Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 22 (2003), No. 3, 517–528

Fixed-Point Properties of Roughly Contractive Mappings

H. X. Phu

Abstract. For given $k \in (0,1)$ and $r > 0$, a self-mapping $T:\, M \to M$ is said to be r-roughly k-contractive provided

$$\|Tx - Ty\| \le k\,\|x-y\| + r \quad (x,y \in M).$$

To state fixed-point properties of such a mapping, the self-Jung constant $J_s(X)$ is used, which is defined as the supremum of the ratio $2\,r_{\operatorname{conv} S}(S)/\operatorname{diam} S$ over all non-empty, non-singleton and bounded subsets S of some normed linear space X, where $r_{\operatorname{conv} S}(S) = \inf_{x\in \operatorname{conv} S}\sup_{y\in S}\|x-y\|$ is the self-radius of S and $\operatorname{diam} S$ is its diameter. If M is a closed and convex subset of some finite-dimensional normed space X and if $T:\, M \to M$ is r-roughly k-contractive, then for all $\varepsilon > 0$ there exists $x^* \in M$ such that

$$\|x^* - Tx^*\| < \tfrac{1}{2}\,J_s(X)\,r + \varepsilon.$$

If $\dim X = 1$, or X is some two-dimensional strictly convex normed space, or X is some Euclidean space, then there is $x^* \in M$ satisfying $\|x^* - Tx^*\| \le \frac{1}{2}\,J_s(X)\,r$.

Keywords: *Roughly contractive mapping, fixed-point theorem, rough invariance, self-Jung constant*

AMS subject classification: Primary 47H10, secondary 54H25

1. Introduction

Let X be a finite-dimensional normed linear space. For given $M \subset X$, $k \in (0,1)$ and $r > 0$, a mapping $T:\, M \to M$ is said to be *r-roughly k-contractive* provided

$$\|Tx - Ty\| \le k\,\|x-y\| + r \qquad (x,y \in M). \tag{1.1}$$

We introduced this notion in [17, 18, 21] as a generalization of contractive mappings considered in the well-known Banach fixed-point theorem [2], where

---

H. X. Phu: Institute of Mathematics, P.O. Box 631 Bo Ho, 10000 Hanoi, Vietnam
hxphu@thevinh.ncst.ac.vn

ISSN 0232-2064 / $ 2.50

some roughness degree r is added to the right-hand side of the inequality. Such mappings may arise in quite natural ways. For example, if a given k-contractive mapping T_0 defined by $\|T_0x - T_0y\| \le k\,\|x-y\|$ for all $x, y \in M$ cannot be determined exactly but it is only approximated by T, where $r = 2\max_{x\in M}\|T_0x - Tx\|$ denotes the double maximal approximation error, then

$$\begin{aligned}\|Tx - Ty\| &\le \|T_0x - Tx\| + \|Tx - Ty\| + \|Ty - T_0y\| \\ &\le k\,\|x-y\| + r\end{aligned}$$

for all $x, y \in M$, i.e. T is r-roughly k-contractive.

Actually, such mappings were considered independently by Kirk [10], but his main attention was devoted to so-called h–non-expansive mappings defined by

$$\|Tx - Ty\| \le \max\{\|x-y\|, h\} \qquad (x, y \in M) \tag{1.2}$$

and to Hölder continuous mappings.

Since roughly contractive mappings cannot always possess fixed points, we have to consider so-called γ-*fixed* or γ-*invariant points* defined by $\|x^* - Tx^*\| \le \gamma$ for some $\gamma > 0$, as already done for discontinuous mappings by Klee [12], Cromme and Diener [5 - 6], Bula [4], and Kirk [10]. In [16 - 18], we determined the best invariant degree γ of roughly contractive mappings in Minkowski and Euclidean spaces.

We now take advantage of the self-Jung constant $J_s(X)$ defined by (2.2) below to get a better result for non-Euclidean n-dimensional normed spaces. In Section 2, the self-Jung constant is used to estimate the distance between some set S and any point $z \in \operatorname{conv} S \setminus S$. This is applied in Section 3 to state the following fixed point property of an r-roughly k-contractive mapping on a closed and convex subset M of some n-dimensional normed space X:

$$\forall \varepsilon > 0\ \exists x^* \in M: \quad \|x^* - Tx^*\| < \frac{1}{2}\,J_s(X)\,r + \varepsilon.$$

Moreover, if

dim $X = 1$,
or X is some two-dimensional strictly convex normed space,
or X is some Euclidean space,

then there exists $x^* \in M$ such that

$$\|x^* - Tx^*\| \le \frac{1}{2}\,J_s(X)\,r$$

(Theorem 3.3).

