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A Note on the Poincaré Inequality for Convex Domains

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Abstract. In this article a proof for the Poincaré inequality with explicit constant for convex domains is given. This proof is a modification of the original proof [5], which is valid only for the two-dimensional case.

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1. Introduction

The classical proof for the Poincaré inequality

$$\|u\|_{L^2(\Omega)} \leq c_\Omega \|\nabla u\|_{L^2(\Omega)},$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $u \in H^1(\Omega)$ with vanishing mean value over Ω, is based on the compact embedding of $H^1(\Omega)$ in $L^2(\Omega)$ which is valid under quite general assumptions on Ω (cf. [6]). However, the constant c_Ω depends on the domain Ω, and the proof based on compactness does not provide insight into this dependency.

For practical purposes it is important to know an explicit expression for this constant (see, for example, [2, 7]). Therefore, the special case of convex domains is interesting, since in [5] this constant is proved to be $\frac{d}{\pi}$, where d is the diameter of Ω. Though this proof is elegant, it contains a mistake in the case $n \geq 3$. The same mistake can also be found in [1], in which the L^1-estimate is considered.

The goal of this article is to fix this gap (see Remark 3.3). Luckily, the constant $\frac{d}{\pi}$ in the Poincaré inequality remains valid.

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2. The one-dimensional case

We first prove the Poincaré inequality for the one-dimensional case. In fact we will prove a generalization which the multi-dimensional case can be reduced to.

Lemma 2.1. *Let $m \in \mathbb{N}$ and ρ be a non-negative concave function on the interval $[0, L]$. Then, for all $u \in H^1(0, L)$ satisfying*

$$\int_0^L \rho^m(x)u(x)\,dx = 0, \tag{2.1}$$

there holds

$$\int_0^L \rho^m(x)|u(x)|^2 dx \le \frac{L^2}{\pi^2}\int_0^L \rho^m(x)|u'(x)|^2 dx. \tag{2.2}$$

Furthermore, the constant $\frac{L^2}{\pi^2}$ is optimal.

Proof.

(a) Let us first assume that ρ is strictly positive and twice differentiable. Then each non-zero function v minimizing the quotient

$$\frac{\int_0^L \rho^m(x)|u'(x)|^2 dx}{\int_0^L \rho^m(x)|u(x)|^2 dx} \tag{2.3}$$

and satisfying (2.1) must satisfy the Sturm-Liouville system (cf. [3])

$$\left.\begin{aligned} [\rho^m v']' + \lambda\rho^m v = 0 \\ v'(0) = v'(L) = 0 \end{aligned}\right\} \tag{2.4}$$

where λ is the minimum of quotient (2.3). After dividing (2.4) by ρ^m and differentiating, we introduce the new variable $w = \rho^{m/2}v'$ and obtain

$$\left.\begin{aligned} w'' + \frac{m}{2}\left[\frac{\rho''}{\rho} - \left(1 + \frac{m}{2}\right)\frac{(\rho')^2}{\rho^2}\right] w + \lambda w = 0 \\ w(0) = w(L) = 0 \end{aligned}\right\}.$$

Since ρ is concave, $\rho'' \le 0$. Hence, $w'' + (\lambda - a)w = 0$, where a is a non-negative function. Integration by parts leads to

$$\lambda \ge \frac{\int_0^L |w'(x)|^2 dx}{\int_0^L |w(x)|^2 dx}.$$

The last quotient is bounded by the first eigenvalue of the vibrating string with fixed ends, which gives $\lambda \geq \frac{\pi^2}{L^2}$.

(b) If ρ is a non-negative concave function, we may represent it as the L^∞-limit of strictly positive concave C^2-functions ρ_k (cf. [4]). From Part (a) one has

$$\int_0^L \rho_k^m(x)|\hat{u}(x)|^2 dx \leq \frac{L^2}{\pi^2} \int_0^L \rho_k^m(x)|u'(x)|^2 dx$$

where $\hat{u}(x) = u(x) - \overline{u}$ and

$$\overline{u} = \frac{\int_0^L \rho_k^m(x) u(x) dx}{\int_0^L \rho_k^m(x)\, dx}.$$

Hence,

$$\int_0^L \rho_k^m(x)|u(x)|^2 dx \leq \frac{L^2}{\pi^2} \int_0^L \rho_k^m(x)|u'(x)|^2 dx + \overline{u} \int_0^L \rho_k^m(x) u(x)\, dx.$$

In the limit $k \to \infty$ we obtain (2.2).

(c) To see that the constant $\frac{L^2}{\pi^2}$ is optimal, choose $\rho^m \equiv 1$, $L = 1$ and $u(x) = \cos(\pi x)$. Then $\int_0^1 \rho^m(x) u(x)\, dx = 0$ and

$$\frac{\int_0^1 \rho^m(x)|u(x)|^2 dx}{\int_0^1 \rho^m(x)|u'(x)|^2 dx} = \frac{1}{\pi^2} \frac{\int_0^1 \cos^2(\pi x)\, dx}{\int_0^1 \sin^2(\pi x)\, dx} = \frac{1}{\pi^2}.$$

