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Maximum Local Lyapunov Dimension Bounds the Box Dimension. Direct Proof for Invariant Sets on Riemannian Manifolds

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Abstract. For a C^1 map ϕ on a Riemannian manifold and for a compact invariant set K it is proven that the maximal local Lyapunov dimension of ϕ on K bounds the box dimension of K from above. A version for Hilbert spaces is also presented. The introduction of an adapted Riemannian metric provides in a certain sense an optimal upper bound for the box dimension of the Lorenz attractor.

Keywords: *Box dimension, Lyapunov dimension, singular value function*

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1. Introduction

In the seminal paper by Douady and Oesterlé [3] it has been shown that the maximal local Lyapunov dimension of a C^1 map ϕ on a compact invariant set $K \subset \mathbb{R}^m$ bounds the Hausdorff dimension of K from above (see also Il'yashenko [9]). This result has been generalized to Hilbert spaces [8, 17] and to Riemannian manifolds [10, 13]. Il'yashenko conjectured that this upper bound is in fact an upper bound for the box dimension which majorizes the Hausdorff dimension. For a C^1 map and a compact invariant set this conjecture has been verified by Hunt [7], where the author uses the on $\mathbb{R}^m$ equivalent definition of the box dimension via a grid covering. For twice continuously Frechét-differentiable maps in a separable Hilbert space, Blinchevskaya and Ilyashenko [1] extended this result by showing that a compact invariant set has box dimension not exciting k if the maps contracts k-volumina. In the present paper we give a direct proof of Il'yashenko's conjecture for a C^1 map on a Riemannian manifold. We also present a version for Hilbert spaces. The

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dimension estimates [1, 7] mentioned above turn out as special cases. Moreover, our method of proof is perhaps simpler than those referred to.

Recall that, in various kind of applications, for chaotic attractors their box dimension is of higher significance than their Hausdorff dimension. One example are embedding strategies for dynamical systems with a high-dimensional phase space which answers the question how many degrees of freedom for a model system are sufficient to represent the essential dynamics faithfully. If for such a system we have given an attractor of box dimension d, Sauer, Yorke and Casdagli [15] show that in "almost all cases" it can be mapped injectively via a linear transformation into $\mathbb{R}^n$ provided $n > 2d$. A counterexample by I. Kan in the appendix of [15] points up that the box dimension may not be replaced by the Hausdorff dimension. Another example are noisy systems where the volume of the attractor scales with the magnitude of the noise, with a scaling factor depending on the box dimension of the noiseless attractor [14].

In the sequel we consider an m-dimensional Riemannian C^3 manifold M. Let $U \subset M$ be an open set and let $\phi: U \to M$ be a C^1 map. Given $p \in U$, we consider the *singular values* $\sigma_1(d_p\phi) \geq \ldots \geq \sigma_m(d_p\phi) \geq 0$ of the tangent map $d_p\phi: T_pM \to T_pM$ which are defined as the eigenvalues of the linear operator $((d_p\phi)^* d_p\phi)^{1/2}$, where $(d_p\phi)^*$ denotes the adjoint of $d_p\phi$. We introduce the *singular value function* of $d_p\phi$ of order d which is defined by

$$\begin{aligned} \omega_0(d_p\phi) &= 1 && \text{for } d = 0 \\ \omega_d(d_p\phi) &= \sigma_1(d_p\phi) \cdots \sigma_{\lfloor d \rfloor}(d_p\phi) \sigma_{\lfloor d \rfloor + 1}(d_p\phi)^{d - \lfloor d \rfloor} && \text{for } d \in (0, m]. \end{aligned}$$

Here $\lfloor d \rfloor$ denotes the the largest integer less than d. We denote by $\dim_{\mathrm{L}}(\phi, p)$ the *local Lyapunov dimension* of ϕ at p which is defined to be the largest number $d \in (0, m]$ for which $\omega_d(d_p\phi) \geq 1$. If $\sigma_1(d_p\phi) < 1$, we set $\dim_{\mathrm{L}}(\phi, p) = 0$. Note that the functions $p \mapsto \sigma_i(d_p\phi)$ $(i = 1, \ldots, m)$ are continuous on U. The function $p \mapsto \dim_{\mathrm{L}}(\phi, p)$ is continuous on U except at a point u which satisfies $\sigma_1(d_u\phi) = 1$, where it is still upper semi-continuous. For a compact set $K \subset M$ we introduce the notation

$$\dim_{\mathrm{L}}(\phi, \widetilde{K}) = \sup_{p \in \widetilde{K}} \dim_{\mathrm{L}}(\phi, p).$$

We denote by $\dim_{\mathrm{B}} K$ the *box dimension* of K.

Our main result is the following

Theorem 1. *Let M be a Riemannian C^3 manifold. Let $U \subset M$ be an open set and let $\phi: U \to M$ be a C^1 map. For compact sets $K, \widetilde{K} \subset U$ which satisfy $K \subset \phi^t(K) \subset \widetilde{K}$ for all $t \in \mathbb{N}$ we have $\dim_{\mathrm{B}} K \leq \dim_{\mathrm{L}}(\phi, \widetilde{K})$.*

We will prove Theorem 1 in Chapter 3.

Remark 1. Note that for an invariant set K, i. e. if $\phi(K) = K$, the dimension bound in Theorem 1 can be shown using the result by Hunt [7] and an embedding result by Whitney [19]. Let us sketch this idea. For a compact m-dimensional C^k manifold Whitney [19] proves the existence of a C^k embedding $i\colon M \to \mathbb{R}^{2m+1}$. Note that in particular the box dimensions of $K \subset M$ and of $i(K)$ coincide. For the C^1 map $\phi\colon U \to M$ we find a C^1 extension of the lifted map $\widetilde{\phi} := i \circ \phi \circ i^{-1}$ onto an open neighborhood of $i(K) \subset \mathbb{R}^{2m+1}$. By [7: Theorem 1] we have $\dim_{\mathrm{B}} K \le \dim_{\mathrm{L}}(\widetilde{\phi}, i(K))$. Using Lyapunov-type functions (compare, for instance, [11]) one can show the existence of a number $l \in \mathbb{N}$ satisfying $\dim_{\mathrm{L}}(\widetilde{\phi}^l, i(K)) \le d$ provided $\dim_{\mathrm{L}}(\phi, K) \le d$ for some $d \in (0, m]$. This proves the statement.

