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Michael Selection Problem in Hyperconvex Metric spaces

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Abstract. In the present paper, the Michael selection problem is researched in hyperconvex metric spaces. Our results show that the answer is "yes" for hyperconvex metric spaces and that the lower semicontinuity of the multi-valued mapping can be weakened. Moreover, as an application of our selection theorem, a fixed point theorem for locally-uniform weak lower semicontinuous multi-valued mappings is given.

Keywords: *Hyperconvex metric spaces, lower semicontinuity, sub-admissible sets, quasi-lower semicontinuity, locally-uniform weak lower semicontinuity*

AMS subject classification: 47H04, 47H10, 52A99, 54H25, 58E35

1. Introduction and preliminaries

In 1956, Michael [7] first researched the continuous selection problem and obtained the following now well known theorem:

Theorem 1.1. *Let Y be a Banach space and X a paracompact topological space. If $F : X \to 2^Y$ is a lower semicontinuous set-valued mapping with non-empty closed convex values, then F has a continuous selection, i.e. there exists a continuous mapping $f : X \to Y$ such that $f(x) \in F(x)$ for all $x \in X$.*

The following open problem is due also to E. Michael and appeared in [8].

Michael selection problem. *Does Theorem* 1.1 *remain true if Y is a non-locally convex complete, metrizable topological linear space ?*

Up to the present, this question is still open.

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We would like to point that in 1989 and 2000 Sine [10] and Khamsi et al. [6] researched the non-expansive selection problem for non-expansive multi-valued mappings in hyperconvex metric spaces, respectively. Motivated by their idea, in the present paper our main purpose is to research the above Michael selection problem in hyperconvex metric spaces. Our result shows that the answer is "yes" if Y is a hyperconvex metric space and that the lower semicontinuity of the mapping F can be weakened. Moreover, as an application of our selection, a fixed point theorem for locally-uniform weak lower semicontinuous multi-valued mappings is given.

To begin with we explain the notion of a hyperconvex metric space introduced by Aronszajn and Panitchpakdi [1] and related concepts on hyperconvex metric spaces.

Definition 1.1. A metric space (M, d) is called *hyperconvex* if, for any collections of points $\{x_\alpha\}_{\alpha\in I} \subset M$ and non-negative reals $\{r_\alpha\}_{\alpha\in I}$ with $d(x_\alpha, x_\beta) \le r_\alpha + r_\beta$ for all $\alpha, \beta \in I$,

$$\bigcap_{\alpha\in I} \overline{B}(x_\alpha, r_\alpha) \neq \emptyset$$

where

$$B(x,r) = \big\{y \in M :\, d(x,y) < r\big\}$$
$$\overline{B}(x,r) = \big\{y \in M :\, d(x,y) \le r\big\}.$$

Definition 1.2. Let (M, d) be a metric space and $A \subset M$ a non-empty subset. Then we set

$$\mathrm{co}(A) = \cap\big\{B : B \ \text{ is a closed ball such that } A \subset B\big\}$$

and

$$A + r = \cup_{a\in A}\overline{B}(a, r) \qquad (r \ge 0).$$

If A is an intersection of some closed balls, we will say A is an *admissible subset* of M. A is called *sub-admissible* if, for each finite subset D of A, $\mathrm{co}(D) \subset A$.

Sine [11] pointed that if A is an admissible subset of a hyperconvex metric space, then so is $A + r$.

Let X be a topological space. We denote by 2^X the family of all susets of X. If $A \subset X$, we shall denote by $\mathrm{cl}(A)$ the closure of A.

Definition 1.3. Let X, Y be two topological spaces and $T :\, X \to 2^Y$ a multi-valued mapping.

(1) T is called *lower semicontinuous* if for each $x \in X$ and each open set $V \subset Y$ with $T(x) \bigcap V \neq \emptyset$ there exists an open neighborhood U of x such that $T(z) \bigcap V \neq \emptyset$ for each $z \in U$.

