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A Note on μ-Stable Surfaces with Prescribed Constant Mean Curvature

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Abstract. Using a generalized stability condition we give an upper bound of the principle curvatures of certain constant mean curvature surfaces. This implies a theorem of Bernstein type.

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1. Introduction

Let $B = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 < 1\}$ denote the open unit disc, $\overline{B} \subset \mathbb{R}^2$ its topological closure. We consider immersions $X \in C^{3+\alpha}(B, \mathbb{R}^3) \cap C^0(\overline{B}, \mathbb{R}^3)$ $(0 < \alpha < 1)$ of prescribed constant mean curvature $H(X) \equiv h_0 \in [0, +\infty)$. Introducing conformal parameters $(u, v) \in B$, such an immersion satisfies the nonlinear system

$$\left.\begin{aligned} \triangle X(u,v) &= 2h_0(X_u \wedge X_v) \\ |X_u|^2 = W &= |X_v|^2 \\ X_u \cdot X_v^t &= 0 \ \text{ in } B \end{aligned}\right\}.$$

Here $W = |X_u \wedge X_v| > 0$ denotes the surface element with the usual cross product $\wedge$ between two vectors in $\mathbb{R}^3$. Finally, by

$$N(u,v) = \frac{X_u(u,v) \wedge X_v(u,v)}{|X_u(u,v) \wedge X_v(u,v)|}$$

we denote the spherical mapping of the surface $X = X(u, v)$.

Definition 1.1. The immersion $X \in C^{3+\alpha}(B, \mathbb{R}^3) \cap C^0(\overline{B}, \mathbb{R}^3)$ $(0 < \alpha < 1)$ of constant mean curvature $h_0 \in [0, +\infty)$ is of *class* $\mathcal{C}_\mu(B, \mathbb{R}^3)$ if the following conditions are satisfied:

(i) X satisfies the above nonlinear system.

(ii) X has finite Dirichlet integral $\iint_B |\nabla X(u, v)|^2 dudv < +\infty$.

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(iii) X is *μ-stable* with a real number $\mu > 0$, namely

$$\iint_B |\nabla\varphi(u,v)|^2 dudv \geq \mu \iint_B (2h_0^2 - K)W\varphi(u,v)^2 dudv$$

for all test functions $\varphi \in C_0^\infty(B,\mathbb{R})$, where $K = K(u,v)$ is the Gaussian curvature of the immersion $X = X(u,v)$.

We are dealing with immersions $X \in \mathcal{C}_\mu(B,\mathbb{R}^3)$ $(\mu > 0)$. In Section 2 we control the modulus of projection of such an immersion. In Section 3 we show how to realize μ-stability if

$$\iint_B (2h_0^2 - K)W\,dudv < \omega_0$$

is satisfied with a real constant $\omega_0 \in (0,4\pi)$. Finally, we derive an upper bound for the principle curvatures of the immersions applying [11: heorem 1], and we get a result of Bernstein type.

2. Projectivity

We prove the following

Lemma 2.1. *Let $X \in \mathcal{C}_\mu(B,\mathbb{R}^3)$ with $\mu \in (1,2)$. For $w_0 \in B$ and real $\nu \in (0, 1-|w_0|)$ we assume*

$$\psi^*(u,v) := N(u,v)\cdot(0,0,1)^t > \frac{2}{\mu} - 1 \qquad \text{for all } (u,v) \in \partial B_\nu(w_0)$$

where $\partial B_\nu(w_0)$ is the boundary of $B_\nu(w_0) = \{w \in \mathbb{R}^2 : |w - w_0| < \nu\}$. Then the inequality

$$N(u,v)\cdot(0,0,1)^t \geq \frac{2}{\mu} - 1 \qquad \text{for all } (u,v) \in \overline{B}_\nu(w_0)$$

holds true. In particular, $X|_{\overline{B}_\nu(w_0)}$ can be represented as a graph.

