Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 22 (2003), No. 3, 609–617

A Wave Equation with Fractional Damping

N.-e. Tatar

Abstract. We consider a wave equation with an internal damping represented by a fractional derivative of lower order than one. An exponential growth result is proved in presence of a source of polynomial type. This result improves an earlier one where the initial data are supposed to be very large in some norm. A new argument based on a new functional is proposed.

Keywords: *Exponential growth, fractional derivative, internal damping*

AMS subject classification: 35L20, 35L70

1. Introduction

We are interested in the fractional differential problem

$$\left.\begin{array}{lr} u_{tt} + \partial_t^{1+\alpha} u = \Delta u + |u|^{p-1}u & (x \in \Omega, t > 0) \\ u(x,t) = 0 & (x \in \Gamma, t > 0) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & (x \in \Omega) \end{array}\right\} \qquad (1)$$

where $p > 1$, $-1 < \alpha < 1$, u_0 and u_1 are given functions, Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary Γ and $\partial_t^{1+\alpha}$ is Caputo's fractional derivative of order $1+\alpha$ (see [18: Chapter 2.4.1]) defined by

$$\partial_t^{1+\alpha} w(t) = \begin{cases} I^{-\alpha} \frac{d}{dt} w(t) & \text{if } -1 < \alpha < 0 \\ I^{1-\alpha} \frac{d^2}{dt^2} w(t) & \text{if } 0 < \alpha < 1 \end{cases} \qquad (2)$$

where $I^\beta \;\; (\beta > 0)$ is the fractional integral

$$I^\beta w(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} w(s)\, ds. \qquad (3)$$

Nasser-eddine Tatar: King Fahd Univ. of Petr. & Min., Dept. Math. Sci., P.O. Box 1446, Dhahran 31261, Saudi Arabia; tatarn@kfupm.edu.sa

ISSN 0232-2064 / $ 2.50

See also [4, 17, 18, 20] for more on fractional derivatives and applications.In particular, in Control theory, it is known that noise is amplified by the differentiation process. To attenuate this noise one is lead to use derivatives of lower order.

Problem (1) was first studied for $\alpha = \frac{1}{2}$ by Lokshin in [11] and Lokshin and Rok in [12]. Then, it has been considered for $-1 < \alpha < 1$ by Matignon et al. [13]. The authors have managed to replace the hereditary equation by a non-hereditary system for which the standard methods, such as the Galerkin method and LaSalle's invariance principle, apply. For the well posedness we refer the reader to this reference (see also [8] for more on existence results).

Let us mention here that the case $\alpha = 0$ corresponds to an internal damping. This damping competes with the polynomial source. As a result, it was proved (see [5, 14, 16, 19]) that solutions exist globally in time when the initial data are in a *stable* set and blow up in a finite time when the initial data are in an *unstable* set. It is also known that in the linear case ($p = 1$), we have global existence even without any type of dissipation.

The wave equation without damping corresponds to the case where $\alpha = -1$. This case has been extensively studied by many authors (see, for instance [1 - 3, 7, 7, 10, 21]). It has been proved, in particular, that solutions blow up in finite time for sufficiently large initial data (in some sense) and also for small initial data provided that the exponent p lies in some critical range.

In this paper we improve an earlier result by the present author with M. Kirane in [9]. There, for sufficiently large initial data (in some sense), it has been shown that the solution is unbounded provided that the initial data are very large in some norm. In fact, an exponential growth result was proved. Here we relax considerably this condition on the initial data. So the space of initial data is enlarged. To this end we present a different argument based on a new functional while the previous proof makes use of the Hardy-Littlewood-Sobolev inequality and some "convolution" inequalities.