(2) If (Y, d) is a metric space, then:

(a) T is called *quasi-lower semicontinuous* if for each $x \in X$ and each $\varepsilon > 0$ there exists a point $y \in T(x)$ and a neighborhood $U(x)$ of x such that, for each $z \in U(x)$, $d(y, T(z)) < \varepsilon$.

(b) T is called *locally-uniformly weak lower semicontinuous* if T is quasi-lower semicontinuous and for each $x \in X$ there exists an open neighborhood $N(x)$ of x such that, for each $\varepsilon > 0$ and each $y \in Y$, there is a $\delta > 0$ with the following property:

$$\forall\, z \in N(x), \exists\, r > 0 : \quad \emptyset \neq B(y,r) \cap T_\delta(z) \subset T_\mu(z) + \varepsilon \quad (\mu > 0)$$

where

$$T_\eta(z) = \Big\{ y \in Y : \exists\, U \in \mathcal{N}(z) \text{ such that}, \forall a \in U, d(y, T(a)) < \eta \Big\}$$

where $\mathcal{N}(z)$ is the family of all open neighborhoods of z.

Remark 1.1. As Y is a normed linear space, the above concepts were given by Deutsch and Kenderov [2] and by Przeslawski and Rybinski [9], respectively.

Remark 1.2. Obviously, if T is a lower semicontinuous multi-valued mapping with non-empty values, then T must be locally-uniformly weak lower semicontinuous.

2. Main results

To begin, we give two proximate selection theorems.

Theorem 2.1. *Let X be a paracompact topological space, (M, d) a hyperconvex metric space and Y a non-empty sub-admissible subset of M. Further, let $T : X \to 2^Y$ be a multi-valued mapping such that:*

(i) *For each $x \in X$, $T(x)$ is a non-empty sub-admissible subset of M.*

(ii) *T is quasi-lower semicontinuous.*

Then for each $\varepsilon > 0$ there exists a continuous mapping $f : X \to M$ such that $d(f(x), T(x)) < \varepsilon$ for all $x \in X$.

Proof. Since (M, d) is a hyperconvex metric space, by [5: Proposition 1/Conclusion 1] there exists an index set I and an isometric embedding from M into $l^\infty(I)$. We will identify M with the isometric embedding image set in $l^\infty(I)$. Since hyperconvexity is preserved by isometry, by [5: Proposition 1/Conclusion 4] there exists a non-expansive retract $r : l^\infty(I) \to M$.

For each $\varepsilon > 0$, by condition (ii) we know that, for each $x \in X$, there exist a point $y(x) \in T(x)$ and an open neighborhood $N(x)$ of x such that $d(y(x), T(z)) < \varepsilon$ for all $z \in N(x)$. Since X is paracompact, there exists a local finite open refinement $\mathcal{R} = \{U_\alpha\}_{\alpha \in D}$ of the family $\{N(x)\}_{x \in X}$. Hence for each $\alpha \in D$, there exists a point $x_\alpha \in X$ such that $U_\alpha \subset N(x_\alpha)$. Consequently, $d(y(x_\alpha), T(z)) < \varepsilon$ for all $z \in U_\alpha$. Let $\{f_\alpha\}_{\alpha \in D}$ be a partition of the continuous unity corresponding to the covering $\mathcal{R}$ of X. We can thus define a mapping

$$g :\, X \to l^\infty(I), \quad g(x) = \sum_{\alpha \in D} f_\alpha(x) y(x_\alpha) \ \ (x \in X).$$

Since $\mathcal{R}$ is a local finite open covering of X and $\{f_\alpha\}_{\alpha \in D}$ is a partition of the continuous unity corresponding to $\mathcal{R}$, g is a well-defined continuous mapping. Now let $f = r \circ g$. Then $f :\, X \to M$ is continuous. For each $x \in X$, let

$$\sigma(x, \mathcal{R}) = \big\{U \in \mathcal{R} :\, x \in U\big\}.$$

Then $\sigma(x, \mathcal{R})$ is finite, and hence, let

$$\sigma(x, \mathcal{R}) = \big\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\big\}.$$

