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A Class of Linear Integral Equations and Systems with Sum and Difference Kernel

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Abstract. By means of Fourier transform and Cauchy integral techniques a complete investigation of a class of linear integral equations and corresponding systems of equations of cross-correlation type in the Lebesgue spaces L^1 and L^2 is performed. Integral equations of first and second kind are reduced to explicitly solvable Riemann-Hilbert problems for a holomorphic function in the upper half-plane and the system of equations to conjugacy problems for a sectionally holomorphic function, where in the case of a finite interval also the analytic continuation of the solutions to the lower half-plane can be carried out in explicit way. Further, a resolvent representation of the solution to the integral equation and its adjoint equation is derived.

Keywords: *Integral equations of correlation type, boundary value problems for holomorphic functions, explicit solutions*

AMS subject classification: Primary 45B05, 45E10, 45F05, secondary 30E25

1. Introduction

Cross-correlation technique is an important method for investigating real- and complex-valued signals in general signal theory [11]. In this paper we study related linear integral equations of the second kind with a given input signal as kernel and the sum of both-cross-correlations of it with the wanted signal as solution in the cases of the half-axis and of a finite interval, respectively. Equation (2.1) on the finite interval $(0, T)$ can be considered as a special case of equation (2.2) on the half-axis for a right-hand side with support on $[0, T]$ (measurement of the output signal on $[0, T]$) but with the additional requirement that also only solutions (wanted signals) with support on $[0, T]$ are taken into account. Therefore, a separate treatment of equation (2.1) for finite T is given.

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Equations (2.1) - (2.2) also occur in the theory of Markov processes. The homogeneous equation (2.2) with a probability density as kernel is Feller's equation [3] who obtains its solution by eyes and verification. In a systematic way Berkovič [2] found this solution by reducing Feller's equation (as well as the general inhomogeneous equation) via Fourier transformation to a boundary value problem of Carleman type for a holomorphic function in the upper half-plane.

Our approach generalizes the Fourier transform method in the manner like in [13, 14]. Using a modified (mixed) form of the Fourier transformation we reduce the integral equations (2.1) - (2.2) directly to the well-known Rieman-Hilbert problem for a holomorphic function in the upper half-plane and the corresponding system of two integral equations (6.1) - (6.2) to a conjugacy problem of Riemann (or Hilbert) type for a sectionally holomorphic function. Due to the analyticity of the data in these problems their solutions can be given in a form simpler in comparison to the general theory. In the case of equation (2.1) and the system of equations (6.1) - (6.2) with finite T the holomorphic functions in the upper half-plane have analytic continuations to the lower half-plane which also can be constructed in an explicit way. This yields the explicit solutions to the equations of the second kind.

The integral equation (2.1) in the real case can be considered as linearization via the Fréchet derivative of the finite autocorrelation equation of second kind (Percus-Yevick equation). In this context Nussbaum [18] determined the spectrum of the integral operator in equation (2.1). We further remark that the system of equations (6.1) - (6.2) for $T = \infty$ with a special class of real kernels is studied by Arabadzhyan [1] by the method of successive approximations.

In the case of integral equations of the first kind (5.1) - (5.2) some additional assumptions are required for the existence of the solution. Further, we represent the solution of equation (2.1) for finite T (formally) in resolvent form which leads to the explicit solution of the adjoint equation, too. For convenience, we deal with equations in L^2-spaces but make a remark to the case of L^1-spaces in the Appendix at the end of the paper.

The plan of the paper is as follows. In Section 2 we state the integral equations and the mixed Fourier transformations. The solutions of equations (2.1) - (2.2) are derived in Section 3 in the regular case and in Section 4 in the singular case. In Section 5 we briefly deal with corresponding equations of the first kind (5.1) - (5.2). The more complex case of system of two equations (6.1) - (6.2) is treated for the regular and singular case in Sections 6 and 7, respectively. Finally, in Section 8 some examples are worked out in detail. The resolvent form of the solution is derived in the Appendix.

2. Statement of equations

For $T > 0$, we deal with the *linear integral equations*

$$p(t) - \int_0^{T-t} \overline{k(s)} p(s+t)\, ds - \int_0^{T-t} k(s+t) \overline{p(s)}\, ds = g(t) \qquad (2.1)$$

on $(0,T)$ for a complex-valued solution $p \in L^2(0,T)$, given complex-valued kernel $k \in L^1(0,T)$ and right-hand side $g \in L^2(0,T)$ and with equation

$$p(t) - \int_0^{\infty} \overline{k(s)} p(s+t)\, ds - \int_0^{\infty} k(s+t) \overline{p(s)}\, ds = g(t) \qquad (2.2)$$

on $\mathbb{R}_+ = (0,\infty)$ for complex-valued functions $p \in L^2(\mathbb{R}_+)$, kernel $k \in L^1(\mathbb{R}_+)$ and right-hand side $g \in L^2(\mathbb{R}_+)$. These equations contain the corresponding real equations with real-valued p, k and g as important particular cases. Further, we remark that equation (2.1) can be considered as special case of equation (2.2) for $k = g = 0$ in (T,∞) where only solutions p with $p = 0$ in (T,∞) are looked on.

Equation (2.1) can be written in the form $p - Ap = g$ with *integral operator* A defined by

$$(Ap)(t) = \int_t^{T} \overline{k(s-t)} p(s)\, ds + \int_0^{T-t} k(s+t) \overline{p(s)}\, ds$$

on $(0,T)$. By means of Young's inequality, $\|Ap\|_2 \le 2\|k\|_1 \|p\|_2$ follows where $\|\cdot\|_2$ and $\|\cdot\|_1$ are the norms in $L^2(0,T)$ and $L^1(0,T)$, respectively. Therefore, for $k \in L^1(0,T)$ the operator A is bounded in $L^2(0,T)$. Further, the operator A can be represented as sum of two convolution operators, and by a well-known theorem on the compactness of convolution operators with summable kernels in L^p, $1 \le p \le \infty$ (cf. [17: Chapter 2/Section 2.5]) A is compact in $L^2(0,T)$, too. Moreover, if in addition $k \in L^2(0,T)$, then the operator A has finite double-norm, i.e. it is of Hilbert-Schmidt type. Therefore, Fredholm theorems hold for equation (2.1) with $k \in L^1(0,T)$.

The homogeneous adjoint equation to (2.1) is given by (cp. [8: Section 11])

$$q(t) = \int_0^{t} k(t-s) q(s)\, ds + \int_0^{T-t} k(s+t) \overline{q(s)}\, ds \qquad (2.3)$$

on $(0,T)$. Necessary and sufficient solvability conditions for equation (2.1) have the form

$$\operatorname{Re} \int_0^{T} g(t) \overline{q_j(t)}\, dt = 0 \qquad (j = 1, ..., n) \qquad (2.4)$$

where $\{q_j\}_{j=1}^n$ is a complete system of linearly independent (with respect to linear combinations with real coefficients) solutions of equation (2.3). In the following the validity of the Fredholm theorems are shown by our constructive solution method in a direct manner without relying on the theory.

To equation (2.1) we apply the *mixed Fourier transformation* $\mathcal{F}_1$ defined by

$$(\mathcal{F}_1 h)(x) = \mathrm{Re} \int_0^T e^{ixt} h(t)\, dt \qquad (x \in \mathbb{R}) \tag{2.5}$$

mapping complex functions $h \in L^2(0,T)$ into real quadratic summable functions $\mathcal{F}_1 h$ on $\mathbb{R}$. From $\mathcal{F}_1 h = \gamma$ for real $\gamma \in L^2(0,T)$ of form (2.5) with complex $h \in L^2(0,T)$ we obtain the inversion formula

$$h(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-ixt} \gamma(x)\, dx \qquad (0 < t < T). \tag{2.6}$$

Namely, we have

$$\begin{aligned} \mathrm{Re} \int_0^T e^{ixt} h(t)\, dt &= \frac{1}{2} \int_0^T \left[e^{ixt} h(t) + e^{-ixt} \overline{h(t)} \right] dt \\ &= \frac{1}{2} \int_{-T}^T e^{ixt} h(t)\, dt \\ &= \frac{1}{2} \int_{-\infty}^{\infty} e^{ixt} h(t)\, dt \\ &= \gamma(x) \end{aligned}$$

for $x \in \mathbb{R}$ if we put $h(t) = \overline{h(-t)}$ for $t \in (-T, 0)$ and $h(t) = 0$ outside $[-T, T]$. Hence (2.6) follows. An analogous inversion formula of the same form is valid for the transformation

$$(\mathcal{F}_2 h)(x) = i\, \mathrm{Im} \int_0^T e^{ixt} h(t)\, dt \qquad (x \in \mathbb{R}) \tag{2.7}$$

to be used later in Section 6, too.

For $h = Ap$ with $p \in L^2(0,T)$ the relation

$$\mathcal{F}_1(Ap)(x) = \mathrm{Re}\left[\overline{K(x)} F(x)\right] \qquad (x \in \mathbb{R}) \tag{2.8}$$

with finite Fourier transforms

$$\left.\begin{aligned} F(x) &= \int_0^T p(t) e^{ixt}\, dt \qquad (x \in \mathbb{R}) \\ K(x) &= \int_0^T k(t) e^{ixt}\, dt \qquad (x \in \mathbb{R}) \end{aligned}\right\} \quad (x \in \mathbb{R}) \tag{2.9}$$

holds. The proof follows by a simple calculation: we have

$$
\begin{aligned}
&\int_0^T e^{ixt}\Big[\int_0^{T-t}\overline{k(s)}p(s+t)\,ds+\int_0^{T-t}k(s+t)\overline{p(s)}\,ds\Big]dt\\
&\quad=\int_0^T\Big[\int_0^{T-s}e^{ixt}p(s+t)\,dt\cdot\overline{k(s)}+\int_0^{T-s}e^{ixt}k(s+t)\,dt\cdot\overline{p(s)}\Big]ds\\
&\quad=\int_0^T\Big[\int_s^T e^{ix\sigma}p(\sigma)\,d\sigma\cdot e^{-ixs}\overline{k(s)}+\int_s^T e^{ix\sigma}k(\sigma)\,d\sigma\cdot e^{-ixs}\overline{p(s)}\Big]ds\\
&\quad=\int_0^T\Big[\int_0^s e^{-ix\sigma}\overline{k(\sigma)}\,d\sigma\cdot e^{ixs}p(s)+\int_s^T e^{ix\sigma}k(\sigma)\,d\sigma\cdot e^{-ixs}\overline{p(s)}\Big]ds
\end{aligned}
$$

and

$$
\begin{aligned}
&\operatorname{Re}\int_0^T e^{ixt}(Ap)(t)\,dt\\
&\quad=\frac{1}{2}\Big[\int_0^T e^{ixt}(Ap)(t)\,dt+\overline{\int_0^T e^{ixt}(Ap)(t)\,dt}\Big]\\
&\quad=\frac{1}{2}\Big[\int_0^T e^{-ix\sigma}\overline{k(\sigma)}\,d\sigma\cdot\int_0^T e^{ixs}p(s)\,ds\\
&\qquad+\int_0^T e^{ix\sigma}k(\sigma)\,d\sigma\cdot\int_0^T e^{-ixs}\overline{p(s)}\,ds\Big]\\
&\quad=\frac{1}{2}\big[\,\overline{K(x)}F(x)+K(x)\overline{F(x)}\,\big]\\
&\quad=\operatorname{Re}\big[\,\overline{K(x)}F(x)\big].
\end{aligned}
$$

Relation (2.8) holds true for $T=\infty$ with the Fourier transforms

$$
\left.\begin{aligned}
F(x)&=\int_0^\infty p(t)e^{ixt}\,dt\\
K(x)&=\int_0^\infty k(t)e^{ixt}dt
\end{aligned}\right\}\quad (x\in\mathbb{R}). \tag{2.10}
$$

Remark. The equation

$$
p_1(t)-\int_0^{T-t}\overline{k(s)}p_1(s+t)\,ds+\int_0^{T-t}k(s+t)\overline{p_1(s)}\,ds=g_1(t)
$$

on $(0,T)$ can be reduced to equation (2.1) putting $p_1=ip$ and $g_1=ig$. In the real case it may be more straightforward to treat this equation directly by applying the Fourier sine transformation to it. (Remind that transformation (2.5) reduces to the Fourier cosine transformation in the real case.)

3. Method of solution – the regular case

Applying the transformation $\mathcal{F}_1$ to equation (2.1), in view of (2.8) we obtain the condition

$$\operatorname{Re}\big[(1-\overline{K(x)})F(x)\big] = G(x) \qquad (x \in \mathbb{R}) \tag{3.1}$$

where $G = \mathcal{F}_1 g \in L^2(\mathbb{R})$, i.e.

$$G(x) = \operatorname{Re} \int_0^T e^{ixt} g(t)\, dt \qquad (x \in \mathbb{R}). \tag{3.2}$$

By the assumption $g \in L^2(0,T)$, the real-valued function G is continuous on $\mathbb{R}$ and vanishes for $x \to \pm\infty$. The complex Fourier transforms of $p \in L^2(0,T)$ and $k \in L^1(0,T)$

$$\left.\begin{aligned} F(z) &= \int_0^T p(t) e^{izt} dt \\ K(z) &= \int_0^T k(t) e^{izt} dt \end{aligned}\right\} \qquad (z = x+iy) \tag{3.3}$$

are entire functions of exponential type which are bounded and vanish at infinity on $\operatorname{Im} z \geq 0$ and have continuous values on $\mathbb{R}$.

We construct the *Fourier transform* F of the solution p to equation (2.1) in two steps. At first we derive F on the upper half-plane $\operatorname{Im} z > 0$ as solution of the Riemann-Hilbert problem with boundary condition (3.1) on $\mathbb{R}$, and then we perform an analytic continuation of F across the real axis into the lower half-plane $\operatorname{Im} z < 0$. The Riemann-Hilbert problem (3.1) is equivalent to the *conjugacy problem* [4, 7]

$$\Phi^+(x) = A(x)\Phi^-(x) + H(x) \qquad (x \in \mathbb{R}) \tag{3.4}$$

for the sectionally holomorphic function

$$\Phi(z) = \begin{cases} F(z) & \text{in } \operatorname{Im} z > 0 \\ -\overline{F(\overline{z})} & \text{in } \operatorname{Im} z < 0 \end{cases} \tag{3.5}$$

satisfying the symmetry relation $\Phi(\overline{z}) = -\overline{\Phi(z)}$ and the limiting relation $\Phi(\infty) = 0$ where

$$A(x) = \frac{1-K(x)}{1-\overline{K(x)}}, \qquad H(x) = \frac{2G(x)}{1-\overline{K(x)}} \tag{3.6}$$

satisfy the limiting relations $A(\pm\infty) = 1$ and $H(\pm\infty) = 0$.

In the *regular case* we have

$$1 - K(x) \neq 0 \qquad (x \in \mathbb{R}). \tag{3.7}$$

We introduce the index

$$\kappa = \frac{1}{2\pi}\left[\arg(1 - K(x))\right]_{\mathbb{R}} \tag{3.8}$$

which is finite and equal to the number of zeros of the function $K_1(z) = 1 - K(z)$ in the upper half-plane. Hence $\kappa \geq 0$. We remark that in [4] the index is defined by expression (3.8) with $1 - K(x)$ replaced by $A(x)$, i.e. by 2κ. We prefer to work with κ equal to the number of zeros of K_1 in $\operatorname{Im} z > 0$ which is in analogy to the real case.

In the *case* $\kappa = 0$ the homogeneous problem (3.4) has only the trivial solution $\Phi = 0$ satisfying $\Phi(\infty) = 0$. The solution of the non-homogeneous problem (3.4) is given by

$$\Phi(z) = \begin{cases} \Phi^+(z) & \text{in } \operatorname{Im} z > 0 \\ \Phi^-(z) & \text{in } \operatorname{Im} z < 0 \end{cases} \tag{3.9}$$

where

$$\begin{aligned} \Phi^+(z) \equiv F(z) &= K_1(z)\Psi(z) \\ \Phi^-(z) &= K_2(z)\Psi(z) \end{aligned} \tag{3.10}$$

with $K_1(z) = 1 - K(z), K_2(z) = \overline{K_1(\overline{z})} = 1 - \overline{K(\overline{z})}$ and

$$\Psi(z) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{G(\xi)}{B(\xi)}\,\frac{d\xi}{\xi - z}, \qquad B(x) = K_1(x)K_2(x). \tag{3.11}$$

In view of (3.7) the real function $B(x) = |1 - K(x)|^2$ is positive on $\mathbb{R}$ with $B(\pm\infty) = 1$. From (3.10) - (3.11) and the Plemelj-Sochozky formula we obtain F on $\mathbb{R}$ as

$$F(x) = \Phi^+(x) = \frac{G(x)}{K_2(x)} + K_1(x)\frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{G(\xi)}{B(\xi)}\,\frac{d\xi}{\xi - x} \tag{3.12}$$

which by $G \in L^2(\mathbb{R})$ is in $L^2(\mathbb{R})$, too.

