(t)$ of problem (1) - (2) satisfies $y(b)+\sum_{k=1}^{p}c_ky(t_k)=x_1$.

The following lemmas are crucial in the proof of our main theorem:

Lemma 3.1 [19]. *Let I be a compact real interval and X be a Banach space. Moreover, let F be a multi-valued map satisfying hypothesis* (H2) *and let Γ be a linear continuous mapping from $L^1(I,X)$ to $C(I,X)$. Then the operator*

$$\Gamma\circ S_F: C(I,X)\to P_{b,cl,c}(C(I,X)),\quad y\longmapsto(\Gamma\circ S_F)(y)=\Gamma(S_{F,y})$$

is a closed graph operator in $C(I,X)\times C(I,X)$.

Lemma 3.2 [20]. *Let* X *be a Banach space and* $N : X \to P_{b,cl,c}(X)$ *be an upper semicontinuous condensing map. If the set*

$$\Omega = \big\{ y \in X : \lambda y \in N(y) \ \ \textit{for some } \lambda > 1 \big\}$$

is bounded, then N *has a fixed point.*

Lemma 3.3 [15: p. 36]. *Suppose that* $\varphi_1, \varphi_2 \in C([0,b],\mathbb{R})$ *and* $\varphi_3 \in L^1([0,b],\mathbb{R})$ *with* $\varphi_3(t) \ge 0$ *a.e. on* $[0,b]$ *and* $\varphi_1(t) \le \varphi_2(t) + \int_0^t \varphi_3(s)\varphi_1(s)\,ds$. *Then*

$$\varphi_1(t) \le \varphi_2(t) + \int_0^t \varphi_3(s)\varphi_2(s) \exp\left(\int_s^t \varphi_3(\tau)\,d\tau \right) ds.$$

Theorem 3.1. *Assume that hypotheses* (H1) - (H6) *are satisfied. Then problem* $(1)-(2)$ *is non-locally controllable on* $[0,b]$.

Proof. Using hypothesis (H5) for an arbitrary function $y(\cdot)$ define the control

$$\begin{aligned} u_y(t) = W^{-1}\Bigg[& x_1 - \sum_{k=1}^{p} c_k y(t_k) - U_y(t,0)\Theta y_0 \\ & + \sum_{k=1}^{p} c_k U_y(t,0)\Theta \int_0^{t_k} U_y(t_k,s) g(s)\,ds - \int_0^t U_y(t,s) g(s)\,ds \Bigg](t) \end{aligned}$$

where

$$g \in S_{F,y} = \Big\{ g \in L^1([0,b],E) : g(t) \in F(t,y(t)) \ \text{ for a.e. } t \in [0,b] \Big\}.$$

We shall now show that, when using this control, the operator

$$N : C([0,b],E) \to \mathcal{P}(C([0,b],E))$$

defined by

$$N(y) = \left\{ h \in C([0,b],E) \ \middle| \ h(t) = \left\{ \begin{array}{l} U_y(t,0)\Theta y_0 \\ -\sum_{k=1}^{p} c_k U_y(t,0)\Theta \int_0^{t_k} U_y(t_k,s) g(s)\,ds \quad (g \in S_{F,y}) \\ +\int_0^t U_y(t,s)\big[g(s) + (Bu)(s)\big]\,ds \end{array} \right. \right\}$$

has a fixed point. This fixed point is then a solution of system (1) - (2).

It is obvious that $x_1 - \sum_{k=1}^{p} c_k y(t_k) \in (Ny)(b)$. We shall show that N is completely continuous with bounded, closed, convex values and it is upper semicontinuous. The proof will be given in several steps.

`Step 1`: *$N(y)$ is convex for each $y \in C([0,b],E)$.* This is trivial, since $S_{F,y}$ is convex. However, for completness we present the proof: Let h_1, h_2 belong to $N(y)$. Then there exist $g_1, g_2 \in S_{F,y}$ such that, for each $t \in [0,b]$,

$$\left.\begin{aligned} h_i(t) = U_y(t,0)\Theta y_0 - \sum_{k=1}^{p} c_k U_y(t,0)\Theta \int_0^{t_k} U_y(t_k,s) g_i(s)\,ds \\ + \int_0^t U_y(t,s)\big[(Bu_y)(s) + g_i(s)\big]\,ds \end{aligned}\right. \qquad (i = 1,2).$$

Let $0 \le k \le 1$. Then, for each $t \in [0,b]$,

$$\begin{aligned} &\big(\alpha h_1 + (1-\alpha)h_2\big)(t) \\ &\quad = U_y(t,0)By_0 \\ &\qquad - \sum_{k=1}^{p} c_k U_y(t,0)B \int_0^{t_k} U_y(t_k,s)\big[\alpha g_1(s) + (1-\alpha) g_2(s)\big]\,ds \\ &\qquad + \int_0^t U_y(t,s)\big[\alpha g_1(s) + (1-\alpha)g_2(s)\big]\,ds + \int_0^t U_y(t,s)(Bu_y)(s)\,ds. \end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values), then $\alpha h_1 + (1-\alpha)h_2 \in N(y)$.