Consequently,

$$\begin{aligned} f(x) = r[g(x)] &= r\left[\sum_{k=1}^{n} f_{\alpha_k}(x) y(x_{\alpha_k})\right] \\ &\in r\big(\mathrm{conv}\{y(x_{\alpha_1}), \ldots, y(x_{\alpha_n})\}\big) \\ &\subset \mathrm{co}\big(\{y(x_{\alpha_1}), \ldots, y(x_{\alpha_n})\}\big) \end{aligned}$$

where $\mathrm{conv}\{y(x_{\alpha_1}), \ldots, y(x_{\alpha_n})\}$ is the convex hull of $\{y(x_{\alpha_1}), \ldots, y(x_{\alpha_n})\}$ in $l^\infty(I)$. For each $i \in \{1, 2, \ldots, n\}$, since $d(y(x_{\alpha_i}), T(x)) < \varepsilon$, there exists a point $z_i \in T(x)$ such that $d(y(x_{\alpha_i}), z_i) < \varepsilon$. Since again $T(x)$ is a sub-admissible subset of M, $\mathrm{co}(\{z_1, z_2, \ldots, z_n\}) \subset T(x)$. Let $z(x) = r\big[\sum_{k=1}^{n} f_{\alpha_k}(x) z_k\big]$. Then $z(x) \in T(x)$ and

$$d(f(x), z(x)) \le \sum_{k=1}^{n} f_{\alpha_k}(x)\, d(y(x_{\alpha_k}), z_k) < \varepsilon.$$

Hence $d(f(x), T(x)) < \varepsilon$. This completes the proof ∎

From the proof of Theorem 2.1 we know that if X is compact, then we have the following theorem.

Theorem 2.2. *Let X be a compact topological space, (M,d) a hyperconvex metric space and Y a non-empty sub-admissible subset of M. Further, let $T: X \to 2^Y$ be a multi-valued mapping such that:*

(i) *For each $x \in X$, $T(x)$ is a non-empty sub-admissible subset of M.*

(ii) *T is quasi-lower semicontinuous.*

Then for each $\varepsilon > 0$ there exists a continuous mapping $f = r \circ g$ such that $d(f(x), T(x)) < \varepsilon$ for all $x \in X$, where $r: l^\infty(I) \to M$ is a non-expansive retract, I is some index set such that M is isometrically embedded into $l^\infty(I)$ and $g: X \to l^\infty(I)$ is continuous such that $g(X)$ is contained in a polytope $P = \operatorname{conv}\{y_1, \dots, y_n\}$ of $l^\infty(I)$, $\{y_1, \dots, y_n\} \subset T(X)$.

Theorem 2.3. *Let X be a paracompact topological space, (M,d) a hyperconvex metric space and Y a non-empty sub-admissible subset of M. Further, let $T: X \to 2^Y$ be a multi-valued mapping such that:*

(i) *For each $x \in X$, $T(x)$ is a non-empty closed sub-admissible subset of M.*

(ii) *T is lower semicontinuous.*

Then there exists a continuous mapping $f: X \to M$ such that $f(x) \in T(x)$ for all $x \in X$.

Proof. By Theorem 2.1 there exists a continuous mapping $f_1: X \to Y$ such that

$$d(f_1(x), T(x)) < \frac{1}{2} \qquad (x \in X).$$

Let

$$T_1(x) = T(x) \cap B\Big(f_1(x), \frac{1}{2}\Big) \qquad (x \in X).$$

Then, for all $x \in X$, $T_1(x)$ is a non-empty sub-admissible subset of M. Moreover, by [3: p. 348/Lemma 1] we know that $T_1: X \to 2^Y$ is lower semicontinuous. Consequently, by Theorem 2.1 there exists a continuous mapping $f_2: X \to Y$ such that

$$d(f_2(x), T_1(x)) < \frac{1}{2^2} \qquad (x \in X).$$

Hence

$$\left.\begin{aligned} d(f_2(x), T(x)) &\le d(f_2(x), T_1(x)) < \frac{1}{2^2} \\ d(f_1(x), f_2(x)) &< \frac{1}{2} + \frac{1}{2^2} \end{aligned}\right\} \qquad (x \in X).$$