2. Distance estimation by using self-Jung constant

For a bounded set $S \subset X$,

$$\operatorname{diam} S = \sup_{x,y \in S} \|x-y\| \quad \text{and} \quad r_A(S) = \inf_{x\in A} \sup_{y \in S} \|x-y\| \tag{2.1}$$

are the *diameter* and the *relative radius* of S with respect to A. In particular, $r_X(S)$ and $r_{\operatorname{conv} S} S$ are called its *absolute radius* and *self-radius*, respectively.

The ratio between absolute radius and diameter was studied by Jung [8] who investigated Euclidean spaces. Later on, Bohnenblust [3], Leichtweiß [13], and Grünbaum [7] considered this problem in Minkowski spaces. These works started a research direction of determining the supremum of this ratio which is called Jung's constant (see, e.g., [1, 19, 20] and references therein).

In this paper, we are interested in a similar one, namely the *self-Jung constant* of X defined by

$$J_s(X) = \sup \left\{ \frac{2\, r_{\operatorname{conv} S}(S)}{\operatorname{diam} S} \;\middle|\; S \subset X \text{ is } \begin{bmatrix} \text{bounded} \\ \text{non-empty} \\ \text{non-singleton} \end{bmatrix} \right\}. \tag{2.2}$$

Obviously,

$$\begin{gathered} \frac{1}{2} \operatorname{diam} S \le r_X(S) \le r_{\operatorname{conv} S}(S) \le \operatorname{diam} S \\ 1 \le J_s(X) \le 2. \end{gathered} \tag{2.3}$$

For n-dimensional spaces, we have

$$J_s(\ell_2^n) = \left(\frac{2\,n}{n+1} \right)^{1/2} \tag{2.4}$$

which follows from the classical results of Jung [8] and Klee [11] (see [14]), and

$$J_s(X) \le \frac{2n}{n+1} \qquad \text{if } \dim X = n \tag{2.5}$$

(see [1]).

Let us now use the self-Jung constant to estimate the distance between a set S and any point $z \in \operatorname{conv} S \setminus S$.

Proposition 2.1. *Let S be a bounded set of some finite-dimensional normed space X and $z \in \operatorname{conv} S \setminus S$. Then there exists $s \in S$ such that*

$$\|z - s\| \le \frac{1}{2}\, J_s(X)\, \operatorname{diam} S.$$

Proof. Since $z \in \operatorname{conv} S \setminus S$, there exists a set $S_k = \{x_1, \dots, x_k\} \subset S$ of $k \geq 2$ linearly independent points such that $z \in \operatorname{ri}(\operatorname{conv} S_k)$, where $\operatorname{ri} A$ denotes the relative interior of A.

Let us prove by induction

$$\min_{1 \leq i \leq k} \|z - x_i\| \leq \frac{1}{2} J_s(X) \operatorname{diam} S. \tag{2.6}$$

For $k = 2$, $\|x_1 - x_2\| \leq \operatorname{diam} S$ and (2.3) yield

$$\min \big\{ \|z - x_1\|, \|z - x_2\| \big\} \leq \frac{1}{2} \|x_1 - x_2\| \leq \frac{1}{2} J_s(X) \operatorname{diam} S.$$

Assuming now that (2.6) is true for $2 \leq k \leq l$, we have to show it for $k = l+1$. Consider the relative center set of S_k with respect to the compact set $\operatorname{conv} S_k$

$$\mathcal{C}_{\operatorname{conv} S_k}(S_k) = \Big\{ x \in \operatorname{conv} S_k : \sup_{y \in S_k} \|x - y\| = r_{\operatorname{conv} S_k}(S_k) \Big\}$$

which is obviously non-empty. For any fixed $c \in \mathcal{C}_{\operatorname{conv} S_k}(S_k)$, (2.1) and (2.2) imply

$$\begin{aligned} \max_{1 \leq i \leq k} \|c - x_i\| &\leq r_{\operatorname{conv} S_k}(S_k) \\ &\leq \frac{1}{2} J_s(X) \ \operatorname{diam} S_k \\ &\leq \frac{1}{2} J_s(X) \operatorname{diam} S. \end{aligned} \tag{2.7}$$