Proof. In addition to $\psi^* = \psi^*(u,v)$ we define

$$\psi(u,v) = \psi^*(u,v) - \omega \qquad \big((u,v) \in \overline{B}_\nu(w_0),\ \omega = \tfrac{2}{\mu} - 1\big).$$

Because $\triangle N + 2qN = 0$ with $q = (2h_0^2 - K)W > 0$, we obtain

$$\triangle\psi^* = -2q\psi^* = -2q\psi - 2q\omega = \triangle\psi \quad \text{in } B_\nu(w_0).$$

Set

$$\psi^-(u,v) = \min(\psi(u,v), 0).$$

It remains to prove $\psi^- \equiv 0$. Since $\psi|_{\partial B} > 0$, there exists a $\varrho \in (0,\nu)$ with $\operatorname{supp}\psi^- \subset B_\varrho(w_0)$. That means $\psi^- \in H^{1,2}(B_\varrho(w_0),\mathbb{R}) \cap C_0^0(B_\varrho(w_0),\mathbb{R})$, where

$$\nabla\psi^- = \begin{cases} 0 & \text{if } \psi \geq 0 \\ \nabla\psi & \text{if } \psi < 0. \end{cases}$$

Partial integration yields (we set $B^* = B_\varrho(w_0)$ and omit $dudv$)

$$\iint_{B^*} |\nabla\psi^-|^2 = -\iint_{B^*} \psi^- \triangle\psi$$
$$= \mu \iint_{B^*} q|\psi^-|^2 + (2-\mu)\iint_{B^*} q|\psi^-|^2 + 2\omega \iint_{B^*} q\psi^-.$$

For $\chi \in C_0^\infty(B^*, \mathbb{R})$ and $\varepsilon \in \mathbb{R}$, we consider the admissible test function

$$\varphi(u,v) = \psi^-(u,v) + \varepsilon\chi(u,v), \qquad \big(\chi \in C_0^\infty(B^*, \mathbb{R})\big).$$

The μ-stability condition implies

$$\iint_{B^*} |\nabla\psi^-|^2 + 2\varepsilon \iint_{B^*} \nabla\psi^- \cdot \nabla\chi + \varepsilon^2 \iint_{B^*} |\nabla\chi|^2$$
$$\geq \mu \iint_{B^*} q|\psi^-|^2 + 2\mu\varepsilon \iint_{B^*} q\psi^-\chi + \mu\varepsilon^2 \iint_{B^*} q\chi^2.$$

Therefore we have

$$2\varepsilon \iint_{B^*} \nabla\psi^- \cdot \nabla\chi + \varepsilon^2 \iint_{B^*} |\nabla\chi|^2 dudv$$
$$\geq (\mu-2) \iint_{B^*} q|\psi^-|\,|\psi^-| + 2\omega \iint_{B^*} q|\psi^-|$$
$$+ 2\mu\varepsilon \iint_{B^*} q\psi^-\chi + \mu\varepsilon^2 \iint_{B^*} q\chi^2.$$

Because $-1-\omega \leq \psi^- \leq 0$ and $\mu - 2 \leq 0$, we deduce

$$2\varepsilon \iint_{B^*} \nabla\psi^- \cdot \nabla\chi + \varepsilon^2 \iint_{B^*} |\nabla\chi|^2 dudv$$
$$\geq (1+\omega)(\mu-2) \iint_{B^*} q|\psi^-| + 2\omega \iint_{B^*} q|\psi^-|$$
$$+ 2\mu\varepsilon \iint_{B^*} q\psi^-\chi + \mu\varepsilon^2 \iint_{B^*} q\chi^2$$
$$= 2\mu\varepsilon \iint_{B^*} q\psi^-\chi + \mu\varepsilon^2 \iint_{B^*} q\chi^2$$

taking $(1+\omega)(\mu-2) + 2\omega = 0$ into account (note $\omega = 2\mu^{-1} - 1$). Therefore

$$2\varepsilon \iint_{B^*} (\nabla\psi^- \cdot \nabla\chi - \mu q\psi^-\chi + \varepsilon^2 \iint_{B^*} (|\nabla\chi|^2 - \mu q\chi^2) \geq 0$$

holds true for all $\varepsilon \in \mathbb{R}$, and we find

$$\iint_{B^*} (\nabla\psi^- \cdot \nabla\chi - \mu q\psi^-\chi) = 0$$

for all $\chi \in C_0^\infty(B^*, \mathbb{R})$. Because $X = X(u,v)$ is real analytic, the same holds true for $q = q(u,v)$. By the Lemma of Weyl, $\psi^- = \psi^-(u,v)$ for $(u,v) \in B^*$ is real analytic, and $\psi^-(u,v) \equiv 0$ because it is already zero on a strip of non-vanishing measure ∎

Remark 2.2. This result is motivated by [10: Hilfssatz 6] where the case $\mu = 2$ is investigated. Therefore, the result is also true for $\mu \geq 2$.