`Step 2`: *N is bounded on bounded sets of $C([0,b],E)$.* Set $B_r = \{y \in C([0,b],E) : \|y\|_\infty \le r\}$. Then if $h \in N(y)$, there exists $g \in S_{F,y}$ such that

$$\begin{aligned} h(t) = U_y(t,0)\Theta y_0 & \\ - \sum_{k=1}^{p} c_k U_y(t,0)\Theta \int_0^{t_k} U_y(t_k,s) g(s)\,ds & \qquad (t \in [0,b]). \\ + \int_0^t U_y(t,s)\big[Bu_y(s) + g(s)\big]\,ds & \end{aligned}$$

We observe that

$$\begin{aligned} |u_y(t)| \le M_2 \Bigg[& |x_1| + \sum_{k=1}^{p} |c_k|\,|y(t_k)| + M\|\Theta\|_{B(E)}|y_0| \\ & + \sum_{k=1}^{p} c_k M^2 \|\Theta\|_{B(E)} \int_0^{t_k} p(s)\psi(|y(s)|)d\,s + M\int_0^b p(s)\psi(|y(s)|)\,ds\Bigg] \\ =: \widehat{M}. & \end{aligned}$$

Then, by hypothesis (H4) and by the above inequality,

$$\begin{aligned} |h(t)| \le M\|\Theta\|_{B(E)}|y_0| & \\ + M^2\|\Theta\|_{B(E)} \sum_{k=1}^{p} |c_k| \int_0^{t_k} p(t)\psi(r)\,dt & \qquad (t \in [0,b]). \\ + M\int_0^t p(s)\psi(r)\,ds + bMM_1\widehat{M} & \end{aligned}$$

Therefore, for each $h \in N(B_r)$,

$$\begin{aligned}\|h\|_\infty &\leq M\|\Theta\|_{B(E)}|y_0| \\ &\quad + M^2\|\Theta\|_{B(E)}\sum_{k=1}^{p}|c_k|\int_0^{t_k} p(t)\psi(r)\,dt \\ &\quad + M\int_0^b p(s)\psi(r)\,ds + bMM_1\widehat{M} \\ &=: \ell.\end{aligned}$$

`Step 3`: *N sends bounded sets into equicontinuous sets of $C([0,b],E)$.* Indeed, let $\tau_1, \tau_2 \in [0,b]$ with $\tau_1 < \tau_2$ and let B_r be a bounded set in $C([0,b],E)$. Then

$$\begin{aligned}&|h(\tau_2) - h(\tau_1)| \\ &\quad \leq \|U_y(\tau_2,0) - U_y(\tau_1,0)\|_{B(E)}\|\Theta\|_{B(E)}|y_0| \\ &\qquad + \sum_{k=1}^{p} c_k\|\Theta\|_{B(E)}\|U_y(\tau_2,0) - U_y(\tau_1,0)\|_{B(E)} \\ &\qquad \times \int_0^{t_k}\|U_y(t_k,s)\|_{B(E)}|g(s)|\,ds \\ &\qquad + \int_0^{\tau_1}\|U_y(\tau_2,s) - U_y(\tau_1,s)\|_{B(E)}|Bu_y(s) + g(s)|\,ds \\ &\qquad + \int_{\tau_1}^{\tau_2} U_y(\tau_2,s)|Bu_y(s) + g(s)|\,ds \\ &\quad \leq \|U_y(\tau_2,0) - U_y(\tau_1,0)\|_{B(E)}\|\Theta\|_{B(E)}|y_0| \\ &\qquad + M\sum_{k=1}^{p} c_k\|\Theta\|_{B(E)}\|(U_y(\tau_2,0) - U_y(\tau_1,0))\|_{B(E)} \\ &\qquad \times \int_0^{t_k} p(s)\psi(|y(s)|)\,ds \\ &\qquad + M_1\widehat{M}\int_0^{\tau_1}\|U_y(\tau_2,s) - U_y(\tau_1,s)\|_{B(E)}ds \\ &\qquad + \int_0^{\tau_1}\|U_y(\tau_2,s) - U_y(\tau_1,s)\|_{B(E)}p(s)\psi(|y(s)|)\,ds \\ &\qquad + M_1\widehat{M}\int_{\tau_1}^{\tau_2}\|U_y(\tau_2,s)\|_{B(E)}ds \\ &\qquad + \int_{\tau_1}^{\tau_2}\|U_y(\tau_2,s)\|_{B(E)}p(s)\psi(|y(s)|)\,ds.\end{aligned}$$

Therefore, $N(B_r)$ is relatively compact.

`Step 4`: *$U_u(t,s)$ is continuous with respect to u, i.e. $\|u_n - u^*\|_\infty \to 0$ implies $\|U_{u_n} - U_{u^*}\|_\infty \to 0$ as $n \to \infty$.* Indeed, let $\|u_n - u^*\|_\infty \to 0$. Then there exists $r > 0$

such that $\|u_n\|_\infty, \|u_*\|_\infty \le r$. Moreover, if $s \le t$ (analogously, if $t < s$), we have

$$\begin{aligned}\|U_{u_n} - U_{u^*}\|_\infty &\le \int_s^t \|U_{u_n}(w,s)\|_{B(E)}\|A_{u_n}(w) - A_{u^*}(w)\|_{B(E)}dw \\ &\quad + \int_s^t \|A_{u^*}\|_\infty \|U_{u_n}(w,s) - U_{u^*}(w,s)\|_{B(E)}dw \\ &\le M\int_s^t \|A_{u_n}(w) - A_{u^*}(w)\|_{B(E)}dw \\ &\quad + \int_s^t \|A_{u^*}\|_\infty \|U_{u_n}(w,s) - U_{u^*}(w,s)\|_{B(E)}dw.\end{aligned}$$

