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On a Quaternionic Reformulation of Maxwell's Equations for Chiral Media and its Applications

V.V. Kravchenko and H. Oviedo

Abstract. A quaternionic reformulation of the Maxwell equations for chiral media is proposed. Integral representations for solutions are constructed. A complete solution of the extendability problem for the electromagnetic fields in chiral media is obtained. Maxwell's equations for inhomogeneous chiral media are studied also and some classes of solutions for slowly changing media are obtained.

Keywords: *Maxwell's equations, chiral media, quaternionic analysis*

AMS subject classification: 30G35, 78A25

1. Introduction

Chiral media represent a class of media responding with both magnetic and electric polarization to electric or magnetic excitation and are frequently encountered in nature. A multitude of organic molecules such as the DNA molecule exhibit chiral properties at different frequencies, the pupil of the eye is a chiral medium, and for many other examples we refer the reader to [5, 13 - 16]. At present chiral materials are manufactured in several laboratories (see, e.g., [3, 15]) and are used in different branches of engineering.

In this work we apply the technique of quaternionic analysis for the study of Maxwell's equations for chiral media. Considering homogeneous chiral media we obtain the complete solution of the extendability problem for the electromagnetic field, we show how from the quaternionic Stokes formula the electromagnetic energy-balance equation for chiral media is obtained, and using the quaternionic Cauchy integral theorem we arrive at a new integral form of Maxwell's equations for chiral media.

Both authors: Escuela Sup. de Ing. Mecánica y Eléctrica del Inst. Polit. Nac., C.P. 07738, D.F. México; vkravche@maya.esimez.ipn.mx and hoviedo@ipn.mx

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For slowly changing inhomogeneous chiral media we show that the corresponding Maxwell equations can be diagonalized precisely as in the homogeneous case, and this diagonalization is used for obtaining solutions for stratified chiral media.

2. Preliminaries

We denote the algebra of complex quaternions by $\mathbb{H}(\mathbb{C})$. The elements of $\mathbb{H}(\mathbb{C})$ are represented in the form $q = \sum_{k=0}^{3} q_k i_k$ where $q_k \in \mathbb{C}$, $i_0 = 1$ and $i_k \ (k = 1, 2, 3)$ are standard quaternionic imaginary units: $i_k^2 = -1, i_1 i_2 = -i_2 i_1 = i_3, i_2 i_3 = -i_3 i_2 = i_1, i_3 i_1 = -i_1 i_3 = i_2$. The complex imaginary unit i commutes with $i_k \ (k = 0, 1, 2, 3)$. We will use also the vector representation of complex quaternions. Namely, any $q \in \mathbb{H}(\mathbb{C})$ can be represented in the form $q = \mathrm{Sc}(q) + \mathrm{Vec}(q)$, where $\mathrm{Sc}(q) = q_0$ and $\mathrm{Vec}(q) = \sum_{k=1}^{3} q_k i_k$. The complex quaternions of the form $q = \mathrm{Vec}(q)$ are called purely vectorial, and we identify them with the vectors from $\mathbb{C}^3 : \vec{q} = \mathrm{Vec}(q)$. The complex quaternion $\overline{q} = \mathrm{Sc}(q) - \mathrm{Vec}(q) = q_0 - \vec{q}$ is called conjugate to q. Note that $\overline{p \cdot q} = \overline{q} \cdot \overline{p}$. We will need also the usual complex conjugation $q^* = \sum_{k=0}^{3} q_k^* i_k = \sum_{k=0}^{3} (\mathrm{Re}\, q_k - i \mathrm{Im}\, q_k) i_k$.

It is convenient to use the notations ${}^pMq = p \cdot q$ and $M^p q = q \cdot p$ for the operators of multiplication from the left- and right-hand sides, respectively. Note that the scalar product $\langle \vec{p}, \vec{q} \rangle$ of $\vec{p}$ and $\vec{q}$ can be represented as $\langle \vec{p}, \vec{q} \rangle = -\frac{1}{2}({}^{\vec{p}}M + M^{\vec{p}})\vec{q}$.

We will consider $\mathbb{H}(\mathbb{C})$-valued functions defined in some domain $\Omega \subset \mathbb{R}^3$. On the space $C^1(\Omega; \mathbb{H}(\mathbb{C}))$ the well known Moisil-Theodoresco operator D [12] is defined by the expression

$$D = i_1 \frac{\partial}{\partial x_1} + i_2 \frac{\partial}{\partial x_2} + i_3 \frac{\partial}{\partial x_3}.$$

For $f \in C^1(\Omega; \mathbb{H}(\mathbb{C}))$ the expression Df can be rewritten in the form

$$Df = -\mathrm{div}\vec{f} + \mathrm{grad} f_0 + \mathrm{rot}\vec{f} \tag{1}$$

where the differential operators are defined in the usual way (for instance, $\mathrm{grad} f_0 = \left(i_1 \frac{\partial}{\partial x_1} + i_2 \frac{\partial}{\partial x_2} + i_3 \frac{\partial}{\partial x_3}\right) f_0)$. Equality (1) means that $\mathrm{Sc}(Df) = -\mathrm{div}\vec{f}$ and $\mathrm{Vec}(Df) = \mathrm{grad} f_0 + \mathrm{rot}\vec{f}$. An important property of D is the factorization $D^2 = -\Delta$, where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$.

Let us consider the operators

$$D_{\pm\alpha} = D \pm \alpha$$

where $\alpha \in \mathbb{C}$. Purely vectorial solutions of the equation

$$(D+\alpha)f = 0 \tag{2}$$

are known as Beltrami fields (see, e.g., [13]) as well as force-free fields (see, e.g., [4, 18]). In general, we consider quaternionic solutions of (2), and the reduction of our results to the case of purely vectorial solutions allows us to analyze these important classes of physical fields by methods of quaternionic analysis.

Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain with closed piecewise smooth boundary $\Gamma = \partial\Omega$. We will need the following well known facts (see, e.g., [12: pp. 66 - 67]).

Theorem 1 (Quaternionic Stokes formula). *If* $f, g \in C^1(\Omega; \mathbb{H}(C)) \cap C(\overline{\Omega}; \mathbb{H}(C))$, *then* $\int_\Gamma g \cdot \vec{n} \cdot f \, d\Gamma = \int_\Omega (D_r[g] \cdot f + g \cdot D[f]) \, dx$ *where* D_r *is defined as* $D_r g = \sum_{k=1}^3 \partial_k g i_k$.

Theorem 2 (Quaternionic Cauchy's integral theorem). *If the function* $f \in C^1(\Omega; \mathbb{H}(C)) \cap C(\overline{\Omega}; \mathbb{H}(C))$ *satisfies* (2) *in* Ω, *then* $\int_\Gamma \vec{n} f \, d\Gamma = -\alpha \int_\Omega f \, dx$.

The equality

$$\Delta + \alpha^2 = -(D+\alpha)(D-\alpha) = -D_\alpha D_{-\alpha} \tag{3}$$

holds. Suppose that Θ_α is a fundamental solution of the Helmholtz operator, i.e. $(\Delta + \alpha^2)\Theta_\alpha = \delta$. Using (3) we construct the distributions

$$\mathcal{K}_{\pm\alpha} = -(D \mp \alpha)\Theta_\alpha \tag{4}$$

which are fundamental solutions of $D_{\pm\alpha}$, i.e. $D_{\pm\alpha}\mathcal{K}_{\pm\alpha} = \delta$.