Remark 2. Theorem 1 improves the estimate in [2: Corollary 5] which shows its efficiency especially for low-dimensional objects.

We present a version of Theorem 1 for Hilbert spaces which can be shown with no more than technical alterations (cp., for instance, the approach in [17]). One could instead consider a finite- or infinite-dimensional Riemannian manifold which is modeled on a Hilbert space and thus unifying both results. Our arguments, being of local nature, would carry through, but at the expense of notational simplicity.

Let us introduce for a linear bounded operator L in a Hilbert space H the singular value function ω_d which is given by

$$\begin{aligned} &\omega_0(L) = 1 && \text{for } d = 0 \\ &\omega_d(L) = \omega_{\lfloor d \rfloor}(L)^{1-d+\lfloor d \rfloor}\omega_{\lfloor d \rfloor+1}(L)^{d-\lfloor d \rfloor} && \text{for } d > 0, \end{aligned}$$

where $\omega_k(L) = \| \wedge^k L\|$ for $k \in \mathbb{N}$. Here $\wedge^k L$ denotes the k-th exterior product of L. Recall that if L is compact, then $\omega_k(L)$ equals the product of the k-th largest eigenvalues of $(L^*L)^{1/2}$ (for more details see [17: Section V.1]).

Theorem 2. *Let H be a Hilbert space and let $U \subset H$ be an open set. Let $\Phi\colon U \to H$ be a continuously Frechét differentiable map with Frechét derivative Φ' and let $K, \widetilde{K} \subset U$ be two compact sets satisfying $K \subset \Phi^t(K) \subset \widetilde{K}$ for all $t \in \mathbb{N}$. Then*

$$\dim_{\mathrm{B}} K \le \inf \left\{ d : \sup_{p \in \widetilde{K}} \omega_d(\Phi'(p)) < 1 \right\}.$$

Remark 3. Note that Theorem 2 weakens the prerequisite to the singular value function in [17: Theorem 3.2]. Theorem 2 weakens the differentiability assumptions in [1: Theorem 2]. A detailed analysis of attractors for a number of dissipative partial differential equations can be found e.g. in [17] where, however, only a weaker form of differentiability is discussed. We point out that for a large class of systems (for instance, of evolutionary systems) continuous Frechét differentiability can indeed be guaranteed.

2. Preliminary results

Let us first recall some definitions in order to fix notations. Let K be a compact set in a metric space M. For a number $\varepsilon > 0$ denote by $N_\varepsilon(K)$ the smallest number of ε-balls which are needed to cover K. For $d \geq 0$ we call

$\mu(K, d, \varepsilon) = N_\varepsilon(K)\varepsilon^d$ the *capacitive* (d, ε)*-measure* of K

$\mu(K, d) = \limsup_{\varepsilon \to 0+0} \mu(K, d, \varepsilon)$ the *upper capacitive* d*-measure* of K

$\dim_{\mathrm{B}} K = \inf\{d \geq 0 : \mu(K, d) = 0\}$ the *box dimension* of K.

Recall that the Hausdorff dimension $\dim_{\mathrm{H}} K$ is defined similarly except in using covers with balls of varying radii less or equal ε, and that always $\dim_{\mathrm{H}} K \leq \dim_{\mathrm{B}} K$.

Notice that the capacitive (d, ε)-measure is not monotonous in ε which is the main reason why estimating techniques for the Hausdorff dimension (e.g. [3]) do not carry over immediately to the box dimension. To circumvent this we use the following result from [6].

Lemma 1. *Let M be a metric space. If for a compact set $K \subset M$ and for numbers $d \geq 0, \varepsilon' > 0$ and $0 < D < 1$ we have $\mu(K, d, D\varepsilon) \leq \mu(K, d, \varepsilon)$ for every $\varepsilon \in (0, \varepsilon']$, then* $\dim_{\mathrm{B}} K \leq d$.

Proof. Let $r \in (0, \varepsilon')$ be chosen arbitrarily. Because of $D < 1$ there exists a number $j \in \mathbb{N}$ for which $D^j\varepsilon' \leq r < D^{j-1}\varepsilon'$. Therefore,

$$\mu(K, d, r) = N_r(K) r^d < N_{D^j\varepsilon'}(K)\,(D^{j-1}\varepsilon')^d = D^{-d}\mu(K, d, D^j\varepsilon'). \tag{1}$$

Setting $\varepsilon = D^{j-1}\varepsilon', \ldots, \varepsilon = \varepsilon'$ we obtain $\mu(K, d, r) < D^{-d}\mu(K, d, \varepsilon')$. Since K is compact, $\mu(K, d, \varepsilon')$ is finite. Thus, $\mu(K, d, r)$ is uniformly bounded from above for all $r < \varepsilon'$ which implies $\dim_{\mathrm{B}} K \leq d$ ∎

Let for the following E and E' be two m-dimensional Euclidean spaces and let $\mathcal{E} \subset E$ be an ellipsoid. Denote by $\sigma_1(\mathcal{E}) \geq \ldots \geq \sigma_m(\mathcal{E}) \geq 0$ the singular values of $\mathcal{E}$, i.e. the lengths of its half-axes. Correspondingly to the singular value function of a linear map, for $d \in [0, m]$ we introduce the *singular value function* of $\mathcal{E}$ *order* d by

$$\begin{aligned} &\omega_0(\mathcal{E}) = 1 && \text{for } d = 0 \\ &\omega_d(\mathcal{E}) = \sigma_1(\mathcal{E}) \cdots \sigma_{\lfloor d \rfloor}(\mathcal{E})\sigma_{\lfloor d \rfloor + 1}(\mathcal{E})^{d - \lfloor d \rfloor} && \text{for } d \in (0, m]. \end{aligned}$$

We call $\mathcal{E}$ *degenerated* if $\sigma_i(\mathcal{E}) = 0$ for some $i \in \{1, \ldots, m\}$. Note that for the ε-ball $B(O, \varepsilon)$ centered in the origin of E and for a linear map L: $E \to E'$ we have $\omega_d(LB(O, \varepsilon)) = \omega_d(L)\varepsilon^d$.