Applying Lemma 3.3, we obtain

$$\begin{aligned}\|U_{u_n} - U_{u^*}\|_\infty &\le M\int_s^t \|A_{u_n}(w) - A_{u^*}(w)\|_{B(E)}dw \\ &\quad + M\int_s^t \|A_{u^*}(w)\|_{B(E)}\left[\int_s^t \|A_{u_n}(\tau) - A_{u^*}(\tau)\|_{B(E)}d\tau\right] \\ &\quad \times \exp\left(\int_w^t \|A_{u^*}(z)\|_{B(E)}dz\right)dw \\ &\le bM\|A_{u_n} - A_{u^*}\|_\infty + b^2M\|A_{u^*}\|_\infty\|A_{u_n} - A_{u^*}\|_\infty \exp(b\|A_{u^*}\|_\infty) \\ &\le \|A_{u_n} - A_{u^*}\|_\infty Mb\big(1 + br_1\exp(br_1)\big).\end{aligned}$$

`Step 5`: *N has a closed graph.* Indeed, let $y_n \to y^*$ and $h_n \in N(y_n)$ with $h_n \to h^*$. We shall prove that $h^* \in N(y^*)$. The inclusion $h_n \in N(y_n)$ means that there exists $g_n \in S_{F,y_n}$ such that

$$\begin{aligned}h_n(t) &= U_{y_n}(t,0)\Theta y_0 \\ &\quad - \sum_{k=1}^p c_kU_{y_n}(t,0)\Theta\int_0^{t_k} U_{y_n}(t_k,s)g_n(s)\,ds \qquad (t \in [0,b]) \\ &\quad + \int_0^t U_{y_n}(t,s)\big[g_n(s) + (Bu_{y_n})(s)\big]ds\end{aligned}$$

where

$$\begin{aligned}u_{y_n}(t) = W^{-1}\Bigg[&x_1 - \sum_{k=1}^p c_ky_n(t_k) - U_{y_n}(b,0)\Theta y_0 \\ &+ \sum_{k=1}^p c_kU_{y_n}(b,0)\Theta\int_0^{t_k} U_{y_n}(t_k,s)g_n(s)\,ds - \int_0^b U_{y_n}(b,s)g_n(s)\,ds\Bigg](t).\end{aligned}$$

We must prove that there exists $g^* \in S_{F,y^*}$ such that

$$\begin{aligned}h^*(t) &= U_{y^*}(t,0)\Theta y_0 \\ &\quad - \sum_{k=1}^p c_kU_{y^*}(t,0)\Theta\int_0^{t_k} U_{y^*}(t_k,s)g^*(s)\,ds \qquad (t \in [0,b]) \\ &\quad + \int_0^t U_{y^*}(t,s)\big[g^*(s) + (Bu_{y^*})(s)\big]ds\end{aligned}$$

where

$$u_{y^*}(t) = W^{-1}\Big[x_1 - \sum_{k=1}^{p} c_k y^*(t_k) - U_{y^*}(b,0)\Theta y_0 + \sum_{k=1}^{p} c_k U_{y^*}(b,0)\Theta \int_0^{t_k} U_{y^*}(t_k,s)g^*(s)\,ds - \int_0^b U_{y^*}(b,s)g^*(s)\,ds\Big](t).$$

Set

$$\overline{u}_y(t) = W^{-1}\Big[x_1 - \sum_{k=1}^{p} c_k y(t_k) - U_y(b,0)\Theta y_0\Big].$$

Since W^{-1} is continuous, then $\overline{u}_{y_n}(t) \to \overline{u}_{y_*}(t)$ for $t \in [0,b]$. Clearly, we have

$$\Big\|\Big(h_n - U_{y_n}(t,0)\Theta y_0 - \int_0^t U_{y_n}(t,s)(B\overline{u}_{y_n})(s)\,ds\Big) - \Big(h_* - U_{y^*}(t,0)\Theta y_0 - \int_0^t U_{y^*}(t,s)(B\overline{u}_{y^*})(s)ds\Big)\Big\|_\infty \longrightarrow 0$$

as $n \to \infty$. Consider the operator

$$\Gamma :\ L^1([0,b],E) \to C([0,b],E)$$

defined by

$$g \to \Gamma(g)(t) = \int_0^t U_y(t,s)g(s)\,ds - \sum_{k=1}^{p} c_k U_y(t,0)\Theta \int_0^{t_k} U_y(t_k,s)g(s)\,ds + \int_0^t U_y(t,s)BW^{-1}\Big(\int_0^b U_y(b,w)g(w)\,dw\Big)ds.$$

We can see that the operator Γ is linear and continuous. Indeed, one has

$$\|(\Gamma g)\|_\infty \le \overline{M}\|g\|_{L^1}$$

where $\overline{M}$ is given by

$$\overline{M} = M + M^2\|\Theta\|_{B(E)} \sum_{k=1}^{p} |c_k| + bM^2 M_1 M_2.$$

From Lemma 3.1 it follows that $\Gamma \circ S_F$ is a closed graph operator. Moreover,

$$h_n(t) - U_{y_n}(t,0)\Theta y_0 - \int_0^t U_{y_n}(t,s)(Bu_{y_n})(s)\,ds \in \Gamma(S_{F,y_n}).$$

Since $y_n \to y^*$, it follows from Lemma 3.1 that

$$\begin{aligned} h^*(t) - U_{y^*}(t,0)\Theta y_0 = &\int_0^t U_{y^*}(t,s)g^*(s)\,ds \\ &- \sum_{k=1}^p c_k U_{y^*}(t,0)\Theta \int_0^{t_k} U_{y^*}(t_k,s)g^*(s)\,ds \\ &+ \int_0^t U_{y^*}(t,s)(Bu_{y^*})(s)\,ds \end{aligned}$$

for some $g^* \in S_{F,y^*}$. Therefore, N is a completely continuous multi-valued map, upper semicontinuous with convex closed values.