If $z = c$, then (2.6) follows from (2.7). Otherwise, the ray from c through z cuts the boundary $\operatorname{conv} S_k \setminus \operatorname{ri}(\operatorname{conv} S_k)$ at some point $z' \in \operatorname{conv} S_{k'}$ where $S_{k'} = \big\{ x_{i_1}, \dots, x_{i_{k'}} \big\} \subset S_k$ and $k' \leq k - 1 = l$. If $z' \in S_{k'}$, then $z \in [c, z']$ yields

$$\|z - z'\| \leq \|c - z'\| \leq r_{\operatorname{conv} S_k}(S_k) \leq \frac{1}{2} J_s(X) \operatorname{diam} S.$$

If $z' \notin S_{k'}$, then $z' \in \operatorname{conv} S_{k'} \setminus S_{k'}$. By the inductive assumption, there is some $y \in S_{k'} \subset S$ such that $\|z' - y\| \leq \frac{1}{2} J_s(X) \operatorname{diam} S$. Therefore, it follows from $z \in [c, z']$ and (2.7) that

$$\|z - y\| \leq \max \big\{ \|c - y\|, \|z' - y\| \big\} \leq \frac{1}{2} J_s(X) \operatorname{diam} S$$

which completes our proof ∎

Note that the assumption $z \in \operatorname{conv} S \setminus S$ in the previous proposition and in the following one means at least $\operatorname{diam} S > 0$.

Proposition 2.2. *Suppose X is some two-dimensional strictly convex normed space or some Euclidean space, $S = \{x_1, \dots, x_k\} \subset X$, and $z \in$* $\operatorname{conv} S \setminus S$. *Then either*

$$\min_{1\le i\le k} \|z - x_i\| < r_{\operatorname{conv} S}(S) \le \frac{1}{2}\, J_s(X) \operatorname{diam} S$$

or

$$\|z - x_i\| = r_{\operatorname{conv}\ S}(S) \le \frac{1}{2}\, J_s(X) \operatorname{diam} S \qquad (i = 1, \dots, k).$$

Proof. By (2.2), we have to prove by induction that

$$\min_{1\le i\le k} \|z - x_i\| \ge r_{\operatorname{conv}\ S}(S) \tag{2.8}$$

implies

$$\|z - x_i\| = r_{\operatorname{conv}\ S}(S) \qquad (i = 1, \dots, k). \tag{2.9}$$

If $\dim S = 1$, then all points of S lie in some segment, say for instance, in the segment $[x_1, x_k]$ connecting x_1 and x_k. Then $r_{\operatorname{conv} S}(S) = \frac{1}{2} \operatorname{diam} S = \frac{1}{2}\|x_1 - x_k\|$ and

$$\min_{1\le i\le k} \|z - x_i\| < r_{\operatorname{conv} S}(S) \qquad \text{if } z \ne \frac{1}{2}\,(x_1 + x_k).$$

Therefore, (2.8) implies $z = \frac{1}{2}(x_1 + x_k)$ and $\max_{1\le i\le k} \|z - x_i\| \le r_{\operatorname{conv} S}(S)$. Hence, (2.9) follows from (2.8).

Assume now that the assertion is true for $\dim S \le l$, and (2.8) holds for some set $S = \{x_1, \dots, x_k\}$ with $\dim S = l + 1 \ge 2$. We have to show (2.9) now. Due to [11: Corollary 3], $r_X(S) = r_{\operatorname{conv} S}(S)$ and the absolute center set

$$\mathcal{C}_X(S) = \Big\{x \in X : \sup_{y\in S} \|x - y\| = r_X(S)\Big\}$$

is a singleton contained in $\operatorname{conv} S$, say $\mathcal{C}_X(S) = \{c\}$. If $c = z$, then similarly as above, (2.9) follows from (2.8). If $c \ne z \in \operatorname{ri}(\operatorname{conv} S)$, then the ray L from c through z cuts the boundary $\operatorname{conv} S \setminus \operatorname{ri}(\operatorname{conv} S)$ at $z' \in \operatorname{conv} S_l$ for some $S_l \subset S$ with $\dim S_l \le l$. If $z' \in S_l$, then $z \in [c, z']$ and $c \ne z$ yield