Remark 2.3. To realize a modulus of continuity for the spherical mapping $N = N(u,v)$ of the immersion, we decided on a well known regularity result of Weyl (cp. [8: Section 4.2]). Furthermore, in [6] a maximum principle is established which refers to the positivity of the first eigenvalue of an elliptic operator. This maximum principle could be invested for an alternative proof of the above lemma in the case of strictly stable immersions. Anyway, it seems that we can not abandon any suitable stability condition.

3. A result of Ruchert type

Let $\triangle^*$ denote the Laplace-Beltrami operator on $S^2 = \{Z \in \mathbb{R}^3 : |Z| = 1\}$. The proof of the next result follows the lines of [9].

Proposition 3.1. *Let $X \in C^{3+\alpha}(B,\mathbb{R}^3) \cap C^0(\overline{B},\mathbb{R}^3)$ $(0 < \alpha < 1)$ be an immersion of constant mean curvature $h_0 \in [0,+\infty)$, given in conformal parameters $(u,v) \in B$. We assume that*

$$Q := \iint_B (2h_0^2 - K)W \, dudv < \omega_0 \in (0, +4\pi)$$

holds true. Let $S^2_\omega \subset S^2$ be a spherical cap with $\operatorname{Area} S^2_\omega = \omega_0$, and let $\mu > 0$ be the first eigenvalue of $\triangle^*$ with respect to the Dirichlet problem

$$\left.\begin{aligned} \triangle^*\psi + \lambda\psi &= 0 \qquad \text{on } S^2_\omega \\ \psi &= 0 \qquad \text{on } \partial S^2_\omega \end{aligned}\right\}.$$

Then the surface is μ-stable with this number $\mu > 0$.

Proof. The Gaussian curvature $K = K(u,v)$ of the surface satisfies

$$K(u,v) = -\frac{1}{W}\,\triangle(\log\sqrt{W}).$$

Set $\chi = 2h_0^2 - K$. For the Gaussian curvature $\widehat{K} = \widehat{K}(u,v)$ with respect to the metric

$$(\widehat{g}_{ij})_{i,j=1,2} \qquad \text{with } \begin{cases} \widehat{g}_{11} = \chi W = \widehat{g}_{22} \\ \widehat{g}_{12} = 0 = \widehat{g}_{21} \end{cases}$$

one finds

$$\chi\widehat{K} = K - \frac{1}{2W}\,\triangle(\log\chi).$$

As shown in [9: Lemma 2.3], $\widehat{K} \leq 1$ in B. Let $\widehat{\triangle}$ denote the Laplacian with respect to $(\widehat{g}_{ij})_{i,j=1,2}$, and $\widehat{\lambda}_1 > 0$ means the first eigenvalue of the problem

$$\left.\begin{aligned} \widehat{\triangle}\varphi + \lambda\varphi &= 0 \qquad \text{in } B \\ \varphi &= 0 \qquad \text{on } \partial B \end{aligned}\right\}.$$

Let $S^2_\omega \subset S^2$ with $\omega \in (0, 4\pi)$ be a spherical cap with Area $S^2_\omega = Q$, and let $\lambda_1^* > 0$ be the first eigenvalue of the spherical Laplacian with respect to

$$\left.\begin{aligned} \triangle^*\varphi^* + \lambda^*\varphi^* = 0 \quad & \text{in } \ S^2_\omega \\ \varphi^* = 0 \quad & \text{on } \ \partial S^2_\omega \end{aligned}\right\}.$$