2. Exponential growth

Let us define the classical energy by

$$E(t) = \int_\Omega \Big\{ \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} |u|^{p+1} \Big\} dx$$

and the modified energy by

$$E_\varepsilon(t) = \int_\Omega \Big\{ \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 - \varepsilon u u_t - \frac{1}{p+1} |u|^{p+1} \Big\} dx \tag{4}$$

for some $\varepsilon > 0$. **Theorem.** *Let* $u = u(x,t)$ *be a regular solution of problem* (1) *with* $-1 < \alpha < 0$ *and* $p > 1$. *If the initial data* u_0 *and* u_1 *are such that* $E_\varepsilon(0) < 0$, *then* $u(x,t)$ *grows up exponentially in the* L_{p+1}*-norm.*

Proof. Let us multiply $(1)_1$ by $u_t - \varepsilon u$ and integrate over Ω. We get

$$\begin{aligned}\frac{d}{dt}\int_\Omega \Big\{\frac{1}{2}u_t^2 + \frac{1}{2}|\nabla u|^2 - \varepsilon u u_t - \frac{1}{p+1}|u|^{p+1}\Big\}dx \\ + \frac{1}{\Gamma(-\alpha)}\int_\Omega u_t \int_0^t (t-s)^{-(\alpha+1)}u_t(s)\,ds dx \\ = \varepsilon \int_\Omega |\nabla u|^2 dx + \frac{\varepsilon}{\Gamma(-\alpha)}\int_\Omega u \int_0^t (t-s)^{-(\alpha+1)}u_t(s)\,dsdx \\ - \varepsilon\int_\Omega u_t^2 dx - \varepsilon\int_\Omega |u|^{p+1}dx.\end{aligned}$$

Recalling definition (4) of E_ε, we see that

$$\begin{aligned}&\frac{dE_\varepsilon(t)}{dt} + \frac{1}{\Gamma(-\alpha)}\int_\Omega u_t\int_0^t (t-s)^{-(\alpha+1)}u_t(s)\,dsdx \\ &= \varepsilon\int_\Omega |\nabla u|^2 dx \\ &+ \frac{\varepsilon}{\Gamma(-\alpha)}\int_\Omega u\int_0^t (t-s)^{-(\alpha+1)}u_t(s)\,dsdx \\ &- \varepsilon\int_\Omega u_t^2 dx - \varepsilon\int_\Omega |u|^{p+1}dx.\end{aligned} \tag{5}$$

Next, we define the auxiliary functional $F = F_{\alpha,\beta,\sigma}$ by

$$F = \int_0^t \int_\Omega G_{\alpha,\beta}(t-s)e^{-\sigma\varepsilon s}|u_t|^2 dxds \tag{6}$$

with

$$G_{\alpha,\beta}(t) = e^{\beta t}\int_t^{+\infty} e^{-\beta s}s^{-(2\alpha+3)}ds \tag{7}$$

where $\beta > 0$ and $\varepsilon > 0$ are constants which will be precised later on. Since there is no risk of confusion in the notation, we will drop the subscripts of F and G for convenience.

The differentiation of $F(t)$ in (6) with respect to t gives

$$\begin{aligned}\frac{dF(t)}{dt} &= \int_\Omega G_\beta(0)e^{-\sigma\varepsilon t}|u_t|^2 dx \\ &+ \int_0^t\int_\Omega \Big\{ -(t-s)^{-(2\alpha+3)} \\ &+ \beta e^{\beta(t-s)}\int_{t-s}^{+\infty} e^{-\beta z}z^{-(2\alpha+3)}dz\Big\}e^{-\sigma\varepsilon s}|u_t|^2 dxds.\end{aligned} \tag{8}$$

Observe that

$$G(0)=\int_0^{+\infty} e^{-\beta s}s^{-(2\alpha+3)}ds=\beta^{2(\alpha+1)}\Gamma(2\alpha+4).$$

Then relation (8) becomes

$$\begin{aligned}\frac{dF(t)}{dt}&=\beta^{2(\alpha+1)}\Gamma(2\alpha+4)e^{-\sigma\varepsilon t}\int_\Omega |u_t|^2dx\\&-\int_0^t\int_\Omega (t-s)^{-(2\alpha+3)}e^{-\sigma\varepsilon s}|u_t|^2dxds+\beta F(t).\end{aligned}\tag{9}$$