`Step 6:` *The set* $\Omega = \{y \in C([0,b],E) : \lambda y \in N(y) \ \ \text{for some } \lambda > 1\}$ *is bounded.* Indeed, let $y \in \Omega$. Then $\lambda y \in N(y)$ for some $\lambda > 1$. Thus there exists $g \in S_{F,y}$ such that

$$\begin{aligned} y(t) = &\lambda^{-1} U_y(t,0)\Theta y_0 \\ &- \lambda^{-1} \sum_{k=1}^p c_k U_y(t,0)\Theta \int_0^{t_k} U_y(t_k,s)g(s)\,ds \\ &+ \lambda^{-1} \int_0^t U_y(t,s)g(s)ds + \lambda^{-1} \int_0^t U_y(t,s)BW^{-1} \\ &\times \Bigg[x_1 - \sum_{k=1}^p c_k y(t_k) - U_y(b,0)\Theta y_0 + \sum_{k=1}^p c_k U_y(b,0)\Theta \\ &\times \int_0^{t_k} U_y(t_k,w)g(w)dw - \int_0^b U_y(b,w)g(w)dw\Bigg](s)\,ds \end{aligned}$$

for $t \in [0,b]$. This implies by hypothesis (H4) that, for each $t \in [0,b]$,

$$\begin{aligned} |y(t)| \le &M\|\Theta\|_{B(E)}|y_0| \\ &+ M^2\|\Theta\|_{B(E)} \sum_{k=1}^p |c_k| \int_0^{t_k} p(t)\psi(|y(s)|)\,dt \\ &+ M\int_0^t p(s)\psi(|y(s)|)\,ds + bMM_1M_2\Bigg[|x_1| + \sum_{k=1}^p |c_k|\,|y(t_k| \\ &+ M\|\Theta\|_{B(E)}|y_0| + \sum_{k=1}^p M^2\|\Theta\|_{B(E)} \int_0^{t_k} p(s)\psi(|y(s)|) \\ &+ M\int_0^b p(s)\psi(|y(s)|)\,ds\Bigg] \\ \le &M\|\Theta\|_{B(E)}|y_0| \\ &+ M^2\|\Theta\|_{B(E)} \sum_{k=1}^p |c_k| \int_0^{t_k} p(s)\psi(|y(s)|)\,dt \\ &+ M\int_0^t p(s)\psi(|y(s)|)\,ds + bMM_1\widehat{M}. \end{aligned}$$

Let us take the right-hand side of the above inequality as $v(t)$. Then

$$\begin{aligned} v(0) &= M\|\Theta\|_{B(E)}|y_0| \\ &\quad + M^2\|\Theta\|_{B(E)}\sum_{k=1}^{p}|c_k|\int_0^{t_k} p(t)\psi(|y(s)|)\,dt + bMM_1\widehat{M} \\ |y(t)| &\le v(t) \quad (t\in[0,b]) \\ v'(t) &= Mp(t)\psi(|y(t)|) \quad (t\in[0,b]). \end{aligned}$$

Using the non-decreasing character of ψ we get

$$v'(t) \le Mp(t)\psi(v(t)) \qquad (t\in[0,b]).$$

This implies

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \le M\int_0^b p(s)ds < \int_{v(0)}^{\infty} \frac{du}{\psi(u)} \qquad (t\in[0,b]).$$

The above inequality implies that there exists a constant d such that $v(t) \le d$ $(t\in[0,b])$ and hence $\|y\|_\infty \le d$, where d depends only on the functions p and ψ. This shows that Ω is bounded.

Set $X = C([0,b],E)$. As a consequence of Lemma 3.2, we deduce that N has a fixed point, and therefore system (1) - (2) is non-locally controllable on $[0,b]$ ∎

3.1 The non-convex case. In this subsection we consider problem (1) -(2) with a nonconvex-valued right-hand side.

Let (X,d) be a metric space indused by the normed space $(X,|\cdot|)$. Consider the operator $H_d:\, P(X)\times P(X) \to \mathbb{R}_+\cup\{\infty\}$ given by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\}$$

where $d(A,b) = \inf_{a\in A} d(a,b)$ and $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(P_{b,cl}(X),H_d)$ is a metric space and $(P_{cl}(X),H_d)$ is a generalized (complete) metric space [17].

Definition 3.3. A multi-valued operator $N:\, X\to P_{cl}(X)$ is called

a) γ-*Lipschitz* if, for some $\gamma>0$, $H_d(N(x),N(y)) \le \gamma d(x,y)$ for each $x,y\in X$

b) *contraction* if it is γ-Lipschitz with $\gamma<1$

c) *having a fixed point* if there is $x\in X$ so that $x\in N(x)$.