Now assume that $f_1, f_2, \dots, f_n: X \to Y$ has been found such that, for each $x \in X$,

$$\begin{aligned} d(f_k(x), f_{k+1}(x)) &< \frac{1}{2^k} + \frac{1}{2^{k+1}} \quad (k = 1, 2, \dots, n-1) \\ d(f_k(x), T(x)) &< \frac{1}{2^k} \qquad\qquad\quad (k = 1, 2, \dots, n). \end{aligned}$$

Let
$$T_n(x) = T(x) \cap B\Big(f_n(x), \frac{1}{2^n}\Big) \qquad (x \in X).$$
Then by Theorem 2.1 there exists a continuous mapping $f_{n+1} : X \to Y$ such that
$$d(f_{n+1}(x), T_n(x)) < \frac{1}{2^{n+1}} \qquad (x \in X).$$
Hence
$$\left.\begin{aligned} d(f_{n+1}(x), T(x)) &\le d(f_{n+1}(x), T_n(x)) < \frac{1}{2^{n+1}} \\ d(f_n(x), f_{n+1}(x)) &< \frac{1}{2^n} + \frac{1}{2^{n+1}} \end{aligned}\right\} \qquad (x \in X).$$
Consequently, we can find a sequence $\{f_n\}_{n\ge 1}$ of continuous functions $f_n : X \to Y$ such that
$$d(f_n(x), f_{n+1}(x)) < \frac{1}{2^n} + \frac{1}{2^{n+1}} \tag{2.1}$$
$$d(f_n(x), T(x)) < \frac{1}{2^n} \tag{2.2}$$
for all $x \in X$ and $n \ge 1$. By (2.1) we know that $\{f_n(x)\}_{n\ge 1}$ is a uniformly Cauchy sequence in M. Since each hyperconvex space is complete, there is a mapping $f : X \to M$ such that $f_n(x) \to f(x)$ for all $x \in X$. Since again f_n is continuous, the mapping $f : X \to M$ is continuous, too. Moreover, by (2.2) and the closeness of $T(x)$ we know that $f(x) \in T(x)$ for all $x \in X$. This completes the proof ∎

Theorem 2.4. *Let X be a paracompact topological space, (M, d) a hyperconvex metric space and Y a non-empty sub-admissible subset of M. Further, let $T : X \to 2^Y$ be a multi-valued mapping such that:*

(i) *For each $x \in X$, $T(x)$ is a non-empty closed sub-admissible subset of M.*

(ii) *T is locally-uniform weak lower semicontinuous.*

Then there exists a continuous mapping $f : X \to M$ such that $f(x) \in T(x)$ for all $x \in X$.

Proof. For each $r > 0$ and each $x \in X$, we denote
$$T_r(x) = \Big\{y \in Y : \exists\, U \in \mathcal{N}(x) \text{ such that, } \forall\, a \in U,\ d(y, T(a)) < r\Big\}$$
and
$$T_0(x) = \cap_{r>0} T_r(x)$$

where $\mathcal{N}(x)$ is the family of all open neighborhoods of x. We first prove the following several Facts 1 - 6:

Fact 1: If $0 < r_1 < r_2$, then $\mathrm{cl}[T_{r_1}(x)] \subset T_{r_2}(x)$ for all $x \in X$, where $\mathrm{cl}[T_{r_1}(x)]$ is the closure of $F_{r_1}(x)$. Indeed, for each $x \in X$ and each $y \in \mathrm{cl}[T_{r_1}(x)]$ there is a sequence $\{y_n\}_{n\geq 1}$ in $T_{r_1}(x)$ such that $y_n \to y$. Hence there exists an n_0 such that $d(y, y_{n_0}) < r_2 - r_1$. Since $y_{n_0} \in T_{r_1}(x)$, there exists an open neighborhood $N(x)$ of x such that $d(y_{n_0}, T(z)) < r_1$ for all $z \in N(x)$. Hence

$$d(y, T(z)) \leq d(y, y_{n_0}) + d(y_{n_0}, T(z)) < r_2 \qquad (z \in N(x)).$$

This shows $y \in T_{r_2}(x)$. Hence $\mathrm{cl}[T_{r_1}(x)] \subset T_{r_2}(x)$.