Now, we consider the functional

$$H(t)=e^{-\sigma\varepsilon t}E_\varepsilon(t)+\mu F(t)\qquad (t\geq 0)\tag{10}$$

for some $\mu>0$ to be determined. Its derivative with respect to t is, according to (5) and (9), equal to

$$\begin{aligned}\frac{dH(t)}{dt}&=-\sigma\varepsilon e^{-\sigma\varepsilon t}E_\varepsilon(t)\\&+e^{-\sigma\varepsilon t}\Big\{-\frac{1}{\Gamma(-\alpha)}\int_\Omega u_t\int_0^t (t-s)^{-(\alpha+1)}u_t(s)\,dsdx\\&+\varepsilon\int_\Omega|\nabla u|^2dx-\varepsilon\int_\Omega |u_t|^2dx\\&+\frac{\varepsilon}{\Gamma(-\alpha)}\int_\Omega u\int_0^t (t-s)^{-(\alpha+1)}u_t(s)\,dsdx-\varepsilon\int_\Omega |u|^{p+1}dx\Big\}\\&+\mu\Big\{\beta^{2(\alpha+1)}\Gamma(2\alpha+4)e^{-\sigma\varepsilon t}\int_\Omega|u_t|^2dx\\&-\int_0^t\int_\Omega (t-s)^{-(2\alpha+3)}e^{-\sigma\varepsilon s}|u_t|^2dxds+\beta F(t)\Big\}.\end{aligned}$$

Using definition (4) of E_ε, we may write

$$\begin{aligned}\frac{dH(t)}{dt}&=-\Big(\frac{\sigma\varepsilon}{2}+\varepsilon-\mu\beta^{2(\alpha+1)}\Gamma(2\alpha+4)\Big)e^{-\sigma\varepsilon t}\int_\Omega |u_t|^2dx\\&-\Big(\frac{\sigma\varepsilon}{2}-\varepsilon\Big)e^{-\sigma\varepsilon t}\int_\Omega|\nabla u|^2dx+\sigma\varepsilon^2e^{-\sigma\varepsilon t}\int_\Omega u_tu\,dx\\&-\Big(\varepsilon-\frac{\sigma\varepsilon}{p+1}\Big)e^{-\sigma\varepsilon t}\int_\Omega |u|^{p+1}dx\\&-\mu\int_0^t\int_\Omega (t-s)^{-(2\alpha+3)}e^{-\sigma\varepsilon s}|u_t|^2dxds\\&+\frac{\varepsilon e^{-\sigma\varepsilon t}}{\Gamma(-\alpha)}\int_\Omega u\int_0^t (t-s)^{-(\alpha+1)}u_t(s)\,dsdx\\&-\frac{e^{-\sigma\varepsilon t}}{\Gamma(-\alpha)}\int_\Omega u_t\int_0^t (t-s)^{-(\alpha+1)}u_t(s)\,dsdx+\mu\beta F(t).\end{aligned}\tag{11}$$

By the generalized Young inequality and the Poincaré inequality, we clearly have

$$\int_\Omega u_t u\,dx \le \frac{1}{4\varepsilon}\int_\Omega |u_t|^2 dx + \varepsilon C_p \int_\Omega |\nabla u|^2 dx \tag{12}$$

where C_p is the Poincaré constant. The seventh term in the right-hand side of (11) may be handled in the following manner. First note that

$$\begin{aligned} &e^{-\sigma\varepsilon t}\int_\Omega u_t \int_0^t (t-s)^{-(\alpha+1)} u_t(s)\,dsdx \\ &\quad = e^{-\frac{\sigma\varepsilon}{2}t}\int_\Omega u_t \int_0^t (t-s)^{-(\alpha+1)} e^{-\frac{\sigma\varepsilon}{2}(t-s)} e^{-\frac{\sigma\varepsilon}{2}s} u_t(s)\,dsdx. \end{aligned}$$