$$\|z - z'\| < \|c - z'\| \le r_X(S) = r_{\operatorname{conv} S}(S),$$

a contradiction to (2.8). Hence, $z' \in \operatorname{conv} S_l \setminus S_l$ and $S_l \cap L = \emptyset$. Consequently, for all $y \in S_l$, the function $g(x) = \|x - y\|$ is strictly convex on L, which implies by $\|c - y\| \le r_X(S)$, $\|z - y\| \ge r_{\operatorname{conv} S}(S) = r_X(S)$ and $z' \in L \setminus [c, z]$ that

$$\|z' - y\| > r_X(S) \ge r_X(S_l) = r_{\operatorname{conv} S_l}(S_l) \qquad (y \in S_l),$$

i.e. (2.8) is satisfied for z' and S_l instead of z and S while (2.9) fails, a contradiction to the inductive assumption.

If $c \neq z \notin \mathrm{ri}(\mathrm{conv}\, S)$, then $z \in \mathrm{conv}\, S \setminus S$ implies $z \in \mathrm{ri}(\mathrm{conv}\, S_l)$ for some $S_l \subset S$ with $\dim S_l \le l$. By the inductive assumption, it follows from

$$\min_{y \in S_l} \|z - y\| \ge \min_{y \in S} \|z - y\| \ge r_{\mathrm{conv}\, S}(S) = r_X(S) \ge r_X(S_l)$$

and $\dim S_l \le l$ that

$$\|z - y\| = r_X(S) = r_X(S_l) \qquad (y \in S_l).$$

Therefore, by the strict convexity of the normed space X and $\|c - y\| \le r_X(S) = r_X(S_l)$ we have

$$\left\| \tfrac{1}{2}(c + z) - y \right\| < r_X(S_l) \qquad (y \in S_l),$$

a contradiction to the definition of $r_X(S_l)$ ∎

3. Fixed-point theorems

Following the Banach fixed-point theorem [2], to investigate the invariant property of an r-roughly k-contractive mapping $T : M \to M$, we consider the iteration

$$\left. \begin{aligned} x_0 &\in M \\ x_{i+1} &= T x_i \quad (i \ge 0) \end{aligned} \right\}. \tag{3.1}$$

Without assuming M to be closed and convex, we proved in [18] the following γ-fixed-point theorem.

Theorem 3.1. *Let (M, d) be a metric space and let $T : M \to M$ be an r-roughly k-contractive mapping, i.e. $d(Tx, Ty) \le k\, d(x, y) + r$ for all $x, y \in M$, where $r > 0$ and $k \in (0, 1)$ are given. Suppose $x_0 \in M$ and $a := d(x_0, T x_0) - \frac{r}{1-k} > 0$.*

- **(a)** *If $\gamma > \frac{r}{1-k}$ and $i \ge \log_k \left((\gamma - \frac{r}{1-k}) a^{-1} \right)$, then x_i determined by (3.1) is a γ-invariant point under T, i.e. $d(x_i, T x_i) \le \gamma$.*
- **(b)** *If $x^* \in M$ is a cluster point of the sequence (x_i), then it is a γ-invariant point under T with $\gamma = \frac{r}{1-k}$.*
- **(c)** *For every $\gamma > 0$, the set I_γ of all γ-invariant points (of T) is bounded. If $\gamma \ge \frac{r}{1-k}$, then I_γ is invariant under T, i.e. $T I_\gamma \subset I_\gamma$.*

Consequently, if M is a compact metric space or if it is a closed subset of some finite-dimensional metric space, then each r-roughly k-contractive mapping $T : M \to M$ admits at least one γ-invariant point with $\gamma = \frac{r}{1-k}$.