The fixed point set of the multi-valued operator N will be denoted by $\mathrm{Fix}\,N$.

Our considerations here are based on the following fixed point theorem for contraction multi-valued operators given by Covitz and Nadler in 1970 [11] (see also Deimling [12: Theorem 11.1]).

Lemma 3.4. *Let (X, d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then* $\operatorname{Fix} N \neq \emptyset$.

We will need the following assumptions:

(A1) $F : [0,b] \times E \to P_{cp}(E)$ has the property that $F(\cdot, y) : [0,b] \to P_{cl}(E)$ is measurable for each $y \in E$.

(A2) $H_d\big(F(t,y), F(t,\overline{y})\big) \leq l(t)|y - \overline{y}|$ for a.e. $t \in [0,b]$ and $y, \overline{y} \in E$, where $l \in L^1([0,b], \mathbb{R}_+)$ and $d(0, F(t,0)) \leq l(t)$ for a.e. $t \in [0,b]$.

Remark 3.3. From assumption (A2),

$$\begin{aligned}\|F(t,y(t))\|_{\mathcal{P}} &\leq \|F(t,y(t)) - F(t,0)\|_{\mathcal{P}} + \|F(t,0)\|_{\mathcal{P}} \\ &\leq \ell(t)|y(t)| + \ell(t) \\ &\leq (1 + \sup\nolimits_{t\in[0,b]}|y(t)|)\ell(t).\end{aligned}$$

for each $t \in [0,b]$.

Now, we are able to state and prove our main result for this section.

Theorem 3.2. *Assume that hypotheses* (H1), (H3), (H5) *and* (A1) - (A2) *are satisfied. Then problem* (1) – (2) *is non-locally controllable on* $[0,b]$, *provided*

$$\begin{gathered} C_0 d\|\Theta\|_{B(E)}|y_0| \\ + \sum_{k=1}^{p} |c_k| M^2 \|\Theta\|_{B(E)} + \sum_{k=1}^{p} |c_k| M^2 \|\Theta\|_{B(E)} L(t_k) \\ + C_0 d M_1 b Q + M M_1 b K + M + M L(b) < 1 \end{gathered}$$

where $L(t) = \int_0^t \ell(s)\,ds$.

Proof. Using hypothesis (H5), for an arbirtary function $y(\cdot)$ define the control

$$\begin{aligned} u_y(t) = W^{-1}\Bigg[x_1 &- \sum_{k=1}^{p} c_k y(t_k) - U_y(b,0)\Theta y_0 \\ &+ \sum_{k=1}^{p} c_k U_y(b,0)\Theta \int_0^{t_k} U_y(t_k,s) g(s)\,ds - \int_0^b U_y(b,s) g(s)\,ds \Bigg](t) \end{aligned}$$

where

$$g \in S_{F,y} = \Big\{ g \in L^1([0,b], E) : g(t) \in F(t, y(t)) \ \text{ for a.e. } t \in [0,b] \Big\}.$$

We shall then show that, when using this control, the operator

$$N : C([0,b], E) \to \mathcal{P}(C([0,b], E))$$

defined by

$$N(y)=\left\{h\in C([0,b],E)\;\middle|\; h(t)=\begin{cases} U_y(t,0)\Theta y_0\\ -\sum_{k=1}^{p}c_kU_y(t,0)\Theta\\ \times\int_0^{t_k}U_y(t_k,s)g(s)\,ds\\ +\int_0^t U_y(t,s)[g(s)+(Bu_y)(s)]ds\end{cases}\quad (g\in S_{F,y})\right\}$$

has a fixed point. This fixed point is then a solution of problem(1) - (2).

Clearly, $y_1-\sum_{k=1}^{p}c_ky(t_k)\in N(y)(b)$. We shall show that N satisfies the assumptions of Lemma 3.4. The proof will be given in two steps:

`Step 1`: *$N(y)\in P_{cl}(C[0,b],E)$ for each $y\in C([0,b],E)$.* Indeed, let $(y_n)_{n\geq 0}\in N(y)$ be such that $y_n\to\tilde{y}$ in $C([0,b],E)$. Then $\tilde{y}\in C([0,b],E)$ and there exist $g_n\in S_{F,y}$ such that

$$\begin{aligned} y_n(t)=U_y(t,0)\Theta y_0&\\ -\sum_{k=1}^{p}c_kU_y(t,0)\Theta\int_0^{t_k}U_y(t_k,s)g_n(s)\,ds&\qquad (t\in[0,b]).\\ +\int_0^t U_y(t,s)\big[g_n(s)+(Bu_y)(s)\big]ds& \end{aligned}$$

From the fact that F has compact values and from assumption (A2) we may pass to a subsequence if necessary to get $g_n\to g$ in $L^1([0,b],E)$ and hence $g\in S_{F,y}$. Then

$$\begin{aligned} y_n(t)\to\tilde{y}(t)=U_y(t,0)\Theta y_0&\\ -\sum_{k=1}^{p}c_kU_y(t,0)\Theta\int_0^{t_k}U_y(t_k,s)g(s)\,ds&\qquad (t\in[0,b]).\\ +\int_0^t U_y(t,s)\big[g(s)+(Bu_y)(s)\big]ds& \end{aligned}$$

So $\tilde{y}\in N(y)$.