The main technique in the proof of Theorem 1 is to study in first approximation the deformation of small balls under the map ϕ. Further, the capacitive measure of the image of such a ball is estimated in terms of singular value functions using certain covering strategies. We reformulate [3: Lemma 3] which is true for a non-degenerated ellipsoid $\mathcal{E}$ only, in order to take into consideration possibly degenerated ellipsoids. (The case of degeneracy is of interest if the considered map ϕ is non-injective.)

Lemma 2. *Let $\mathcal{E}$ be an ellipsoid in an m-dimensional Euclidean space E. Let $\theta, \kappa, \varsigma$ be positive numbers, $d \in (0, m]$ and $\kappa \leq \theta^d$. Suppose that $\omega_s(\mathcal{E})\varsigma^{d-s} \leq \kappa$ for any $s = 0, \ldots, \lfloor d \rfloor$, $\omega_d(\mathcal{E}) \leq \kappa$ and $\sigma_1(\mathcal{E}) \leq \theta$. Then for any $\eta > 0$ the sum $\mathcal{E} + B(O, \eta)$ is contained in an ellipsoid $\mathcal{E}' \subset E$ which satisfies*

$$\omega_d(\mathcal{E}') \leq \left(1 + \left(\frac{\theta^{\lfloor d \rfloor}}{\kappa}\right)^{\frac{1}{d - \lfloor d \rfloor}} \eta\right)^d \kappa$$

$$\sigma_{\lfloor d \rfloor + 1}(\mathcal{E}') \leq \left(1 + \left(\frac{\theta^{\lfloor d \rfloor}}{\kappa}\right)^{\frac{1}{d - \lfloor d \rfloor}} \eta\right) \max\left\{\varsigma, \sigma_{\lfloor d \rfloor + 1}(\mathcal{E})\right\}.$$

Proof. We "fatten" the ellipsoid $\mathcal{E}$ as follows: If $\sigma_{\lfloor d \rfloor + 1}(\mathcal{E}) < \varsigma$, we replace the singular values $\sigma_{\lfloor d \rfloor + 1}(\mathcal{E}), \ldots, \sigma_n(\mathcal{E})$ by ς. These values determine a non-degenerate ellipsoid which contains $\mathcal{E}$ and for which [3: Lemma 3] can be applied ∎

With Lemma 2 and with methods in [17: Section V.3.2] we obtain

Lemma 3. *We keep the assumptions of Lemma 2. Then for any number $\eta > 0$ the set $\mathcal{E} + B(O, \eta)$ can be covered by $\left[\frac{2^{\mathrm{d}}\kappa}{\widetilde{\varsigma}^{\mathrm{d}}}\right]$ balls with radius*

$$\left(1 + \left(\frac{\theta^{\lfloor d \rfloor}}{\kappa}\right)^{\frac{1}{d - \lfloor d \rfloor}} \eta\right) \widetilde{\varsigma} \sqrt{\lfloor d \rfloor + 1}$$

where we set $\widetilde{\varsigma} = \max\left\{\varsigma, \sigma_{\lfloor d \rfloor + 1}(\mathcal{E})\right\}$.

3. Proof of Theorem 1

The proof can be sketched as follows: We consider the non-trivial case that $\dim_{\mathrm{L}}(\phi, \widetilde{K}) < m$ and choose an arbitrary number $d \in \left(\dim_{\mathrm{L}}(\phi, \widetilde{K}), m\right)$. For sufficiently small $\varepsilon > 0$ we fix a cover $\mathcal{U}$ of K consisting of finitely many ε-balls. We choose a sufficiently small number $\gamma \in (0, 1)$ and construct a cover $\mathcal{G}$ of K consisting of so-called "filial" balls each of radius $R \approx \gamma\varepsilon$. Here γ depends only on the oscillation of the singular values $\sigma_{\lfloor d \rfloor + 1}(d_p\phi)$ for $p \in \widetilde{K}$. For this homogeneous cover $\mathcal{G}$ we prove

$$\sum_{S \in \mathcal{G}} \mu(S, d, R) < \sum_{S \in \mathcal{U}} \mu(S, d, \varepsilon)$$

and apply Lemma 1.

First we prove the following lemma in which we set $\widehat{K} = \overline{\cup_{t \geq 0} \phi^t(K)}$.

Lemma 4. *Let $d \in (0, m)$. Assume that*

$$2(8\sqrt{\lfloor d \rfloor + 1})^d \omega_d(d_p\phi) < 1 \qquad (p \in \widehat{K}).$$

Then there exist numbers $\varepsilon_0 > 0$ and $\alpha, \beta \in (0,1)$ and a uniformly continuous function $\sigma\colon \widehat{K} \to (\alpha, \beta)$ such that for any $l \in \mathbb{N}$ the following holds:

For every ball $B(q, \varepsilon)$ with $\varepsilon \in (0, \varepsilon_0)$ and $q \in \phi^l(K)$ there exists a family of filial balls $\mathcal{F}^{(1)}(B(q,\varepsilon))$, each with radius $\sigma(q)\varepsilon$ and center point in $\phi^{l+1}(K)$, whose union covers $\phi(B(q,\varepsilon)) \cap \phi^{l+1}(K)$. For the minimal number $N(q)$ of balls in $\mathcal{F}^{(1)}(B(q,\varepsilon))$ we have $N(q) \leq \frac{1}{2(\sigma(q))^d}$.