In general there is no smaller invariant degree γ as given in Theorem 3.1 if M is not assumed to be convex. This fact was shown in [18] by considering the mapping $T: M_1 \cup M_2 \to M_1 \cup M_2$ defined by

$$Tx = \begin{cases} \frac{r}{2} - kx & \text{if } x \in M_1 = \big(-\infty, -\frac{r}{2(1-k)}\big) \\ -\frac{r}{2} - kx & \text{if } x \in M_2 = \big(\frac{r}{2(1-k)}, \infty\big). \end{cases}$$

This mapping is r-roughly k-contractive but has no γ-invariant points with $\gamma \leq \frac{r}{1-k}$. If T is extended continuously to the closure

$$\operatorname{cl} M_1 \cup \operatorname{cl} M_2 = \Big(-\infty, -\frac{r}{2(1-k)}\Big] \cup \Big[\frac{r}{2(1-k)}, \infty\Big),$$

i.e. $T\frac{-r}{2(1-k)} = \frac{r}{2(1-k)}$ and $T\frac{r}{2(1-k)} = \frac{-r}{2(1-k)}$, then $\frac{-r}{2(1-k)}$ and $\frac{r}{2(1-k)}$ are γ-invariant with $\gamma = \frac{r}{1-k}$ and there exists no γ-invariant point with $\gamma < \frac{r}{1-k}$.

Theorem 3.1 says that for all $\gamma > \frac{r}{1-k}$ there exists an $x \in M$ such that $d(x, Tx) \leq \gamma$. Consequently,

$$\inf\{d(x, Tx) : x \in M\} \leq \frac{r}{1-k}.$$

This inequality was shown by Kirk [10]. Note that the infimum $\frac{r}{1-k}$ is not necessarily attainable as shown by the above example.

For convex M, in [16, 18] we obtained the following result.

Theorem 3.2. *Let $T: M \to M$ be an r-roughly k-contractive mapping on a closed and convex subset M of some n-dimensional normed space X. If* $\dim X = 1$*, then there exists $x^* \in M$ such that*

$$\|x^* - Tx^*\| \leq \tfrac{1}{2}\, r. \tag{3.2}$$

If $\dim X \geq 2$*, then for all $\varepsilon > 0$ there exists $x^* \in M$ such that*

$$\|x^* - Tx^*\| < \tfrac{n}{n+1}\, r + \varepsilon. \tag{3.3}$$

If, in addition, the normed space X is strictly convex, then there exists $x^* \in M$ such that

$$\|x^* - Tx^*\| < \tfrac{n}{n+1}\, r. \tag{3.4}$$

If X is the n-dimensional Euclidean space, then there exists $x^* \in M$ such that

$$\|x^* - Tx^*\| \leq \big(\tfrac{n}{2(n+1)}\big)^{1/2} r. \tag{3.5}$$

It is worth mentioning that iteration (3.1) is not suitable to approximate γ-invariant points with $\gamma < \frac{r}{1-k}$ even if they exist, as pointed out in [18] by considering

$$Tx = \begin{cases} \frac{r}{2} - kx & \text{if } x \leq 0 \\ -\frac{r}{2} - kx & \text{if } x > 0. \end{cases}$$

For any $\gamma \geq \frac{1}{2}\, r$, each x satisfying $-\frac{\gamma - \frac{r}{2}}{1+k} \leq x \leq \frac{\gamma - \frac{r}{2}}{1+k}$ is a γ-invariant point of this r-roughly k-contractive mapping $T : \mathbb{R} \to \mathbb{R}$. But, for any starting point $x_0 \in \mathbb{R}$, the sequence (x_i) determined by (3.1) has only two cluster points $x^- = -\frac{r}{2(1-k)}$ and $x^+ = \frac{r}{2(1-k)}$, which satisfy $Tx^- = x^+$, $Tx^+ = x^-$, and $|x^- - Tx^-| = |x^+ - Tx^+| = |x^- - x^+| = \frac{r}{1-k}$.

Let us now use the self-Jung constant $J_s(X)$ to improve Theorem 3.2.