In the *case* $\kappa > 0$ the function K_1 has the zeros z_j $(j = 1, ..., \kappa)$ in $\operatorname{Im} z > 0$ counted according to their multiplicities. The solutions of the *homogeneous problem* (3.4) can be easily constructed by analytic continuation writing the conjugacy condition in the form

$$\frac{R(x)\Phi^+(x)}{K_1(x)} = \frac{R(x)\Phi^-(x)}{K_2(x)} \qquad (x \in \mathbb{R}) \tag{3.13}$$

with polynomial of degree 2κ

$$R(z) = \prod_{j=1}^{\kappa} (z - z_j)(z - \overline{z}_j) \tag{3.14}$$

satisfying $R(\overline{z}) = \overline{R(z)}$. From (3.13) the general solution of the homogeneous problem (3.4) has the form

$$\Phi_0(z) = \begin{cases} \Phi_0^+(z) & \text{in } \operatorname{Im} z > 0 \\ \Phi_0^-(z) & \text{in } \operatorname{Im} z < 0 \end{cases} \tag{3.15}$$

where

$$\Phi_0^+(z) \equiv F_0(z) = \frac{K_1(z)}{R(z)}\, iP^0_{2\kappa-1}(z) \qquad (\operatorname{Im} z > 0) \tag{3.16}$$

$$\Phi_0^-(z) = \frac{K_2(z)}{R(z)}\, iP^0_{2\kappa-1}(z) \qquad (\operatorname{Im} z < 0) \tag{3.17}$$

with a polynomial $P^0_{2\kappa-1}$ of degree $2\kappa-1$ which in view of the relation $\Phi_0(\overline{z}) = -\overline{\Phi_0(z)}$ has (arbitrary) real coefficients.

Further, as a particular solution Φ_1 of the *non-homogeneous problem* (3.4) in the case $\kappa > 0$ we can take the same solution (3.9) as in the case $\kappa = 0$. This yields the general solution F of the non-homogeneous problem (3.1) in the upper half-plane in the form

$$F(z) = \frac{K_1(z)}{R(z)}\, iP_{2\kappa-1}(z) + K_1(z)\Psi(z) \qquad (\operatorname{Im} z > 0) \tag{3.18}$$

where $K_1(z) = 1 - K(z)$, R and Ψ are given by (3.14) and (3.11), respectively, and $P_{2\kappa-1}$ is a polynomial of degree $2\kappa - 1$ with real coefficients. Because of additional conditions resulting from the analytic continuation of F into the lower half-plane below we have to choose the polynomials $P^0_{2\kappa-1}$ and $P_{2\kappa-1}$ of the homogeneous and non-homogeneous problem (3.1) as different, in general. The values of F on $\mathbb{R}$ are given by the Plemelj-Sochozky formula in the form (cp. (3.12))

$$F(x) = \frac{K_1(x)}{R(x)}\, iP_{2\kappa-1}(x) + \frac{G(x)}{K_2(x)} + K_1(x)\frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{G(\xi)}{B(\xi)}\, \frac{d\xi}{\xi - x}. \tag{3.19}$$

In the *second step* we continue F analytically across $\mathbb{R}$ into the lower half-plane. From (3.16) we have for the homogeneous problem (3.1)

$$F_0(z) = \frac{K_1(z)}{R(z)}\, iP^0_{2\kappa-1}(z) \qquad (\operatorname{Im} z < 0) \tag{3.20}$$

and from (3.18), observing the Plemelj-Sochozky formulae, for the non-homogeneous problem (3.1) we have

$$F(z) = \frac{K_1(z)}{R(z)}\, iP_{2\kappa-1}(z) + \frac{2G(z)}{K_2(z)} + K_1(z)\Psi(z) \qquad (\operatorname{Im} z < 0) \qquad (3.21)$$

with real polynomials $P^0_{2\kappa-1}$ and $P_{2\kappa-1}$ of degree $2\kappa - 1$, funtion Ψ defined by (3.11), and function

$$G(z) = \frac{1}{2}\int_0^T \left[g(t)e^{izt} + \overline{g(t)}e^{-izt}\right] dt \qquad (3.22)$$

satisfying $G(\overline{z}) = \overline{G(z)}$.

The functions F_0 and F must be finite in the points $\overline{z}_j \quad (j = 1, ..., \kappa)$ which are the zeros of the functions R and K_2 in the lower half-plane. Let us assume here that z_j are simple zeros of K_1 in the upper half-plane, shifting the general case of multiple zeros to the Appendix. Then solution (3.20) of the homogeneous problem (3.1) should fulfil the conditions

$$K_1(\overline{z}_j)P^0_{2\kappa-1}(\overline{z}_j) = 0 \qquad (j = 1, ..., \kappa) \qquad (3.23)$$

for the 2κ real coefficients of $P^0_{2\kappa-1}$. If $\overline{z}_j$ is a zero of K_1, the corresponding condition (3.23) is fulfilled. Otherwise, for $K_1(\overline{z}_j) \neq 0$, it is equivalent to the condition $P^0_{2\kappa-1}(\overline{z}_j) = 0$ or, what is the same, to $P^0_{2\kappa-1}(z_j) = 0$. So, if

$$K_1(\overline{z}_j) = \begin{cases} 0 & \text{for } j = 1, ..., \kappa_0 \\ \neq 0 & \text{for } j = \kappa_0 + 1, ..., \kappa \end{cases} \qquad (0 \leq \kappa_0 \leq \kappa), \qquad (3.24)$$

conditions (3.23) are equivalent to the $\kappa - \kappa_0$ complex of conditions

$$P^0_{2\kappa-1}(z_j) = 0 \qquad (j = \kappa_0 + 1, ..., \kappa) \qquad (3.25)$$

leaving free $n = 2\kappa_0$ real parameters in the polynomial $P^0_{2\kappa-1}$. I.e., for the homogeneous problem (3.1) there remain $n = 2\kappa_0$ linearly independent solutions (over the real field). The polynomial $P^0_{2\kappa-1}$ in the general solution $F_0(z)$ of the homogeneous problem (3.1) has the form

$$P^0_{2\kappa-1}(z) = \prod_{j=\kappa_0+1}^{\kappa} (z - z_j)(z - \overline{z}_j)Q^0_{2\kappa_0-1}(z) \qquad (3.26)$$

where $Q^0_{2\kappa_0-1}$ is an arbitrary real polynomial of degree $2\kappa_0 - 1$.

For solution (3.21) of the non-homogeneous problem (3.1), the conditions

$$K_1(\overline{z}_j)P_{2\kappa-1}(\overline{z}_j) = 2i\,\overline{c}_j\,G(\overline{z}_j) \qquad (j = 1, ..., \kappa) \tag{3.27}$$

have to be satisfied with constants $\overline{c}_j = \lim_{z\to\overline{z}_j}\left[\frac{R(z)}{K_2(z)}\right] \neq 0$. These conditions split up into the κ_0 complex of *solvability conditions* for the non-homogeneous problem (3.1)

$$G(z_j) = 0 \qquad (j = 1, ..., \kappa_0) \tag{3.28}$$

and the $\kappa - \kappa_0$ complex of conditions for $P_{2\kappa-1}$

$$i\,P_{2\kappa-1}(z_j) = 2d_j\,G(z_j) \qquad (j = \kappa_0 + 1, ..., \kappa) \tag{3.29}$$

with (non-vanishing) constants $d_j = \frac{c_j}{K_2(z_j)} = \frac{1}{K_2(z_j)}\lim_{z\to z_j}\frac{R(z)}{K_1(z)}$. If $P^1_{2\kappa-1}$ is a particular real polynomial satisfying conditions (3.29), then the polynomial $P_{2\kappa-1}$ in formula (3.21) for the general solution of the non-homogeneous problem (3.1) is given by $P_{2\kappa-1} = P^0_{2\kappa-1} + P^1_{2\kappa-1}$ with $P^0_{2\kappa-1}$ defined by (3.26).

By (3.22), the solvability conditions (3.28) for the non-homogeneous problem (3.1) are of form (2.4) with the $n = 2\kappa_0$ linearly independent functions

$$\left.\begin{aligned} q_j^{(1)} &= e^{-iz_jt} + e^{-i\overline{z}_jt} \\ q_j^{(2)} &= i\big[e^{-iz_jt} - e^{-i\overline{z}_jt}\big] \end{aligned}\right. \qquad (j = 1, ..., \kappa_0).$$

It is easy to verify that these functions are solutions of the homogeneous adjoint integral equation (2.3) if $\int_0^T k(t)e^{iz_jt}dt = \int_0^T k(t)e^{i\overline{z}_jt}dt = 1$.

It remains to prove that the function F defined by (3.18) and (3.21) with values (3.19) on $\mathbb{R}$ can be represented as a finite Fourier integral of form (3.3) with $p \in L^2(0,T)$. In view of $G \in L^2(\mathbb{R})$, from (3.19) we have $F \in L^2(\mathbb{R})$. Further, F is an entire function of exponential type since G and K_1 are such ones and $\frac{1}{K_2}$ is bounded at infinity on $\operatorname{Im} z \leq 0$. Finally, for applying the Paley-Wiener theorem [6: Chapter 6/Section E] we show that the parameters

$$\begin{aligned} a &= \limsup_{y\to\infty}\tfrac{1}{y}\ln|F(-iy)| \\ b &= \limsup_{y\to\infty}\tfrac{1}{y}\ln|F(iy)| \end{aligned}$$

satisfy the inequalities $b \leq 0$ and $a \leq T$. Indeed, from (3.18) and (3.21), for $y > 0$ we have

$$\begin{aligned} |F(iy)| &\leq |K_1(iy)|\Big[\frac{|P_{2\kappa-1}(iy)|}{|R(iy)|} + |\Psi(iy)|\Big] \\ |F(-iy)| &\leq |K_1(-iy)|\Big[\frac{|P_{2\kappa-1}(-iy)|}{|R(-iy)|} + |\Psi(-iy)|\Big] + \frac{2|G(-iy)|}{|K_2(-iy)|}. \end{aligned}$$

For sufficiently large y (≥ 1) the estimations

$$\begin{aligned}
&|K_1(iy)| \leq C_0, \quad |K_1(-iy)| \leq C_0 e^{yT}, \quad |K_2(-iy)| \geq \frac{1}{C_0}\\
&\Big|\frac{P_{2\kappa-1}(iy)}{R(iy)}\Big| \leq C_1, \quad \Big|\frac{P_{2\kappa-1}(-iy)}{R(-iy)}\Big| \leq C_2\\
&2|G(-iy)| \leq \int_0^T |g(t)|\,dt \cdot (e^{yT}+1) \leq D_0\, e^{yT}
\end{aligned}$$

with $C_0 = 1 + \int_0^T |k(t)|dt$ and some positive constants C_1, C_2, D_0 hold, and

$$|\Psi(\pm iy)|^2 \leq \frac{1}{\pi^2}\int_{-\infty}^{\infty} \frac{G^2\xi)}{B^2(\xi)}\,d\xi \cdot \int_{-\infty}^{\infty} \frac{d\xi}{\xi^2+y^2} \leq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{G^2(\xi)}{B^2(\xi)}\,d\xi < \infty$$

since $G \in L^2(\mathbb{R})$ and $B(x) \geq d > 0$ on $\mathbb{R}$. Therefore, for sufficiently large y we obtain the inequalities $|F(iy)| \leq A$ and $|F(-iy)| \leq C + D\, e^{yT}$ with some positive constants A, C, D from which $b \leq 0$ and $a \leq T$ follow. Hence, in view of $(2.9)_1$ and the unique invertibility of the transformation $\mathcal{F}_1$,

$$p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(x)e^{-ixt}dx \qquad (0 < t < T) \tag{3.30}$$

with F given by (3.19) is the general solution of equation (2.1).

Theorem 1. *Let be $k \in L^1(0,T)$ and $g \in L^2(0,T)$, let the function $K_1(z) = 1 - \int_0^T k(t)e^{itz}dt$ $(z = x+iy)$ satisfy the condition $K_1(x) \neq 0$ on $\mathbb{R}$ and let it possess κ simple zeros z_j $(j = 1,...,\kappa)$ in $\operatorname{Im} z > 0$ where for $j = 1,...,\kappa_0$ $(0 \leq \kappa_0 \leq \kappa)$ also $\overline{z}_j$ is a zero of K_1. Then:*

(i) *The homogeneous equation (2.1) has $n = 2\kappa_0$ linearly independent solutions over the real field given by (3.30) with*

$$F_0(x) = \frac{K_1(x)}{R(x)}\, iP^0_{2\kappa-1}(x), \quad R(x) = \prod_{j=1}^{\kappa}(x - z_j)(x - \overline{z}_j) \tag{3.31}$$

where the real polynomial $P^0_{2\kappa-1}$ of form (3.26) has zeros in the points z_j $(j = \kappa_0+1,...,\kappa)$ for which $K_1(\overline{z}_j) \neq 0$.

(ii) *The non-homogeneous equation (2.1) is solvable if and only if z_j $(j = 1,...,\kappa_0)$ are zeros of the function $\int_0^T \big[g(t)e^{izt} + \overline{g(t)}e^{-izt}\big]dt$. The general solution of equation (2.1) is given by (3.30), where F is defined by (3.19) and the real polynomial $P_{2\kappa-1}$ of degree $2\kappa-1$ obeys the $2(\kappa-\kappa_0)$ real conditions (3.29) in the points z_j $(j = \kappa_0+1,...,\kappa)$.*

Corollary 1. *In the general case of zeros* ζ_k $(k = 1, ..., r)$ *of* K_1 *with multiplicities* ν_k *in* $\operatorname{Im} z > 0$ *the function* R *in* (3.14) *is defined by* (A.1), *and conditions* (3.25), (3.28) *and* (3.29) *must be replaced by conditions* (A.5), (A.7) *and* (A.3), *respectively, in the Appendix.*

Corollary 2. *For real-valued kernel* k *and right-hand side* g *in* (2.1) *we have the relations* $K(-z) = \overline{K(\overline{z})}$ *leading to* $K_\alpha(-z) = \overline{K_\alpha(\overline{z})}$ $(\alpha = 1, 2)$, *and* $G(z) = \int_0^T g(t) \cos zt \, dt$ *leading to* $G(-z) = \overline{G(\overline{z})} = G(z)$ *and real values* $G(x)$ *on* $\mathbb{R}$. *Looking for real-valued solutions* p *of* (2.1), *the function* F *must satisfy the relation* $F(-z) = \overline{F(\overline{z})}$. *The solution* p *is then given by*

$$p(t) = \operatorname{Re}\left(\frac{1}{\pi} \int_0^\infty F(x) e^{-ixt} dx\right) \tag{3.32}$$

on $(0, T)$. *The* κ *simple zeros* z_j *of* K_1 *in* $\operatorname{Im} z > 0$ *are divided into* N_0 *zeros of the form* $z_j = iy_j$ $(y_j > 0;\ j = 1, ..., N_0)$ *and* $2N$ *zeros of the form* $z_j = \pm x_j + iy_j$ $(x_j, y_j > 0;\ j = N_0 + 1, ..., N_0 + N)$ *so that* $\kappa = N_0 + 2N$. *The function* R *in* (3.14) *is given by*

$$R(z) = \prod_{j=1}^{N_0} (z^2 + y_j^2) \prod_{j=N_0+1}^{N_0+N} (z^2 - z_j^2)(z^2 - \overline{z}_j^2) \tag{3.33}$$

satisfying the relation $R(-z) = \overline{R(\overline{z})} = R(z)$ *with real values* $R(x)$ *on* $\mathbb{R}$.