Let us assume that $\operatorname{Im}\alpha \geq 0$. Then the fundamental solution of the Helmholtz operator is chosen as $\Theta_\alpha(x) = -\frac{e^{i\alpha|x|}}{4\pi|x|}$. From (4) we obtain

$$\mathcal{K}_{\pm\alpha}(x) = -\mathrm{grad}\Theta_\alpha(x) \pm \alpha\Theta_\alpha(x) = \Big(\pm \alpha + \frac{x}{|x|^2} - i\alpha\frac{x}{|x|}\Big)\Theta_\alpha(x)$$

where $x = \sum_{k=1}^3 x_k i_k$. Let us introduce the operators

$$\begin{aligned} T_\alpha f(x) &= \int_\Omega \mathcal{K}_\alpha(x-y) f(y)\, d\Omega_y && (x \in \mathbb{R}^3) \\ K_\alpha f(x) &= -\int_\Gamma \mathcal{K}_\alpha(x-y)\vec{n}(y) f(y)\, d\Gamma_y && (x \in \mathbb{R}^3 \backslash \Gamma) \\ S_\alpha f(x) &= -2\int_\Gamma \mathcal{K}_\alpha(x-y)\vec{n}(y) f(y)\, d\Gamma_y && (x \in \Gamma), \end{aligned}$$

but also

$$P_\alpha = \tfrac{1}{2}(I + S_\alpha)$$
$$Q_\alpha = I - P_\alpha$$

where $\Gamma = \partial\Omega$ is a closed Liapunov surface in $\mathbb{R}^3$, the boundary of a bounded domain Ω, $\vec{n} = \sum_{k=1}^{3} n_k i_k$ is the outward unit normal on Γ and I is the identity operator. In the following theorem we summarize important properties of the above integral operators which show that T_α, K_α and S_α can be considered as natural spatial generalizations of the complex T-operator, the Cauchy-type operator and the operator of singular integration, respectively.

Theorem 3 (see [12]).

1) *Borel-Pompeiu formula: If* $f \in C^1(\Omega; \mathbb{H}(\mathbb{C})) \cap C(\overline{\Omega}; \mathbb{H}(\mathbb{C}))$, *then* $K_{\pm\alpha} f + T_{\pm\alpha} D_{\pm\alpha} f = f$ *in* Ω.

2) *Cauchy integral formula: If* $f \in C^1(\Omega; \mathbb{H}(\mathbb{C})) \cap C(\overline{\Omega}; \mathbb{H}(\mathbb{C})) \cap \ker D_{\pm\alpha}(\Omega)$, *then* $f = K_{\pm\alpha} f$ *in* Ω.

3) *Plemelj-Sokhotski formulas: If* $f \in C^{0,\varepsilon}(\Gamma)$ *for* $0 < \varepsilon \le 1$, *then* $\lim_{\Omega \ni x \to y \in \Gamma} K_{\pm\alpha} f(x) = P_{\pm\alpha} f(y)$ *and* $\lim_{\overline{\Omega} \not\ni x \to y \in \Gamma} K_{\pm\alpha} f(x) = -Q_{\pm\alpha} f(y)$ *for any* $y \in \Gamma$.

4) *Involutiveness of the operator of singular integration: If* $f \in C^{0,\varepsilon}(\Gamma)$ *for* $0 < \varepsilon \le 1$, *then* $S^2_{\pm\alpha} f = f$.

Remark 4. The Cauchy integral formula is also valid for unbounded domains. In this case f must also fulfill the radiation condition $\big(1 \pm i\frac{x}{|x|}\big) \cdot f(x) = o\big(\frac{1}{|x|}\big)$ $(|x| \to \infty)$ uniformly for all directions (see [11, 17]). Then $f(x) = -K_{\pm\alpha} f(x)$ for all $x \in \mathbb{R}^3 \backslash \overline{\Omega}$.

Now let us consider the boundary value problem discussed in [10].

Problem 5. *Given two complex quaternionic functions* $v \in C^{0,\varepsilon}(\Gamma; \mathbb{H}(\mathbb{C}))$ *and* $g \in C(\overline{\Omega}; \mathbb{H}(\mathbb{C}))$, *find a function* $f \in C^1(\Omega; \mathbb{H}(\mathbb{C})) \cap C^{0,\varepsilon}(\overline{\Omega}; \mathbb{H}(\mathbb{C}))$ $(0 < \varepsilon < 1)$ *such that*

$$\left.\begin{aligned} D_\alpha f &= g \quad \text{in } \Omega \\ f|_\Gamma &= v \end{aligned}\right\}. \tag{5}$$

We introduce a new function $u = f - T_\alpha g$. Note that $T_\alpha g \in W^1_p(\Omega)$ for any $p > 1$ (see [20]) and that, due to Sobolev's embedding theorem (see, e.g., [21: p. 287]), $T_\alpha g \in C^{0,\varepsilon}(\overline{\Omega})$. If f solves problem (5), then u is a solution of the boundary value problem

$$\left.\begin{aligned} D_\alpha u &= 0 \quad \text{in } \Omega \\ u|_\Gamma &= w \end{aligned}\right\}$$

where $w(x) = v(x) - T_\alpha g(x)$ for $x \in \Gamma$. The solution of this problem exists if and only if (see [12: p. 113]) the function w fulfills the condition $w = S_\alpha w$ on Γ. In other words,

$$v - T_\alpha g = S_\alpha v - S_\alpha T_\alpha g \quad \text{on } \Gamma. \tag{6}$$

If this condition is fulfilled, then the solution is $u = K_\alpha w = K_\alpha(v - T_\alpha g)$.

Consider the expression $K_\alpha T_\alpha g$. From Borel-Pompeiu's formula,

$$K_\alpha T_\alpha g = (I - T_\alpha D_\alpha)T_\alpha g = 0. \tag{7}$$

Thus $u = K_\alpha v$. Moreover, (7) gives us that $P_\alpha T_\alpha g = 0$ on Γ, that is $T_\alpha g = -S_\alpha T_\alpha g$, and condition (6) can be rewritten as $v - S_\alpha v = 2T_\alpha g$ on Γ or

$$Q_\alpha v = T_\alpha g \quad \text{on } \Gamma. \tag{8}$$

Now, returning to boundary value Problem 5, we obtain its solution in the form

$$f = K_\alpha v + T_\alpha g \tag{9}$$

under the necessary and sufficient condition (8). In fact, such a function obviously satisfies $(5)_1$, and due to the Plemelj-Sokhotski formulas and (8), we obtain on the boundary the required equality $(5)_2$: $f|_\Gamma = P_\alpha v + (T_\alpha g)|_\Gamma = P_\alpha v + Q_\alpha v = v$. Thus the following fact is true.