Then, by the generalized Young inequality, we find

$$\begin{aligned} &e^{-\sigma\varepsilon t}\int_\Omega u_t \int_0^t (t-s)^{-(\alpha+1)} u_t(s)\,dsdx \\ &\quad \le \frac{\varepsilon\Gamma(-\alpha)}{2} e^{-\sigma\varepsilon t}\int_\Omega |u_t|^2 dx \\ &\qquad + \frac{1}{2\varepsilon\Gamma(-\alpha)}\int_\Omega \left(\int_0^t (t-s)^{-(\alpha+1)} e^{-\frac{\sigma\varepsilon}{2}(t-s)} e^{-\frac{\sigma\varepsilon}{2}s} u_t(s)\,ds\right)^2 dx. \end{aligned}$$

Here we have multiplied as in [15] by $e^{-\frac{\sigma\varepsilon}{2}s}e^{\frac{\sigma\varepsilon}{2}s}$. Using the decomposition $\alpha+1 = -\frac{1}{2} + (\alpha+\frac{3}{2})$, we obtain by the Hölder inequality

$$\begin{aligned} &e^{-\sigma\varepsilon t}\int_\Omega u_t \int_0^t (t-s)^{-(\alpha+1)} u_t(s)\,dsdx \\ &\quad \le \frac{\varepsilon\Gamma(-\alpha)}{2} e^{-\sigma\varepsilon t}\int_\Omega |u_t|^2 dx \\ &\qquad + \frac{1}{2\Gamma(-\alpha)\sigma\varepsilon^3}\int_\Omega\int_0^t (t-s)^{-(2\alpha+3)} e^{-\sigma\varepsilon s}|u_t|^2 dsdx. \end{aligned} \tag{13}$$

Similarly, we obtain for the sixth term in the right-hand side of (11)

$$\begin{aligned} &e^{-\sigma\varepsilon t}\int_\Omega u \int_0^t (t-s)^{-(\alpha+1)} u_t(s)\,dsdx \\ &\quad \le \delta C_p e^{-\sigma\varepsilon t}\int_\Omega |\nabla u|^2 dx \\ &\qquad + \frac{1}{4\delta\sigma\varepsilon^2}\int_\Omega\int_0^t (t-s)^{-(2\alpha+3)} e^{-\sigma\varepsilon s}|u_t|^2 dsdx \quad (\delta > 0). \end{aligned} \tag{14}$$

Taking into account (12) - (14) in (11), we infer that

$$\begin{aligned}\frac{dH(t)}{dt} \leq & -\Big[\frac{\sigma\varepsilon}{2} + \varepsilon - \mu\beta^{2(\alpha+1)}\Gamma(2\alpha+4) - \frac{1}{4}\sigma\varepsilon - \frac{\varepsilon}{2}\Big]e^{-\sigma\varepsilon t}\int_\Omega |u_t|^2 dx \\ & - \varepsilon\Big[\frac{\sigma}{2} - \varepsilon\Big(1 + \sigma\varepsilon^2 C_p + \frac{C_p\delta}{\Gamma(-\alpha)}\Big)\Big]e^{-\sigma\varepsilon t}\int_\Omega |\nabla u|^2 dx \\ & - \Big[\mu - \frac{1}{4\sigma^2\varepsilon^2\Gamma(-\alpha)}\Big(\frac{2}{\varepsilon\Gamma(-\alpha)} + \frac{\varepsilon}{\delta}\Big)\Big] \\ & \times \int_0^t \int_\Omega (t-s)^{-(2\alpha+3)} e^{-\sigma\varepsilon s}|u_t|^2 dx ds \\ & - \varepsilon\Big(1 - \frac{\sigma}{p+1}\Big)e^{-\sigma\varepsilon t}\int_\Omega |u|^{p+1} dx + \mu\beta F(t).\end{aligned}$$