The solutions F_0 *of the homogeneous problem* (3.1) *defined by* (3.16), (3.20) *fulfil the relation* $F_0(-z) = \overline{F_0(\overline{z})}$ *if the real polynomial* $P^0_{2\kappa-1}$ *has only odd powers so that we have now* κ *free real parameters in it. Further, the* κ_0 *zeros* $\overline{z}_j$ $(j = 1, ..., \kappa_0)$ *of* K_1 *in* $\operatorname{Im} z < 0$ *are divided into* ν_0 $(0 \le \nu_0 \le N_0)$ *zeros of the form* $\overline{z}_j = -iy_j$ $(y_j > 0;\ j = 1, ..., \nu_0)$ *and* 2ν $(0 \le \nu \le N)$ *zeros of the form* $\overline{z}_j = \pm x_j - iy_j$ $(x_j, y_j > 0;\ j = N_0 + 1, ..., N_0 + \nu)$ *so that* $\kappa_0 = \nu_0 + 2\nu$. *Therefore, conditions* (3.25) *split up into the* $N_0 - \nu_0$ *real conditions*

$$P^0_{2\kappa-1}(iy_j) = 0 \qquad (j = \nu_0 + 1, ..., N_0) \tag{3.34}$$

and the $N - \nu$ *complex conditions*

$$P^0_{2\kappa-1}(x_j + iy_j) = 0 \qquad (j = N_0 + \nu + 1, ..., N_0 + N), \tag{3.35}$$

i.e. together there are $N_0 - \nu_0 + 2(N - \nu) = \kappa - \kappa_0$ *real conditions. That means, the homogeneous problem* (3.1) *has* $n = \kappa_0$ *linearly independent real solutions in the real case.*

The solvability conditions (3.28) *for the non-homogeneous problem* (3.1) *take the form*

$$\begin{aligned} G(iy_j) &= 0 \qquad (j = 1, ..., \nu_0) \\ G(x_j + iy_j) &= 0 \qquad (j = N_0 + 1, ..., N_0 + \nu) \end{aligned} \tag{3.36}$$

which are equivalent to the $n = \kappa_0 = \nu_0 + 2\nu$ *real conditions*

$$\begin{aligned} \int_0^T g(t)q_j(t)\,dt = 0 \qquad & (j = 1, ..., \nu_0) \\ \int_0^T g(t)q_j^{+,-}(t)\,dt = 0 \qquad & (j = N_0 + 1, ..., N_0 + \nu) \end{aligned} \tag{3.37}$$

where $q_j(t) = \cosh y_j t$ *and* $q_j^{+,-}(t) = \begin{cases} \cos x_j t \cdot \cosh y_j t \\ \sin x_j t \cdot \sinh y_j t \end{cases}$. *In the real case the function* Ψ *in* (3.11) *satisfies the relation* $\Psi(-z) = \overline{\Psi(\overline{z})}$, *and the polynomial* $P_{2\kappa-1}$ *in solution* (3.18), (3.21) *of the non-homogeneous problem* (3.1) *contains only odd powers* (*like* $P^0_{2\kappa-1}$) *and conditions* (3.29) *reduce to* $\kappa - \kappa_0$ *real ones.*

In the general real case with multiple zeros of K_1 *the function* R *is defined by* (A.8), *and conditions* (3.34) – (3.37) *must be replaced by conditions* (A.5) *with* (A.9) *and* (A.10) – (A.11), *respectively, of the Appendix.*

For *equation* (2.2) the solution can be derived in an analogous manner. But F is now a holomorphic function in $\operatorname{Im} z > 0$ only and it has to be constructed only there. Therefore, we obtain the Fourier transforms F_0 and F by formulas (3.16) and (3.18) with boundary values (3.19) where $P_{2\kappa-1}$ is an arbitrary real polynomial of degree $2\kappa - 1$. Functions (3.16) and (3.18) are bounded on $\operatorname{Im} z \geq 0$ and the boundary values (3.19) are lying in $L^2(\mathbb{R})$. Hence, a corresponding Paley-Wiener theorem [9: Theorem 8] proves the representation of F_0 and F in the form of Fourier integral (2.11). Then

$$p(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(x) e^{-ixt} dx \qquad (t > 0) \tag{3.38}$$

represents a solution of equation (2.2).

Theorem 2. *Let be* $k \in L^1(\mathbb{R}_+)$ *and* $g \in L^2(\mathbb{R}_+)$, *let the function* $K_1(z) = 1 - \int_0^\infty k(t)e^{izt}dt \quad (z = x + iy)$ *satisfy the condition* $K_1(x) \neq 0$ *on* $\mathbb{R}$ *and let it possess* κ *zeros* $z_j \quad (j = 1, ..., \kappa)$ (*counted according to their multiplicities*) *in* $\operatorname{Im} z > 0$. *Then:*

(i) *The homogeneous equation* (2.2) *has* $n = 2\kappa$ *linearly independent solutions over the real field given by* (3.38) *with* (3.31) *where* $P^0_{2\kappa-1}$ *is an arbitrary real polynomial of degree* $2\kappa - 1$.

(ii) *The non-homogeneous equation* (2.2) *is always solvable and has the general solution* (3.38) *where* F *is defined by* (3.19) *with an arbitrary real polynomial* $P_{2\kappa-1}$ *of degree* $2\kappa - 1$.

Corollary 3. *In the case of real-valued kernel* k *and right-hand side* g *we have* $n = \kappa$ *linearly independent real solutions* p *of* (2.2). *The general solution* p *is given by formula* (3.32) *in* $t > 0$ *where* F *is defined by* (3.19) *with real polynomial* $P_{2\kappa-1}$ *of degree* $2\kappa - 1$ *having odd powers only.*

4. Method of solution – the singular case

In the *singular case* of equation (2.1) the entire function $K_1(z) = 1 - K(z)$ with limits $K_1(\pm\infty) = 1$ on $\mathbb{R}$ has finitely many zeros $x_m \quad (m = 1, ..., \rho)$ of integer order n_m on $\mathbb{R}$:

$$K_1(z) = \Pi(z)K_0(z), \quad \Pi(z) = \prod_{m=1}^{\rho} (z - x_m)^{n_m} \tag{4.1}$$

where $K_0(x) \neq 0 \ \ (x \in \mathbb{R})$ and $K_0(x) \sim x^{-N}$ with $N = \sum_{m=1}^{\rho} n_m \ \ (\geq 1)$ for $x \to \pm\infty$. Then the coefficients in the conjugacy problem (3.4) take the form

$$A(x) = \frac{K_0(x)}{\overline{K_0(x)}}, \quad H(x) = \frac{2G_0(x)}{\overline{K_0(x)}} \tag{4.2}$$

with real-valued function $G_0 = \frac{G}{\Pi}$.

Observing the asymptotic behaviour of K_0 at infinity, the general solution of the *homogeneous problem* (3.4) with (4.2) is given by

$$F_0(z) = \frac{K_0(z)}{R(z)}\, iP^0_{2\kappa-1+N}(z) \tag{4.3}$$

where κ is again the number of zeros z_j of the function K_1 in $\operatorname{Im} z > 0$, R is defined by (3.14) and $P^0_{2\kappa-1+N}$ is a polynomial of degree $2\kappa - 1 + M$ with real coefficients. If the zeros $z_j \quad (j = 1, ..., \kappa)$ of K_1 are simple, the polynomial $P^0_{2\kappa-1+N}$ must satisfy the $\kappa - \kappa_0$ complex conditions

$$P^0_{2\kappa-1+N}(z_j) = 0 \qquad (j = \kappa_0 + 1, ..., \kappa) \tag{4.4}$$

in z_j with $K_1(\overline{z}_j) \neq 0$, leaving free $n = 2\kappa + N - 2(\kappa - \kappa_0) = N + 2\kappa_0$ real parameters in $P^0_{2\kappa-1+N}$. That means, the homogeneous problem (3.1) has $n = N + 2\kappa_0 \ \ (\geq N)$ linearly independent (over the real field) solutions, where $\kappa_0 \ \ (0 \leq \kappa_0 \leq \kappa)$ is the number of (simple) zeros $z_j \ \ (j = 1, ..., \kappa_0)$ of K_1 in $\operatorname{Im} z > 0$ which are also zeros of $K_1(\overline{z})$.

For the *non-homogeneous problem*, at first from $G_0 = \frac{G}{\Pi}$ we have the N real solvability conditions

$$G^{(k)}(x_m) = 0 \qquad (k = 0, ..., n_m - 1;\ m = 1, ..., \rho). \tag{4.5}$$

Further, the function G has to fulfill the κ_0 complex conditions (3.28).

The Fourier transform F of the solution p of the non-homogeneous problem (3.1) can be given in the form

$$F(z) = \begin{cases} \frac{K_0(z)}{R(z)}\, iP_{2\kappa-1+N}(z) + K_0(z)\Psi_0(z) & \text{if } \operatorname{Im} z > 0 \\ \frac{K_0(z)}{R(z)}\, iP_{2\kappa-1+N}(z) + \frac{2G_0(z)}{\widetilde{K}_0(z)} + K_0(z)\Psi_0(z) & \text{if } \operatorname{Im} z < 0 \end{cases}$$

where $G_0(z) = \frac{G(z)}{\Pi(z)}, \widetilde{K}_0(z) = \overline{K_0(\overline{z})}$,

$$\Psi_0(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \Big[\Big(\frac{z+i}{\xi+i}\Big)^N + \Big(\frac{z-i}{\xi-i}\Big)^N\Big] \frac{G_0(\xi)}{|K_0(\xi)|^2}\,\frac{d\xi}{\xi - z}$$

and $P_{2\kappa-1+N}$ is a real polynomial of degree $2\kappa-1+N$ satisfying the conditions (cp. (3.29))

$$i\,P_{2\kappa-1+N}(z_j) = 2d_j\,G_0(z_j) \qquad (j = \kappa_0+1,...,\kappa) \tag{4.6}$$

with constants $d_j = \frac{1}{\widetilde{K}_0(z_j)} \lim_{z\to z_j} \frac{R(z)}{K_0(z)}$. We remark that for even $N = 2M$ the function Ψ_0 can be given in the simpler form

$$\Psi_0(z) = (z^2+1)^M \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{G_0(\xi)}{(\xi^2+1)^M |K_0\xi)|^2}\,\frac{d\xi}{\xi - z}.$$

The function F has the values

$$\begin{aligned} F(x) = {} & \frac{K_0(x)}{R(x)}\, iP_{2\kappa-1+N}(x) \\ & + \frac{G_0(x)}{\overline{K_0(x))}}\, K_0(x) \frac{1}{\pi i}\int_{\infty}^{\infty} Q(x,\xi)\frac{G_0(\xi)}{|K_0(\xi)|^2}\,\frac{d\xi}{\xi - x} \end{aligned} \tag{4.7}$$

on $\mathbb{R}$ where $Q(x,\xi) = \frac{1}{2}\big[\big(\frac{x+i}{\xi+i}\big)^N + \big(\frac{x-i}{\xi-i}\big)^N\big]$, respectively $Q(x,\xi) = \big[\frac{x^2+1}{\xi^2+1}\big]^M$ for $N = 2M$.

The Paley-Wiener condition can be proved as in the regular case above.

Theorem 3. *Let be $k \in L^1(0,T), g \in L^2(0,T)$, let the function $K_1(z) = 1 - K(z)$ with K defined in $(3.3)_2$ has form (4.1) with $K_0(x) \neq 0$ on $\mathbb{R}$ and let K_0 possess the κ simple zeros z_j $(j = 1,...,\kappa)$ in $\operatorname{Im} z > 0$ where for $j = 1,...,\kappa_0$ $(0 \le \kappa_0 \le \kappa)$ also $\overline{z}_j$ is a zero of K_0. Then:*

(i) *The homogeneous equation (2.1) has $n = N + 2\kappa_0$, with N given in (4.1), linearly independent solutions over the real field given by (3.30) with*

$$F_0 = \frac{K_0}{R}\, iP^0_{2\kappa-1+N}, \quad R(x) = \prod_{j=1}^{\kappa}(x - z_j)(x - \overline{z}_j) \tag{4.8}$$

where the real polynomial $P^0_{2\kappa-1+N}$ *of degree* $2\kappa-1+N$ *has zeros in the points* z_j $(j=\kappa_0+1,...,\kappa)$ *for which* $K_0(\overline{z}_j)\neq 0$.

(ii) *The non-homogeneous equation* (2.1) *is solvable if and only if the* N *real conditions* (4.5) *and the* κ_0 *complex conditions* (3.28) *are fulfilled. The general solution of equation* (2.1) *is given by* (3.30), *where* F *is defined by* (4.7) *and the real polynomial* $P_{2\kappa-1+N}$ *of degree* $2\kappa-1+N$ *obeys the* $2(\kappa-\kappa_0)$ *real conditions* (4.6) *in the points* z_j $(j=\kappa_0+1,...,\kappa)$.

Corollary 4. *In the general case of zeros* ζ_k $(k=1,...,r)$ *of* K_1 *with multiplicities* ν_k *in* $\operatorname{Im} z>0$ *the function* R *in* (3.14) *is again defined by* (*A*.1) *and conditions* (4.4), (3.28) *and* (4.6) *must be replaced by conditions* (*A*.5), (*A*.7 *and* (*A*.3), *respectively, in the Appendix, with* $P^0_{2\kappa-1+N}$ *and* $P_{2\kappa-1+N}$ *instead of* $P^0_{2\kappa-1}$ *and* $P_{2\kappa-1}$ *in* (*A*.5) *and* (*A*.3), *respectively.*

Corollary 5. *For real-valued kernel* k *in* (2.1) *it follows* $K_1(-x)=\overline{K_1(x)}$ *so that with* x_k (>0) *also* $-x_k$ *is a zero of* K_1 *on* $\mathbb{R}$. *Therefore, the polynomial* Π *in* (4.1) *takes the form*

$$\Pi(z)=z^{n_0}\prod_{k=1}^{\rho_1}\big[(z-x_k)(z+x_k)\big]^{n_k}\qquad (x_k>0)$$

and K_0 *with* $K_0(x)\neq 0$ $(x\in\mathbb{R})$ *satisfies* $K_0(-z)=(-1)^{n_0}\overline{K_0(\overline{z})}$ *and* $K_0(x)\sim x^{-N}$ *with* $N=n_0+2\sum_{k=1}^{\rho_1}n_k$ *for* $x\to\pm\infty$. *Arranging the zeros* z_j *of* K_1 *in* $\operatorname{Im} z>0$ *as in Corollary* 2, *again* R *is defined by (3.33) satisfying* $R(-z)=\overline{R(\overline{z})}$. *The solution* F_0 *of the homogeneous problem* (3.1) *defined by* (4.3) *fulfils the condition* $F_0(-z)=\overline{F_0(\overline{z})}$ *for real solutions* p *of* (2.1) *if the real polynomial* $P^0_{2\kappa-1+M}$ *has only odd powers for even* n_0 *and only even powers for odd* n_0. *Hence, we have* $\kappa+\left[\frac{N+1}{2}\right]$ *free parameters in* $P^0_{2\kappa-1+M}$. *Further,* $P^0_{2\kappa-1+M}$ *has to satisfy the* $\kappa-\kappa_0$ *real conditions* $(3.34)-(3.35)$. *That means, the homogeneous problem* (3.1) *has in the singular real case* $n=\kappa_0+\left[\frac{N+1}{2}\right]$ *linearly independent real solutions.*

Solvability conditions for the non-homogeneous problem (3.1) *with* (4.1) *in the real case are given by conditions* (4.5) *in the points* $x_0=0,\pm x_k$ $(k=1,...,\rho_1)$, *respectively, and the* κ_0 *real conditions* (3.36). *But for real right-hand side* g *in* (2.1) *the function* G *defined by* (3.2) *is even such that conditions* (4.5) *in the points* $-x_k$ $(k=1,...,\rho_1)$ *are left out and there remain* $n=\kappa_0+\left[\frac{N+1}{2}\right]$ *linearly independent real conditions. The general real solution of the non-homogeneous equation* (2.1) *is given by formula* (3.32) *where* F *is defined by* (4.7) *with real polynomial* $P_{2\kappa-1+N}$ *of degree* $2\kappa-1+N$ *possessing for even* N *only odd powers and for odd* N *only even powers* (*like* $P^0_{2\kappa-1+N}$).

For *equation* (2.2), where $T=\infty$, the function K_1 on $\mathbb{R}$ can have infinitely many zeros and zeros of non-integer order. Avoiding further lengthy discussions we restrict ourselves to the most important case of finitely many zeros of integer order such that again (4.1) holds. Then, proceeding as above, we obtain

Theorem 4. *Let be* $k \in L^1(\mathbb{R}_+), g \in L^2(\mathbb{R}_+)$, *let the function* $K_1(z) = 1 - \int_0^\infty k(t)e^{izt}dt$ *have form* (4.1) *where* K_0 *possesses* κ *zeros* $z_j \quad (j = 1, \dots, \kappa)$ (*counted according to their multiplicities*) *in* $\operatorname{Im} z > 0$ *and let the function* G *given by* (3.2) *be sufficiently smooth near the points* $x_m \quad (m = 1, \dots, \rho)$. *Then:*

(i) *The homogeneous equation* (2.2) *has* $n = N + 2\kappa$ *linearly independent solutions over the real field given by* (3.38) *with* (4.8) *where* $P^0_{2\kappa-1+N}$ *is an arbitrary real polynomial of degree* $2\kappa - 1 + N$.

(ii) *The non-homogeneous equation* (2.2) *is solvable for any right-hand side* g *satisfying the* N *real conditions* (4.5) *and has the general solution* (3.38) *where* F *is defined by* (4.7) *with arbitrary real polynomial* $P_{2\kappa-1+N} = P^0_{2\kappa-1+N}$ *of degree* $2\kappa - 1 + N$.