Theorem 6. *The solution of Problem* 5 *exists and has form* (9) *if and only if condition* (8) *is fulfilled.*

3. Electromagnetic fields in chiral media

The complex amplitudes $\widetilde{E}$ and $\widetilde{H}$ of a time-harmonic electromagnetic field in a homogeneous isotropic chiral medium satisfy the usual system of Maxwell equations

$$\left.\begin{aligned} \operatorname{div}\widetilde{E}(x) &= \tfrac{\rho(x)}{\varepsilon} \\ \operatorname{rot}\widetilde{E}(x) &= i\omega\widetilde{B}(x) \\ \operatorname{rot}\widetilde{H}(x) &= -i\omega\widetilde{D}(x) + \widetilde{j}(x) \\ \operatorname{div}\widetilde{H}(x) &= 0 \end{aligned}\right\} \tag{10}$$

where ω is the frequency, ε is the permittivity of the medium, ρ and $\widetilde{j}$ are the charge density and the current density, respectively. The particularity of chiral media is reflected in the corresponding constitutive relations. The vectors $\widetilde{D}$ and $\widetilde{B}$ are related to $\widetilde{E}$ and $\widetilde{H}$ via the Drude-Born-Fedorov relations (see, e.g., [13, 15, 16])

$$\left.\begin{aligned} \widetilde{D}(x) &= \varepsilon\big[\widetilde{E}(x) + \beta \mathrm{rot} \widetilde{E}(x)\big] \\ \widetilde{B}(x) &= \mu\big[\widetilde{H}(x) + \beta \mathrm{rot} \widetilde{H}(x)\big] \end{aligned}\right\} \tag{11}$$

where μ is the permeability and β is the chirality measure of the medium. Using these constitutive relations, the Maxwell equations (10) can be written as

$$\left.\begin{aligned} \mathrm{rot}\widetilde{E}(x) &= i\omega\mu\big[\widetilde{H}(x) + \beta \mathrm{rot} \widetilde{H}(x)\big] \\ \mathrm{rot}\widetilde{H}(x) &= -i\omega\varepsilon\big[\widetilde{E}(x) + \beta \mathrm{rot} \widetilde{E}(x)\big] + \widetilde{j}(x) \end{aligned}\right\}.$$

Introducing the notations

$$\vec{E} = \tfrac{1}{\sqrt{\mu}} \cdot \widetilde{E}, \quad \vec{H} = \tfrac{1}{\sqrt{\varepsilon}} \cdot \widetilde{H}, \quad \vec{j} = \tfrac{1}{\sqrt{\varepsilon}} \cdot \widetilde{j}$$

we obtain the equations

$$\left.\begin{aligned} \mathrm{rot}\vec{E}(x) &= ik\big[\vec{H}(x) + \beta \mathrm{rot} \vec{H}(x)\big] \\ \mathrm{rot}\vec{H}(x) &= -ik\big[\vec{E}(x) + \beta \mathrm{rot} \vec{E}(x)\big] + \vec{j}(x) \end{aligned}\right\} \tag{12}$$

where $k = \omega\sqrt{\varepsilon\mu}$ in electromagnetic theory is known as the wave number. As the chiral medium distinguishes waves of opposing circular polarization and even their respective propagation speeds in general are different, it is natural to consider two wave numbers corresponding to propagation of left-handed and right-handed waves. This pair of wave numbers is introduced in Section 4 (see Remark 8). Note that from $(10)_1$ and $(12)_2$ the continuity equation

$$\tfrac{\rho}{\sqrt{\mu\varepsilon}} = -\tfrac{i}{k}\mathrm{div}\vec{j} \tag{13}$$

follows.

4. Maxwell's equations for chiral media in quaternionic form

Following [6], let us consider the purely vectorial biquaternionic functions

$$\begin{aligned} \Phi &= \vec{E} + i\vec{H} \\ \Psi &= \vec{E} - i\vec{H}. \end{aligned} \tag{14}$$

We have

$$D\Phi = -\tfrac{\rho}{\sqrt{\mu\varepsilon}} + \mathrm{rot}\vec{E} + i\mathrm{rot}\vec{H}.$$

Using (12) we obtain

$$
\begin{aligned}
D\Phi &= -\tfrac{\rho}{\sqrt{\mu\varepsilon}} + ik(\vec{H} + \beta \mathrm{rot}\vec{H}) + i\big[-ik(\vec{E} + \beta \mathrm{rot}\vec{E}) + \vec{j}\big] \\
&= -\tfrac{\rho}{\sqrt{\mu\varepsilon}}(1 - k\beta) + k(\vec{E} + i\vec{H}) + k\beta(D\vec{E} + iD\vec{H}) + i\vec{j}.
\end{aligned}
$$

That is,

$$
D\Phi = -\tfrac{\rho}{\sqrt{\mu\varepsilon}}(1 - k\beta) + k\Phi + k\beta D\Phi + i\vec{j}.
$$

Thus the complex quaternionic function Φ satisfies the equation

$$
\Big(D - \frac{k}{1 - k\beta}\Big)\Phi = -\frac{\rho}{\sqrt{\mu}\varepsilon} + i\frac{\vec{j}}{1 - k\beta}. \tag{15}
$$

By analogy we obtain for Ψ the equation

$$
\Big(D + \frac{k}{1 + k\beta}\Big)\Psi = -\frac{\rho}{\sqrt{\mu}\varepsilon} - i\frac{\vec{j}}{1 + k\beta}. \tag{16}
$$

Introducing the notations

$$
\begin{aligned}
\alpha_1 &= \tfrac{k}{1+k\beta} \\
\alpha_2 &= \tfrac{k}{1-k\beta}
\end{aligned} \tag{17}
$$

and using (13) we rewrite equations (15) - (16) as

$$
\begin{aligned}
(D - \alpha_2)\Phi &= \frac{i}{k}(\alpha_2\vec{j} + \mathrm{div}\vec{j}) \\
(D + \alpha_1)\Psi &= -\frac{i}{k}(\alpha_1\vec{j} - \mathrm{div}\vec{j}).
\end{aligned} \tag{18}
$$

In this way we proved the following

Proposition 7. *Let $\vec{E}$ and $\vec{H}$ be solutions of the Maxwell equations* (12)*. Then the pair of purely vectorial biquaternionic functions Φ and Ψ defined by* (14) *are solutions of equations* (18)*, and vice versa, if two purely vectorial biquaternionic functions Φ and Ψ are solutions of equations* (18)*, then the vectors*

$$
\begin{aligned}
\vec{E} &= \tfrac{1}{2}(\Phi + \Psi) \\
\vec{H} &= \tfrac{1}{2i}(\Phi - \Psi)
\end{aligned} \tag{19}
$$

are solutions of equations (12)*.*

Note that in a sourceless situation equations (18) take the form

$$
\begin{aligned}
(D + \alpha_1)\Psi &= 0 \\
(D - \alpha_2)\Phi &= 0.
\end{aligned} \tag{20}
$$

Remark 8. Introducing functions (14) we diagonalize the Maxwell equations (12) and in this way there follows the Bohren transformation procedure (see, e.g., [16]). Nevertheless, the interpretation of Φ and Ψ not as solutions of the equations

$$\operatorname{rot}\Psi + \alpha_1\Psi = 0$$
$$\operatorname{rot}\Phi - \alpha_2\Phi = 0$$

but of quaternionic equations (20) which possess much better properties allows us to obtain some results given below which were not obtained with traditional methods of vector analysis. The numbers α_1 and α_2 play the role of two different wave numbers corresponding to the propagation of electromagnetic waves of opposing circular polarizations. From (17) it can be observed that when the chirality measure of a medium β is equal to zero, then the numbers α_1 and α_2 coincide with the usual wave number k.

5. Integral representations for solutions of Maxwell's equations in chiral media

After having established in the preceding section a simple relation between solutions of Maxwell's equations for chiral media and solutions of quaternionic equations (18) we can use the statements of Section 2 in order to obtain corresponding results for the vectors of the electromagnetic field.