Fact 2: $T_0(x) = \bigcap_{\varepsilon>0} \mathrm{cl}[T_\varepsilon(x)]$ for all $x \in X$. Indeed, for each $y \in \bigcap_{\varepsilon>0} \mathrm{cl}[T_\varepsilon(x)]$, if $y \notin T_0(x)$, then there is a $r > 0$ such that $y \notin T_r(x)$. Consequently, by Fact 1, $y \notin \mathrm{cl}[T_\varepsilon(x)]$ as $0 < \varepsilon < r$. This contradicts that $y \in \bigcap_{\varepsilon>0} \mathrm{cl}[T_\varepsilon(x)]$.

Fact 3: $T_0(x)$ is sub-admissible for all $x \in X$. Indeed, for each $r > 0$ and each $x \in X$, if $A = \{a_1, a_2, \ldots, a_n\}$ is a finite subset of $T_r(x)$, then there exists an open neighborhood $N(x)$ of x such that, for each $z \in N(x)$, $d(a_i, T(z)) < r$ for all $i \in \{1, 2, \ldots, n\}$. Consequently, for each $z \in N(x)$ there exists a finite subset $B = \{b_1, b_2, \ldots, b_n\}$ of $T(z)$ and a $0 < \varepsilon < r$ such that $d(a_i, b_i) \leq \varepsilon$ for all $i \in \{1, 2, \ldots, n\}$. Hence $A \subset \mathrm{co}(B) + \varepsilon$, and hence $\mathrm{co}(A) \subset \mathrm{co}(B) + \varepsilon \subset T(z) + \varepsilon$. Therefore, $\mathrm{co}(A) \subset T_r(x)$. This shows that $T_r(x)$ is sub-admissible, and hence $T_0(x)$ is sub-admissible.

Fact 4: For each $x_0 \in X$, $T_0(x_0) \neq \emptyset$ and there exists an open neighborhood $N(x_0)$ of x_0 with the property that, for each $\varepsilon > 0$ and each $y \in Y$, there is a $\delta > 0$ such that $d(y, T_0(x)) < d(y, T_\delta(x)) + 2\varepsilon$ for all $x \in N(x_0)$. Indeed, since T is locally-uniform weak lower semicontinuous, for each fixed $x_0 \in X$ there exists an open neighborhood $N(x_0)$ of x_0 with the property that, for each $\varepsilon > 0$ and each $y \in X$, there is a $\delta > 0$ such that for each $x \in N(x_0)$ there exists a $r > 0$ satisfying

$$\emptyset \neq B(y, r) \cap T_\delta(x) \subset T_\mu(x) + \varepsilon \qquad (\mu > 0).$$

Consequently, for $\frac{\varepsilon}{2}$ there exists $\delta_1 > 0$ and $r_1 > 0$ such that

$$\emptyset \neq B(y, r_1) \cap T_{\delta_1}(x) \subset T_\mu(x) + \frac{\varepsilon}{2} \qquad (\mu > 0).$$

Let $r_0 = d(y, T_\delta(x)) + \frac{\varepsilon}{2}$. Then $B(y, r_0) \cap T_\delta(x) \neq \emptyset$. Hence there exists a point y_1 such that

$$y_1 \in B(y, r_0) \cap B(y, r_1) \cap T_\delta(x) \subset T_\mu(x) + \frac{\varepsilon}{2} \quad (\mu > 0)$$

and hence $d(y_1, T_\mu(x)) < \frac{\varepsilon}{2}$ for all $\mu > 0$.