Corollary 6. *In the real singular case we have for homogeneous equation* (2.2) $n = \kappa + \left[\frac{N+1}{2}\right]$ *linearly independent solutions of form* (3.32) *with* $T = \infty$ *and* (4.8) *where the real polynomial* $P^0_{2\kappa-1+N}$ *of degree* $2\kappa - 1 + N$ *has only odd powers if* N *is even and only even powers if* N *is odd. The general solution of the non-homogeneous equation* (2.2) *is given by* (3.32) *with* $T = \infty$ *and expression* (4.7) *for* F *where* $P_{2\kappa-1+N} = P^0_{2\kappa-1+N}$ *and the* $\left[\frac{N+1}{2}\right]$ *solvability conditions* (4.5) *in the points* $x_0 = 0$ *and* $x_k > 0 \quad (k = 1, \dots, \rho_1)$ *must be satisfied.*

5. Equations of the first kind

We briefly deal with the *equations of the first kind*

$$\int_0^{T-t} \overline{k(s)} p(s+t)\, ds + \int_0^{T-t} k(s+t) \overline{p(s)}\, ds = g(t) \tag{5.1}$$

$$\int_0^{\infty} \overline{k(s)} p(s+t)\, ds + \int_0^{\infty} k(s+t) \overline{p(s)}\, ds = g(t) \tag{5.2}$$

on $(0, T)$ and $\mathbb{R}_+$, respectively, where we assume that $k, g \in L^2(0, T)$ and $k \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+), g \in L^2(\mathbb{R}_+)$, respectively. We are looking for quadratic summable solutions p again. Applying the transformation $\mathcal{F}_1$ to equations (5.1) and (5.2), the condition

$$\operatorname{Re}\left[\overline{K(x)} F(x)\right] = G(x) \qquad (x \in \mathbb{R}) \tag{5.3}$$

is obtained where again $G = \mathcal{F}_1 g \in L^2(\mathbb{R})$. The complex Fourier transforms F of p and K of k are holomorphic functions in $\operatorname{Im} z > 0$, bounded on $\operatorname{Im} z \geq 0$

and vanishing at infinity in $\operatorname{Im} z > 0$, and for $T < 0$ they are entire functions of exponential type.

As in Section 3, at first we construct the holomorphic function F in $\operatorname{Im} z > 0$ as a solution of the Riemann-Hilbert problem (5.3) reducing this one to the conjugacy problem

$$\Phi^+(x) = A_0(x)\Phi^-(x) + H_0(x) \qquad (x \in \mathbb{R})$$

for the sectionally holomorphic function Φ defined by (3.5) and satisfying the relation $\Phi(\overline{z}) = -\overline{\Phi(z)}$ with $\Phi(\infty) = 0$, where $A_0 = \frac{K}{K_0}$ and $H_0 = \frac{2G}{K_0}$ with $K_0(x) = \overline{K(x)}$ (in this section we partly use other notations as in the sections above).

For the *homogeneous equations* (5.1) and (5.2) we have the condition

$$K_0(x)\Phi^+(x) = K(x)\Phi^-(x) \qquad (x \in \mathbb{R}). \tag{5.4}$$

Problem (5.4) has the solution

$$\Phi(z) = \begin{cases} C\,iK(z) & \text{in } \operatorname{Im} z > 0 \\ C\,iK_0(z) & \text{in } \operatorname{Im} z < 0 \end{cases} \tag{5.5}$$

with arbitrary $C \in \mathbb{R}$ and $K_0(z) = \overline{K(\overline{z})}$. This gives the obvious solution $p = Cik$ to the homogeneous equations (5.1) and (5.2). There are further solutions

$$\Phi(z) = \begin{cases} i\,P_N(z)K(z) & \text{in } \operatorname{Im} z > 0 \\ i\,P_N(z)K_0(z) & \text{in } \operatorname{Im} z < 0 \end{cases}$$

of (5.4) where P_N is a real polynomial of degree $N \geq 1$ if the kernel k is N times differentiable, with $k^{(n)}(0) = k^{(n)}(T) = 0 \quad (n = 0, ..., N-1)$ and $k^{(N)} \in L^2(0,T)$ for equation (5.1) and $k^{(n)}(0) = \lim_{t\to\infty} k^{(n)}(t) = 0 \quad (n = 0, ..., N-1)$ and $k^{(n)} \in L^2(\mathbb{R}_+) \quad (n = 0, ..., N)$ for equation (5.2). The corresponding solutions p of the homogeneous equations (5.1) and (5.2) are

$$p(t) = i\sum_{n=0}^{N} a_n i^n k^{(n)}(t) \qquad (a_n \in \mathbb{R}) \tag{5.6}$$

if $P_N(z) = \sum_{n=0}^{N} a_n z^N$.

Finally, let K have the zeros $z_j \quad (j = 1, 2, ...)$ of order $\nu_j \geq 1$ in $\operatorname{Im} z > 0$ where $\overline{z}_j$ is a zero of order $\mu_j \geq 0$ of K in $\operatorname{Im} z < 0$. Then

$$\Phi(z) = \begin{cases} i\,\frac{P_{2N}(z)}{R(z)}\,K(z) & \text{in } \operatorname{Im} z > 0 \\ i\,\frac{P_{2N}(z)}{R(z)}\,K_0(z) & \text{in } \operatorname{Im} z < 0, \end{cases}$$

where $R(z) = \prod_{j=1}^{r}(z-z_j)^{\nu_j}(z-\overline{z}_j)^{\nu_j}$ and $N = \sum_{j=1}^{r}\nu_j$, with arbitrary $r \geq 1$ zeros z_j yields a solution of (5.4). The corresponding holomorphic function $F(z) = i\,\frac{P_{2N}(z)}{R(z)}\,K(z) \quad (z \in \mathbb{C})$ is regular in $\operatorname{Im} z < 0$ if the real polynomial P_{2N} satisfies the complex conditions

$$P_{2N}^{(l)}(z_j) = 0 \qquad \begin{cases} l = 0, ..., \max(\nu_j - \mu_j, 0) \\ j = 1, ..., r \end{cases} \tag{5.7}$$

which leave free

$$\rho = (2N+1) - 2\sum_{j=1}^{r}\max(\nu_j - \mu_j, 0) = 1 + 2\sum_{j=1}^{r}\min(\nu_j, \mu_j) \geq 1 \tag{5.8}$$

(real) coefficients in P_{2N}. Therefore, for any combination of r zeros z_j with multiplicities ν_j of K in $\operatorname{Im} z > 0$ we obtain the solutions

$$p(t) = \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{P_{2N}(x)K(x)}{R(x)}\,e^{-ixt}dx$$

for $T = \infty$ with an arbitrary real polynomial P_{2N} of degree $2N = 2\sum_{j=1}^{r}\nu_j$.

For $T < \infty$ the polynomial P_{2N} has to fulfil conditions (5.7) giving ρ, defined by (5.8), linearly independent (with respect to the real field) solutions. In particular, if the orders ν_j and μ_j of the zeros z_j and $\overline{z}_j$ of K are equal, we have $2N+1$ solutions as for $T = \infty$, and if $\mu_j = 0$, we come back to the solution $p = Cik$ from (5.5).

If the kernel k is real-valued and we are looking for real solutions p, too, the function F must satisfy the relation $\overline{F(z)} = F(-\overline{z})$ such that the polynomial P_{2N} could have only odd powers. In particular, solution (5.5) falls out and in (5.6) we have the coefficients $a_n = 0$ for even n. Further, for zeros of the form $z_j = iy_j \;\; (y_j > 0)$ conditions (5.7) are real and for zeros of the form $z_j = \pm x_j + iy_j \;\; (y_j > 0)$ conditions (5.7) for fixed j coincide. Therefore, if we take $r = s_0 + 2s$ zeros z_j of K in $\operatorname{Im} z > 0$ with $z_j = iy_j \;\; (y_j > 0;\, j = 1, ..., s_0)$ and $z_j = x_j + iy_j \;\; (x_j, y_j > 0;\, j = s_0+1, ..., s_0+s)$, we have $\max_{1 \leq j \leq s_0}(\nu_j - \mu_j, 0)$ and $2\max_{s_0+1 \leq j \leq s_0+s}(\nu_j - \mu_j, 0)$ real conditions (5.7) yielding

$$\begin{aligned}\rho_0 &= N - \sum_{j=1}^{s_0}\max(\nu_j - \mu_j, 0) - 2\sum_{j=s_0+1}^{s_0+s}(\nu_j + \mu_j, 0) \\ &= \sum_{j=1}^{r}\min(\nu_j, \mu_j) \geq 0\end{aligned}$$

linearly independent real solutions of the homogeneous equation (5.1) and N real solutions of the homogeneous equation (5.2).

In the case of the *non-homogeneous equations* (5.1) and (5.2) we assume that $K(x) \neq 0 \ \ (x \in \mathbb{R})$ and for equation (5.1) also that K has no zeros in the finite half-plane $\operatorname{Im} z > 0$. We restrict ourselves to a discussion of the (formal) solution (cp. (3.18), (3.21))

$$F(z) = \begin{cases} K(z)\Psi_0(z) & \text{in } \operatorname{Im} z > 0 \\ K(z)\Psi_0(z) + \frac{2G(z)}{K_0(z)} & \text{in } \operatorname{Im} z < 0 \end{cases} \tag{5.9}$$

for $T < \infty$, and $F(z) = K(z)\Psi_0(z) \ \ (\operatorname{Im} z > 0)$ for $T = \infty$ where $K_0(z) = \overline{K(\overline{z})}$, G is again defined by (3.22), and

$$\Psi_0(z) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{G(\xi)}{B_0(\xi)}\,\frac{d\xi}{\xi - z}, \qquad B_0(x) = K(x)K_0(x) = |K(x)|^2.$$

We further assume that $\frac{G}{B_0} \in L^2(\mathbb{R})$. In virtue of this assumption the values of F on $\mathbb{R}$

$$F(x) = \frac{G(x)}{K_0(x)} + K(x)\frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{G(\xi)}{B_0(\xi)}\,\frac{d\xi}{\xi - x} \tag{5.10}$$

define a function in $L^2(\mathbb{R})$ since K is bounded by $k \in L^1$.

For equation (5.1) we have to show that (5.9) represents an entire function of exponential type which satisfies the Paley-Wiener condition for representation (3.3). The first statement is fulfilled if $\left|\frac{G(z)}{K(z)}\right| \le C\exp[A|z|] \ \ (C, A > 0;\ \operatorname{Im} z \ge 0)$ for large $|z|$ which we assume. For this an estimation of the form $|K(z)| \ge C_0 \exp[-A_0|z|] \ \ (C_0, A_0 > 0;\ \operatorname{Im} z \ge 0)$ for large $|z|$ is sufficient. For the parameter

$$b = \limsup_{y\to\infty} \frac{1}{y}\,\ln|F(iy)| = \limsup_{y\to\infty} \frac{1}{y}\,\ln|K(iy)\Psi_0(iy)|$$

we have $b \le 0$ in view of the relations $|K(iy)|, |\Psi_0(iy)| = O(1)$ for $y \to \infty$ (cp. Section 3). For the estimation of the parameter

$$a = \limsup_{y\to\infty} \frac{1}{y}\,\ln|F(-iy)| = \limsup_{y\to\infty} \frac{1}{y}\,\ln\left|K(-iy)\Psi_0(-iy) + \frac{2G(-iy)}{K_0(-iy)}\right|$$

we use the relations (cp. Section 3 again)

$$\left.\begin{aligned} |K(-iy)| &= O(e^{yT}) \\ |\Psi_0(-iy)| &= O(1) \\ |G(-iy)| &= O(e^{yT}) \end{aligned}\right\} \qquad (y \to \infty)$$

to obtain

$$|F(-iy)| = O(e^{yT})\Big[\frac{1}{|K_0(-iy)|} + 1\Big].$$

Hence we have $a \le T$ if additionally we assume $\lim_{y\to\infty} \frac{1}{y} \ln |K(iy)| \ge 0$ which is, for instance, satisfied if $|K(iy)| \ge C_1 y^{-\beta}$ $(C_1, \beta > 0)$ for large y. In particular, this is fulfilled with $\beta = 1$ for a real kernel k satisfying $k(t) \ge \delta > 0$ in $[0, T]$.

For equation (5.2) we ensure that $F(z) = K(z)\Psi_0(z)$ is a bounded function on $\operatorname{Im} z \ge 0$ assuming in addition to $\frac{G}{B_0} \in L^2(\mathbb{R})$ that $\frac{G}{B_0}$ is a Hölder-continuous function in $\mathbb{R}$ with $|\frac{G(x)}{B_0(x)}| \le \mathrm{Const}\, |x|^{-\alpha}$ $(\alpha > \frac{1}{2})$ for large $|x|$.

The solution p for equations (5.1) - (5.2) is then given by formula (3.30) with F defined in (5.10). In particular, if k and g are real-valued functions such that $K(-x) = \overline{K(x)}$, $B(-x) = \overline{B(x)}$ and $G(-x) = G(x) = \overline{G(x)}$, the function F satisfies also the relation $F(-x) = \overline{F(x)}$ and p is a real-valued solution which can be obtained from formula (3.32).

6. System of equations – the regular case

Finally, we treat the *system of equations*

$$p_\alpha(t) - (A_\alpha p)(t) = g_\alpha(t) \qquad (\alpha = 1, 2) \tag{6.1}$$

on $(0, T)$ for complex-valued $p = (p_1, p_2) \in L^2(0, T) \times L^2(0, T)$ where

$$(A_{1,2}p)(t) = \int_0^{T-t} \overline{k_{1,2}(s)}\, p_{1,2}(s+t)\, ds + \int_0^{T-t} k_{2,1}(s+t)\, \overline{p_{2,1}(s)}\, ds \tag{6.2}$$

and the corresponding system for $T = \infty$. Again we assume that $k_{1,2} \in L^1(0, T)$ and $g_{1,2} \in L^2(0, T)$. To equations (6.1) - (6.2) we apply the *mixed Fourier transformation*

$$(\widehat{\mathcal{F}}h)(x) = \int_0^T \big[e^{ixt} h_1(t) + e^{-ixt}\overline{h_2(t)}\big]\, dt \tag{6.3}$$

which maps a pair of complex functions $h = (h_1, h_2) \in L^2(0, T) \times L^2(0, T)$ into complex functions $\widehat{\mathcal{F}}h \in L^2(0, T)$. It follows

$$\widehat{\mathcal{F}}h = \mathcal{F}_1(h_1 + h_2) + \mathcal{F}_2(h_1 - h_2) \tag{6.4}$$

with $\mathcal{F}_1, \mathcal{F}_2$ defined by (2.5), (2.7) and mapping quadratic summable complex functions into real and purely imaginary quadratic summable functions, respectively. As in Section 2 this yields the inversion formula $\widehat{\mathcal{F}}(h_1, h_2) = \gamma$ for $\gamma \in L^2(0,T)$ of form (6.3) as

$$\begin{aligned} h_1(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt}\gamma(x)\,dx \\ h_2(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt}\overline{\gamma(x)}\,dx \end{aligned} \tag{6.5}$$

in $(0,T)$. Formulae (6.3) and (6.5) are valid for $T=\infty$, too.

For $h = (h_1, h_2) = (A_1 p, A_2 p)$ with $p = (p_1, p_2) \in L^2(0,T) \times L^2(0,T)$ the relation

$$\widehat{\mathcal{F}}(A_1 p, A_2 p)(x) = \overline{K_1(x)}F_1(x) + K_2(x)\overline{F_2(x)} \tag{6.6}$$

with $F_\alpha(x) = \int_0^T p_\alpha(t)e^{ixt}dt$ and $K_\alpha(x) = \int_0^T k_\alpha(t)e^{ixt}dt$ $(\alpha = 1,2)$ on $\mathbb{R}$ holds. The proof of (6.6) is quite analogous to the one for the similar relation (2.8) in Section 2 and is omitted. Relation (6.6) is also valid for $T = \infty$ with Fourier transforms $F_\alpha(x) = \int_0^\infty p_\alpha(t)e^{ixt}dt$ and $K_\alpha(x) = \int_0^\infty k_\alpha(t)e^{ixt}dt$ $(\alpha = 1,2)$ on $\mathbb{R}$.