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a Liapunov boundary Γ. Then using the Borel-Pompeiu formula in Theorem 3/item 1) we obtain for Φ and Ψ the equations

$$\begin{aligned}\Psi(x) &= -\frac{i}{k}T_{\alpha_1}\left[\alpha_1\vec{j}(x) - \operatorname{div}\vec{j}(x)\right] + K_{\alpha_1}\Psi(x)\\ \Phi(x) &= \frac{i}{k}T_{-\alpha_2}\left[\alpha_2\vec{j}(x) + \operatorname{div}\vec{j}(x)\right] + K_{-\alpha_2}\Phi(x)\end{aligned} \qquad (x \in \Omega).$$

Taking into account (19) and denoting $T_1 = T_{\alpha_1}, T_2 = T_{-\alpha_2}, K_1 = K_{\alpha_1}, K_2 = K_{-\alpha_2}, \Theta_1 = \Theta_{\alpha_1}$ and $\Theta_2 = \Theta_{-\alpha_2}$ we obtain for the vectors of the electromagnetic field in Ω the integral representations

$$\left.\begin{aligned}\vec{E} &= -\frac{i}{2k}T_1(\alpha_1\vec{j} - \operatorname{div}\vec{j}) + \frac{1}{2}K_1(\vec{E} - i\vec{H})\\ &\quad + \frac{i}{2k}T_2(\alpha_2\vec{j} + \operatorname{div}\vec{j}) + \frac{1}{2}K_2(\vec{E} + i\vec{H})\\ \vec{H} &= \frac{1}{2k}T_1(\alpha_1\vec{j} - \operatorname{div}\vec{j}) - \frac{1}{2i}K_1(\vec{E} - i\vec{H})\\ &\quad + \frac{1}{2k}T_2(\alpha_2\vec{j} + \operatorname{div}\vec{j}) + \frac{1}{2i}K_2(\vec{E} + i\vec{H})\end{aligned}\right\} \tag{21}$$

or, in a more explicit form,

$$\left.\begin{aligned}\vec{E}(x) = &-\frac{i}{2k}\int_\Omega \Big[\mathcal{K}_{\alpha_1}(x-y)\big(\alpha_1\vec{j}(y) - \mathrm{div}\vec{j}(y)\big)\\ &- \mathcal{K}_{-\alpha_2}(x-y)\big(\alpha_2\vec{j}(y) + \mathrm{div}\vec{j}(y)\big)\Big]dy\\ &- \frac{1}{2}\int_\Gamma \Big(\mathcal{K}_{\alpha_1}(x-y)\vec{n}(y)\big[\vec{E}(y) - i\vec{H}(y)\big]\\ &+ \mathcal{K}_{-\alpha_2}(x-y)\vec{n}(y)\big[\vec{E}(y) + i\vec{H}(y)\big]\Big)d\Gamma_y\\ \vec{H}(x) = &\frac{1}{2k}\int_\Omega \Big(\mathcal{K}_{\alpha_1}(x-y)\big[\alpha_1\vec{j}(y) - \mathrm{div}\vec{j}(y)\big]\\ &+ \mathcal{K}_{-\alpha_2}(x-y)\big[\alpha_2\vec{j}(y) + \mathrm{div}\vec{j}(y)\big]\Big)dy\\ &+ \frac{1}{2i}\int_\Gamma \Big(\mathcal{K}_{\alpha_1}(x-y)\vec{n}(y)\big[\vec{E}(y) - i\vec{H}(y)\big]\\ &- \mathcal{K}_{-\alpha_2}(x-y)\vec{n}(y)\big[\vec{E}(y) + i\vec{H}(y)\big]\Big)d\Gamma_y\end{aligned}\right\}\quad (x\in\Omega) \tag{22}$$

The vector parts herein are the Stratton-Chu formulas for chiral media which for a sourceless situation can be found, for example, in [1]; the scalar parts represent a certain kind of the Gauss formula (see the explanation in [12: p. 120]), and are identities if the vectors $\vec{E}$ and $\vec{H}$ satisfy the Maxwell equations (12). Thus (21) represent a quaternionic form of the Stratton-Chu formulas and give us the possibility to reconstruct the solutions $\vec{E}$ and $\vec{H}$ of equations (12) inside the domain by their values on the boundary.

Now let us consider the following question. When do we know that a pair of vectors $\vec{e}$ and $\vec{h}$ defined on the boundary are boundary values of a pair of vectors $\vec{E}$ and $\vec{H}$, solutions of equations (12)? In other words, how can we guarantee that a pair of vectors $\vec{e}$ and $\vec{h}$ are extendable from Γ into Ω in such a way that their extensions $\vec{E}$ and $\vec{H}$ satisfy equations (12) and coincide with $\vec{e}$ and $\vec{h}$ on Γ? We formulate this question as the following

Problem 9. *Let $\Gamma = \partial\Omega$ be a closed Liapunov surface on which the vectors $\vec{e}$ and $\vec{h}$ satisfying the Hölder condition are given. Find two vectors $\vec{E}$ and $\vec{H}$ satisfying the Maxwell equations* (12) *in Ω and on the boundary Γ coinciding with $\vec{e}$ and $\vec{h}$: $\vec{E}|_\Gamma = \vec{e}$ and $\vec{H}|_\Gamma = \vec{h}$.*

Taking into account (14), we obtain that Problem 9 is equivalent to the following two problems:

$(18)_1$ in Ω with boundary condition $\vec{\Phi}|_\Gamma = \vec{e} + i\vec{h}$

and

(18)$_2$ in Ω with boundary condition $\vec{\Psi}|_\Gamma = \vec{e} - i\vec{h}$.

Applying Theorem 6 to each of these two problems and using (19) we arrive at the following result.

Theorem 10. *The solution of Problem* 9 *exists and has the form*

$$\left.\begin{aligned} \vec{E} &= -\frac{i}{2k} T_1(\alpha_1 \vec{j} - \operatorname{div}\vec{j}) + \frac{1}{2} K_1(\vec{e} - i\vec{h}) \\ &\quad + \frac{i}{2k} T_2(\alpha_2 \vec{j} + \operatorname{div}\vec{j}) + \frac{1}{2} K_2(\vec{e} + i\vec{h}) \\ \vec{H} &= \frac{1}{2k} T_1(\alpha_1 \vec{j} - \operatorname{div}\vec{j}) - \frac{1}{2i} K_1(\vec{e} - i\vec{h}) \\ &\quad + \frac{1}{2k} T_2(\alpha_2 \vec{j} + \operatorname{div}\vec{j}) + \frac{1}{2i} K_2(\vec{e} + i\vec{h}) \end{aligned}\right\}$$