For y_1 and $\frac{\varepsilon}{2^2}$ there exists $\delta_2 > 0$ and $r_2 > 0$ such that

$$\emptyset \neq V_{r_2}(y_1) \cap T_{\delta_2}(x) \subset T_\mu(x) + \frac{\varepsilon}{2^2} \qquad (\mu > 0).$$

Since $y_1 \in T_{\delta_2}(x) + \frac{\varepsilon}{2}$, we have $B(y_1, \frac{\varepsilon}{2}) \cap T_{\delta_2}(x) \neq \emptyset$. Hence there exists a point y_2 such that

$$y_2 \in B\Big(y_1, \frac{\varepsilon}{2}\Big) \cap B(y_1, r_2) \cap T_{\delta_2}(x) \subset T_\mu(x) + \frac{\varepsilon}{2^2} \qquad (\mu > 0).$$

Therefore, $d(y_1, y_2) < \frac{\varepsilon}{2}$ and $d(y_2, T_\mu(x)) < \frac{\varepsilon}{2^2}$ for all $\mu > 0$. Now assume that $y_1, y_2, \ldots, y_n$ has been found such that

$$\left.\begin{aligned} d(y_k, y_{k+1}) &< \frac{\varepsilon}{2^k} \qquad (k = 1, 2, \ldots, n-1) \\ d(y_k, T_\mu(x)) &< \frac{\varepsilon}{2^k} \qquad\qquad (k = 1, 2, \ldots, n) \end{aligned}\right. \qquad (\mu > 0).$$

For y_n and $\frac{\varepsilon}{2^{n+1}}$ there exists $\delta_{n+1} > 0$ and $r_{n+1} > 0$ such that

$$\emptyset \neq B(y_n, r_{n+1}) \cap T_{\delta_{n+1}}(x) \subset T_\mu(x) + \frac{\varepsilon}{2^{n+1}} \qquad (\mu > 0).$$

Since $y_n \in T_{\delta_{n+1}}(x) + \frac{\varepsilon}{2^n}$, we have $B(y_n, \frac{\varepsilon}{2^n}) \cap T_{\delta_{n+1}}(x) \neq \emptyset$. Hence there exists a point y_{n+1} such that

$$y_{n+1} \in B(y_n, \frac{\varepsilon}{2^n}) \cap B(y_n, r_{n+1}) \cap T_{\delta_{n+1}}(x) \subset T_\mu(x) + \frac{\varepsilon}{2^{n+1}} \qquad (\mu > 0).$$

Therefore,

$$\left.\begin{aligned} d(y_n, y_{n+1}) &< \frac{\varepsilon}{2^n} \\ d(y_{n+1}, T_\mu(x)) &< \frac{\varepsilon}{2^{n+1}} \end{aligned}\right. \qquad (\mu > 0).$$

Consequently, by induction we obtain a sequence $\{y_n\}_{n \geq 1}$ such that

$$\left.\begin{aligned} d(y_n, y_{n+1}) &< \frac{\varepsilon}{2^n} \\ d(y_n, T_\mu(x)) &< \frac{\varepsilon}{2^n} \end{aligned}\right. \qquad (\mu > 0, n \geq 1).$$

For each $n \in \mathbb{N}$, by $d(y_n, T_{\frac{\varepsilon}{2^n}}(x)) < \frac{\varepsilon}{2^n}$ there exists a point $z_n \in T(x)$ such that $d(y_n, z_n) < \frac{\varepsilon}{2^{n-1}}$. Since again $d(y_n, y_{n+1}) < \frac{\varepsilon}{2^n}$, $\{z_n\}_{n \geq 1}$ is a Cauchy sequence in $T(x)$ and $d(y_1, y_n) < \varepsilon$. By the completeness of $T(x)$ there is a point $y_0 \in T(x)$ such that $z_n \to y_0$. Hence $y_n \to y_0$ and $d(y_1, y_0) \leq \varepsilon$. Since

$d(y_n, T_\mu(x)) < \frac{\varepsilon}{2^n}$ for all $\mu > 0$ and $n \geq 1$, we have $d(y_0, T_\mu(x)) = 0$ for all $\mu > 0$. Hence $y_0 \in \text{cl}(T_\mu(x))$, and hence $y_0 \in T_0(x)$ by Fact 2. Therefore,

$$\begin{aligned} d(y, T_0(x)) &\leq d(y, y_0) \\ &\leq d(y, y_1) + d(y_1, y_0) \\ &< r_0 + \varepsilon \\ &= d(y, T_\delta(x)) + \frac{\varepsilon}{2} + \varepsilon \\ &< d(y, T_\delta(x)) + 2\varepsilon. \end{aligned}$$

.