By (6.6) the system of equations (6.1) - (6.2) is reduced to the problem of finding the Fourier transforms F_α $(\alpha = 1,2)$ of the solutions p_α satisfying the condition

$$[1 - \overline{K_1(x)}]F_1(x) + [1 - K_2(x)]\overline{F_2(x)} = G(x) \qquad (x \in \mathbb{R}) \tag{6.7}$$

where $G = \widehat{\mathcal{F}}g \in L^2(0,T)$ for $g = (g_1, g_2)$, i.e. $G(x) = \int_0^T \left[g_1(t)e^{ixt} + \overline{g_2(x)}e^{-ixt}\right]dt$ $(x \in \mathbb{R})$. We look on (6.7) as a *Riemann-Hilbert type problem* for the bounded holomorphic functions $F_\alpha(z) = \int_0^T e^{izt}p_\alpha(t)\,dt$ $(z = x+iy)$ in the upper half-plane vanishing at infinity. Introducing the sectionally holomorphic function

$$\Phi(z) = \begin{cases} F_1(z) & \text{if } \operatorname{Im} z > 0 \\ -\overline{F_2(\overline{z})} & \text{if } \operatorname{Im} z < 0, \end{cases}$$

(6.7) writes as the conjugacy condition for Φ with $\Phi(\infty) = 0$ (cp. (3.4))

$$\Phi^+(x) = A(x)\Phi^-(x) + H(x) \qquad (x \in \mathbb{R}) \tag{6.8}$$

where

$$A(x) = \frac{1 - K_2(x)}{1 - \overline{K_1(x)}}, \quad H(x) = \frac{G(x)}{1 - \overline{K_1(x)}}. \tag{6.9}$$

For simplicity of presentation, here we restrict ourselves to the *regular case* where

$$1 - K_\alpha(x) \neq 0 \qquad (x \in \mathbb{R};\ \alpha = 1,2) \tag{6.10}$$

shifting the singular case to the next section. The index of problem (6.8) with (6.10) is defined by $\kappa = \kappa_1 + \kappa_2$ with $\kappa_\alpha = \frac{1}{2\pi}\big[\arg(1 - K_\alpha(x))\big]_{\mathbb{R}}$ $(\alpha = 1, 2)$ where κ_α are finite non-negative numbers equal to the numbers of zeros of the functions $L_\alpha(z) = 1 - K_\alpha(z)$ with $K_\alpha(z) = \int_0^T k_\alpha(t)e^{izt}dt$ $\quad (z = x + iy)$ in the upper half-plane.

In the *case* $\kappa = 0$ the homogeneous problem (6.8) has only the solution $\Phi = 0$ satisfying $\Phi(\infty) = 0$. The solution of the inhomogeneous problem (6.8) is given by

$$\Phi(z) = \begin{cases} \Phi^+(z) & \text{in } \operatorname{Im} z > 0 \\ \Phi^-(z) & \text{in } \operatorname{Im} z < 0 \end{cases} \qquad \text{where } \begin{cases} \Phi^+(z) = F_1(z) = L_2(z)\Psi(z) \\ \Phi^-(z) = -\overline{F_2(\overline{z})} = M_1(z)\Psi(z) \end{cases}$$

with $L_\alpha(z) = 1 - K_\alpha(z), M_\alpha(z) = \overline{L_\alpha(\overline{z})} = 1 - \overline{K_\alpha(\overline{z})}$ and

$$\Psi(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{G(\xi)}{B(\xi)}\,\frac{d\xi}{\xi - z}, \qquad B(x) = M_1(x)L_2(x). \tag{6.11}$$

By (6.10) we have $B(x) \neq 0$ on $\mathbb{R}$ with $B(\pm\infty) = 1$. Further,

$$\begin{aligned} F_1(x) = \Phi^+(x) &= \frac{1}{2}\frac{G(x)}{M_1(x)} + \frac{L_2(x)}{2\pi i}\int_{-\infty}^{\infty} \frac{G(\xi)}{B(\xi)}\,\frac{d\xi}{\xi - x} \\ F_2(x) = -\overline{\Phi^-(x)} &= \frac{1}{2}\frac{\overline{G(x)}}{M_2(x)} + \frac{L_1(x)}{2\pi i}\int_{-\infty}^{\infty} \frac{\overline{G(\xi)}}{\overline{B(\xi)}}\,\frac{d\xi}{\xi - x} \end{aligned} \tag{6.12}$$

on $\mathbb{R}$, where $\overline{B(x)} = L_1(x)M_2(x)$. The complete functions F_α $(\alpha = 1, 2)$ are then defined by

$$\begin{aligned} F_1(z) &= \begin{cases} L_2(z)\Psi(z) & \text{if } \operatorname{Im} z > 0 \\ \frac{G(z)}{M_1(z)} + L_2(z)\Psi(z) & \text{if } \operatorname{Im} z < 0 \end{cases} \\ F_2(z) &= \begin{cases} -L_1(z)\overline{\Psi(\overline{z})} & \text{if } \operatorname{Im} z > 0 \\ \overline{G(\overline{z})}M_2(z) + L_1(z)\overline{\Psi(\overline{z})} & \text{if } \operatorname{Im} z < 0 \end{cases} \end{aligned} \tag{6.13}$$

where $G(z) = \int_0^T \big[g_1(t)e^{izt} + \overline{g_2(t)}e^{-izt}\big]dt$. As in the case of equation (2.1) the functions F_α obey the Paley-Wiener conditions so that

$$p_\alpha(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt}F_\alpha(x)\,dx, \tag{6.14}$$

with $F_\alpha \in L^2(\mathbb{R})$ given by (6.12), represent the solutions of system (6.1) - (6.2) in the case $\kappa = 0$.

In the *case* $\kappa > 0$ the functions L_1 and L_2 have zeros $z_j \quad (j = 1, ..., r)$ with multiplicities n_j and zeros $\zeta_k \quad (k = 1, ..., \rho)$ with multiplicities ν_k, respectively, in the upper half-plane where $\kappa_1 = \sum_{j=1}^{r} n_j$ and $\kappa_2 = \sum_{k=1}^{\rho} \nu_k$. The homogeneous problem (6.8) can be written in the form

$$R(x)\,\frac{\Phi^+(x)}{L_2(x)} = R(x)\,\frac{\Phi^-(x)}{M_1(x)} \qquad (x \in \mathbb{R}) \tag{6.15}$$

with polynomial $R(z) = \prod_{j=1}^{r}(z - \overline{z}_j)^{n_j} \cdot \prod_{k=1}^{\rho}(z - \zeta_k)^{\nu_k}$ of degree κ. From (6.15) we obtain the general solution of the homogeneous problem (6.8)

$$\Phi_0(z) = \begin{cases} \Phi_0^+(z), & \Phi_0^+(z) = F_1^0(z) = \frac{L_2(z)}{R(z)}\,P_{\kappa-1}^0(z) \quad (\operatorname{Im} z > 0) \\ \Phi_0^-(z), & \Phi_0^-(z) = -\overline{F_2^0(\overline{z})} = \frac{M_1(z)}{R(z)}\,P_{\kappa-1}^0(z) \quad (\operatorname{Im} z < 0) \end{cases}$$

or $F_2^0(z) = -\frac{L_1(z)}{R_0(z)}\,Q_{\kappa-1}^0(z) \;\; (\operatorname{Im} z > 0)$ with an arbitrary complex polynomial $P_{\kappa-1}^0$ of degree $\kappa - 1$, and with $R_0(z) = \overline{R(\overline{z})}$ and $Q_{\kappa-1}^0(z) = \overline{P_{\kappa-1}^0(\overline{z})}$. The general solution of the inhomogeneous problem (6.8) can be given as

$$\begin{aligned} \Phi^+(z) &= F_1(z) = \frac{L_2(z)}{R(z)}\,P_{\kappa-1}(z) + L_2(z)\Psi(z) && (\operatorname{Im} z > 0) \\ \Phi^-(z) &= -\overline{F_2(\overline{z})} = \frac{M_1(z)}{R(z)}\,P_{\kappa-1}(z) + M_1(z)\Phi(z) && (\operatorname{Im} z < 0) \end{aligned} \tag{6.16}$$

where $P_{\kappa-1}$ is a complex polynomial $P_{\kappa-1}$ of degree $\kappa - 1$ and Ψ is defined by (6.11) again. From the last expression the formula for F_2 in $\operatorname{Im} z > 0$

$$F_2(z) = -\frac{L_1(z)}{R_0(z)}\,Q_{\kappa-1}(z) - L_1(z)\overline{\Psi(\overline{z})} \tag{6.17}$$

follows with $Q_{\kappa-1}(z) = \overline{P_{\kappa-1}(\overline{z})}$. The values of $F_\alpha \quad (\alpha = 1, 2)$ on $\mathbb{R}$ are

$$\begin{aligned} F_1(x) &= \frac{L_2(x)}{R(x)}\,P_{\kappa-1}(x) + \frac{1}{2}\,\frac{G(x)}{M_1(x)} + \frac{L_2(x)}{2\pi i}\int_{-\infty}^{\infty}\frac{G(\xi)}{B(\xi)}\,\frac{d\xi}{\xi - x} \\ F_2(x) &= -\frac{L_1(x)}{R_0(x)}\,Q_{\kappa-1}(x) + \frac{1}{2}\,\frac{\overline{G(x)}}{M_2(x)} + \frac{L_1(x)}{2\pi i}\int_{-\infty}^{\infty}\frac{\overline{G(\xi)}}{\overline{B(\xi)}}\,\frac{d\xi}{\xi - x}. \end{aligned} \tag{6.18}$$

Analytic continuation yields for the functions F_α^0 and $F_\alpha \quad (\alpha = 1, 2)$ in the lower half-plane $\operatorname{Im} z < 0$ the expressions

$$\begin{aligned} F_1^0(z) &= \frac{L_2(z)}{R(z)}\,P_{\kappa-1}^0(z) \\ F_2^0(z) &= -\frac{L_1(z)}{R_0(z)}\,Q_{\kappa-1}^0(z) \\ F_1(z) &= \frac{L_2(z)}{R(z)}\,P_{\kappa-1}(z) + \frac{G(z)}{M_1(z)} + L_2(z)\Psi(z) \\ F_2(z) &= -\frac{L_1(z)}{R_0(z)}\,Q_{\kappa-1}(z) + \frac{\overline{G(\overline{z})}}{M_2(z)} - L_1(z)\overline{\Psi(\overline{z})}. \end{aligned} \tag{6.19}$$

The functions F_α^0 and F_α must be regular in the points $\overline{z}_j \quad (j = 1, ..., r)$ and $\overline{\zeta}_k \quad (k = 1, ..., \rho)$, respectively. This implies the conditions

$$\big[L_2(z)P_{\kappa-1}^0(z)\big]^{(l)}(\overline{z}_j) = 0 \tag{6.20}$$

$$\big[L_2(z)P_{\kappa-1}(z) + \tfrac{R(z)}{M_1(z)}\, G(z)\big]^{(l)}(\overline{z}_j) = 0 \tag{6.21}$$

for $l = 0, ..., n_j - 1$ and $j = 1, ..., r$, and the conditions

$$\big[L_1(z)Q_{\kappa-1}^0(z)\big]^{(l)}(\overline{\zeta}_k) = 0 \tag{6.22}$$

$$\big[-L_1(z)Q_{\kappa-1}(z) + \tfrac{R_0(z)}{M_2(z)}\, \overline{G(\overline{z})}\big]^{(l)}(\overline{\zeta}_k) = 0 \tag{6.23}$$

for $l = 0, ..., \nu_k - 1$ and $k = 1, ..., \rho$. Now let $\overline{z}_j$ be a zero of order $m_j \geq 0$ of L_2 and $\overline{\zeta}_k$ a zero of order $\mu_k \geq 0$ of L_1, i.e.

$$L_2^{(l)}(\overline{z}_j) = 0 \quad \begin{cases} l = 0, ..., m_j - 1 \\ j = 1, ..., r \end{cases}, \quad L_2^{(m_j)}(\overline{z}_j) \neq 0$$

$$L_1^{(l)}(\overline{\zeta}_k) = 0 \quad \begin{cases} l = 0, ..., \mu_k - 1 \\ k = 1, ..., \rho \end{cases}, \quad L_1^{(\mu_k)}(\overline{\zeta}_k) \neq 0.$$

Then conditions (6.20) and (6.22) are equivalent to

$$\begin{aligned} &[P_{\kappa-1}^0(z)]^{(l)}(\overline{z}_j) = 0 \qquad &&\begin{cases} l = 0, ..., \max(n_j - m_j, 0) - 1 \\ j = 1, ..., r \end{cases} \\ &[P_{\kappa-1}^0(z)]^{(l)}(\zeta_k) = 0 \qquad &&\begin{cases} l = 0, ..., \max(\nu_k - \mu_k, 0) - 1 \\ k = 1, ..., \rho. \end{cases} \end{aligned} \tag{6.24}$$

These relations, for

$$\kappa_0 = \sum_{j=1}^{r} \min(n_j, m_j) + \sum_{k=1}^{\rho} \min(\nu_k, \mu_k), \tag{6.25}$$

represent $\kappa - \kappa_0$ linearly independent complex conditions, i.e. in the polynomial $P_{\kappa-1}^0$ there remain κ_0 free complex parameters. That is, if $\kappa_0 \geq 1$, we have $n = \kappa_0$ linearly independent (over the complex field) solutions of the form

$$\begin{aligned} F_1^0(x) &= \frac{L_2(x)}{R(x)}\, P_{\kappa-1}^0(x) \\ F_2^0(x) &= -\frac{L_1(x)}{\overline{R(x)}}\, \overline{P_{\kappa-1}^0(x)} \end{aligned} \qquad (x \in \mathbb{R}) \tag{6.26}$$

to the homogeneous problem (6.7).

Conditions (6.21) and (6.23) for the inhomogeneous problem (6.7) divide into the κ_0 complex conditions

$$\begin{aligned} G^{(l)}(\overline{z}_j) = 0 \qquad & \begin{cases} l = 0, ..., \min(n_j, m_j) - 1 \\ j = 1, ..., r \end{cases} \\ G^{(l)}(\zeta_k) = 0 \qquad & \begin{cases} l = 0, ..., \min(\nu_k, \mu_k) - 1 \\ k = 1, ..., \rho \end{cases} \end{aligned} \tag{6.27}$$

and the $\kappa - \kappa_0$ complex conditions (6.21) and (6.23) for

$$l = \begin{cases} \min(n_j, \mu_j), ..., n_j - 1 & (j = 1, ..., r) \\ \min(\nu_k, \mu_k), ..., \nu_k - 1 & (k = 1, ..., \rho), \end{cases}$$

respectively, for the κ complex coefficients of $P_{\kappa-1}$. Hereby, (6.23) is equivalent to

$$\Big[- M_1(z) P_{\kappa-1}(z) + \frac{R(z)}{L_2(z)}\, G(z) \Big]^{(l)}(\zeta_k) = 0 \tag{6.28}$$

for $l = 0, ..., \nu_k - 1$ and $k = 1, ..., \rho$.

Since the functions F_α^0 and F_α again satisfy the Paley-Wiener conditions and the transformation $\widehat{F}$ is injective we obtain

Theorem 5. *Let be $k_\alpha \in L^1(0,T)$ and $g_\alpha \in L^2(0,T)$, let the functions $L_\alpha(z) = 1 - \int_0^T k_\alpha(t) e^{izt} dt$ satisfy the conditions $L_\alpha(x) \neq 0$ on $\mathbb{R}$ $(\alpha = 1, 2)$, and let L_1 possess zeros z_j $(j = 1, ..., r)$ with multiplicities n_j and L_2 zeros ζ_k $(k = 1, ..., \rho)$ with multiplicities ν_k in $\operatorname{Im} z > 0$. Then:*

(i) *The homogeneous system (6.1)–(6.2) has $n = \kappa_0$ linearly independent complex solutions where κ_0 is defined by (6.25). The solutions are given by (6.14) with F_α^0 defined in (6.26) where the polynomial $P_{\kappa-1}^0$ of degree $\kappa - 1$ with $\kappa = \kappa_1 + \kappa_2, \kappa_1 = \sum_{j=1}^r n_j$ and $\kappa_2 = \sum_{k=1}^\rho \nu_k$, satisfies the $\kappa - \kappa_0$ conditions (6.24).*

(ii) *The inhomogeneous system (6.1) – (6.2) is solvable if and only if the κ_0 solvability conditions (6.27) are satisfied. The general solution of system (6.1) – (6.2) is then given by (6.14) with F_α defined by (6.18) where the polynomial $P_{\kappa-1}$ of degree $\kappa - 1$ obeys the $\kappa - \kappa_0$ conditions (6.21), (6.28) in the points $\overline{z}_j$ and ζ_k, respectively.*

Corollary 7. *For real-valued functions h_α and g_α we are looking for real solutions*

$$p_\alpha(t) = \operatorname{Re}\Big(\frac{1}{\pi} \int_0^\infty F_\alpha(x) e^{-ixt} dx \Big) \tag{6.29}$$

of system (6.1) – (6.2) where $F_\alpha(-z) = \overline{F_\alpha(\overline{z})}$ $(\alpha = 1, 2)$ is required.