Proof. We introduce the notation $\omega_d(\phi, K) = \max_{p\in K} \omega_d(d_p\phi)$. Choose a number $h > \omega_d(\phi, \widehat{K})$ satisfying

$$\left(8\sqrt{\lfloor d \rfloor + 1}\right)^d h < \frac{1}{2} \tag{3}$$

and an open set $V \subset U$ containing $\widehat{K}$ and which is itself contained within a compact set $A \subset U$ satisfying $\omega_d(\phi, A) < h$. Further, choose a number $\kappa < 1$ satisfying $\omega_d(\phi, A) \leq \kappa < h$ and a number $\theta > 0$ for which $\kappa \leq \theta^d$ and $\omega_1(\phi, A) \leq \theta$ hold and set $C = \left(\frac{\theta^{\lfloor d \rfloor}}{\kappa}\right)^{\frac{1}{d - \lfloor d \rfloor}}$. At last, choose $\varsigma > 0$ satisfying

$$\omega_s(\phi, \widehat{K})\varsigma^{d-s} \leq \kappa \qquad (s = 0, \ldots, \lfloor d \rfloor). \tag{4}$$

The equation

$$(1 + C\eta)^d \kappa = h \tag{5}$$

uniquely determines a number $\eta > 0$. Since

$$\sup_{p\in A} \sigma_{\lfloor d \rfloor + 1}(d_p\phi) \leq \omega_d(\phi, A)^{\frac{1}{d}} \leq \kappa^{\frac{1}{d}}$$

we have

$$(1 + C\eta) \sup_{p \in A} \sigma_{\lfloor d \rfloor + 1}(d_p\phi) \leq h^{\frac{1}{d}}. \tag{6}$$

Denote by $\exp_q\colon T_qM \to M$ the exponential map at a point $q \in \widetilde{K}$. Since $\exp_q$ is a smooth map which satisfies $\|d_{O_q} \exp_q\| = 1$, for every point $q \in M$ there is a number $\delta_q > 0$ such that $\|d_v \exp_q\| \leq 2$ for any $v \in B(O_q, \delta_q)$. Further, since $\widetilde{K}$ is compact, there is a number $\delta_0 = \min_{q\in\widetilde{K}} \delta_q > 0$ and, consequently,

$$\rho(\exp_q v_1, \exp_q v_2) \leq 2\|v_1 - v_2\|_{T_qM}$$

for any $q \in \widetilde{K}$ and any $v_1, v_2 \in B(O_q, \delta_0)$. Here $\rho(\cdot,\cdot)$ denotes the geodesic distance on M and $\tau_u^q\colon T_uM \to T_qM$ denotes the isometric operator defined by the parallel transport along the geodesic for points lying sufficiently close to each other. Let $\varepsilon_0 > 0$ be so small such that:

1) For every $q \in \widehat{K}$, the set $B(q, 2\varepsilon_0)$ is contained in V.
2) $\varepsilon_0(1+\theta+\eta) \le \delta_0$.
3) $\|\tau_{\phi(w)}^{\phi(q)} \circ d_w\phi \circ \tau_q^w - d_q\phi\| \le \eta$ for all $w, q \in V$ with $\rho(w,q) \le \varepsilon_0$.

Then every ball $B(q,\varepsilon)$ $(\varepsilon \le \varepsilon_0)$ which intersects $\widehat{K}$ is contained in V. Taylor's formula for the differentiable map ϕ gives for any $u \in B(q,\varepsilon)$ the estimate

$$\begin{aligned} &\big\| \exp_{\phi(q)}^{-1} \phi(u) - d_q\phi(\exp_q^{-1} u)\big\| \\ &\qquad \le \sup_{w\in B(q,\varepsilon)} \big\|\tau_{\phi(w)}^{\phi(q)} \circ d_w\phi \circ \tau_q^w - d_q\phi\big\| \, \|\exp_q^{-1} u\| \end{aligned}$$

which together with property 3) implies the relation

$$\phi(B(q,\varepsilon)) \subset \exp_{\phi(q)} \big(d_q\phi\, B(O_q,\varepsilon) + B(O_{\phi(q)}, \eta\varepsilon)\big)\,. \tag{7}$$

Because of the choice of ς in (4) and because of Lemma 3, for every point $q \in \phi^l(K)$ the set $d_q\phi\, B(q,\varepsilon) + B(O_{\phi(q)},\eta\varepsilon)$ can be covered by $\big[\frac{2^d\kappa}{\widetilde{\varsigma}^d}\big]$ balls of radius $\sqrt{\lfloor d\rfloor+1}(1+C\eta)\widetilde{\varsigma}\varepsilon$ where $\widetilde{\varsigma} = \max\{\varsigma, \sigma_{\lfloor d\rfloor+1}(d_q\phi)\}$. Here the cover can be evidently chosen in such a way that any ball is contained in a ball of radius $(1+\theta+\eta)\varepsilon$ centered at $O_{\phi(q)}$, which follows from $\omega_1(d_q\phi) < \theta$ and from (6) and (3). Hence, by (7) and property 2), the set $\phi(B(q,\varepsilon))$ can be covered by $\big[\frac{2^d\kappa}{\widetilde{\varsigma}^d}\big]$ balls of radius $2\sqrt{\lfloor d\rfloor+1}(1+C\eta)\widetilde{\varsigma}\varepsilon$. For this cover any ball intersecting $\phi^{l+1}(K)$ can be replaced by a ball which is centered at a point in $\phi^{l+1}(K)$ and with twice the radius.