if and only if the two equalities

$$\left.\begin{aligned} \vec{e}(x) &= -\frac{i}{2k} \int_\Omega \Big\{ \mathcal{K}_{\alpha_1}(x-y)\big[\alpha_1 \vec{j}(y) - \operatorname{div}\vec{j}(y)\big] \\ &\quad - \mathcal{K}_{-\alpha_2}(x-y)\big[\alpha_2 \vec{j}(y) + \operatorname{div}\vec{j}(y)\big] \Big\} dy \\ &\quad - \frac{1}{2} \int_\Gamma \Big(\mathcal{K}_{\alpha_1}(x-y)\vec{n}(y)\big[\vec{e}(y) - i\vec{h}(y)\big] \\ &\quad + \mathcal{K}_{-\alpha_2}(x-y)\vec{n}(y)\big[\vec{e}(y) + i\vec{h}(y)\big] \Big) d\Gamma_y \\ \vec{h}(x) &= \frac{1}{2k} \int_\Omega \Big(\mathcal{K}_{\alpha_1}(x-y)\big[\alpha_1 \vec{j}(y) - \operatorname{div}\vec{j}(y)\big] \\ &\quad + \mathcal{K}_{-\alpha_2}(x-y)\big[\alpha_2 \vec{j}(y) + \operatorname{div}\vec{j}(y)\big] \Big) dy \\ &\quad + \frac{1}{2i} \int_\Gamma (\mathcal{K}_{\alpha_1}(x-y)\vec{n}(y)\big[\vec{e}(y) - i\vec{h}(y)\big] \\ &\quad - \mathcal{K}_{-\alpha_2}(x-y)\vec{n}(y)\big[\vec{e}(y) + i\vec{h}(y)\big] \Big) d\Gamma_y \end{aligned}\right\} \qquad (23)$$

for any $x \in \Gamma$ *are fulfilled.*

Remark 11. The vector parts of equalities (23) reflect the quite well known (for the non-chiral media see, e.g., [19]) necessary condition that $\vec{e}$ and $\vec{h}$ must be boundary values of the corresponding Stratton-Chu integrals, but this is not a sufficient condition. The scalar parts give us the lacking information. Remaining in the framework of three-dimensional vector calculus it is not easy to find the reason for the necessity of the scalar parts from (23), but it becomes obvious in quaternionic terms.

Note that a similar result can be obtained for unbounded domains also. In this case the radiation conditions $\left(1 \pm i\frac{x}{|x|}\right) \cdot f(x) = o\left(\frac{1}{|x|}\right)$ $(|x| \to \infty)$ in Remark 4 turn to be the Silver-Müller conditions for the electromagnetic field (see [7, 10]).

6. Electromagnetic energy balance in chiral media

We show that the electromagnetic energy balance equation is a simple corollary of the quaternionic Stokes theorem. In addition, a useful vector identity for the electromagnetic field in a chiral medium is obtained as vector part of the Stokes theorem.

Theorem 12 (Energy balance in chiral media). *Let* $\{\vec{E}, \vec{H}\} \subset C^1(\Omega) \cap C(\overline{\Omega})$ *be a solution of equations* (12) *and let* $\Gamma = \partial\Omega$ *be a closed piecewise smooth surface. Then the equality*

$$\begin{aligned} -\int_\Gamma \langle[\vec{E} \times \vec{H}^*], \vec{n}\rangle d\Gamma \\ = \frac{ik^*}{1 - k^{*2}\beta^{*2}} \int_\Omega \langle \vec{E}, \vec{E}^* \rangle \, dx - \frac{ik}{1 - k^2\beta^2} \int_\Omega \langle \vec{H}, \vec{H}^* \rangle \, dx \\ + \frac{1}{1 - k^{*2}\beta^{*2}} \int_\Omega \langle \vec{E}, \vec{j}^* \rangle \, dx - \frac{ik\beta}{1 - k^2\beta^2} \int_\Omega \langle \vec{H}^*, \vec{j} \rangle \, dx \end{aligned} \tag{24}$$

is valid. Besides, the vector equality

$$\begin{aligned} \int_\Gamma \vec{E}\langle \vec{n}, \vec{H}^* \rangle \, d\Gamma + \int_\Gamma \vec{H}^* \langle \vec{E}, \vec{n} \rangle \, d\Gamma - \int_\Gamma \vec{n} \langle \vec{E}, \vec{H}^* \rangle \, d\Gamma \\ = 2i\mathrm{Im}\left(\frac{k^2\beta}{1 - k^2\beta^2}\right) \int_\Omega [\vec{E} \times \vec{H}^*] \, dx + \frac{ik\beta}{1 - k^2\beta^2} \int_\Omega [\vec{j} \times \vec{H}^*] \, dx \\ - \frac{1}{1 - k^{*2}\beta^{*2}} \int_\Omega [\vec{E} \times \vec{j}^*] \, dx - \frac{1}{\sqrt{\mu\varepsilon}} \int_\Omega \rho \vec{H}^* dx \end{aligned} \tag{25}$$

holds.

Proof. Using $(1), (10)_1$ and $(10)_4$ we obtain the equalities

$$D\vec{E} = -\tfrac{\rho}{\sqrt{\mu\varepsilon}} + \mathrm{rot}\vec{E}$$
$$D\vec{H} = \mathrm{rot}\vec{H}.$$

From (12) we have

$$D\vec{E} = -\frac{\rho}{\sqrt{\mu\varepsilon}} + \mathrm{rot}\vec{E} = -\frac{\rho}{\sqrt{\mu\varepsilon}} + ik(\vec{H} + \beta\,\mathrm{rot}\vec{H})$$
$$D\vec{H} = -ik(\vec{E} + \beta\,\mathrm{rot}\vec{E}) + \vec{j}.$$

Then

$$\begin{aligned} D\vec{E} &= -\frac{\rho}{\sqrt{\mu\varepsilon}} + ik\vec{H} + ik\beta D\vec{H} \\ &= -\frac{\rho}{\sqrt{\mu\varepsilon}} + ik\vec{H} + k^2\beta\vec{E} + k^2\beta^2 \mathrm{rot}\vec{E} + ik\beta\vec{j} \\ &= -\frac{\rho}{\sqrt{\mu\varepsilon}} + ik\vec{H} + k^2\beta\vec{E} + k^2\beta^2\Big(D\vec{E} + \frac{\rho}{\sqrt{\mu\varepsilon}}\Big) + ik\beta\vec{j}. \end{aligned}$$

We can write the last equality in the more convenient form

$$D\vec{E} = -\frac{\rho}{\sqrt{\mu\varepsilon}} + \frac{1}{1-k^2\beta^2}\big(k^2\beta\vec{E} + ik\vec{H} + ik\beta\vec{j}\big). \tag{26}$$

Analogously we have

$$\begin{aligned} D\vec{H} &= -ik(\vec{E} + \beta \mathrm{rot}\vec{E}) + \vec{j} \\ &= -ik\vec{E} + k^2\beta\vec{H} + k^2\beta^2 D\vec{H} + \vec{j}. \end{aligned}$$

Taking the complex conjugation of the last equality yields

$$D\vec{H}^* = \frac{1}{1-k^{*2}\beta^{*2}}\big(ik^*\vec{E}^* + k^{*2}\beta^*\vec{H}^* + \vec{j}^*\big). \tag{27}$$

From Theorem 1 we have

$$\int_\Gamma \vec{E}\vec{n}\vec{H}^* d\Gamma = \int_\Omega \big(\overline{D\vec{E}} \cdot \vec{H}^* + \vec{E} \cdot D\vec{H}^*\big) dx.$$

Using (26) - (27) we obtain

$$\begin{aligned} \int_\Gamma \vec{E}\vec{n}\vec{H}^* d\Gamma = &-\frac{1}{1-k^2\beta^2}\int_\Omega \big(k^2\beta\vec{E} + ik\vec{H} + ik\beta\vec{j}\big)\vec{H}^* dx \\ &+ \frac{1}{1-k^{*2}\beta^{*2}}\int_\Omega \vec{E}(ik^*\vec{E}^* + k^{*2}\beta^*\vec{H}^* + \vec{j}^*\big) dx \\ &- \int_\Omega \frac{\rho}{\sqrt{\mu\varepsilon}}\vec{H}^* dx. \end{aligned}$$

Now the scalar part of this equality gives us (24) and the vector part corresponds to (25) ∎

The following fact is an analogue of Theorem 2 for the electromagnetic field in a chiral medium.