(i) *We divide the zeros of* L_α *in* $\operatorname{Im} z > 0$ *into*

$$z_j = iy_j \qquad (y_j > 0;\ j = 1, ..., r_1)$$
$$\zeta_k = i\eta_k \qquad (\eta_k > 0;\ k = 1, ..., \rho_1)$$

and

$$z_j = \pm x_j + iy_j \qquad (x_j, y_j > 0; j = r_1 + 1, ..., r)$$
$$\zeta_k = \pm\xi_k + i\eta_k \qquad (\xi_k, \eta_k > 0;\ k = \rho_1 + 1, ..., \rho)$$

and put

$$N_1 = \sum_{j=1}^{r_1} n_j, \quad N_2 = \sum_{k=1}^{\rho_1} \nu_k, \quad M_1 = \sum_{j=r_1+1}^{r} n_j, \quad M_2 = \sum_{h=\rho_1+1}^{\rho} \nu_k$$

such that $\kappa_\alpha = N_\alpha + 2M_\alpha \ \ (\alpha = 1, 2)$*. Then the function*

$$R(z) = \prod_{j=1}^{r_1} (z + iy_j)^{n_j} \prod_{j=r_1+1}^{r} \big[(z - x_j + iy_j)(z + x_j + iy_j)\big]^{n_j}$$
$$\times \prod_{k=1}^{\rho_1} (z - i\eta_k)^{\nu_k} \prod_{k=\rho_1+1}^{\rho} \big[(z - \xi_k - i\eta_k)(z + \xi_k - i\eta_k)\big]^{\nu_k}$$

satisfies $R(-z) = (-1)^N \overline{R(\overline{z})}$ *with* $N = N_1 + N_2$*. Therefore, the solutions* F^0_α $(\alpha = 1, 2)$ *of the homogeneous problem* (6.7) *defined by* (6.16) – (6.17) *and* (6.19) *fulfil the relations* $F^0_\alpha(-z) = \overline{F^0_\alpha(\overline{z})}$ *if the polynomial* $P^0_{\kappa-1}$ *obeys the relation* $P^0_{\kappa-1}(-z) = (-1)^N \overline{P^0_{\kappa-1}(\overline{z})}$*. Further, conditions* (6.24) *for* $P^0_{\kappa-1}$ *are equivalent to* $\kappa - \kappa_0$ *real conditions where*

$$\kappa_0 = \sum_{j=1}^{r_1} \min(n_j, m_j) + \sum_{k=1}^{\rho_1} \min(\nu_k, \mu_k)$$
$$+ 2\Big[\sum_{j=r_1+1}^{r} \min(n_j, m_j) + \sum_{k=\rho_1+1}^{\rho} \min(\nu_k, \mu_k)\Big]. \tag{6.30}$$

So we have $n = \kappa_0$ *real solutions to the homogeneous system of equations* (6.1) – (6.2)*.*

(ii) *In the same way, the solvability conditions* (6.27) *represent* κ_0 *real conditions on the right-hand sides* g_α $(\alpha = 1, 2)$ *of the inhomogeneous system* (6.1) – (6.2)*. The solutions* p_α *of the inhomogeneous system* (6.1) – (6.2) *are given by means of formulas* (6.14), (6.18) *where we can replace* $R_0(x), \overline{G(x)}, \overline{B(x)}$ *correspondingly by* $(-1)^N R(-x), G(-x), B(-x)$ *and where* $P_{\kappa-1}$ *satisfies* $\overline{P_{\kappa-1}(x)} = (-1)^N P_{\kappa-1}(-x)$ *and* (6.21), (6.28)*.*

Corollary 8.

(i) *In the case* $T = \infty$, *under the regularity assumption* (6.10), *there exist* $n = \kappa$ *linearly independent complex (in the real case: real) solutions of the homogeneous system of equations* (6.1) – (6.2) *given by* (6.14) *with* F_α^0 *defined in* (6.26), *with an arbitrary complex polynomial* $P_{\kappa-1}^0$ *of degree* $\kappa - 1$ $\big($*or, in the real case, by* (6.29) *with* (6.26) *and a polynomial* $P_{\kappa-1}^0$ *of degree* $\kappa - 1$ *satisfying* $\overline{P_{\kappa-1}^0(x)} = (-1)^N P_{\kappa-1}^0(-x)\big)$.

(ii) *The inhomogeneous system* (6.1) – (6.2) *is always solvable and its solutions are given by* (6.14), *respectively* (6.29), *with* F_α *defined by formulae* (6.18) *with* $P_{\kappa-1} = P_{\kappa-1}^0$.

7. System of equations – the singular case

In the *singular case* of system (6.1) - (6.2) the conjugacy problem (6.8) is of singular (exceptional) type in general where the coefficient $A(x)$ can have zeros and poles on $\mathbb{R}$ (cf. [5: Chapter 3] and [10: Section 10.4]). Let the entire functions $L_\alpha(z) = 1 - K_\alpha(z) \quad (\alpha = 1, 2)$ have finitely many zeros $x_m \quad (m = 1, ..., \rho_1)$ of integer order n_m and $\xi_j \quad (j = 1, ..., \rho_2)$ of integer order ν_j, respectively, on $\mathbb{R}$. Then the representations

$$L_1(z) = \Pi_1(z) L_{1,0}(z), \quad \Pi_1(z) = \prod_{m=1}^{\rho_1} (z - x_m)^{n_m} \tag{7.1}$$

$$L_2(z) = \Pi_2(z) L_{2,0}(z), \quad \Pi_2(z) = \prod_{j=1}^{\rho_2} (z - \xi_j)^{\nu_j} \tag{7.2}$$

hold where $L_{\alpha,0}(x) \neq 0 \quad (x \in \mathbb{R})$ and $L_{\alpha,0}(x) \sim x^{-N_\alpha} \quad (x \to \pm\infty)$ with $N_1 = \sum_{m=1}^{\rho_1} n_m$ and $N_2 = \sum_{j=1}^{\rho_2} \nu_j$.

The *conjugacy problem* (6.8) has the coefficients

$$A(x) = \Pi(x) \frac{L_{2,0}(x)}{M_{1,0}(x)}, \quad H(x) = \frac{G_1(x)}{M_{1,0}(x)} \tag{7.3}$$

where $\Pi(z) = \frac{\Pi_2(z)}{\Pi_1(z)}$, $M_{1,0}(x) = \overline{L_{1,0}(x)}$ and $G_1(x) = \frac{G(x)}{\Pi_1(x)}$. Now let be $\xi_k = x_k \quad (k = 1, ..., \rho)$ and $\xi_j \neq x_m$ for $j, m > \rho$ where $0 \leq \rho \leq \min(\rho_1, \rho_2)$. Then

$$\Pi(z) = \prod_{k=1}^{\rho} (z - x_k)^{\nu_k - n_k} \frac{\prod_{j=\rho+1}^{\rho_2} (z - \xi_j)^{\nu_j}}{\prod_{m=\rho+1}^{\rho_1} (z - x_m)^{n_m}}.$$

Further, let be $\nu_k \geq n_k$ $(k = 1, ..., \rho_0; 0 \leq \rho_0 \leq \rho)$ and $\nu_k < n_k$ $(k = \rho_0 + 1, ..., \rho)$. Then, finally, $\Pi = \frac{\Pi_{2,0}}{\Pi_{1,0}}$ where

$$\Pi_{1,0}(z) = \prod_{k=\rho_0+1}^{\rho} (z - x_k)^{n_k - \nu_k} \prod_{m=\rho+1}^{\rho_1} (z - x_m)^{n_m}$$

is a (real) polynomial of degree $N_{1,0} = \sum_{k=\rho_0+1}^{\rho} (n_k - \nu_k) + \sum_{m=\rho+1}^{\rho_1} n_m$ and

$$\Pi_{2,0}(z) = \prod_{k=1}^{\rho_0} (z - x_k)^{\nu_k - n_k} \prod_{j=\rho+1}^{\rho_2} (z - \xi_j)^{\nu_j}$$

is a (real) polynomial of degree $N_{2,0} = \sum_{k=1}^{\rho_0} (\nu_k - n_k) + \sum_{j=\rho+1}^{\rho_2} \nu_j$. There holds $N_1 - N_{1,0} = N_2 - N_{2,0} = \sum_{k=1}^{\rho} \min(n_k, \nu_k)$.

Further, let the functions L_α $(\alpha = 1, 2)$ have zeros z_j $(j = 1, ..., r_1)$ with multiplicities d_j and zeros ζ_k $(k = 1, ..., r_2)$ with multiplicities δ_k, respectively, in $\operatorname{Im} z > 0$. We put $\kappa_1 = \sum_{j=1}^{r_1} d_j$ and $\kappa_2 = \sum_{k=1}^{r_2} \delta_k$ and set $\kappa = \kappa_1 + \kappa_2$ (≥ 0). Then the *homogeneous problem* (6.8) with (7.3) can be written in the form

$$R(x)\Pi_{1,0}(x) \, \frac{\Phi^+(x)}{L_{2,0}(x)} = R(x)\Pi_{2,0}(x) \, \frac{\Phi^-(x)}{M_{1,0}(x)} \qquad (x \in \mathbb{R})$$

with polynomial $R(z) = \prod_{j=1}^{r_1} (z - \overline{z}_j)^{d_j} \prod_{k=1}^{r_2} (z - \zeta_k)^{\delta_k}$ of degree κ. In view of the asymptotic behaviour of L_α at infinity the general solution to the homogeneous problem (6.8) with (7.3) is given by

$$F_1^0(z) = \frac{L_{2,0}(z)}{R(z)} \, \frac{P^0_{\kappa+\lambda-1}(z)}{\Pi_{1,0}(z)}$$
$$F_2^0(z) = \frac{L_{1,0}(z)}{R_0(z)} \, \frac{Q^0_{\kappa+\lambda-1}(z)}{\Pi_{2,0}(z)}$$

where $R_0(z) = \overline{R(\overline{z})}$ and $P^0_{\kappa+\lambda-1}$ is a complex polynomial of degree $\kappa + \lambda - 1$ with

$$\lambda = N_1 + N_{2,0} = N_2 + N_{1,0} = \sum_{m=\rho_0+1}^{\rho_1} n_m + \sum_{k=1}^{\rho_0} \nu_k + \sum_{j=\rho+1}^{\rho_2} \nu_j \tag{7.4}$$

and $Q^0_{\kappa+\lambda-1}(z) = \overline{P^0_{\kappa+\lambda-1}(\overline{z})}$. Regularity of F^0_α in the points x_m, ξ_j on $\mathbb{R}$ requires that $P^0_{\kappa+\lambda-1} = \Pi_{1,0}\Pi_{2,0}P^0_{\kappa+\lambda_0-1}$ with a complex polynomial $P^0_{\kappa+\lambda-1}$ of degree $\kappa + \lambda_0 - 1$,

$$\lambda_0 = \lambda - [N_{1,0} + N_{2,0}] = \sum_{k=1}^{\rho} \min(n_k, \nu_k) \tag{7.5}$$

such that

$$F_1^0(z) = \frac{L_{2,0}(z)}{R(z)}\,\Pi_{2,0}(z) P^0_{\kappa+\lambda_0-1}(z)$$
$$F_2^0(z) = -\frac{L_{1,0}(z)}{R_0(z)}\,\Pi_{1,0}(z) Q^0_{\kappa+\lambda_0-1}(z).$$

Finally, regularity of F_α^0 in the points $\overline{z}_j$ and $\overline{\zeta}_k$ in $\operatorname{Im} z < 0$ leads to the conditions

$$\begin{aligned} &\left[L_{2,0}(z)P^0_{\kappa+\lambda_0-1}(z)\right]^{(l)}(\overline{z}_j) = 0 \qquad \begin{cases} l = 0, ..., d_j - 1 \\ j = 1, ..., r_1 \end{cases} \\ &\left[L_{1,0}(z)Q^0_{\kappa+\lambda_0-1}(z)\right]^{(l)}(\overline{\zeta}_k) = 0 \qquad \begin{cases} l = 0, ..., \delta_k - 1 \\ k = 1, ..., r_2). \end{cases} \end{aligned} \tag{7.6}$$

Now let $\overline{z}_j$ be a zero of order $m_j \geq 0$ of $L_{2,0}$ and $\overline{\zeta}_k$ a zero of order $\mu_k \geq 0$ of $L_{1,0}$. Then conditions (7.6) are equivalent to

$$\begin{aligned} &[P^0_{\kappa+\lambda_0-1}(z)]^{(l)}(\overline{z}_j) = 0 \qquad \begin{cases} l = 0, ..., \max(d_j - m_j, 0) - 1 \\ j = 1, ..., r_1 \end{cases} \\ &[P^0_{\kappa+\lambda_0-1}(z)]^{(l)}(\zeta_k) = 0 \qquad \begin{cases} l = 0, ..., \max(\delta_k - \mu_k, 0) - 1 \\ k = 1, ..., r_2. \end{cases} \end{aligned} \tag{7.7}$$

These relations represent $\kappa - \kappa_0$ linearly independent complex conditions with

$$\kappa_0 = \sum_{j=1}^{r_1} \min(d_j, m_j) + \sum_{k=1}^{r_2} \min(\delta_k, \mu_k). \tag{7.8}$$

Therefore, the homogeneous problem (6.7) has exactly $n = \kappa_0 + \lambda_0$ linearly independent (over the complex field) solutions of the form

$$\begin{aligned} F_1^0(x) &= \frac{L_{2,0}(x)}{R(x)}\,\Pi_{2,0}(x) P^0_{\kappa+\lambda_0-1}(x) \\ F_2^0(x) &= -\frac{L_{1,0}(x)}{\overline{R(x)}}\Pi_{1,0}(x)\overline{P_{\kappa+\lambda_0-1}(x)} \end{aligned} \tag{7.9}$$

where the complex polynomial $P^0_{\kappa+\lambda_0-1}$ of degree $\kappa + \lambda_0$ satisfies the $\kappa - \kappa_0$ conditions (7.7).

In the *case* $T = \infty$ the solutions of the homogeneous system (6.1) - (6.2) with (7.1) - (7.2) are given by (7.9) with an arbitrary complex polynomial $P^0_{\kappa+\lambda_0-1}$ of degree $\kappa + \lambda_0$. Further, in the *real case* the polynomial $P^0_{\kappa+\lambda_0-1}$ must satisfy the symmetry relation $\overline{P^0_{\kappa+\lambda_0-1}(x)} = (-1)^{N_0} P^0_{\kappa+\lambda_0-1}(-x)$ where

$N_0 = \widetilde{N}_1 + \widetilde{N}_2 + \min(n_0, \nu_0)$, with n_0, ν_0 the multiplicities of the zero $x_0 = 0$ of L_1, L_2, respectively, and $\widetilde{N}_\alpha$ $(\alpha = 1, 2)$ the sums of multiplicities of the zeros of L_α on the positive imaginary axis (cf. Corollary 7). And conditions (7.7) reduce to $\kappa - \kappa_0$ real conditions so that we have $n = \kappa_0 + \lambda_0$ real solutions to the homogeneous system of equations (6.1) - (6.2) in the real singular case where κ_0 defined by (7.8) has the form of (6.30) again.

For the *inhomogeneous system* (6.1) - (6.2), from (6.8) with (7.3) it follows that in the points x_k $(k = 1, ..., \rho_0)$ the function $G_1 = \frac{G}{\Pi_1}$ and in the points x_k $(k = \rho_0 + 1, ..., \rho)$ the function $G_2 = \frac{G}{\Pi_2}$ must be bounded. This yields λ_0 solvability conditions

$$G^{(l)}(x_k) = 0 \qquad \begin{cases} l = 0, ..., \min(n_k, \nu_k) - 1 \\ k = 1, ..., \rho. \end{cases} \tag{7.10}$$

Further, the function G has to fulfill κ_0 conditions

$$\begin{aligned} G^{(l)}(\overline{z}_j) = 0 \qquad & \begin{cases} l = 0, ..., \min(d_j, m_j) - 1 \\ j = 1, ..., r_1 \end{cases} \\ G^{(l)}(\zeta_k) = 0 \qquad & \begin{cases} l = 0, ..., \min(\delta_k, \mu_k) - 1 \\ k = 1, ..., r_2 \end{cases} \end{aligned} \tag{7.11}$$

(cp. (6.27)).