Fact 5: $T_0(x) \subset T(x)$ for all $x \in X$. Indeed, for each $y_0 \in T_0(x)$ we have $y_0 \in T_{\frac{1}{n}}(x)$ for all $n \in \mathbb{N}$. Hence $d(y_0, T(x)) < \frac{1}{n}$ for all $n \in \mathbb{N}$, and hence $y_0 \in T(x)$. This shows that $T_0(x) \subset T(x)$ for all $x \in X$.

Fact 6: $T_0 : X \to 2^Y$ is lower semicontinuous. Indeed, for each $x_0 \in X$ and each open set $G \subset Y$, if $T_0(x_0) \cap G \neq \emptyset$, then there exist a point $y \in T_0(x_0) \cap G$ and an $\varepsilon > 0$ such that $B(y, 2\varepsilon) \subset G$. By Fact 4 there exists an open neighborhood $N(x_0)$ of x_0 and a $\delta > 0$ such that

$$d(y, T_0(x)) < d(y, T_\delta(x)) + 2\varepsilon \qquad (x \in N(x_0)).$$

Since $y \in T_0(x_0) \subset T_\delta(x_0)$, there exists an open neighborhood $N_1(x_0) \subset N(x_0)$ of x_0 such that $d(y, T(z)) < \delta$ for all $x \in N_1(x_0)$. Since again $N_1(x_0)$ is an open neighborhood of x for all $x \in N_1(x_0)$, we have $y \in T_\delta(x)$ for all $x \in N_1(x_0)$. Hence $d(y, T_\delta(x)) = 0$ for all $x \in N_1(x_0)$, and hence $d(y, T_0(x)) < d(y, T_\delta(x)) + 2\varepsilon = 2\varepsilon$ for all $x \in N_1(x_0)$. Consequently, $T_0(x) \cap G \neq \emptyset$ for all $x \in N_1(x_0)$. This shows that $T_0 : X \to 2^Y$ is lower semicontinuous.

Summing up the above facts we get that $T_0 : X \to 2^Y$ is a lower semicontinuous multi-valued mapping with non-empty closed sub-admissible values and $T_0(x) \subset T(x)$ for each $x \in X$. Hence by virtue of Theorem 2.3 there exist a continuous mapping $f : X \to Y$ such that $f(x) \in T_0(x) \subset T(x)$ for all $x \in X$ ∎

As an application of Theorem 2.4 we get the following fixed point theorem.

Theorem 2.5. *Let (Y, d) be a hyperconvex metric space and X be a non-empty compact sub-admissible subset of Y. Further, let $T : X \to 2^Y$ be a locally-uniform weak lower semicontinuous multi-valued mapping such that:*

(i) *For each $x \in X$, $T(x)$ is a non-empty closed sub-admissible subset of Y*

(ii) *For each $x \in X$ with $x \notin T(x)$ and each $z \in T(x)$, there exists $\alpha \in (0,1)$ such that*

$$X \cap \overline{B}(x, \alpha d(x,z)) \cap \overline{B}(z, (1-\alpha) d(x,z)) \neq \emptyset.$$

Then T has a fixed point, i.e. there exists a point $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$.

Proof. By Theorem 2.4 there exists a continuous mapping $f : X \to Y$ such that $f(x) \in T(x)$ for all $x \in X$. For each $x \in X$, set

$$G(x) = \big\{ y \in X : d(y, f(y)) \le d(x, f(y)) \big\}.$$

Then $G : X \to 2^Y$ is a multi-valued mapping with non-empty closed values. We claim that G is a metric KKM (see [5]) mapping. Otherwise, there exists a finite subset $A \subset X$ and a point $y \in \mathrm{co}(A)$ such that $y \notin \bigcup_{x \in A} G(x)$. Since X is a sub-admissible subset of Y, $\mathrm{co}(A) \subset X$, and hence

$$d(x, f(y)) < d(y, f(y)) \qquad (x \in A).$$

Let $\varepsilon > 0$ be such that

$$d(x, f(y)) \le d(y, f(y)) - \varepsilon \qquad (x \in A).$$

Then

$$A \subset X \cap \overline{B}\big(f(y), d(y, f(y)) - \varepsilon\big).$$

Hence

$$\mathrm{co}(A) \subset X \cap \overline{B}\big(f(y), d(y, f(y)) - \varepsilon\big).$$

Consequently,

$$d(y, f(y)) \le d(y, f(y)) - \varepsilon.$$

This is a contradiction.