This inequality may also be written as

$$\begin{aligned}\frac{dH(t)}{dt} \leq & \, \sigma\varepsilon H(t) - \varepsilon\Big[\sigma - (1 + \sigma\varepsilon^2 C_p + \frac{C_p\delta}{\Gamma(-\alpha)}\Big)\Big]e^{-\sigma\varepsilon t}\int_\Omega |\nabla u|^2 dx \\ & - \Big[\sigma\varepsilon + \varepsilon - \mu\beta^{2(\alpha+1)}\Gamma(2\alpha+4) - \frac{1}{4}\sigma\varepsilon - \frac{\varepsilon}{2}\Big]e^{-\sigma\varepsilon t}\int_\Omega |u_t|^2 dx \\ & - \Big[\mu - \frac{1}{4\sigma^2\varepsilon^2\Gamma(-\alpha)}\Big(\frac{2}{\varepsilon\Gamma(-\alpha)} + \frac{\varepsilon}{\delta}\Big)\Big] \\ & \times \int_0^t \int_\Omega (t-s)^{-(2\alpha+3)} e^{-\sigma\varepsilon s}|u_t|^2 dx ds \\ & - \varepsilon\Big(1 - \frac{2\sigma}{p+1}\Big)e^{-\sigma\varepsilon t}\int_\Omega |u|^{p+1} dx + \sigma\varepsilon^2 e^{-\sigma\varepsilon t}\int_\Omega u_t u\, dx \\ & + \mu(\beta - \sigma\varepsilon)F(t).\end{aligned} \tag{15}$$

To get (15), we have added and subtracted $\sigma\varepsilon H(t)$ in the right-hand side of the previous inequality.

Finally, we apply (12) to the term $\int_\Omega u_t u\, dx$ in (15) to obtain

$$\begin{aligned}\frac{dH(t)}{dt} \leq & \, \sigma\varepsilon H(t) \\ & - \varepsilon\Big[\sigma - \Big(1 + 2\sigma\varepsilon^2 C_p + \frac{C_p\delta}{\Gamma(-\alpha)}\Big)\Big]e^{-\sigma\varepsilon t}\int_\Omega |\nabla u|^2 dx \\ & - \frac{1}{2}\Big[\sigma\varepsilon + \varepsilon - 2\mu\beta^{2(\alpha+1)}\Gamma(2\alpha+4)\Big]e^{-\sigma\varepsilon t}\int_\Omega |u_t|^2 dx \\ & - \Big[\mu - \frac{1}{4\sigma^2\varepsilon^2\Gamma(-\alpha)}\Big(\frac{2}{\varepsilon\Gamma(-\alpha)} + \frac{\varepsilon}{\delta}\Big)\Big] \\ & \times \int_0^t \int_\Omega (t-s)^{-(2\alpha+3)} e^{-\sigma\varepsilon s}|u_t|^2 dx ds \\ & - \varepsilon\Big(1 - \frac{2\sigma}{p+1}\Big)e^{-\sigma\varepsilon t}\int_\Omega |u|^{p+1} dx + \mu(\beta - \sigma\varepsilon)F(t).\end{aligned} \tag{16}$$

Choosing $\delta = (p-1)\frac{\Gamma(-\alpha)}{4C_p}$, inequality (16) reduces to

$$\begin{aligned}\frac{dH(t)}{dt} &\le \sigma\varepsilon H(t) - \varepsilon\Big[\sigma - (2\sigma\varepsilon^2 C_p + \frac{p+3}{4}\Big)\Big]e^{-\sigma\varepsilon t}\int_\Omega |\nabla u|^2 dx \\ &- \frac{1}{2}\Big[\sigma\varepsilon + \varepsilon - 2\mu\beta^{2(\alpha+1)}\Gamma(2\alpha+4)\Big]e^{-\sigma\varepsilon t}\int_\Omega |u_t|^2 dx \\ &- \Big[\mu - \frac{1}{2\sigma^2\varepsilon^2\Gamma^2(-\alpha)}\Big(\frac{1}{\varepsilon} + \frac{2\varepsilon C_p}{p-1}\Big)\Big] \\ &\times \int_0^t \int_\Omega (t-s)^{-(2\alpha+3)} e^{-\sigma\varepsilon s}|u_t|^2 dx ds \\ &- \varepsilon\Big(1 - \frac{2\sigma}{p+1}\Big)e^{-\sigma\varepsilon t}\int_\Omega |u|^{p+1}dx + \mu(\beta - \sigma\varepsilon)F(t).\end{aligned} \tag{17}$$