The Fourier transforms of the solutions $p = (p_1, p_2)$ to system (6.1) - (6.2) can be given in the form

$$\begin{aligned} F_1(z) &= \frac{L_{2,0}(z)}{\Pi_{1,0}(z)} \Big[\frac{P_{\kappa+\lambda-1}(z)}{R(z)} + \hat{\Psi}(z)\Big] \qquad (\mathrm{Im}\, z > 0) \\ F_2(z) &= -\frac{L_{1,0}(z)}{\Pi_{2,0}(z)} \Big[\frac{Q_{\kappa+\lambda-1}(z)}{R_0(z)} + \overline{\hat{\Psi}(\overline{z})}\Big] \qquad (\mathrm{Im}\, z > 0) \end{aligned}$$

where $P_{\kappa+\lambda-1}$ is a complex polynomial of degree $\kappa + \lambda - 1$, $Q_{\kappa+\lambda-1}(z) = \overline{P_{\kappa+\lambda-1}(\overline{z})}$ and

$$\widehat{\Psi}(z) = \frac{1}{4\pi i} \int_{-\infty}^{\infty} \Big[\Big(\frac{z+i}{\xi+i}\Big)^{\hat{N}} + \Big(\frac{z-i}{\xi-i}\Big)^{\hat{N}}\Big] \frac{G(\xi)}{\Pi_0(\xi) B(\xi)} \frac{d\xi}{\xi - z}$$

with $B(x) = M_{1,0}(x) L_{2,0}(x)$, $\widehat{N} = N_1 + N_2 - \lambda_0$ and the polynomial

$$\Pi_0(x) = \frac{\Pi_1(x)}{\Pi_{1,0}(x)} = \frac{\Pi_2(x)}{\Pi_{2,0}(x)} = \prod_{k=1}^{\rho} (x - x_k)^{\min(n_k, \nu_k)}.$$

of degree λ_0. We point out that $\frac{G}{\Pi_0} \in L^2(\mathbb{R})$ due to conditions (7.10).

The polynomial $P_{\kappa+\lambda-1}$ must fulfil $\lambda-\lambda_0 = N_{1,0}+N_{2,0}$ conditions that F_1 and F_2 are regular in the zeros of $\Pi_{1,0}$ and $\Pi_{2,0}$ on $\mathbb{R}$, respectively, and $\kappa-\kappa_0$ conditions, κ_0 defined in (7.8), that the analytic continuations of F_α $(\alpha = 1,2)$ in $\operatorname{Im} z < 0$ are regular in the conjugate points $\overline{z}_j$ and $\overline{\zeta}_k$ to the zeros z_j and ζ_k of L_α. The last conditions are left out in the *case* $T = \infty$ where we assume that G and K_α are sufficiently smooth near the zeros x_m and ξ_j, respectively, of L_α on $\mathbb{R}$.

Finally, we write out the values of F_α $(\alpha = 1,2)$ on $\mathbb{R}$ as

$$\begin{aligned} F_1(x) &= \frac{L_{2,0}(x)}{\Pi_{1,0}(x)}\Big[\frac{P_{\kappa+\lambda-1}(x)}{R(x)} + \widehat{\Psi}(x)\Big] + \frac{1}{2}\,\frac{G(x)}{\Pi_1(x)M_{1,0}(x)} \\ F_2(x) &= -\frac{L_{1,0}(x)}{\Pi_{2,0}(x)}\Big[\frac{\overline{P_{\kappa+\lambda-1}(x)}}{\overline{R(x)}} + \overline{\widehat{\Psi}(x)}\Big] + \frac{1}{2}\,\frac{\overline{G(x)}}{\Pi_2(x)M_{2,0}(x)} \end{aligned} \tag{7.12}$$

where $M_{\alpha,0}(x) = \overline{L_\alpha(0,x)}$ and

$$\widehat{\Psi}(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \widehat{Q}(x,\xi)\,\frac{G(\xi)}{\Pi_0(\xi)B(\xi)}\,\frac{d\xi}{\xi - x} \qquad (x \in \mathbb{R})$$

with $\widehat{Q}(x,\xi) = \frac{1}{2}\big[(\frac{x+i}{\xi+i})^{\widehat{N}} + (\frac{x-i}{\xi-i})^{\widehat{N}}\big]$. The formulae for F_α in the *real case* are analogous.

In view of formulae (7.12) the $\lambda - \lambda_0$ *conditions* in the zeros of $\Pi_{1,0}$ and $\Pi_{2,0}$ can be expressed in the form

$$\Big[\frac{P_{\kappa+\lambda-1}(x)}{R(x)} + \widehat{\Psi}(x) + \frac{1}{2}\,\frac{G(x)}{\Pi_0(x)B(x)}\Big]^{(l)}(x_m) = 0 \tag{7.13}$$

for

$$l = \begin{cases} 0,...,n_m - \nu_m - 1 & (m = \rho_0 + 1,...,\rho) \\ 0,...,n_m - 1 & (m = \rho + 1,...,\rho_1) \end{cases}$$

and

$$\Big[\frac{P_{\kappa+\lambda-1}(x)}{R(x)} + \widehat{\Psi}(x) - \frac{1}{2}\,\frac{G(x)}{\Pi_0(x)B(x)}\Big]^{(l)}(\xi_j) = 0 \tag{7.14}$$

for

$$l = \begin{cases} 0,...,\nu_j - n_j - 1 & (j = 1,...,\rho_0) \\ 0,...,\nu_j - 1 & (j = \rho + 1,...,\rho_2). \end{cases}$$

The $\kappa - \kappa_0$ conditions in the points $\overline{z}_j$ and $\overline{\zeta}_k$ can be written as (cp. (6.21) and (6.28))

$$\Big[L_{2,0}(z)P_{\kappa+\lambda-1}(z) + \frac{R(z)}{M_{1,0}(z)}\,\frac{G(z)}{\Pi_0(z)}\Big]^{(l)}(\overline{z}_j) = 0 \tag{7.15}$$

for $l = \min(d_j, m_j),...,d_j - 1$ $(j = 1,...,r_1)$ and

$$\Big[- M_{1,0}(z)P_{\kappa+\lambda-1}(z) + \frac{R(z)}{L_{2,0}(z)}\,\frac{G(z)}{\Pi_0(z)}\Big]^{(l)}(\zeta_k) = 0 \tag{7.16}$$

for $l = \min(\delta_k, \mu_k),...,\delta_k - 1$ $(k = 1,...,r_2)$ where $M_{\alpha,0}(z) = \overline{L_{\alpha,0}(\overline{z})}$.

Theorem 6. *Let be* $k_\alpha \in L^1(0,T)$ *and* $g_\alpha \in L^2(0,T)$, *let the functions* $L_\alpha(z) = 1 - \int_0^T k_\alpha(t)e^{izt}dt \quad (\alpha = 1,2)$ *have form* (7.1) – (7.2), *and let the function* L_1 *possess zeros* $z_j \quad (j = 1,...,r_1)$ *with multiplicities* d_j *and* L_2 *possess zeros* $\zeta_k \quad (k = 1,...,r_2)$ *with multiplicities* δ_k *in* $\operatorname{Im} z > 0$. *Then:*

(i) *The homogeneous system* (6.1) – (6.2) *has* $n = \kappa_0 + \lambda_0$ *linearly independent* (*complex; in the real case – real*) *solutions where* κ_0 *is defined by* (7.8) *and* λ_0 *by* (7.5). *The solutions are given by* (6.14) *with* F_α^0 *defined in* (7.9) *where the polynomial* $P^0_{\kappa+\lambda_0-1}$ *of degree* $\kappa + \lambda_0 - 1$ *with* $\kappa = \kappa_1 + \kappa_2$, $\kappa_1 = \sum_{j=1}^{r_1} d_j$ *and* $\kappa_2 = \sum_{k=1}^{r_2} \delta_k$ *satisfies the* $\kappa - \kappa_0$ *conditions* (7.7) (*and in the real case additionally a symmetry relation*).

(ii) *The inhomogeneous system* (6.1) – (6.2) *is solvable if the* $\kappa_0 + \lambda_0$ *solvability conditions* (7.10) – (7.11) *are satisfied. The general solution of system* (6.1) – (6.2) *is then given by* (6.14) *with* F_α *defined by* (7.12) *where the polynomial* $P_{\kappa+\lambda-1}$ *of degree* $\kappa + \lambda - 1$ *with* λ *given by* (7.4) *obeys the* $(\lambda - \lambda_0) + (\kappa - \kappa_0)$ *conditions* (7.13) – (714) *and* (7.15) – (7.16), *respectively* (*and, in the real case, additionally a symmetry relation*).

For $T = \infty$ *there holds* $\kappa_0 = \kappa$ *and conditions* (7.7) *and* (7.15) – (7.16) *are omitted.*

8. Examples

We illustrate the above results by several examples where we restrict ourselves to the real case.

Example 1. (*Generalized Feller equation*). We consider equations (2.1) - (2.2) for a non-negative real-valued kernel $k \in L^1(0,T)$ with $\int_0^T k(t)\,dt \le 1$ divided in the two cases

$$k(t) \ge 0 \;\text{ on } (0,T), \qquad \int_0^T k(t)\,dt < 1 \tag{8.1}$$

$$k(t) \ge 0 \;\text{ on } (0,T), \qquad \int_0^T k(t)\,dt = 1. \tag{8.2}$$

In the second case k is a probability density function with support on $[0,T]$. In this case we denote the finite expectation

$$c = \int_0^T t\,k(t)\,dt > 0 \tag{8.3}$$

assuming also $t\,k \in L^1(\mathbb{R}_+)$ for $T = \infty$. Then the function $k_1(z) = 1 - K(z) = 1 - \int_0^T k(t)e^{izt}\,dt$ has no zeros on $\operatorname{Im} z \ge 0$ in case (8.1) and it has the only zero $x_1 = 0$ on $\mathbb{R}$ which by (8.3) is a simple one in case (8.2).

For (8.1) we have the regular case of equations (2.1) - (2.2) with $\kappa = 0$. Then the homogeneous equations (2.1) - (2.2) possess the trivial solution $p_0 = 0$ only and the solution to the inhomogeneous equations (2.1) - (2.2) is given by its Fourier transform F in (3.12).

For (8.2) we have the singular case of equations (2.1) - (2.2) with $N = 1, \kappa = 0$ and $n_0 = 1$. Therefore, the homogeneous equations (2.1) - (2.2) have the solution p_0 with the Fourier transform given by (4.3) as

$$\begin{aligned} F_0(z) = \frac{K_1(z)}{z}\, iC &= iC \int_0^T k(t)\, \frac{1 - e^{izt}}{z}\, dt \\ &= C \int_0^T \int_t^T k(\tau)\, d\tau\ e^{izt} dt \end{aligned} \qquad (C \in \mathbb{R})$$

observing (8.2), which yields the solution

$$p_0(t) = C \int_t^T k(\tau)\, d\tau \qquad (C \in \mathbb{R}). \tag{8.4}$$

If p_0 should be a density function as k, from $\int_0^T p_0(t)\, dt = 1$ we obtain $C = \frac{1}{c}$ in view of (8.3), i.e.

$$p_0(t) = \frac{1}{c} \int_t^T k(\tau)\, d\tau. \tag{8.5}$$

This, for $T = \infty$, is Feller's solution [3: Chapter 6/Section 11] as derived by Berkovič [2]. The inhomogeneous equation (2.1) is solvable if $G(0) = 0$ by (4.5), i.e. $\int_0^T g(t)\, dt = 0$.

A particular solution p_1 is given by its Fourier transform (cp. (4.7))

$$F_1(x) = \frac{G(x)}{K_1(x)} + \frac{K_1(x)}{x}\, \frac{1}{\pi i} \int_{-\infty}^{\infty} Q(x,\xi)\, \frac{\xi G(\xi)}{|K_1(\xi)|^2}\, \frac{d\xi}{\xi - x}$$

where $G(x) = \int_0^T g(t) \cos xt\, dt$ and $Q(x,\xi) = \frac{x\xi + 1}{\xi^2 + 1}$ and for $T = \infty$ additionally $t\, g \in L^1(\mathbb{R}_+)$ is assumed. (The last assumption can be weakened to $\frac{G(x)}{x} \in L^2(\mathbb{R}_+)$ ensured if $G(x) = O(x^\gamma)$ with $\gamma > \frac{1}{2}$ as $x \to 0$.) We remark that also in the case $T = \infty$ the solution (8.5) belongs to $L^2(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$ due to the assumption $t\, k \in L^1(\mathbb{R}_+)$.

Example 2 (*Generalized Arabadzhyan's system*). In the corresponding system of equations (6.1) - (6.2) for non-negative real-valued kernels $k_\alpha \in L^1(0,T)$ $(\alpha = 1,2)$ with $\int_0^T k_\alpha(t)\, dt \le 1$ we have the three different cases

$$k_\alpha(t) \ge 0 \ \text{ on } (0,T), \qquad \int_0^T k_\alpha(t)\, dt < 1 \tag{8.6}$$

$$k_\alpha(t) \geq 0 \ \text{ on } (0,T), \quad \int_0^T k_1(t)\,dt = 1, \ \int_0^T k_2(t)\,dt < 1 \tag{8.7}$$

$$k_\alpha(t) \geq 0 \ \text{ on } (0,T), \quad \int_0^T k_\alpha(t)\,dt = 1 \tag{8.8}$$

where for $T = \infty$ we additionally assume $t\,g_\alpha, t\,k_\alpha \in L^1(\mathbb{R}_+)$ if $\int_0^\infty k_\alpha(t)\,dt = 1 \ \ (\alpha = 1,2)$.

For (8.6) we have the regular case of system (6.1) - (6.2) with $\kappa = 0$. The homogeneous system (6.1) - (6.2) has only the trivial solution $p = (p_1, p_2) = 0$, and the solution of the inhomogeneous system (6.1) - (6.2) follows by its Fourier transforms $F_\alpha \ \ (\alpha = 1,2)$ in (6.12).

The cases (8.7) - (8.8) are singular. For (8.7) we have $N_1 = N_{1,0} = 1, N_2 = N_{2,0} = 0$ and $\kappa = 0$ such that $\lambda = 1, \lambda_0 = 0$ and $\kappa_0 = 0$. Therefore, the homogeneous system (6.1) - (6.2) has only the trivial solution $p = 0$ and the solution of the inhomogeneous system (6.1) - (6.2) is given by its Fourier transforms $F_\alpha \ \ (\alpha = 1,2)$ in (7.12) where $\Pi_1(x) = \Pi_{1,0}(x) = x, \Pi_2(x) = \Pi_{2,0}(x) = 1, P_{\kappa+\lambda-1} = 0$, and $\widehat{N} = 1$.

For (8.8) it holds $N_1 = N_2 = 1, N_{1,0} = N_{2,0} = 0$ and $\kappa = 0$ such that $\lambda = \lambda_0 = 1$ and $\kappa_0 = 0$. Hence the homogeneous system (6.1) - (6.2) has the solution p^0 with the Fourier transforms $F^0_\alpha \ \ (\alpha = 1,2)$ given by (7.9) as $F^0_1(x) = \frac{K_2(x)}{x}\, iC$ and $F^0_2(x) = \frac{K_1(x)}{x}\, iC \ \ (C \in \mathbb{R})$ leading to

$$\begin{aligned} p^0_1(t) &= C\textstyle\int_t^T k_2(\tau)\,d\tau \\ p^0_2(t) &= C\textstyle\int_t^T k_1(\tau)\,d\tau \end{aligned} \tag{8.9}$$

(cp. (8.4)). In view of (7.10) the inhomogeneous system (6.1) - (6.2) is solvable if the condition $G(0) = 0$, i.e. $\int_0^T [g_1(t) + g_2(t)]\,dt = 0$ is fulfilled, and the general solution of the inhomogeneous system (6.1) - (6.2) is given by its Fourier transforms F_α $(\alpha = 1,2)$ in (7.12) again, where $\Pi_1(x) = \Pi_2(x) = x, \Pi_{1,0}(x) = \Pi_{2,0}(x) = 1, P_{\kappa+\lambda-1}(x) = iC \ \ (C \in \mathbb{R})$ and $\widehat{N} = 1$.

System (6.1) - (6.2) for $T = \infty$ under the above conditions has been studied by Arabadzhyan [1] who gives solution (8.9) for the homogeneous system and, via successive approximations, the solutions $p_\alpha \in L^1(\mathbb{R}_+)$ or $p_\alpha \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ for the inhomogeneous system with $g_\alpha \in L^1(\mathbb{R}_+) \ \ (\alpha = 1,2)$ in the cases (8.6) - (8.7) without solvability condition.