For $u \in U$ we put

$$\sigma(u) = 4\sqrt{\lfloor d\rfloor+1}\,(1+C\eta)\cdot \max\big\{\varsigma, \sigma_{\lfloor d\rfloor+1}(d_u\phi)\big\}.$$

Thus, for any $q \in \phi^l(K)$ the set $\phi(B(q,\varepsilon)) \cap \phi^{l+1}(K)$ can be covered by $N(q)$ balls $B(q_j, \sigma(q)\varepsilon)$ $\big(j = 1,\dots,N(q)\big)$ which are centered at $q_j \in \phi^{l+1}(K)$. Here we have

$$N(q) \le \left[\frac{2^d\kappa}{\sigma(q)^d}\big(4\sqrt{\lfloor d\rfloor+1}(1+C\eta)\big)^d\right] \le \frac{1}{2\sigma(q)^d}$$

where for the second inequality we have used (5) and (3). The function $\sigma\colon U \to \mathbb{R}, u \mapsto \sigma(u)$ is uniformly continuous on the compact set $\widehat{K}$ because of smooth dependence of the singular values of $d_u\phi$ on u. Because of (6) and (3) there exist numbers $\alpha, \beta \in (0,1)$ for which

$$\alpha < \sigma(u) < \beta \qquad (u \in \widehat{K}). \tag{8}$$

This proves the lemma ∎

Now we are ready to give the

Proof of Theorem 1. Let us assume that $\dim_{\mathrm{L}}(\phi, \widetilde{K}) < m$. Let us choose an arbitrary number $d \in (\dim_{\mathrm{L}}(\phi, \widetilde{K}), m)$. Recall that

$$\sup\left\{\omega_d(d_p\phi^t) : p \in \bigcup_{\tau\geq 0} \phi^\tau(K)\right\} \leq \max_{p\in\widetilde{K}} \omega_d(d_p\phi)^t$$

for any natural number t. Hence, for a sufficiently large number $t \in \mathbb{N}$, we have $2(8\sqrt{\lfloor d\rfloor + 1})^d \omega_d(\phi^t, \widehat{K}) < 1$ and thus the prerequisite of Lemma 4 is satisfied for the map $\psi = \phi^t$.

Step 1: We choose an arbitrary finite cover $\mathcal{U} = \cup_{j=1}^J B(p_j, \varepsilon)$ of K with $p_j \in K$.

Step 2: We construct a family of filial covers. Indeed, by Lemma 4, for any ball $B(p_j, \varepsilon)$ we find a family of balls

$$\mathcal{F}^{(1)}(B(p_j,\varepsilon)) = \left\{B(q_i, \sigma(p)\varepsilon)\right\}_{i=1}^{N(p)}$$

which cover the set $\psi(B(p_j,\varepsilon)) \cap \psi(K)$. We call $\mathcal{F}^{(1)}(B(p_j,\varepsilon))$ in accordance with [1] a *family of filial balls* for $B(p_j,\varepsilon)$ *of order* 1. Further, we define a sequence of filial covers recursively by setting

$$\mathcal{F}^{(t)}(B(p_j,\varepsilon)) = \bigcup \left\{\mathcal{F}^{(1)}(S) : S \in \mathcal{F}^{(t-1)}(B(p_j,\varepsilon))\right\} \qquad (2 \leq t \in \mathbb{N}).$$

Let us denote by $r(S)$ the radius of a ball S. For each family of filial balls of $B(p_j,\varepsilon)$ $(p_j \in K)$ of order t we obtain therefore the estimates

$$\begin{aligned} \sum_{S\in\mathcal{F}^{(t)}(B(p_j,\varepsilon))} r(S)^d &= \sum_{S\in\mathcal{F}^{(t-1)}(B(p_j,\varepsilon))} \sum_{S'\in\mathcal{F}^{(1)}(S)} r(S')^d \\ &\leq \sum_{S\in\mathcal{F}^{(t-1)}(B(p_j,\varepsilon))} \frac{r(S)^d}{2} \\ &\leq \frac{\varepsilon^d}{2^t}. \end{aligned} \tag{9}$$

Step 3: We assign certain iteration depths. Indeed, for every point $p \in K$ we fix a prehistory $\{s_0(p), s_1(p), \ldots\}$ with respect to ψ as follows: We set

$$\begin{aligned} &s_0(p) = p \\ &s_i(p) = q \quad (i \geq 1) \text{ for some } q \in \left\{u \in K : \psi(u) = s_{i-1}(p)\right\}. \end{aligned}$$

Further, we choose some number $\gamma \in (0, \frac{1}{2})$ satisfying

$$\frac{2^{-\frac{\log\gamma}{\log\alpha}}}{\alpha^d} < 2^{-(d+2)}. \tag{10}$$

Because of (8), to any point $p \in K$ we can assign a prehistory of finite length $I(p)$ for which the inequalities

$$\alpha\gamma < \sigma(s_1(p)) \cdots \sigma(s_{I(p)}(p)) \le \gamma \tag{11}$$

hold. Because of (8) and (11) we obtain $\alpha\gamma < \beta^{I(p)}$ and $\alpha^{I(p)} < \gamma$, and therefore

$$\frac{\log \alpha\gamma}{\log \beta} > I(p) > \frac{\log \gamma}{\log \alpha} \tag{12}$$

for any $p \in K$. Without loss of generality we assume that γ has been chosen small enough such that $I(p) > 1$ for all $p \in K$. We set $I = \sup_{p \in K} I(p)$ which is finite because of (12).