Theorem 13 (Quaternionic Cauchy's integral theorem for chiral media). *Let* $\{\vec{E}, \vec{H}\} \subset C^1(\Omega; \mathbb{H}(\mathbb{C})) \cap C(\overline{\Omega}; \mathbb{H}(C))$ *satisfy the Maxwell equations*

$$\begin{aligned} \mathrm{rot}\vec{E} &= ik(\vec{H} + \beta \mathrm{rot}\vec{H}) \\ \mathrm{rot}\vec{H} &= -ik(\vec{E} + \beta \mathrm{rot}\vec{E}) \end{aligned}$$

in Ω. *Then the equalities*

$$\begin{aligned} \int_\Gamma [\vec{n} \times \vec{E}]\, d\Gamma &= \frac{(\alpha_2 - \alpha_1)}{2} \int_\Omega \vec{E} dx + \frac{(\alpha_2 + \alpha_1) i}{2} \int_\Omega \vec{H} dx \\ \int_\Gamma [\vec{n} \times \vec{H}]\, d\Gamma &= -\frac{(\alpha_2 + \alpha_1) i}{2} \int_\Omega \vec{E} dx + \frac{(\alpha_2 - \alpha_1)}{2} \int_\Omega \vec{H} dx \end{aligned}$$

and

$$\begin{aligned} \int_\Gamma \langle \vec{n}, \vec{E} \rangle \, d\Gamma &= 0 \\ \int_\Gamma \langle \vec{n}, \vec{H} \rangle \, d\Gamma &= 0 \end{aligned}$$

hold.

Proof. Consider the functions $\vec{\Psi}$ and $\vec{\Phi}$ defined by (14). Applying Theorem 2 we get

$$\begin{aligned} \int_\Gamma \vec{n}\vec{\Psi} d\Gamma &= -\alpha_1 \int_\Omega \vec{\Psi} dx \\ \int_\Gamma \vec{n}\vec{\Phi} d\Gamma &= \alpha_2 \int_\Omega \vec{\Phi} dx. \end{aligned}$$

That is, the pair of equalities

$$\left.\begin{aligned} \int_\Gamma \vec{n}(\vec{E} - i\vec{H}) d\Gamma &= -\alpha_1 \int_\Omega (\vec{E} - i\vec{H})\, dx \\ \int_\Gamma \vec{n}(\vec{E} + i\vec{H}) d\Gamma &= \alpha_2 \int_\Omega (\vec{E} + i\vec{H}) dx \end{aligned}\right\} \tag{28}$$

are valid. Adding and substracting them we obtain

$$\begin{aligned} \int_\Gamma \vec{n}\vec{E} d\Gamma &= \frac{(\alpha_2 - \alpha_1)}{2} \int_\Omega \vec{E} dx + \frac{(\alpha_2 + \alpha_1) i}{2} \int_\Omega \vec{H} dx \\ \int_\Gamma \vec{n}\vec{H} d\Gamma &= \frac{(\alpha_2 + \alpha_1)}{2i} \int_\Omega \vec{E} dx + \frac{(\alpha_2 - \alpha_1)}{2} \int_\Omega \vec{H} dx. \end{aligned}$$

Rewriting this pair of equalities in a vector form finishes the proof ∎

7. Maxwell's equations in inhomogeneous chiral media

In this section we consider all the electromagnetic characteristics of the medium being continuously differentiable functions of spatial coordinates: $\varepsilon = \varepsilon(x)$, $\mu = \mu(x)$ and $\beta = \beta(x)$. Then the Maxwell equations for a time-harmonic electromagnetic field in an inhomogeneous chiral medium have the form

$$\left.\begin{array}{l}\operatorname{rot}\vec{H}(x) = -i\omega\varepsilon(x)\big(\vec{E}(x) + \beta(x)\operatorname{rot}\vec{E}(x)\big) + \vec{j}(x)\\ \operatorname{rot}\vec{E}(x) = i\omega\mu(x)\big(\vec{H}(x) + \beta(x)\operatorname{rot}\vec{H}(x)\big)\end{array}\right\}. \tag{29}$$

Introducing the notations

$$\vec{\varepsilon} = \frac{\operatorname{grad}\sqrt{\varepsilon}}{\sqrt{\varepsilon}}, \quad \vec{\mu} = \frac{\operatorname{grad}\sqrt{\mu}}{\sqrt{\mu}}, \quad \vec{\mathcal{E}} = \sqrt{\varepsilon}\vec{E}, \quad \vec{\mathcal{H}} = \sqrt{\mu}\vec{H}$$

and following arguments from [9] we obtain the pair of equations equivalent to the system above:

$$(D + M^{\vec{\varepsilon}})\vec{\mathcal{E}} - \frac{i}{k\beta}(D + M^{\vec{\mu}})\vec{\mathcal{H}} = -\frac{1}{\beta}\vec{\mathcal{E}} - \frac{\rho}{\sqrt{\varepsilon}} - \frac{i\sqrt{\mu}}{k\beta}\vec{j}$$
$$(D + M^{\vec{\varepsilon}})\vec{\mathcal{E}} - ik\beta(D + M^{\vec{\mu}})\vec{\mathcal{H}} = ik\vec{\mathcal{H}} - \frac{\rho}{\sqrt{\varepsilon}}$$

where as before $k = \omega\sqrt{\varepsilon\mu}$. This pair of equations can be rewritten as

$$(D + M^{\vec{\varepsilon}})\vec{\mathcal{E}} = \frac{k^2\beta\vec{\mathcal{E}}}{1-k^2\beta^2} + \frac{ik\beta\sqrt{\mu}}{1-k^2\beta^2}\vec{j} + \frac{ik\vec{\mathcal{H}}}{1-k^2\beta^2} - \frac{\rho}{\sqrt{\varepsilon}}$$
$$(D + M^{\vec{\mu}})\vec{\mathcal{H}} = \frac{k^2\beta}{1-k^2\beta^2}\vec{\mathcal{H}} - \frac{ik}{1-k^2\beta^2}\vec{\mathcal{E}} + \frac{\sqrt{\mu}}{1-k^2\beta^2}\vec{j}.$$

Denote $\alpha = \frac{k^2\beta}{k^2\beta^2 - 1}$. Introducing the functions

$$\begin{array}{l}\vec{\Phi} = \vec{\mathcal{E}} + i\vec{\mathcal{H}}\\ \vec{\Psi} = \vec{\mathcal{E}} - i\vec{\mathcal{H}}\end{array} \tag{30}$$

we obtain the equalities

$$\begin{array}{l}\Big(D_\alpha + \dfrac{\alpha}{k\beta}\Big)\vec{\Phi} = -k\Big(M^{\frac{\vec{\varepsilon}}{k}}\vec{\mathcal{E}} + iM^{\frac{\vec{\mu}}{k}}\vec{\mathcal{H}} + \alpha\dfrac{i\sqrt{\mu}}{k^3\beta}(k\beta+1)\vec{j} + \dfrac{\rho}{k\sqrt{\varepsilon}}\Big)\\ \Big(D_\alpha - \dfrac{\alpha}{k\beta}\Big)\vec{\Psi} = -k\Big(M^{\frac{\vec{\varepsilon}}{k}}\vec{\mathcal{E}} - iM^{\frac{\vec{\mu}}{k}}\vec{\mathcal{H}} - \alpha\dfrac{i\sqrt{\mu}}{k^3\beta}(1-k\beta)\vec{j} + \dfrac{\rho}{k\sqrt{\varepsilon}}\Big).\end{array} \tag{31}$$