`Step 2`: *$H_d(N(y_1),N(y_2))\leq\gamma\|y_1-y_2\|_\infty$ for each $y_1,y_2\in C([0,b],E)$ where $\gamma<1$.* Indeed, let $y_1,y_2\in C([0,b],E)$ and $h_1\in N(y_1)$. Then there exists $g_1(t)\in F(t,y_1(t))$ such that

$$\begin{aligned} h_1(t)=U_{y_1}(t,0)\Theta y_0&\\ -\sum_{k=1}^{p}c_kU_{y_1}(t,0)\Theta\int_0^{t_k}U_{y_1}(t_k,s)g_1(s)\,ds&\qquad (t\in[0,b]).\\ +\int_0^t U_{y_1}(t,s)\big[g_1(s)+(Bu_{y_1})(s)\big]ds& \end{aligned}$$

From assumption (A2), it follows that

$$H_d\big(F(t,y_1(t)),F(t,y_2(t))\big)\leq l(t)|y_1(t)-y_2(t)|.$$

Hence, there is $w \in F(t, y_2(t))$ such that

$$|g_1(t) - w| \le l(t)|y_1(t) - y_2(t)| \qquad (t \in [0,b]).$$

Consider the operator $U : [0,b] \to \mathcal{P}(E)$ given by

$$U(t) = \Big\{ w \in E : |g_1(t) - w| \le l(t)|y_1(t) - y_2(t)| \Big\}.$$

Since the multi-valued operator $V(t) = U(t) \cap F(t, y_2(t))$ is measurable (see [10: Proposition III.4]), there exists $g_2(t)$ – a measurable selection for V. So, $g_2(t) \in F(t, y_2(t))$ and

$$|g_1(t) - g_2(t)| \le l(t)|y_1(t) - y_2(t)| \qquad (t \in [0,b]).$$

Let us define

$$\begin{aligned} h_2(t) = U_{y_2}(t,0)\Theta y_0 & \\ - \sum_{k=1}^{p} c_k U_{y_2}(t,0)\Theta \int_0^{t_k} U_{y_2}(t_k,s) g_2(s)\, ds & \qquad\qquad (t \in [0,b]). \\ + \int_0^t U_{y_2}(t,s)\big[g_2(s) + (Bu_{y_2})(s)\big] ds & \end{aligned}$$

From Step 4 of the proof of Theorem 3.1,

$$\|U_{y_1} - U_{y_2}\|_\infty \le C_0 \|A_{y_1} - A_{y_2}\|_\infty \qquad \big(y_1, y_2 \in C([0,b], E)\big)$$

where $C_0 = Mb\big(1 + br_1 \exp(br_1)\big)$ and r_1 is real constant. We set

$$C_1 = C_0 d\Big(1 + \sup_{w \in [0,b]} |y_1(w)|\Big) L(b) \qquad \text{where} \;\; L(t) = \int_0^t \ell(s)\, ds.$$

We observe that

$$\begin{aligned} &|u_{y_1}(s) - u_{y_2}(s)| \\ &\quad \le M_2 \Bigg[\sum_{k=1}^{p} |c_k|\, |y_1(t_k) - y_2(t_k)| + \|U_{y_2}(t,0) - U_{y_1}(t,0)\|_{B(E)} \|\Theta\|_{B(E)} |y_0| \\ &\qquad + \sum_{k=1}^{p} |c_k| M \|\Theta\|_{B(E)} \int_0^{t_k} \big|U_{y_2}(t_k,s) g_2(s) - U_{y_1}(t_k,s) g_1(s)\big| ds \\ &\qquad - \int_0^b \big|U_{y_2}(t,s) g_2(s) - U_{y_1}(t,s) g_1(s)\big| ds \Bigg] \end{aligned}$$

$$
\begin{aligned}
&\le M_2\Big[\sum_{k=1}^{p}|c_k|\,\|y_1-y_2\|_\infty + C_0 d\|\Theta\|_{B(E)}|y_0| \\
&\quad + \sum_{k=1}^{p}|c_k|M\|\Theta\|_{B(E)}\Big[C_1\|y_1-y_2\|_\infty + M\int_0^{t_k}|y_1(s)-y_2(s)|\ell(s)\,ds\Big] \\
&\quad + C_1\|y_1-y_2\|_\infty + M\int_0^{b}|y_1(s)-y_2(s)|\ell(s)\,ds\Big] \\
&\le M_2\Big[\sum_{k=1}^{p}|c_k| + C_0 d\|\Theta\|_{B(E)}|y_0| + \sum_{k=1}^{p}c_k C_1 M\|\Theta\|_{B(E)} \\
&\quad + \sum_{k=1}^{p}|c_k|L(t_k)M^2\|\Theta\|_{B(E)} + C_1 + ML(b)\Big]\|y_1-y_2\|_\infty \\
&= K\|y_1-y_2\|_\infty
\end{aligned}
$$

where

$$
\begin{aligned}
K = M_2\Big[&\sum_{k=1}^{p}|c_k| + C_0 d\|\Theta\|_{B(E)}|y_0| + \sum_{k=1}^{p}c_k C_1 M\|\Theta\|_{B(E)} \\
&+ \sum_{k=1}^{p}|c_k|L(t_k)M^2\|\Theta\|_{B(E)} + C_1 + ML(b)\Big]
\end{aligned}
$$