Since $\widehat{K} \le 1$, [1: Propositions 3.3 and 3.16] yield $\lambda_1^* \le \widehat{\lambda}_1$. By assumption, S^2_ω is contained in a spherical cap with first eigenvalue $\mu > 0$. The monotonicity of the first eigenvalue yields $\mu < \lambda_1^*$, and therefore

$$\mu < \lambda_1^* \le \widehat{\lambda}_1 \le \frac{\iint_B |\nabla\varphi|^2 dudv}{\iint_B \varphi^2(2h_0^2 - K)W\, dudv}$$

for all $\varphi \in H^{1,2}(B, \mathbb{R}) \setminus \{0\}$ with $\varphi|_{\partial B} = 0$. The statement follows ∎

Remark 3.2. In [11: Section 5] a curvature estimate for constant mean curvature surfaces is established under the integral condition

$$\iint_B (h_0^2 - K)W\, dudv < 4\pi.$$

The method is based on a comparison surface of Bonnet type and an isoperimetric inequality.

4. An a priori bound for the principle curvatures

Let $X \in \mathcal{C}_\mu(B, \mathbb{R}^3)$ $(\mu > 0)$ represent a geodesic disc $\mathcal{B}_r(X_0)$ of radius $r > 0$ and of center $X_0 = X(0,0)$. In geodesic polar coordinates X takes the form $Z = Z(\varrho, \varphi)$: $[0, r] \times [0, 2\pi] \to \mathbb{R}^3$. For its line element we have

$$\begin{aligned} ds_P^2 &= |Z_\varrho|^2 d\varrho + 2Z\varrho \cdot Z\varphi\, d\varrho d\varphi + |Z_\varphi|^2\, d\varphi \\ &= d\varrho^2 + P(\varrho, \varphi)\, d\varphi^2. \end{aligned}$$

The proofs of the following results can be extracted from the proof of [11: Theorem 3]. We only give the crucial ideas.

Lemma 4.1. *Let $X \in \mathcal{C}_\mu(B, \mathbb{R}^3)$ $(\mu > \frac{1}{2})$ represent a geodesic disc $\mathcal{B}_r(X_0)$. Then we have the area estimate*

$$\mathcal{A}(Z) := \int_0^r \int_0^{2\pi} \sqrt{P(\varrho, \varphi)}\, d\varrho d\varphi \le \frac{2\pi\mu}{2\mu - 1}\, r^2.$$

Proof. Using geodesic polar coordinates $(\varrho, \varphi) \in [0, r] \times [0, 2\pi]$, insert the special test function $\Phi(\varrho) := 1 - r^{-1}\varrho$ $(0 \le \varrho \le r)$ into the μ-stability condition from Definition 1.1. The result follows by partial integration (cp. [7: Proof of Theorem 1]) ∎

Lemma 4.2. *Let $X \in \mathcal{C}_\mu(B, \mathbb{R}^3)$ $(\mu > 0)$. Then for any $\nu \in (0,1)$ we have*

$$\iint_{|w| \le 1-\nu} |\nabla N(u,v)|^2 dudv \le \frac{8\pi}{\mu\nu^2}.$$

Proof. The result follows by inserting a test function $\varphi \in C_0^\infty(B, \mathbb{R})$ with the properties

$$\begin{aligned} \Phi(u,v) &\equiv 1 \qquad \text{in } B_{1-\nu}(0,0) \\ |\nabla\varphi(u,v)| &\le \tfrac{2}{\nu} \qquad \text{in } B \end{aligned}$$

into the μ-stability condition ∎

Lemmata 2.1, 4.1 and 4.2 enable us to apply [11: Theorem 1]. We obtain our main result

Theorem 4.3. *Let $X \in \mathcal{C}_\mu(B, \mathbb{R}^3)$ $(\mu > 1)$ and let it represent a geodesic disc $\mathcal{B}_r(X_0)$. Then there exists a constant $\Theta = \Theta(h_0 r, \mu)$ such that*

$$\kappa_1(0,0)^2 + \kappa_2(0,0)^2 \le \frac{1}{r^2}\,\Theta(h_0 r, \mu)$$

holds true for the principle curvatures κ_1 and κ_2 of the immersion.

Proof. We only present the idea of the proof. For details, we refer to [11].