It is easy to see that $\vec{\varepsilon}/k$ and $\vec{\mu}/k$ are dimensionless magnitudes. For a slowly changing medium the absolute values of these two vectors are much less than one (see, e.g., [2]), and the first two terms on the right-hand side of (31) are negligible. Thus for a slowly changing chiral medium in the absence of sources and currents we obtain the equations

$$\begin{aligned}(D-\alpha_2)\vec{\Phi} &= 0\\ (D+\alpha_1)\vec{\Psi} &= 0\end{aligned} \tag{32}$$

where α_1 and α_2 are defined by (17).

Let α be a complex-valued scalar function defined in $\Omega \subset \mathbb{R}^3$. We consider the equation

$$(D+\alpha(x))u(x) = 0 \qquad \text{in } \Omega \tag{33}$$

where u is an $\mathbb{H}(\mathbb{C})$-valued function. Following [8] (see also [10]) let us suppose that the scalar function ϕ is some solution of the eikonal equation

$$(\nabla\phi)^2 = \alpha^2 \qquad \text{in } \Omega. \tag{34}$$

Then note that the $\mathbb{H}(\mathbb{C})$-valued functions $Q^{\pm} = \alpha \pm \nabla\phi$ are zero divisors in Ω. Let $\eta = e^{\phi}$. Then $\nabla\phi = \frac{\nabla\eta}{\eta}$ and the operator $D+\alpha$ can be rewritten as

$$D+\alpha = \eta(D+Q^{+})\eta^{-1}.$$

Consequently, equation (33) reduces to the equation

$$(D+Q^{+}(x))v(x) = 0 \qquad \text{in } \Omega \tag{35}$$

where $v = \frac{u}{\eta}$. Note that this equation is equivalent to equation (33).

Let us look for the solution of equation (35) in the form

$$v = Q^{-}s \tag{36}$$

where s is an $\mathbb{H}(\mathbb{C})$-valued function. Substituting (36) into (35) we obtain the equation $DQ^{-}s = 0$ for s. Assume that α depends only on one variable: $\alpha = \alpha(x_1)$. Then, e.g., the functions $\phi_1 = i\Theta$ and $\phi_2 = -i\Theta$ are solutions of (34). Here Θ is an antiderivative of α.

Let us consider first the function ϕ_1. We have

$$\begin{aligned}\eta_1(x_1) &= e^{\phi_1(x_1)} = e^{i\Theta(x_1)}\\ Q_1^{\pm}(x_1) &= \alpha(x_1) \pm ii_1\alpha(x_1) = \alpha(x_1)(1 \pm ii_1).\end{aligned}$$

The function v is related with u by $v = e^{-\phi_1}u$, and we are looking for it in the form

$$v(x) = \alpha(x_1)(1 - ii_1)s(x).$$

For the function s we obtain the equation

$$D(\alpha(x_1)(1 - ii_1)s(x)) = 0. \tag{37}$$

Let us denote $f = \alpha \cdot s$ and use the quaternionic representation

$$f = F_1 + F_2 i_2 \qquad \text{where} \quad \begin{cases} F_1 = f_0 + f_1 i_1 \\ F_2 = f_2 + f_3 i_1. \end{cases}$$

We note that F_1 and F_2 commute with $(1 - ii_1)$ and $(1 - ii_1)i_2 = i_2(1 + ii_1)$. Then equation (37) can be rewritten as

$$D\big(F_1(1 - ii_1) + F_2 i_2(1 + ii_1)\big) = 0. \tag{38}$$

Note that $DM^{(1\pm ii_1)} = M^{(1\pm ii_1)}D$. Multiplying (38) from the right-hand side first by $(1-ii_1)$ and then by $(1+ii_1)$ we obtain that equation (38) is equivalent to the system

$$\left.\begin{aligned} D(F_1)(1 - ii_1) = 0 \\ D(F_2 i_2)(1 + ii_1) = 0 \end{aligned}\right\}. \tag{39}$$

The last equation herein can be rewritten in the form $D(F_2)(1-ii_1) = 0$. Thus, F_1 and F_2 must satisfy the same equation. Let us consider the equation $(39)_1$. Its solution obviously has the form

$$F_1(x) = H_1(x) + S_1(x)(1 + ii_1) \tag{40}$$

where S_1 is an arbitrary two-component function and $H_1 = h_0 + h_1 i_1$ satisfies the equation

$$DH_1 = 0. \tag{41}$$

We note that the last term in (40) does not contribute in the final solution of equation (33) because of multiplication by Q^- (see (36)).

In order to solve equation (41) we rewrite it in explicit form

$$(i_1\partial_1 + i_2\partial_2 + i_3\partial_3)(h_0 + h_1 i_1) = 0$$

and obtain that it is equivalent to the system

$$\left.\begin{aligned} \partial_1 h_0 = \partial_1 h_1 = 0 \\ \partial_2 h_0 + \partial_3 h_1 = 0 \\ \partial_3 h_0 - \partial_2 h_1 = 0 \end{aligned}\right\}. \tag{42}$$

From here we have that H_1 is independent of the variable x_1 and is analytic in the usual complex sense with respect to the complex variable $z = x_3 + i_1 x_2$ as $(42)_{2-3}$represent the corresponding Cauchy-Riemann conditions. More precisely, both $\mathrm{Re}H_1 = \mathrm{Re}h_0 + i_1\mathrm{Re}h_1$ and $\mathrm{Im}H_1 = \mathrm{Im}h_0 + i_1\mathrm{Im}h_1$ are analytic with respect to z.

In a similar way, $F_2 = H_2(x_2, x_3)$ is an analytic function with respect to z. Thus,

$$s(x) = \frac{1}{\alpha(x_1)}\big(H_1(x_2, x_3) + H_2(x_2, x_3)i_2\big)$$

and the function

$$\begin{aligned}\widehat{u}_1(x) &= e^{\phi_1(x_1)}\alpha(x_1)(1 - ii_1)s(x)\\ &= e^{i\Theta(x_1)}\big(H_1(x_2, x_3)(1 - ii_1) + H_2(x_2, x_3)i_2(1 + ii_1)\big)\end{aligned}$$

is a solution of equation(33). Moreover, due to the right $\mathbb{H}(\mathbb{C})$-linearity of (33), the function

$$u_1(x) = e^{i\Theta(x_1)}\Big(H_1(x_2, x_3)(1 - ii_1)A_1 + H_2(x_2, x_3)i_2(1 + ii_1)A_2\Big), \qquad (43)$$

where A_1 and A_2 are arbitrary constant complex quaternions, is also a solution.