Step 4: We construct the homogeneous cover $\mathcal{G}$ of K. First, for each point $p \in K$ we construct a ball of radius approximately $\gamma\varepsilon$ containing p as follows: We take the prehistory $\{s_0(p), s_1(p), \ldots\}$ of p and choose some ball in $\mathcal{U}$ which contains the point $s_{I(p)}(p)$ and denote it by $B_{p,I(p)}$. Along the orbit

$$s_{I(p)}(p) \mapsto s_{I(p)-1}(p) \mapsto \cdots \mapsto s_1(p) \mapsto s_0(p) = p$$

of length $I(p)$ we construct balls $B_{p,I(p)-(i+1)}$ $\quad (i = 0, \ldots, I(p) - 1)$ which are defined recursively as follows. The union of filial balls of the family $\mathcal{F}^{(1)}(B_{p,I(p)-i})$ covers the set $\psi(B_{p,I(p)-i}) \cap \psi^i(K)$. Choose $B_{p,I(p)-(i+1)}$ as ball from this cover which contains the point $s_{I(p)-(i+1)}(p)$. We obtain

$$s_{I(p)}(p) \in B_{p,I(p)}, \ldots, s_0(p) = p \in B_{p,0}.$$

We denote by $\widetilde{s}_i(p)$ the center point of the corresponding ball $B_{p,i}$ $\quad (i = 0, \ldots, I(p))$. By construction in the proof of Lemma 4, $\widetilde{s}_i(p) \in \psi^{I(p)-i}(K)$. Since $B_{p,0}$ is an element of a family of filial balls for $B_{p,I(p)}$ of order $I(p)$ we have

$$r(B_{p,0}) = \sigma(\widetilde{s}_{I(p)}(p)) \cdots \sigma(\widetilde{s}_1(p))\, r(B_{p,I(p)}).$$

Since $B_{p,I(p)} \in \mathcal{U}$, we have $r(B_{p,I(p)}) = \varepsilon$ and therefore

$$r(B_{p,0}) = \sigma(\widetilde{s}_{I(p)}(p)) \cdots \sigma(\widetilde{s}_1(p))\varepsilon. \tag{13}$$

Further,

$$\rho\big(s_{I(p)-i}(p), \widetilde{s}_{I(p)-i}(p)\big) \le \varepsilon \qquad \big(i = 0, \ldots, I(p)\big). \tag{14}$$

Now we choose a sub-family $\widetilde{\mathcal{G}} = \{B_{p_l,0}\}_{l=1}^{L}$ of the family $\{B_{p,0}\}_{p \in K}$ such that the union $\cup_{l=1}^{L} B_{p_l,0}$ covers the compact set K and set

$$R = \max_{l=1,\ldots,L} r(B_{p_l,0}). \tag{15}$$

Each ball $B_{p_l,0}$ in $\widetilde{\mathcal{G}}$ with radius $r(B_{p_l,0})$ and center point $p_l = \widetilde{s}_0(p_l)$ can be replaced by the concentric ball with radius R. This gives us a cover

$$\mathcal{G} = \{B(\widetilde{s}_0(p_l), R)\}_{l=1}^{L}$$

of K with balls of equal radius R, where $R \in (\alpha\gamma\varepsilon, \gamma\varepsilon]$ because of (11).

Step 5: We study the oscillation of the radii of balls within the cover $\widetilde{\mathcal{G}}$. For this choose some number $\Delta > 1$ satisfying

$$\Delta^{2dI} < 2. \tag{16}$$

From Lemma 4 we obtain the uniform continuity of the function σ on $\widehat{K}$. Further, by (8) this function is on $\widehat{K}$ uniformly bounded from below by a positive number. Thus, there exists $\varepsilon_1 > 0$ such that

$$\frac{\sigma(p)}{\sigma(q)} \le \Delta \qquad \big(p, q \in \widehat{K}, \rho(p,q) < \varepsilon_1\big). \tag{17}$$

Allowing for change of ε_0, it suffices to use $\varepsilon_1 = \varepsilon_0$ already obtained. From (14), (17) and (11), for every $l = 1, \dots, L$ we conclude

$$\frac{r(B_{p_l,0})}{\varepsilon} = \sigma(\widetilde{s}_1(p_l)) \cdots \sigma(\widetilde{s}_{I(p_l)}(p_l)) \le \Delta^{I(p_l)}\gamma.$$

Analogously we obtain

$$\frac{\varepsilon}{r(B_{p_l,0})} \le \frac{\Delta^{I(p_l)}}{\sigma(s_1(p_l)) \cdots \sigma(s_{I(p_l)}(p_l))} < \frac{\Delta^{I(p_l)}}{\alpha\gamma}.$$

Thus, for any two balls $B_{p_l,0}$ and $B_{p_k,0}$ from the cover $\widetilde{\mathcal{G}}$ we have

$$\frac{r(B_{p_l,0})}{r(B_{p_k,0})} < \frac{\Delta^{I(p_k)}}{\alpha\gamma}\Delta^{I(p_l)}\gamma \le \Delta^{2I}\frac{1}{\alpha}.$$

Finally, for the radius R, from (15)

$$R \le \frac{\Delta^{2I}}{\alpha} r(B_{p_l,0}) \tag{18}$$

follows.

Step 6: We estimate the capacitive measure of K. Recall that $\mu(\cdot, d, R)$ is an outer measure on M. Since K has been covered by balls from $\mathcal{G}$ with equal radius R, we obtain by (18)

$$\mu(K,d,R) \le \sum_{l=1}^{L} \mu\big(B(\widetilde{s}_0(p_l), R), d, R\big) = \sum_{l=1}^{L} R^d \le \frac{\Delta^{2dI}}{\alpha^d} \sum_{l=1}^{L} r(B_{p_l,0})^d. \tag{19}$$

To each ball $B_{p_l,0}$ $(l = 1, \ldots, L)$ we assigned a ball $B(p_j, \varepsilon) \in \mathcal{U}$ such that $B_{p_l,0}$ belongs to the family of filial balls of $B(p_j, \varepsilon)$ of order $I(p_j)$. Consequently, each term in sum in the rightmost term in (19) occurs at most once as term in the sum

$$\sum_{j=1}^{J} \sum_{i=I^*}^{I} \sum_{S \in \mathcal{F}^{(i)}(B(p_j,\varepsilon))} r(S)^d$$

where we have set $I^* = \min_{p \in K} I(p)$. Thus, we obtain

$$\begin{aligned} \mu(K,d,R) &\leq \frac{\Delta^{2dI}}{\alpha^d} \sum_{j=1}^{J} \sum_{i=I^*}^{I} \sum_{S \in \mathcal{F}^{(i)}(B(p_j,\varepsilon))} r(S)^d \\ &\leq \frac{\Delta^{2dI}}{\alpha^d} J \sum_{i=I^*}^{\infty} \frac{1}{2^i} \varepsilon^d \end{aligned}$$

where we have used (9). By (12) and by definition of the number I there holds

$$\frac{\log \gamma}{\log \alpha} < I^* \leq I.$$

From this we deduce for the capacitive measure

$$\mu(K,d,R) < J\, 2^{-\frac{\log \gamma}{\log \alpha}} 2\varepsilon^d \frac{\Delta^{2dI}}{\alpha^d}.$$