If we choose

$$\varepsilon < \min\Big\{1, \frac{1}{C_p}, \Big[\frac{p-1}{2(p+1)C_p}\Big]^{1/2}\Big\},$$

then it is possible to select σ such that

$$\frac{p+3}{4(1-2C_p\varepsilon^2)} < \sigma < \frac{p+1}{2}.$$

This ensures the negativity of the coefficients of $\int_\Omega |\nabla u|^2 dx$ and $\int_\Omega |u|^{p+1} dx$.

Next, assuming μ large enough, namely

$$\mu \ge \frac{1}{2\sigma^2\varepsilon^3\Gamma^2(-\alpha)}\Big(1 + \frac{2\varepsilon^2 C_p}{p-1}\Big)$$

and

$$\beta \le \min\Big\{\sigma\varepsilon, \Big[\frac{\varepsilon}{2\mu\Gamma(2\alpha+4)}\Big]^{\frac{1}{2(\alpha+1)}}\Big\},$$

the remaining coefficients are also negative. Therefore (17) reduces to

$$\frac{dH(t)}{dt} \le \sigma\varepsilon H(t) \qquad (t \ge 0). \tag{18}$$

We assume that

$$H(0) = E_\varepsilon(0) = \int_\Omega \Big\{\frac{1}{2}u_1^2 + \frac{1}{2}|\nabla u_0|^2 - \varepsilon u_0 u_1 - \frac{1}{p+1}|u_0|^{p+1}\Big\}dx$$

is negative. By the Gronwall inequality it is easy to see from (18) that

$$H(t) \le H(0)e^{\sigma\varepsilon t} \qquad (t \ge 0). \tag{19}$$

On the other hand, from the definition of $H(t)$ and (12) (with $\varepsilon = \frac{1}{2}$) we obtain

$$H(t) \geq -\frac{e^{-\sigma\varepsilon t}}{p+1}\int_\Omega |u|^{p+1}dx + \frac{e^{-\sigma\varepsilon t}}{2}\int_\Omega |u_t|^2 dx + \frac{e^{-\sigma\varepsilon t}}{2}\int_\Omega |\nabla u|^2 dx - \frac{\varepsilon e^{-\sigma\varepsilon t}}{2}\int_\Omega |u_t|^2 dx - \frac{\varepsilon C_p e^{-\sigma\varepsilon t}}{2}\int_\Omega |\nabla u|^2 dx$$

or, for $t \geq 0$,

$$H(t) \geq -\frac{e^{-\sigma\varepsilon t}}{p+1}\int_\Omega |u|^{p+1}dx + \frac{e^{-\sigma\varepsilon t}}{2}\int_\Omega [(1-\varepsilon)|u_t|^2 + (1-\varepsilon C_p)|\nabla u|^2]dx.$$

From our choice of ε, it is clear that

$$H(t) \geq -\frac{e^{-\sigma\varepsilon t}}{p+1}\int_\Omega |u|^{p+1}dx. \tag{20}$$

Relations (19) and (20) imply that

$$\int_\Omega |u|^{p+1}dx \geq (p+1)[-H(0)]e^{(2\sigma\varepsilon)t} \qquad (t \geq 0).$$

This completes the proof ∎

Acknowledgment. The author would like to thank King Fahd University of Petroleum and Minerals for its financial support.

References

[1] Glassey, R. T.: *Blow up theorems for nonlinear wave equations.* Math. Z. 132 (1973), 183 – 203.