since

$$
\begin{aligned}
&\int_0^t \big|U_{y_1}(t,s)g_1(s) - U_{y_2}(t,s)g_2(s)\big|ds \\
&\qquad \le \int_0^t \big\|U_{y_1}(t,s) - U_{y_2}(t,s)\big\|_{B(E)}|g_1(s)|\,ds \\
&\qquad\quad + \int_0^t \|U_{y_2}(t,s)\|_{B(E)}|g_1(s)-g_2(s)|\,ds \\
&\qquad \le C_1\|y_1-y_2\|_\infty + M\int_0^t \ell(s)|y_1(s)-y_2(s)|\,ds
\end{aligned}
$$

and

$$
\begin{aligned}
|u_{y_1}| \le M_2\Big[&|x_1| + \sum_{k=1}^{p}c_k|y(t_k)| + M\|B\|_{B(E)}|y_0| \\
&+ \sum_{k=1}^{p}c_k M^2\Big(1+\sup_{w\in[0,b]}|y_1(w)|\Big)L(b) + M\Big(1+\sup_{w\in[0,b]}|y_1(w)|\Big)L(b)\Big] \\
=:\ & Q.
\end{aligned}
$$

Then

$$
\begin{aligned}
&|h_1(t)-h_2(t)| \\
&\quad \le \|U_{y_2}(t,0)-U_{y_1}(t,0)\|_{B(E)}\|\Theta\|_{B(E)}|y_0| \\
&\qquad + M\sum_{k=1}^{p}|c_k|\,\|\Theta\|_{B(E)}\int_0^{t_k}\big|U_{y_2}(t_k,s)g_2(s) - U_{y_1}(t_k,s)g_1(s)\big|ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t \left|U_{y_2}(t,s)Bu_{y_2}(s)-U_{y_1}(t,s)Bu_{y_1}(s)\right|ds\\
&+\int_0^t \left|U_{y_2}(t,s)g_2(s)-U_{y_1}(t,s)g_1(s)\right|ds\\
\le\ & C_0 d\|y_1-y_2\|_\infty\|\Theta\|_{B(E)}|y_0|+\sum_{k=1}^{p}|c_k|M\|\Theta\|_{B(E)}\\
&\times\left[M\|y_1-y_2\|_\infty+M\int_0^{t_k}\ell(s)|y_1(s)-y_2(s)|\,ds\right]\\
&+\left[C_0 dM_1bQ+MM_1bK\right]\|y_1-y_2\|_\infty\\
&+M\|y_1-y_2\|_\infty+M\int_0^t\ell(s)|y_1(s)-y_2(s)|\,ds\\
\le\ & \Bigg[C_0 d\|\Theta\|_{B(E)}|y_0|+\sum_{k=1}^{p}|c_k|M^2\|\Theta\|_{B(E)}\\
&+\sum_{k=1}^{p}|c_k|M^2\|\Theta\|_{B(E)}L(t_k)+C_0dM_1bQ+MM_1bK\\
&+M+ML(b)\Bigg]\|y_1-y_2\|_\infty.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\|h_1-h_2\|_\infty \le\ & \Bigg[C_0 d\|\Theta\|_{B(E)}|y_0|+\sum_{k=1}^{p}|c_k|M^2\|\Theta\|_{B(E)}\\
&+\sum_{k=1}^{p}|c_k|M^2\|\Theta\|_{B(E)}L(t_k)+C_0dM_1bQ+MM_1bK\\
&+M+ML(b)\Bigg]\|y_1-y_2\|_\infty.
\end{aligned}
$$

By an analogous relation, obtained by interchanging the roles of y_1 and y_2, it follows that

$$
\begin{aligned}
&H_d(N(y_1),N(y_2))\\
&\le \Bigg[C_0 d\|\Theta\|_{B(E)}|y_0|+\sum_{k=1}^{p}|c_k|M^2\|\Theta\|_{B(E)}\\
&\quad+\sum_{k=1}^{p}|c_k|M^2\|\Theta\|_{B(E)}L(t_k)+C_0dM_1bQ+MM_1bK\\
&\quad+M+ML(b)\Bigg]\|y_1-y_2\|_\infty.
\end{aligned}
$$

Then N is a contraction and thus, by Lemma 3.3, it has a fixed point y, and thus system (1) - (2) is non-locally controllable on $[0,b]$ ∎

By the help of Schaefer's fixed point theorem combined with the selection theorem of Bressan and Colombo for lower semicontinuous maps with decomposable values, we shall present next an existence result for problem (1) - (2).

Let $F : [0,b] \times E \to \mathcal{P}(E)$ be a multi-valued map with non-empty compact values. Assign to F the multi-valued operator

$$\mathcal{F} : C([0,b],E) \to \mathcal{P}(L^1([0,b],E))$$

by setting

$$\mathcal{F}(y) = \Big\{ w \in L^1([0,b],E) : w(t) \in F(t,y(t)) \text{ for a.e. } t \in [0,b] \Big\}.$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated with F.

Definition 3.4. Let $F : [0,b] \times E \to \mathcal{P}(E)$ be a multi-valued function with non-empty compact values. We say F is of *lower semi-continuous type* if its associated Niemytzki operator $\mathcal{F}$ is lower semi-continous and has non-empty closed and decomposable values.