Example 3 (*Exponential kernel*). We deal with equations (2.1) - (2.2) for the kernel $k(t) = A\,e^{\rho t}$ where $A, \rho \in \mathbb{R}, A > 0$, and $\rho < 0$ in the case of (2.2). There holds

$$K_1(z) = \begin{cases} \frac{\rho + iz + A[1 - e^{(\rho+iz)T}]}{\rho + iz} & \text{if } T < \infty \\ \frac{\rho + iz + A}{\rho + iz} & \text{if } T = \infty. \end{cases}$$

For $T = \infty$ we have the simple zero $z_1 = i(\rho + A)$ of K_1 which is in $\operatorname{Im} z \geq 0$ for $-A \leq \rho < 0$. For $T < \infty$ we put $\zeta = \rho + iz$ and study the zeros of the function $\varphi(\zeta) = e^{\zeta T} - 1 - \frac{\zeta}{A}$ in $\operatorname{Re}\zeta \leq \rho$ corresponding to $\operatorname{Im} z \geq 0$. Under the assumption $ATe^{\rho T} \leq 1$ only real zeros of φ in $\operatorname{Re}\zeta \leq \rho$ can occur. But φ has the real zeros $\xi_2 = 0$ and $\xi_1 \gtreqless 0$ if $AT \lesseqgtr 1$. That means, the function K_1 has the simple zero $z_1 = i(\rho - \xi_1)$ with this real ξ_1. Therefore, we have the three cases $\rho \lesseqgtr \xi_1$ corresponding to $\int_0^T k(t)\,dt \gtreqless 1$ where $\int_0^T k(t)e^{-y_1 t}dt = 1$ with $y_1 = \rho - \xi_1 > 0$ in the third case.

The *Cases* 1 - 2: $\rho \leq \xi_1$ are particular cases of Example 1. In the *Case* 3: $\rho > \xi_1$ for equation (2.1) we have the regular case with $\kappa = 1$ and $\kappa_0 = 0$ so that the homogeneous equation (2.1) has only the trivial solution. The Fourier transform of the solution to the inhomogeneous equation (2.1) is given by formula (3.19) with $P_{2\kappa-1} = Cz$, where $C \in \mathbb{R}$ is determined by condition (3.29) as $-Cy_1 = 2G(iy_1)d_1$ with $d_1 = \frac{2iy_1}{K_2(iy_1)K_1'(iy_1)}$, i.e. $C = \frac{4G(iy_1)}{iK_1(-iy_1)K_1'(iy_1)}$ where $G(iy) = \int_0^T g(t)\cosh yt\,dt$. For equation (2.2) with $\xi_1 = -A$ and $y_1 = A + \rho > 0$ we also have the regular case with $\kappa = 1$ and by (3.31) the solution to the homogeneous equation

$$p_0(t) = \begin{cases} \frac{\rho+A}{2\rho+A}\, e^{-(\rho+A)t} + \frac{\rho}{2\rho+A}\, e^{\rho t} & \text{if } \rho \neq -\frac{A}{2} \\ (1+\rho t)e^{\rho t} & \text{if } \rho = -\frac{A}{2}. \end{cases}$$

The solution to the inhomogeneous equation (2.2) is given by (3.19) with $P_{2\kappa-1} = Cz$ and arbitrary $C \in \mathbb{R}$.

Example 4 (*Linear kernel*). Finally, we consider an example for the regular case of equation (2.1) with a non-trivial solution to the homogeneous equation (*eigenvalue case*), namely equation (2.1) with $T = 1$ and the linear function $k(t) = A + Bt$ $\left(A = \frac{2}{(3-e)(e-1)},\ B = \frac{e-1}{e-3}\right)$. The function $K_1(z) = \frac{z^2+B-Aiz-Be^{iz}+(A+B)ize^{iz}}{z^2}$ has the simple zeros $z_{1,2} = \pm i$. We have the regular case of equation (2.1) with $\kappa = \kappa_0 = 1$. The Fourier transform of the solution p_0 to the homogeneous equation given by (3.16), (3.20) is $F_0(z) = \frac{K_1(z)}{R(z)}\, iCz$ with $C \in \mathbb{R}$ and $R(z) = z^2 + 1$, i.e., choosing $C = 1$, $F_0(z) = \frac{1}{z^2+1}\big[iz + A - (A+B)e^{iz} - B\,\frac{1-e^{iz}}{iz}\big]$. From this the solution

$$p_0(t) = B + \frac{1}{2}(1 + A - B)e^{-t} - \Big(\frac{A}{2} + B\Big)e^{-(1-t)}$$

results. The corresponding solvability condition (3.36) for the inhomogeneous equation is $G(i) = 0$ or $\int_0^1 g(t)\cosh t\,dt = 0$.

9. Appendix

1. Analytic continuation of $\boldsymbol{F}_0$ and $\boldsymbol{F}_1$ in the general case. Let $\zeta_k \;\; (k = 1, ..., r)$ be the different zeros of K_1 in $\operatorname{Im} z > 0$ with multiplicities ν_k where $\kappa = \sum_{k=1}^{r} \nu_k$. Then

$$R(z) = \prod_{k=1}^{r} (z - \zeta_k)^{\nu_k} (z - \overline{\zeta}_k)^{\nu_k} \tag{A.1}$$

and by (3.20) - (3.21) the conditions for the regularity of F_0 and F in the points $\overline{\zeta}_k$ are

$$\big[K_1(z) P^0_{2\kappa-1}(z)\big]^{(l)}(\overline{\zeta}_k) = 0 \quad (l = 0, ..., \nu_k - 1) \tag{A.2}$$

$$\big[K_1(z) i\, P_{2\kappa-1}(z) + \tfrac{R(z)}{K_2(z)}\, G(z)\big]^{(l)}(\overline{\zeta}_k) = 0 \quad (l = 0, ..., \nu_k - 1), \tag{A.3}$$

respectively, for $k = 1, \ldots, r$.

Now let $\overline{\zeta}_k \;\; (k = 1, \ldots, r)$ be zeros of order μ_k of K_1, i.e.

$$K_1^{(l)}(\overline{\zeta}_k) = 0 \qquad (l = 0, ..., \mu_k - 1)$$

and $K_1(\overline{\zeta}_k)^{(\mu_k)} \neq 0$. Then for $\mu_k \geq \nu_k$ condition (A.2) is fulfilled for any $P^0_{2\kappa-1}$ whereas for $\mu_k < \nu_k$ there remain $\nu_k - \mu_k$ complex conditions (A.2) for $l = \mu_k, ..., \nu_k - 1$ which in view of $K_1(\overline{\zeta}_k)^{(\mu_k)} \neq 0$ are equivalent to the conditions

$$[P^0_{2\kappa-1}(z)]^{(l)} = 0 \qquad (l = 0, ..., \nu_k - \mu_k - 1) \tag{A.5}$$

for all k. Taking the two cases together we obtain conditions (A.5) where l runs from 0 to $\max(\nu_k - \mu_k, 0) - 1$ and there are

$$2\sum_{k=1}^{r} \max(\nu_k - \mu_k, 0) = 2(\kappa - \kappa_0), \quad \kappa_0 = \sum_{k=1}^{r} \min(\nu_k, \mu_k) \tag{A.6}$$

linearly independent real conditions. I.e., if $\kappa_0 \geq 1$, then we have $n = 2\kappa_0$ linearly independent solutions to the homogeneous problem (3.1) given by (3.31) with (3.26).

Conditions (A.3) for the inhomogeneous problem (3.1) split up into the conditions $\big[\frac{R(z)}{K_2(z)}\, G(z)\big]^{(l)}(\overline{\zeta}_k) = 0 \;\; (l = 0, ..., \min(\nu_k, \mu_k) - 1)$ which in view of $\overline{c}_k = \lim_{z \to \overline{\zeta}_k} [\frac{R(z)}{K_2(z)}] \neq 0$ are equivalent to the $\min(\nu_k, \mu_k)$ conditions

$$G^{(l)}(\zeta_k) = 0 \qquad (l = 0, ..., \min(\nu_k, \mu_k) - 1) \tag{A.7}$$

and the $\nu_k - \min(\nu_k, \mu_k) = \max(\nu_k - \mu_k, 0)$ conditions (A.3) for $l = \min(\nu_k, \mu_k)$, $..., \nu_k - 1$. Hence we have the $n = 2\kappa_0$ real solvability conditions (A.7) for the inhomogeneous problem (3.1). The remaining $2(\kappa-\kappa_0)$ real conditions (A.3) for the 2κ real coefficients of the polynomial $P_{2\kappa-1}$ leave free $n=2\kappa_0$ real parameters in $P_{2\kappa-1}$ and we again have the decomposition $P_{2\kappa-1}=P^0_{2\kappa-1}+P^1_{2\kappa-1}$ with the polynomial $P^0_{2\kappa-1}$ of form (3.26) and a polynomial $P^1_{2\kappa-1}$ with determined coefficients.

In the case of real functions k and g with real solutions p we have

$$R(z)=\textstyle\prod_{k=1}^{\rho_0}(z^2+y_k^2)^{\nu_k}\prod_{k=N_0+1}^{N_0+\rho}\big[(z^2-\zeta_k^2)(z^2-\overline{\zeta}_k^2)\big]^{\nu_k} \qquad (A.8)$$

where $\zeta_k=iy_k \quad (y_k>0;\ k=1,...,\rho_0)$ and $\zeta_k=\pm x_k+iy_k \quad (x_k,y_k>0;\ k=N_0+1,...,N_0+\rho)$ with $\kappa=N_0+2N_1$, were $N_0=\sum_{k=1}^{\rho_0}\nu_k$ and $N_1=\sum_{k=N_0+1}^{N_0+\rho}\nu_k$. Conditions (A.5) for the real polynomial $P^0_{2\kappa-1}$ with odd powers yield

$$\kappa_0=\textstyle\sum_{k=1}^{\rho_0}\min(\nu_k,\mu_k)+2\sum_{k=N_0+1}^{N_0+\rho}\min(\nu_k,\mu_k) \qquad (A.9)$$

linearly independent real solutions of the homogeneous problem (3.1). The solvability conditions (A.7) for the inhomogeneous problem (3.1) are equivalent to the κ_0 real conditions

$$\textstyle\int_0^T g(t)t^l q_{k,l}(t)\,dt=0 \qquad \begin{cases} l=0,...,\min(\nu_k,\mu_k)-1\\ k=1,...,\rho_0\end{cases} \qquad (A.10)$$

where $q_{k,l}(t)=\begin{cases}\cosh y_k t & \text{for even } l\\ \sinh y_k t & \text{for odd } l\end{cases}$ and

$$\int_0^T g(t)t^l q^{+,-}_{k,l}(t)\,dt=0 \qquad \begin{cases} l=0,...,\min(\nu_k,\mu_k)-1\\ k=N_0+1,...,N_0+\rho\end{cases} \qquad (A.11)$$

where (+ refers to the two upper lines and - to the lower ones)

$$q^{+,-}_{k,l}(t)=\begin{cases}\cos x_k t\cdot\cosh y_k t & \text{for even } l\\ \sin x_k t\cdot\cosh y_k t & \text{for odd } l\\ \sin x_k t\cdot\sinh y_k t & \text{for even } l\\ \cos x_k t\cdot\sinh y_k t & \text{for odd } l.\end{cases}$$

2. Resolvent form of the solution. The solution of equation (2.1) can be represented (formally) by means of a resolvent. We confine ourselves to the regular case with $\kappa=0$. From (3.12) and (3.30) the solution p is given by

$$\begin{aligned}p(t)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{G(x)}{K_2(x)}e^{-ixt}dx+\frac{1}{2\pi^2 i}\int_{-\infty}^{\infty}e^{-ixt}K_1(x)\int_{-\infty}^{\infty}\frac{G(\xi)}{B(\xi)}\frac{d\xi}{\xi-x}\,dx\\ &=\frac{1}{2\pi}\int_{-\infty}^{\infty}G(x)e^{-ixt}dx+\frac{1}{2\pi}\int_{-\infty}^{\infty}G(x)\frac{\overline{K(x)}}{K_2(x)}e^{-ixt}dx\\ &\quad+\frac{1}{2\pi^2 i}\int_{-\infty}^{\infty}\frac{G(\xi)}{B(\xi)}\int_{-\infty}^{\infty}\frac{e^{-ixt}K_1(x)}{\xi-x}dxd\xi\end{aligned}$$

where we have used the relation $\frac{1}{K_2(x)} = 1 + \frac{\overline{K(x)}}{K_2(x)}$ and changed the order of integration. Using the decompositions $K_1(x) = B(x) + K_1(x)\overline{K(x)}$ and $B(x) = B(\xi) + [B(x) - B(\xi)]$ and the relations $\frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{e^{-ixt}}{\xi - x}\,dx = e^{-i\xi t}$ and $g(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} G(x)e^{-ixt}dx$ – the first following from [12: Chapter 5.1] and the second one from (2.6) and (3.2) – we further get

$$\begin{aligned}
p(t) &= g(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} G(x)\frac{\overline{K(x)}}{K_2(x)}e^{-ixt}dx \\
&\quad + \frac{1}{2\pi^2 i}\int_{-\infty}^{\infty}\frac{G(\xi)}{B(\xi)}\int_{-\infty}^{\infty}\frac{e^{-ixt}K_1(x)\overline{K(x)}}{\xi - x}dxd\xi \\
&\quad + \frac{1}{2\pi^2 i}\int_{-\infty}^{\infty}\frac{G(\xi)}{B(\xi)}\int_{-\infty}^{\infty}\frac{e^{-ixt}[B(x) - B(\xi)]}{\xi - x}dxd\xi \\
&= g(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{G(x)}{B(x)}\bigg[K_1(x)\overline{K(x)}e^{-ixt} \\
&\quad - \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{e^{-i\xi t}\{K_1(\xi)\overline{K(\xi)} + B(\xi) - B(x)\}d\xi}{\xi - x}\bigg]dx.
\end{aligned}$$

Observing $G(x) = \frac{1}{2}\int_0^T \left[e^{ixs}g(s) + e^{-ixs}\overline{g(s)}\right]ds$ this yields the formal representation

$$p(t) = g(t) + \int_0^T \Gamma_1(t,s)g(s)\,ds + \int_0^T \Gamma_2(t,s)\overline{g(s)}ds \tag{A.12}$$

on $(0, T)$ with the resolvents

$$\begin{aligned}
\Gamma_1(t,s) &= \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{ixs}\gamma(t,x)\,dx \\
\Gamma_2(t,s) &= \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{-ixs}\gamma(t,x)\,dx
\end{aligned} \tag{A.13}$$

where

$$\begin{aligned}
\gamma(t,x) = \tfrac{1}{B(x)}\big[&K_1(x)\overline{K(x)}e^{-ixt} \\
&- \tfrac{1}{\pi i}\textstyle\int_{-\infty}^{\infty}\tfrac{e^{-i\xi t}\{K_1(\xi)\overline{K(\xi)} + B(\xi) - B(x)\}}{\xi - x}\,d\xi\big].
\end{aligned}$$

From representation (A.12) it follows that in the regular case with $\kappa = 0$ the solution q of the adjoint equation

$$q(t) - \textstyle\int_0^t k(t-s)q(s)\,ds - \int_0^{T-t} k(s+t)\overline{q(s)}\,ds = f(t) \tag{A.14}$$

on $(0, T)$ is given by

$$q(t) = f(t) + \textstyle\int_0^T \overline{\Gamma_1(s,t)}f(s)\,ds + \int_0^T \Gamma_2(s,t)\overline{f(s)}\,ds \tag{A.15}$$

with Γ_1 and Γ_2 from (A.13) (cf. [8: Section 111]). In the case $n = 0$ for $\kappa > 0$ and also in the general case $n \geq 0$ an additional term in expressions (A.12) and (A.15) occurs according to formula (3.19) with (3.29). Further, in the case $n > 0$, from the solvability conditions (3.28) for equation (2.1) by comparison with (2.4) the solutions q_j of the homogeneous equation (A.14), i.e. (2.3), follow and the solvability conditions $\mathrm{Re} \int_0^T f(t)\overline{p_j(t)}\, dt = 0 \quad (j = 1, ..., n)$ for equation (A.14) are obtained by solutions (3.16), (3.20) of the homogeneous problem (3.1). In analogous way the solutions of equations (2.1) and (A.14) in the singular case can be represented. We omit the details.

3. Solutions in L^1-space. If $g \in L^1(0,T)$ in (2.1) - (2.2) or $g_1, g_2 \in L^1(0,T)$ in (6.1), then the derived solutions p or p_1, p_2 for equations (2.1) - (2.2) and system (6.1), respectively, hold now with $p \in L^1(0,T)$ or $p_1, p_2 \in L^1(0,T)$. To prove this statement one has to work in the Wiener algebra $\mathcal{A}$ built up by the transforms of L^1-functions and use the known facts that the Cauchy integral operator (Hilbert transformation) works in $\mathcal{A}$ (cf. [15]) and by the Wiener-Levi theorem [9, 15] corresponding products in formulae (3.12), (3.19) and so on are lying in $\mathcal{A}$, too. The needed Paley-Wiener theorem for functions from $\mathcal{A}$ immediately follows by the theorem of Phragmen-Lindelöf (cf. [6: Chapter 6] as in the proof for the L^2-case there) and the corresponding Paley-Wiener lemma (cf. [17: Chapter 2/Section 5.1]).

For solutions from the L^1-space with weight we refer to [16].

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