Theorem 3.3. *Let $T : M \to M$ be an r-roughly k-contractive mapping on a closed and convex subset M of some n-dimensional normed space X. Then for all $\varepsilon > 0$ there exists $x^* \in M$ such that*

$$\|x^* - Tx^*\| < \frac{1}{2}\, J_s(X)\, r + \varepsilon. \tag{3.6}$$

If $\dim X = 1$, or X is some two-dimensional strictly convex normed space, or X is some Euclidean space, then there exists $x^* \in M$ such that

$$\|x^* - Tx^*\| \leq \frac{1}{2}\, J_s(X)\, r. \tag{3.7}$$

Proof. (a) Take any $x_0 \in M$ and define

$$\widehat{B} = \left\{ x \in X : \|x - x_0\| \leq \hat{r} \right\}$$

where $\hat{r} = \frac{r + \|x_0 - Tx_0\|}{1-k}$. Then $\widehat{M} = M \cap \widehat{B}$ is non-empty, compact, and convex, and T maps $\widehat{M}$ into itself because, for all $x \in \widehat{M}$, (1.1) implies

$$\begin{aligned} \|Tx - x_0\| &\leq \|Tx - Tx_0\| + \|Tx_0 - x_0\| \\ &\leq k\|x - x_0\| + r + (1-k)\hat{r} - r \\ &\leq \hat{r}, \end{aligned}$$

i.e. $Tx \in \widehat{M}$.

(b) Consider $\overline{T} : \widehat{M} \to 2^{\widehat{M}}$ defined by $\overline{T}(x) = \operatorname{conv} \overline{M}(x)$, where

$$\overline{M}(x) = \left\{ y \in \widehat{M} : \exists\, (x_i) \subset \widehat{M} \text{ such that } x_i \to x, Tx_i \to y \right\}. \tag{3.8}$$

For all $x \in \widehat{M}$, $\overline{M}(x)$ is closed and non-empty because $\widehat{M}$ is compact. It is also bounded because $\operatorname{diam} \overline{M}(x) \leq r$ follows from (1.1) and (3.8). Since X

is finite-dimensional, $\overline{M}(x)$ and $\operatorname{conv}\overline{M}(x)$ are compact (see [22: p. 40]). It was shown in [6] that $\overline{T}$ is upper semi-continuous. Therefore, by Kakutani's theorem (see [9, 23]), there exists $\bar{x} \in \widehat{M}$ such that $\bar{x} \in \overline{T}(\bar{x}) = \operatorname{conv}\overline{M}(\bar{x})$.

(c) By Proposition 2.1 and $\operatorname{diam}\overline{M}(\bar{x}) \le r$, there is a $\bar{y} \in \overline{M}(\bar{x})$ such that

$$\|\bar{x} - \bar{y}\| \le \frac{1}{2} J_s(X) \operatorname{diam}\overline{M}(\bar{x}) \le \frac{1}{2} J_s(X)\, r.$$

Due to (3.8), for any $\varepsilon > 0$ there exists a point $x^* \in \widehat{M} \subset M$ such that

$$\|x^* - \bar{x}\| < \frac{\varepsilon}{2} \qquad \text{and} \qquad \|Tx^* - \bar{y}\| < \frac{\varepsilon}{2} \tag{3.9}$$

which yields immediately

$$\|x^* - Tx^*\| \le \|x^* - \bar{x}\| + \|\bar{x} - \bar{y}\| + \|\bar{y} - Tx^*\| < \frac{1}{2} J_s(X)\, r + \varepsilon,$$

i.e. (3.6) holds true.

(d) If $\dim X = 1$, then $J_s(X) = 1$, and therefore, (3.7) follows from (3.2). Assume now that X is a two-dimensional strictly convex normed space or it is the n-dimensional Euclidean space. If $\|\bar{x} - T\bar{x}\| \le \frac{1}{2} J_s(X)\, r$, then (3.7) is fulfilled for $x^* = \bar{x}$. Otherwise, assume

$$\|\bar{x} - T\bar{x}\| > \frac{1}{2} J_s(X)\, r \ge r_{\operatorname{conv} S}(S). \tag{3.10}$$

Since $\bar{x} \in \operatorname{conv}\overline{M}(\bar{x})$, there exists a finite set $\widetilde{M} \subset \overline{M}(\bar{x})$ such that $\bar{x} \in \operatorname{conv}\widetilde{M}$. For $S = \widetilde{M} \cup T\bar{x}$, we have $\bar{x} \in \operatorname{conv} S$, and $\operatorname{diam} S \le r$ follows from (1.1) and (3.8). Consequently, Proposition 2.2 and (3.10) imply that there exists $\bar{y} \in \widetilde{M} \subset \overline{M}(\bar{x}) \subset \widehat{M} \subset M$ satisfying

$$\|\bar{x} - \bar{y}\| < r_{\operatorname{conv} S}(S) \le \frac{1}{2} J_s(X)\, r.$$