Now (16) and (10) imply

$$\mu(K,d,R) < 4\, \frac{2^{-\frac{\log \gamma}{\log \alpha}}}{\alpha^d}\, J\, \varepsilon^d < 2^{-d} J \varepsilon^d. \tag{20}$$

Step 7: We apply Lemma 1. Indeed, the initial cover $\mathcal{U}$ of K of balls of radius ε centered in a point in K has been chosen arbitrarily. Any ball intersecting K of radius $\frac{\varepsilon}{2}$ can be replaced by one which is centered in K and with radius ε. Thus, we can replace the right-hand side in (20) by $\mu(K, d, \frac{\varepsilon}{2})$. All assumptions of Lemma 1 are thus satisfied. From $\mu(K,d,R) < \mu(K,d,\frac{\varepsilon}{2})$ and $R \leq \gamma\varepsilon < \frac{\varepsilon}{2}$ the estimate $\dim_{\mathrm{B}} K \leq d$ follows. This holds for arbitrary $d > \dim_{\mathrm{L}}(\psi, \widetilde{K})$ which proves Theorem 1 ∎

We note that related arguments as, for instance, the consideration of filial covers were used by Blinchevskaya and Ilyashenko in [1].

4. Discussion

Let us consider the long-time behavior of the dynamical system $\phi\colon \mathbb{N}\times K \to K$ generated by the iterates ϕ^t $(t \in \mathbb{N})$ on an invariant set $K \subset U$. We introduce the *global Lyapunov exponents* $\nu_1^u \geq \ldots \geq \nu_m^u$ of ϕ on K which are recursively defined by

$$\nu_1^u + \ldots + \nu_j^u = \lim_{t\to\infty} \frac{1}{t} \log \max_{p\in K} \omega_j(d_p\phi^t) \qquad (j = 1, \ldots, m).$$

The *Lyapunov dimension* of ϕ on K with respect to the global Lyapunov exponents is

$$d_{\mathrm{L}}^u(\phi, K) = k + \frac{\nu_1^u + \ldots + \nu_k^u}{|\nu_{k+1}^u|}$$

where $k \in \{0, \ldots, m-1\}$ denotes the smallest number satisfying $\nu_1^u + \ldots + \nu_{k+1}^u < 0$.

Upper estimates for the Hausdorff dimension in terms of the global Lyapunov exponents have been derived for systems on Riemannian manifolds (see, e.g., [10, 13]), in Hilbert spaces (see, e.g., [5, 17]), and in Banach spaces [18]. Using Theorem 1 and the method of proof of [17: Theorem 3.3] we obtain the following

Theorem 3. *Let M be a Riemannian C^3 manifold. Let $U \subset M$ be an open set, and let $\phi\colon U \to M$ be a C^1 map. For a compact and invariant set $K \subset U$ we have* $\dim_{\mathrm{B}} K \leq d_{\mathrm{L}}^u(\phi, K)$.

Remark 4. Since for invariant sets K the function $t \mapsto \max_{p\in K} \omega_d(d_p\phi^t)$ is sub-exponential (cp. [17]), we have

$$d_{\mathrm{L}}^u(\phi, K) \leq \inf_{t\in\mathbb{N}} \dim_{\mathrm{L}}(\phi^t, K) = \lim_{t\to\infty} \dim_{\mathrm{L}}(\phi^t, K). \tag{21}$$

Further, recall that

$$\inf_{t\to\infty} \dim_{\mathrm{L}}(\phi^t, K) = \sup_{\mu} \dim_{\mathrm{L}}(\phi, \mu)$$

where $\dim_{\mathrm{L}}(\phi, \mu)$ denotes the Lyapunov dimension with respect to the Lyapunov exponents of μ and where the supremum is taken over all invariant ergodic probability measures supported on K (see [10]). Hence, [7: Corollary 2] is a special case of Theorem 3.

In the general case, the dimension bound in Theorem 3 is so hard to compute as the values in (21). This motivates the following investigation of local Lyapunov exponents. In correspondence to the global Lyapunov exponents the *local Lyapunov exponents* $\nu_1(p) \geq \ldots \geq \nu_m(p)$ of ϕ at a point $p \in K$ are defined recursively by

$$\nu_1(p) + \ldots + \nu_j(p) = \limsup_{t\to\infty} \frac{1}{t} \log \omega_j(d_p\phi^t) \qquad (j = 1, \ldots, m).$$

The *local Lyapunov dimension* of ϕ at p with respect to the local Lyapunov exponents is then given by

$$d_{\mathrm{L}}(\phi, p) = k(p) + \frac{\nu_1(p) + \ldots + \nu_{k(p)}(p)}{|\nu_{k(p)+1}(p)|}$$

where $k(p) \in \{0, \ldots, m-1\}$ denotes the smallest number satisfying $\nu_1(p) + \ldots + \nu_{k(p)+1}(p) < 0$.

Remark 5. For an invariant set K the inequality

$$\sup_{p\in K} d_{\mathrm{L}}(\phi, p) \leq d_{\mathrm{L}}^u(\phi, K)$$

has been proven by Eden [4]. He presumed that for a "typical system" there exists always a point p satisfying $d_{\mathrm{L}}(\phi, p) = d_{\mathrm{L}}^u(\phi, K)$ but he refers also to examples for which strict inequality holds. Leonov [12] verifies, for example, the Hénon system and the Lorenz system as typical systems in that sense.