By the compactness of X and [9: Theorem 4], there exists a point $y_0 \in \cap_{x \in X} G(x)$, i.e.

$$d(y_0, f(y_0)) = \inf_{x \in X} d(x, f(y_0)).$$

We claim that $y_0 \in T(y_0)$. Otherwise, since $f(y_0) \in T(y_0)$, by (ii) there exists $\alpha \in (0,1)$ such that

$$X \cap \overline{B}\big(y_0, \alpha\, d(y_0, f(y_0))\big) \cap \overline{B}\big(f(y_0), (1-\alpha) d(y_0, f(y_0))\big) \ne \emptyset.$$

Hence there exists a point $z \in X$ such that

$$\begin{aligned} d(y_0, z) &\le \alpha\, d(y_0, f(y_0)) \\ d(f(y_0), z) &\le (1-\alpha) d(y_0, f(y_0)). \end{aligned}$$

Consequently,

$$d(f(y_0), z) = (1-\alpha) d(y_0, f(y_0)).$$

Indeed, otherwise

$$\begin{aligned} d(y_0, f(y_0)) &= \alpha\, d(y_0, f(y_0)) + (1-\alpha) d(y_0, f(y_0)) \\ &> d(y_0, z) + d(f(y_0), z) \end{aligned}$$

and hence

$$d(y_0, f(y_0)) \le d(y_0, z) + d(f(y_0), z) < d(y_0, f(y_0)).$$

This is a contradiction. Hence

$$\begin{aligned} d(f(y_0), z) &= (1-\alpha) d(y_0, f(y_0)) \\ &= (1-\alpha) \inf_{x \in X} d(x, f(y_0)) \\ &\le (1-\alpha) d(z, f(y_0)). \end{aligned}$$

We get a contradiction. This shows that $y_0 \in T(y_0)$ and the proof is completed ∎

Corollary 2.6. *Let (Y, d) be a hyperconvex metric space and X be a non-empty compact sub-admissible subset of Y. Further, let $T : X \to 2^X$ be a locally-uniform weak lower semicontinuous multi-valued mapping with non-empty closed sub-admissible values. Then T has a fixed point.*

Proof. For each $x \in X$ with $x \notin T(x)$ and each $y \in T(x)$ we have $\mathrm{co}(\{x, y\}) \subset X$ since X is a non-empty sub-admissible subset of Y. Let $co(\{x, y\}) = \cap_{j \in J} \overline{B}(x_j, r_j)$ and take any $\alpha \in (0, 1)$. Since

$$\begin{aligned} d(x, y) &= \alpha\, d(x, y) + (1-\alpha) d(x, y) \\ d(x, x_j) &\le r_j \le r_j + \alpha\, d(x, y) \\ d(y, x_j) &\le r_j \le r_j + (1-\alpha) d(x, y), \end{aligned}$$

by hyperconvexity of Y,

$$\mathrm{co}(\{x, y\}) \cap \overline{B}(x, \alpha\, d(x, y)) \cap \overline{B}(y, (1-\alpha) d(x, y)) \ne \emptyset$$

and hence

$$X \cap \overline{B}(x, \alpha\, d(x, y)) \cap \overline{B}(y, (1-\alpha) d(x, y)) \ne \emptyset.$$

This shows that condition (ii) in Theorem 2.5 is satisfied. Hence the conclusion follows from Theorem 2.5 ∎

Remark 2.1. Horvath [3, 4] studied continuous selection problems and fixed point problems for lower semicontinuous multi-valued mappings in topological spaces with a generalized convexity structure. Our results are different from these corresponding results in [3, 4].

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