By choosing $\varepsilon = \frac{1}{2} J_s(X)\, r - \|\bar{x} - \bar{y}\| > 0$ and $x^* \in \widehat{M} \subset M$ satisfying (3.9), we obtain

$$\begin{aligned} \|x^* - Tx^*\| &\le \|x^* - \bar{x}\| + \|\bar{x} - \bar{y}\| + \|\bar{y} - Tx^*\| \\ &\le \|\bar{x} - \bar{y}\| + \varepsilon \\ &= \frac{1}{2} J_s(X)\, r, \end{aligned}$$

i.e. (3.7) holds true ∎

Note that (3.3) and (3.5) in Theorem 3.2 can be derived from Theorem 3.3 by using the estimation of $J_s(X)$ for Euclidean and Minkowski spaces stated in (2.4) - (2.5).

In general, the invariant degrees given in Theorem 3.3 should be the best ones for r-roughly k-contractive mappings. This can be illustrated by the following

Example 3.1. Let $S = \{x_1, ..., x_{n+1}\}$ be a subset of $n+1$ linearly independent points of the n-dimensional Euclidean space ℓ_2^n, where $\|x_i - x_j\| = r > 0$ for $i \neq j$. Then $M = \operatorname{conv} S$ is an n-dimensional regular simplex in ℓ_2^n, $\bar{x} = \frac{1}{n+1}\sum_{i=1}^{n+1} x_i$ is its unique center, and

$$\|x_i - \bar{x}\| = r_M(M) = \frac{1}{2}\, J_s(\ell_2^n)\, r = \Big(\frac{n}{2(n+1)}\Big)^{1/2} r \quad (1 \leq i \leq n+1).$$

For any $x \in M$ we choose $Tx \in S$ such that $\|x - Tx\| = \max_{s\in S} \|x - s\|$. Then

$$\|Tx - Ty\| \leq \operatorname{diam} M = r \leq k\,\|x - y\| + r \quad (x, y \in M,\ 0 < k < 1)$$

and

$$\|\bar{x} - T\bar{x}\| = r_M(M) = \frac{1}{2}\, J_s(\ell_2^n)\, r < \|x - Tx\| \quad (x \in M \setminus \{\bar{x}\}).$$

Hence, the mapping $T : M \to M$ is r-roughly k-contractive for arbitrary $k \in (0, 1)$ and $\frac{1}{2}\, J_s(\ell_2^n)\, r$ is the smallest invariant degree of T.

4. Concluding remarks

Due to Kirk [10], an r-roughly k-contractive mapping $T : M \to M$ is h–non-expansive for $h = \frac{r}{1-k}$, and if M is a non-empty bounded closed convex subset of a Banach space X, then an h–non-expansive mapping $T : M \to M$ satisfies $\inf\{\|x - Tx\| : x \in K\} \leq h$. This result allows to ensure the existence of γ-invariant points of r-roughly k-contractive mappings only for $\gamma \geq \frac{r}{1-k}$, which is just the same as in Theorem 3.1, where the convexity of M is not required.

By (2.5), if $\dim X = n$, then $J_s(X) \leq \frac{2n}{n+1}$, which implies $\frac{1}{2}\, J_s(X)\, r \leq \frac{nr}{n+1} < \frac{r}{1-k}$, i.e. the invariant degrees given in Theorem 3.3 are better than the ones given in Theorem 3.1, especially for k near 1.

In particular, if $M = [a, b] \subset \mathbb{R}^1$ and if $T : M \to M$ is h–non-expansive, then Kirk [10] showed that there exists $z \in M$ satisfying $|z - Tz| \leq \frac{h}{2}$. This is the best result available for such h–non-expansive mappings. But if this result is applied to an r-roughly k-contractive mapping $T : M \to M$ as an h–non-expansive one for $h = \frac{r}{1-k}$, then we only obtain the invariant degree $\frac{h}{2} = \frac{r}{2(1-k)}$, which is also greater than the invariant degree $\gamma = \frac{r}{2}$ given in Theorems 3.2 - 3.3.