[2] Glassey, R. T.: *Finite time blow up for solutions of nonlinear wave equations.* Math. Z. 177 (1981), 323 – 340.

[3] Glassey, R. T.: *Existence in the large for* $\Box u = F(u)$ *in two space dimensions.* Mat. Z. 178 (1981), 233 – 261.

[4] Gorenflo, R. and S. Vessella: *Abel Integral Equations.*Lect. Notes Math. 1461 (1991).

[5] Ikehata, R. and T. Suzuki: *Stable and unstable sets for evolution equations of parabolic and hyperbolic type.* Hiroshima Math. J. 26 (1996), 475 – 491.

[6] John, F.: *Blow up of solutions of nonlinear wave equations in three space dimensions.* Man. Mat. 28 (1979), 235 – 268.

[7] Keller, J.: *On solutions of nonlinear wave equations.* Comm. Pure Appl. Math. 10 (1957), 523 – 532.

[8] Kilbas, A. A., Bonilla, B. and J. J. Trujillo: *Existence and uniqueness theorem for nonlinear fractional differential equations.*Dem. Math. 33 (2000), 583 – 602.

[9] Kirane, M. and N.-e. Tatar: *Exponential growth for a fractionally damped wave equation.* Z. Anal. Anw. 22 (2003), 167 – 177.

[10] Levine, H. A.: *Instability and nonexistence of global solutions to nonlinear wave equations of the form* $Pu_{tt} = -Au + F(u)$. Trans. Amer. Math. Soc. 192 (1974), 1 – 21.

[11] Lokshin, A. A.: *Wave equation with singular delayed time* (in Russian). Dokl. Akad. Nauk. SSSR 240 (1978), 43 – 46.

[12] Lokshin, A. A. and V. E. Rok: *Fundamental solutions of the wave equation with delayed time* (in Russain). Dokl. Akad. Nauk. SSSR 239 (1978), 1305 – 1308.

[13] Matignon, D., Audounet, J. and G. Montseny: *Energy decay for wave equations with damping of fractional order.* In: Proc. Fourth Intern. Conf. on Math. & Numer. Aspects of Wave Propagation Phenomena, held at June 1 - 5, 1998, at Colorado School of Mines in Golden, Colorado/USA. Published by: INRIA-SIAM, pp. 638 – 640; published also as Techn. Report in: Fract. Diff. Systems – Theory & Appl. Ecole Nat. Sup. des Telecomm. 1998, Tech. Rep. 98C003 (a copy is available from doc@laas.fr).

[14] Matsumura, A.: *Global existence and asymptotics of the solutions of the second-order quasilinear hyperbolic equations with first-order dissipation.* Publ. Res. Inst. Sci. Kyoto Univ. 13 (1977), 349 – 379.

[15] Medved', M.: *A new approach to an analysis of Henry type integral inequalities and their Bihari type versions.* J. Math. Anal. Appl. 214 (1997), 349 – 366.

[16] Nakao, M. and K. Ono: *Global existence to the Cauchy problem of the semilinear wave equation with a nonlinear dissipation.* Funkcialaj Ekvacioj 38 (1995), 417 – 431.

[17] Oldham, K. B. and J. Spanier: *The Fractional Calculus.* New York: Acad. Press 1974.

[18] Podlubny, I.: *Fractional Differential Equations* (Math. in Sci. and Eng.: Vol. 198). New York - London: Acad. Press 1999.

[19] Pucci, P. and J. Serrin: *Global non existence for abstract evolution equations with positive initial energy.* J. Diff. Eqs. 150 (1998), 203 – 214.

[20] Samko, S. G., Kilbas, A. A. and O. I. Marichev: *Fractional Integrals and Derivatives, Theory and Applications.* Amsterdam: Gordon and Breach 1993. [Engl. Trans. from the Russian edition 1987].

[21] Sideris, T. C.: *Nonexistence of global solutions to semilinear wave equations in high dimensions.* J. Diff. Eqs. 52 (1984), 378 – 406.

Received 21.01.2003