Taking the function ϕ_2 as a solution of the eikonal equation (34) and repeating the procedure described above we arrive at another solution

$$u_2(x) = e^{-i\Theta(x_1)}\Big(G_1(x_2, x_3)(1 + ii_1)B_1 + G_2(x_2, x_3)i_2(1 - ii_1)B_2\Big) \qquad (44)$$

of equation (33) where G_1 and G_2, similarly to H_1 and H_2, are analytic functions with respect to z, and B_1, B_2 are arbitrary constant complex quaternions. Thus, the following proposition is valid.

Proposition 14 [8]. *Let $\Theta(x_1)$ be an antiderivative of the function $\alpha(x_1)$, let H_1, H_2 and G_1, G_2 satisfy the Cauchy-Riemann conditions $(42)_{2-3}$, and let A_1, A_2 and B_1, B_2 be arbitrary constant complex quaternions. Then the functions* (43) – (44) *are solutions of the equation*

$$(D + \alpha(x_1))u(x) = 0.$$

From (43) - (44) we obtain the following expressions for $\vec{\Phi}$ and $\vec{\Psi}$:

$$\left.\begin{aligned} \vec{\Phi} &= e^{i\Theta_2(x_1)}\Big(H_1^+(x_2,x_3)(1-ii_1)A_1^+ \\ &\quad + H_2^+(x_2,x_3)i_2(1+ii_1)A_2^+\Big) \\ &\quad + e^{-i\Theta_2(x_1)}\Big(G_1^+(x_2,x_3)(1+ii_1)B_1^+ \\ &\quad + G_2^+(x_2,x_3)i_2(1-ii_1)B_2^+\Big) \\ \vec{\Psi} &= e^{-i\Theta_1(x_1)}\Big(H_1^-(x_2,x_3)(1-ii_1)A_1^- \\ &\quad + H_2^-(x_2,x_3)i_2(1+ii_1)A_2^-\Big) \\ &\quad + e^{i\Theta_1(x_1)}\Big(G_1^-(x_2,x_3)(1+ii_1)B_1^- \\ &\quad + G_2^-(x_2,x_3)i_2(1-ii_1)B_2^-\Big) \end{aligned}\right\} \tag{45}$$

where the functions $H_1^\pm, H_2^\pm$ and $G_1^\pm, G_2^\pm$ satisfy the Cauchy-Riemann conditions $(42)_{2-3}$, Θ_1, Θ_2 are antiderivative of α_1, α_2, respectively, and $A_1^\pm, A_2^\pm$ and $B_1^\pm, B_2^\pm$ are arbitrary constant complex quaternions which must be chosen in such a way that the scalar parts of the expressions on the right-hand sides of (45) be zero.

Let us consider the case $A_2^\pm = B_2^\pm = 0$. Then omitting the subindex "1" we obtain $\vec{\Phi}$ and $\vec{\Psi}$ in the form

$$\left.\begin{aligned} \vec{\Phi} &= e^{i\Theta_2(x_1)}H^+(x_2,x_3)(1-ii_1)A^+ \\ &\quad + e^{-i\Theta_2(x_1)}G^+(x_2,x_3)(1+ii_1)B^+ \\ \vec{\Psi} &= e^{-i\Theta_1(x_1)}H^-(x_2,x_3)(1-ii_1)A^- \\ &\quad + e^{i\Theta_1(x_1)}G^-(x_2,x_3)(1+ii_1)B^- \end{aligned}\right\} \tag{46}$$

where $H^\pm = h_0^\pm + h_1^\pm i_1$ and $G^\pm = g_0^\pm + g_1^\pm i_1$, and the functions $h_0^\pm, h_1^\pm$ and $g_0^\pm, g_1^\pm$ satisfy $(42)_{2-3}$. Note that

$$H^\pm \cdot (1-ii_1) = h^\pm \cdot (1-ii_1) \qquad \text{where } h^\pm = h_0^\pm + ih_1^\pm$$
$$G^\pm \cdot (1+ii_1) = g^\pm \cdot (1+ii_1) \qquad \text{where } g^\pm = g_0^\pm - ig_1^\pm.$$

The scalar parts of the expressions on the right-hand sides in (46) are zero if and only if

$$\mathrm{Sc}((1-ii_1)A^\pm) = \mathrm{Sc}((1+ii_1)B^\pm) = 0$$

which is equivalent to the conditions

$$\begin{aligned} A_0^\pm &= -iA_1^\pm \\ B_0^\pm &= iB_1^\pm \end{aligned} \qquad \text{where } \begin{cases} A^\pm = \sum_{k=0}^3 i_k A_k^\pm \\ B^\pm = \sum_{k=0}^3 i_k B_k^\pm . \end{cases}$$

Under these conditions we obtain

$$(1 - ii_1)A^\pm = a^\pm(i_2 - ii_3) \qquad \text{where } a^\pm = A_2^\pm + iA_3^\pm$$
$$(1 + ii_1)B^\pm = b^\pm(i_2 + ii_3) \qquad \text{where } b^\pm = B_2^\pm - iB_3^\pm.$$

Then, the functions (46) take the form

$$\left.\begin{aligned}\vec{\Phi} &= e^{i\Theta_2(x_1)}h^+(x_2,x_3)a^+(i_2 - ii_3)\\ &\quad + e^{-i\Theta_2(x_1)}g^+(x_2,x_3)b^+(i_2 + ii_3)\\ \Psi &= e^{-i\Theta_1(x_1)}h^-(x_2,x_3)a^-(i_2 - ii_3)\\ &\quad + e^{i\Theta_1(x_1)}g^-(x_2,x_3)b^-(i_2 + ii_3)\end{aligned}\right\}. \tag{47}$$

Thus we obtain the following

Proposition 15. *Let α_1 and α_2 be functions of x_1 only; $h^\pm = h_0^\pm + ih_1^\pm$ and $g^\pm = g_0^\pm - ig_1^\pm$ where the pairs $h_0^\pm, h_1^\pm$ and $g_0^\pm, g_1^\pm$ satisfy the Cauchy-Riemann conditions* $(42)_{2-3}$*, and let $a^\pm$ and $b^\pm$ be arbitrary constant complex numbers. Then the functions* (47) *are solutions of equations* (32)*.*

The corresponding vectors of the electromagnetic field are obtained in the form

$$\begin{aligned}\vec{E} = \frac{1}{2\sqrt{\varepsilon}}\Big[&e^{i\int\alpha_2(x_1)dx_1}h^+(x_2,x_3)a^+(i_2 - ii_3)\\ &+ e^{-i\int\alpha_2(x_1)dx_1}g^+(x_2,x_3)b^+(i_2 + ii_3)\\ &+ e^{-i\int\alpha_1(x_1)dx_1}h^-(x_2,x_3)a^-(i_2 - ii_3)\\ &+ e^{i\int\alpha_1(x_1)dx_1}g^-(x_2,x_3)b^-(i_2 + ii_3)\Big]\\ \vec{H} = \frac{1}{2i\sqrt{\mu}}\Big[&e^{i\int\alpha_2(x_1)dx_1}h^+(x_2,x_3)a^+(i_2 - ii_3)\\ &+ e^{-i\int\alpha_2(x_1)dx_1}g^+(x_2,x_3)b^+(i_2 + ii_3)\\ &- e^{-i\int\alpha_1(x_1)dx_1}h^-(x_2,x_3)a^-(i_2 - ii_3)\\ &- e^{i\int\alpha_1(x_1)dx_1}g^-(x_2,x_3)b^-(i_2 + ii_3)\Big].\end{aligned}$$

They give us solution of (29) for a stratified chiral medium.

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