By using the result of Bula [4] for so-called uniformly w-continuous mappings, we can show that an r-roughly k-contractive mapping $T : M \to M$ possesses γ-invariant points with $\gamma = r + \varepsilon$ for arbitrary small ε, which is obviously greater than $\frac{nr}{n+1} \geq \frac{1}{2} J_s(X)\, r$.

In [15], we obtained similar results for discontinuous self-mappings on compact convex subsets of arbitrary normed spaces. By defining the self-Jung constant analogously, similar results are available for metric spaces.

Acknowledgment. The author thank the referees very much for their valuable and constructive comments.

References

[1] Amir, D.: *On Jung's constant and related constants in normed linear spaces.* Pacific J. Math. 118 (1985), 1 – 15.

[2] Banach, S.: *Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales.* Fund. Math. 3 (1922), 133 – 181.

[3] Bohnenblust, F.: *Convex regions and projections in Minkowski spaces.* Ann. Math. 39 (1938), 301 – 308.

[4] Bula, I.: *On the stability of the Bohl-Brouwer-Schauder theorem.* Nonlin. Anal.: Theory, Method & Appl. 26 (1996), 1859 – 1868.

[5] Cromme, L. J.: *Fixed point theorems for discontinuous functions and applications.* Nonlin. Anal.: Theory, Method & Appl. 30 (1997), 1527 – 1534.

[6] Cromme, L. J. and I. Diener: *Fixed point theorems for discontinuous mapping.* Math. Programming 51 (1991), 257 – 267.

[7] Grünbaum, B.: *On some covering and intersection properties in Minkowski spaces.* Pacific J. Math. 9 (1959), 487 – 494.

[8] Jung, H.: *Über die kleinste Kugel, die eine räumliche Figur einschließt.* J. Reine Angew. Math. 123 (1901), 241 – 257.

[9] Kakutani, S.: *A generalization of Brouwer's fixed point theorem.* Duke Math. J. 8 (1941), 457 – 459.

[10] Kirk, W. A.: *Hölder continuity and minimal displacement.* Numer. Funct. Anal. Optimiz. 19 (1998), 71 – 79.

[11] Klee, V.: *Circumspheres and inner products.* Math. Scand. 8 (1960), 363 – 370.

[12] Klee, V.: *Stability of the fixed-point property.* Coll. Math. 8 (1961), 43 – 46.

[13] Leichtweiß, K.: *Zwei Extremalprobleme der Minkowski-Geometrie.* Math. Z. 62 (1955), 37 – 49.

[14] Maluta, E.: *Uniformly normal structure and related coefficients.* Pacific J. Math. 111 (1984), 357 – 369.

[15] Phu, H. X.: *Fixed-point theorems for discontinuous mappings.* Preprint. Nat. Center for Sci. & Techn. of Vietnam, Inst. Math., Preprint 2002/19.

[16] Phu, H. X.: *Some remarks on circumradii of sets and roughly contractive mappings.* Vietnam J. Math. 31 (2003), 115 – 122.

[17] Phu, H. X. and T. V. Truong: *Some results on approximately fixed points for the class of roughly contractive mappings.* Preprint. Nat. Center for Sci. & Techn. of Vietnam, Inst. Math., Preprint 1999/8.

[18] Phu, H. X. and T. V. Truong: *Invariant property of roughly contractive mappings.* Vietnam J. Math. 28 (2000), 275 – 290.

[19] Pichugov, S. A.: *On Jung's constant of the space* L_p (in Russian). Mat. Zemetki 43 (1988), 604 – 614; English transl.: Math. Notes 43 (1988), 348 – 354.

[20] Routledge, N.: *A result in Hilbert space.* Quart. J. Math. 3 (1952), 12 – 18.

[21] Truong, T. V.: *Some problems on the fixed-points theory* (in Vietnamese). Master thesis. Nat. Center for Sci. & Techn. of Vietnam, Inst. Math. 1998.

[22] Valentine, F. A.: *Convex Sets.* New York etc.: McGraw-Hill Book Comp. 1964.

[23] Zeidler, E.: *Nonlinear Functional Analysis and Its Applications.* Vol. I: *Fixed-Point Theorems.* New York etc.: Springer-Verlag 1986.

Received 02.12.2002, in revised form 02.05.2003