Lemma 3.5 [7]. *Let Y be a separable metric space and $N : Y \to \mathcal{P}(L^1([0,b],E))$ a multi-valued operator which is lower semicontinuous and has non-empty closed and decomposable values. Then N has a continuous selection, i.e. there exists a continuous function* (*single-valued*) *$g : Y \to L^1([0,b],E)$ such that $g(y) \in N(y)$ for every $y \in Y$.*

For our third result let us introduce the following conditions:

(B1) $F : [0,b] \times E \to \mathcal{P}(E)$ is a non-empty compact-valued multi-valued map such that:

a) $(t,y) \mapsto F(t,y)$ is $\mathcal{L} \otimes \mathcal{B}$-measurable

b) $y \mapsto F(t,y)$ is lower semi-continuous for a.e. $t \in [0,b]$

(B2) For each $r > 0$, there exists a function $h_r \in L^1([0,b],\mathbb{R}^+)$ such that

$$\|F(t,y)\|_{\mathcal{P}} := \sup_{v \in F(t,y)} |v| \leq h_r(t)$$

for a.e. $t \in [0,b]$ and $y \in E$ with $|y| \leq r$.

In the proof of our following theorem we will need the next auxiliary result:

Lemma 3.6 [13]. *Let $F : [0,b] \times E \to \mathcal{P}(E)$ be a multi-valued map with non-empty, compact values. Assume conditions* (B1) - (B2) *hold. Then F is of lower semicontinuous type.*

Theorem 3.3 (Non-convex lower semicontinuous case). *Suppose that hapotheses* (H1), (H3) - (H5) *and conditions* (B1) - (B2) *hold. Assume also the following:*

(B3) *For each $t \in [0,b]$, the multi-valued map $F(t,\cdot) : E \to \mathcal{P}(E)$ maps bounded sets into relatively compact sets.*

Then initial value problem $(1) - (2)$ is non-locally controllable on $[0,b]$.

Proof. Conditions (B1) - (B2) imply, by Lemma 3.6, that F is of lower semi-continuous type. Then, from Lemma 3.5, there exists a continuous function $h: C([0,b],E) \to L^1([0,b],E)$ such that $h(y) \in \mathcal{F}(y)$ for all $y \in C([0,b],E)$. We consider the problem

$$y' - A(t,y)y - (Bu)(t) = h(y)(t) \quad (t \in [0,b]) \tag{5}$$

$$y(0) + \sum_{k=1}^{p} c_k y(t_k) = y_0. \tag{6}$$

We remark that if $y \in C([0,b],E)$ is a solution of problem (5) - (6), then y is also a solution to problem (1) - (2).

Transform problem (5) - (6) into a fixed point problem by considering the operator

$$N_1 : C([0,b],E) \to C([0,b],E)$$

defined by:

$$\begin{aligned} N_1(y)(t) = {} & U_y(t,0)\Theta y_0 \\ & - \sum_{k=1}^{p} c_k U_y(t,0)\Theta \int_0^{t_k} U_y(t_k,s)h(y)(s)\,ds \qquad (t \in [0,b]). \\ & + \int_0^t U_y(t,s)\big[h(y)(s) + (Bu)(s)\big]\,ds \end{aligned}$$

We shall show that N_1 is a completely continuous operator. For this, we will show that N_1

is continuous

maps bounded sets into bounded sets in $C([0,b],E)$

maps bounded sets into equicontinuous sets of $C([0,b],E)$

and, finally, that the set

$$\mathcal{E}(N_1) = \Big\{ y \in C([0,b],E) : \ y = \lambda N_1(y) \ \text{ for some } 0 < \lambda < 1 \Big\}$$

is bounded.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C([0,b],E)$. Then

$$\begin{aligned} & \big|N_1(y_n)(t) - N_1(y)(t)\big| \\ & \quad \le \|U_y(t,0) - U_{y_n}(t,0)\|_{B(E)} \|\Theta\|_{B(E)} |y_0| \\ & \qquad + \sum_{k=1}^{p} |c_k| M \|\Theta\|_{B(E)} \Big\{ \int_0^{t_k} \|U_{y_n}(t,s) - U_y(t,s)\|_{B(E)} |h(y_n)(s)| ds \end{aligned}$$

$$
\begin{aligned}
&+ \int_0^{t_k} \|U_y(t,s)\|_{B(E)} |h(y_n)(s) - h(y)(s)|\, ds \Big\} \\
&+ \int_0^{t} \|U_y(t,s)\|_{B(E)} |(Bu_{y_n})(s) - (Bu_y)(s)|\, ds \\
&+ \int_0^{t} \|U_y(t,s) - U_{y_n}(t,s)\|_{B(E)} |(Bu_{y_n})(s)|\, ds \\
&+ \int_0^{t} \|U_y(t,s)\|_{B(E)} |h(y_n)(s) - h(y)(s)|\, ds \\
&+ \int_0^{t} \|U_y(t,s) - U_{y_n}(t,s)\|_{B(E)} |h(y_n)(s)|\, ds \Big].
\end{aligned}
$$

Since the function h is continuous and U_y is continuous, by Step 4 of the proof of Theorem 2.3,

$$
\|N_1(y_n) - N_1(y)\|_\infty \to 0 \qquad (n \to \infty).
$$

The other steps are similar to the corresponding steps of the proof of Theorem 3.1. We omit the details ∎

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