5. The Lorenz system

To handle systems on Riemannian manifolds gives us the freedom also to construct adapted metrics. Recall that, within a class of equivalent metrics, the box dimension of a compact set is the same. However, notice that the local Lyapunov dimensions of a differentiable map strongly depends on the Riemannian metric. This fact can be used to optimize dimension estimates (see, for instance, [2, 13], and [11] for a related approach using adapted Lyapunov functions).

As an example we consider the flow $\phi\colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ of the Lorenz system

$$\left.\begin{aligned} \dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= rx - y - xz \\ \dot{z} &= -bz + xy \end{aligned}\right\} \tag{22}$$

with given parameters $b = \frac{8}{3}, \sigma = 10$ and $r = 28$. The flow is dissipative and has a global attractor $K \subset \mathbb{R}^3$. Since the divergence of the vector field f

given by (22) equals $\operatorname{div} f = -(\sigma + 1 + b) < 0$, the Lorenz system is volume-contracting, hence the Lorenz attractor has Lebesgue measure zero. Leonov [12] shows that the maximal local Lyapunov dimension of the time-1-map ϕ^1 on K equals the local Lyapunov dimension of ϕ^1 at the equilibrium point $p_0 = (0,0,0)$. There he used certain linear transformations and Lyapunov-type functions. We put this approach into the framework of adaption of metrics. For this we introduce the family of matrices

$$S(p) = \exp\Big(\frac{V(p)}{d}\Big) A$$

with

$$A = \begin{pmatrix} a & 0 & 0 \\ -\frac{1}{\sigma}(b-1) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where

$$a = \frac{1}{\sigma}\sqrt{r\sigma + (b-1)(\sigma - b)}$$

and

$$V(p) = \frac{1}{2a\theta}(3-d)\Big(\gamma_1 x^2 + \gamma_2\Big(y^2 + z^2 - x^2\frac{(b-1)^2}{\sigma^2}\Big) + \gamma_3 z\Big)$$

with

$$\theta = 2\sqrt{(\sigma + 1 - 2b)^2 + (2\sigma b)^2}$$

and

$$\begin{aligned} \gamma_2 &= \frac{1}{2a}\ , \gamma_3 = -\frac{4\sigma a}{b} \\ \gamma_1 &= -\frac{1}{2\sigma}\Big[2\gamma_2\frac{r\sigma - (b-1)^2}{\sigma} + \gamma_3 + \frac{2(b-1)}{a\sigma}\Big]. \end{aligned}$$

We consider the metric tensor g on $\mathbb{R}^3$ given at a point $p \in \mathbb{R}^3$ by

$$g(p)(v,w) = \exp\Big(2\frac{V(p)}{d}\Big)\langle A^T A v, w\rangle_{\mathbb{R}^3}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^3}$ is induced by the Euclidean metric. Here d is a positive parameter which will be specified below. Note that with respect to this metric structure the singular value function of order $2+s$ $(s \in (0,1))$ of the map ϕ^1 can be estimated as

$$\begin{aligned} \omega_{2+s}(d_p\phi^1) &\le \exp\int_0^1 \Big[\lambda_1(\phi^\tau(p)) + \lambda_2(\phi^\tau(p)) + s\lambda_3(\phi^\tau(p))\Big] d\tau \\ &\quad + V(\phi^1(p)) - V(p) \\ &=: \Lambda_{2+s}(\phi^1, p) \end{aligned}$$

where $\lambda_1(u) \geq \lambda_2(u) \geq \lambda_3(u)$ denote the eigenvalues of the matrix

$$\frac{1}{2}\big(A\,Df(u)\,A^{-1} + (A\,Df(u)\,A^{-1})^T\big)$$

with $Df(u)$ being the Jacobian of $f(u)$. Note that

$$\lambda_2(p_0) = -b\ , \quad \lambda_{1/3}(p_0) = -\frac{\sigma+1}{2} \pm \frac{1}{2}\sqrt{(\sigma-1)^2 + 4\sigma r}$$

which implies

$$\dim_{\mathrm{L}}(\phi^1, p_0) = 2 + s_0, \quad s_0 = -1 + \frac{2(\sigma+1+b)}{\sigma+1+\sqrt{(\sigma-1)^2+4\sigma r}}$$

and $\Lambda_{2+s}(\phi^1, p_0) \leq 1$ for any $s \geq s_0$. Moreover,

$$\Lambda_{2+s}(\phi^1, p) \leq \Lambda_{2+s_0}(\phi^1, p_0) \qquad (p \in \mathbb{R}^3, s \geq s_0)$$

and thus

$$\dim_{\mathrm{L}}(\phi^1, p_0) = \sup_{p\in\mathbb{R}^3} \dim_{\mathrm{L}}(\phi^1, p).$$

Since p_0 is an equilibrium point, $\dim_{\mathrm{L}}(\phi^1, p_0) = d_{\mathrm{L}}(\phi^1, p_0)$. Hence, the dynamical system generated by (22) on K is typical in the sense of Remark 5. Moreover,

$$\dim_{\mathrm{B}} K \leq \dim_{\mathrm{L}}(\phi^1, p_0) = d^u_{\mathrm{L}}(\phi^1, K) = \inf_{t\in\mathbb{N}} \dim_{\mathrm{L}}(\phi^t, K) \approx 2.401\ . \tag{23}$$

The Lorenz attractor K observed numerically has been the object of several analytic estimates of the box dimension (e.g., Eden, Foias and Temam [5] obtained $\dim_{\mathrm{B}} K \leq 2.408$). Since the local Lyapunov dimension of an equilibrium point is invariant under changes of metrics, estimate (23) is optimal in terms of methods developed in this paper and in [2, 5, 7, 17]. However, numerical investigations suggest $\dim_{\mathrm{B}} K \approx 2.05$. So, Barreto and Hunt [16] conjecture that an upper bound for the box dimension may be obtained in terms of volume expansion rates which are averaged with respect to the natural measure which may challenge further investigations.

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