1. Let $\Gamma(B)$ denote all continuous curves $\gamma : [0,1] \to B$ with $\gamma(0) = (0,0)$ and $\gamma(1) \in \partial B$. Then

$$\inf_{\gamma \in \Gamma(B)} \int_0^1 \left| \frac{d}{dt} X \circ \gamma(t) \right| dt \ge r.$$

Now, set $d = 2\pi\mu(2\mu - 1)^{-1}$. Lemma 4.1 implies $\mathcal{A}(Z) \le dr^2$. Then, by [11: Lemma 1], there is a point $w^* \in B$ with $|w^*| \le 1 - \nu_0$ and $\nu_0 = e^{-4\pi d}$ such that

$$\frac{W(w^*)}{r^2} \ge c_1(\mu) > 0$$

with an a priori constant $c_1 = c_1(\mu)$.

2. The Courant-Lebesgue lemma applied to the estimate of Lemma 4.2 and the projectivity result of Lemma 2.1 ensure the existence of a constant $c_2 = c_2(h_0 r, \mu)$ with the property

$$|r^{-1}X_{uu}| + |r^{-1}X_{vv}| + |r^{-1}X_{uv}| \le c_2(h_0 r, \mu) \qquad \text{in } \; B_\nu(0,0)$$

where $\nu \in (0, \nu_0)$ is choosen sufficiently small due to Lemma 2.1 and Lemma 4.2.

3. The iterative scheme from the proof in [11] yields an a priori constant $c_3 = c_3(h_0 r, \mu)$ such that

$$\frac{W(0,0)}{r^2} \ge c_3(h_0 r, \mu) > 0.$$

4. Investing

$$\kappa_1^2 + \kappa_2^2 = 4h_0^2 - 2K \le \frac{1}{r^2}\{4(h_0 r)^2 + 2r^2|K|\}$$

for the principle curvatures κ_1 band κ_2, as well as

$$K = \frac{(N \cdot X^t_{uu})(N \cdot X^t_{vv}) - (N \cdot X^t_{uv})^2}{W^2}$$

for the Gaussian curvature K, we arrive at (set $Y = r^{-1}X$)

$$\begin{aligned}\kappa_1(0,0)^2 + \kappa_2(0,0)^2 &\le \frac{1}{r^2}\left((4h_0r)^2 + 2\,\frac{|Y_{uu}|\,|Y_{vv}| + |Y_{uv}|^2}{(Wr^{-2})^2}\bigg|_{(0,0)}\right)\\ &\le \frac{1}{r^2}\left(4(h_0r)^2 + \frac{4c_2(h_0r,\mu)^2}{c_3(h_0r,\mu)^2}\right)\\ &=: \frac{1}{r^2}\,\Theta(h_0r,\mu).\end{aligned}$$

This proves the statement ∎

In the case $h_0 = 0$ we immediately obtain the Bernstein-type result

Corollary 4.4. *Let the regular, complete and μ-stable minimal surface $X : \mathbb{R}^2 \to \mathbb{R}^3$ be given and let $\mu > 1$. Then the surface represents a plane in $\mathbb{R}^3$.*

Proof. We have $K \le 0$ for a minimal surface, and therefore $X : \mathbb{R}^2 \to \mathbb{R}^3$ represents a geodesic disc for all $r > 0$ by a theorem of Hadamard. We can apply our main result for $r \to \infty$ ∎

Remark 4.5. This result is proved in [4] even for $\mu \ge 1$. The authors develop methods of complex analysis to investigate complete metrics in $\mathbb{C}$. Compared to our methods, they do not derive Bernstein-type results from curvature estimates.

Remark 4.6. In [5] an adequate μ-stability condition is applied to immersions of minimal surface type. Corollary 4.4 is contained in the Bernstein results of that article with the modification $\mu > \frac{1}{2}$. The smallest possible value of the constant $\mu > 0$ is not known to the author (cp. also the introductory remarks in [4]).

Remark 4.7. Immersions of minimal surface type were originally investigated in [2, 3], where fundamental geometric and analytic properties as well as questions of existence are discussed.

Remark 4.8. In [12] immersions that are stationary for general parametric functionals are investigated, and an a priori bound for the principle curvatures is derived. However, it seems to the author that no Bernstein-type result can be extracted for immersions of minimal surface type. It is an open question how to expand the methods presented here to attack these general functionals.

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