Thus the lemma is proved ∎

3. The n-dimensional case

In the rest of this article we will consider the case $n \geq 2$. By the following lemma we are able to reduce the n-dimensional problem to the one-dimensional case.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain with diameter d. Assume that $u \in L^1(\Omega)$ satisfies $\int_\Omega u(x)\, dx = 0$. Then for any $\delta > 0$ there are disjoint convex domains Ω_i $(i = 1, \ldots, k)$ such that*

$$\overline{\Omega} = \bigcup_{i=1}^k \overline{\Omega}_i, \qquad \int_{\Omega_i} u(x)\, dx = 0$$

and for each Ω_i there is rectangular coordinate system such that

$$\Omega_i \subset \Big\{(x,y) \in \mathbb{R}^n : 0 \le x \le d \text{ and } |y_j| \le \delta, j = 1, \dots, n-1\Big\}.$$

Proof. For each $\alpha \in [0, 2\pi]$ there is a unique hyperplane $H_\alpha \subset \mathbb{R}^n$ with normal $\big(0,\dots,0,\cos(\alpha),\sin(\alpha)\big)$ that divides Ω into two convex sets Ω'_α and Ω''_α of equal volume. Since $I(\alpha) = -I(\alpha+\pi)$, where $I(\alpha) = \int_{\Omega'_\alpha} u(x)\,dx$, by continuity there is α_0 such that $I(\alpha_0) = 0$. Applying this procedure recursively to each of the parts Ω'_{α_0} and Ω''_{α_0}, we are able to subdivide Ω into convex sets Ω_i such that each of the sets is contained between two parallel hyperplanes with normal of the form $\big(0,\dots,0,\cos(\beta),\sin(\beta)\big)$ at distance at most δ, and the average of u vanishes on each of them.

Consider one of these sets. By rotating the coordinate system we can assume that the normal of the enclosing hyperplanes is $(0,\dots,0,1)$. In these coordinates we apply the above arguments using hyperplanes with normals of the form $\big(0,\dots,0,\cos(\alpha),\sin(\alpha),0\big)$. Continuing this procedure we end up with the desired decomposition of Ω ∎

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain with diameter d. Then*

$$\|u\|_{L^2(\Omega)} \le \frac{d}{\pi}\|\nabla u\|_{L^2(\Omega)}$$

for all $u \in H^1(\Omega)$ satisfying $\int_\Omega u(x)\,dx = 0$.

Proof. Let us first assume that u is twice continuously differentiable. According to the previous Lemma 3.1 we are able to decompose Ω into convex subsets Ω_i such that for each Ω_i there is a rectangular coordinate system in which Ω_i is contained in

$$\Big\{(x,y) \in \mathbb{R}^n : 0 \le x \le d_i \text{ and } |y_j| \le \delta, j = 1, \dots, n-1\Big\}.$$

We may assume that the interval $[0, d_i]$ on the x-axis is contained in Ω_i. Let $R(t)$ be the $(n-1)$-volume of the intersection of Ω_i with the hyperplane $x = t$. In polar coordinates $R(t)$ can be written in the form

$$R(t) = \int_{\mathbb{S}^{n-2}} \int_0^{\rho(t,\omega)} r^{n-2} dr d\omega = \frac{1}{n-1} \int_{\mathbb{S}^{n-2}} \rho^{n-1}(t,\omega)\,d\omega$$

where $\rho(t,\omega)$ is the distance of the boundary point of Ω_i at (t,ω) to the x-axis. Since Ω_i is convex, ρ is a concave function with respect to t.

From the smoothness of u it can be seen that there are constants c_1, c_2 and c_3 such that

$$\left| \int_{\Omega_i} u(x,y)\, dxdy - \int_0^{d_i} u(x,0)R(x)\, dx \right| \le c_1 |\Omega_i| \delta \tag{3.2}$$

$$\left| \int_{\Omega_i} \left| \frac{\partial u}{\partial x}(x,y) \right|^2 - \int_0^{d_i} \left| \frac{\partial u}{\partial x}(x,0) \right|^2 R(x)\, dx \right| \le c_2 |\Omega_i| \delta \tag{3.3}$$

$$\left| \int_{\Omega_i} |u(x,y)|^2 dxdy - \int_0^{d_i} |u(x,0)|^2 R(x)\, dx \right| \le c_3 |\Omega_i| \delta. \tag{3.4}$$

Let $\omega \in \mathbb{S}^{n-2}$. Since $u(\cdot, 0) \in H^1(0, d_i)$, we can apply Lemma 2.1 to $\hat{u}_\omega(x) := u(x,0) - \overline{u}_\omega$, where

$$\overline{u}_\omega := \frac{\int_0^{d_i} u(x,0)\rho^{n-1}(x,\omega)\, dx}{\int_0^{d_i} \rho^{n-1}(x,\omega)\, dx}.$$

Hence,

$$\int_0^{d_i} |\hat{u}_\omega(x)|^2 \rho^{n-1}(x,\omega)\, dx \le \frac{d_i^2}{\pi^2} \int_0^{d_i} \left| \frac{\partial u}{\partial x}(x,0) \right|^2 \rho^{n-1}(x,\omega)\, dx.$$

Applying Fubini's theorem we obtain

$$\begin{aligned}
&\frac{d_i^2}{\pi^2} \int_0^{d_i} \left| \frac{\partial u}{\partial x}(x,0) \right|^2 R(x)\, dx \\
&\quad = \frac{1}{n-1} \int_{\mathbb{S}^{n-2}} \frac{d_i^2}{\pi^2} \int_0^{d_i} \left| \frac{\partial u}{\partial x}(x,0) \right|^2 \rho^{n-1}(x,\omega)\, dxd\omega \\
&\quad \ge \frac{1}{n-1} \int_{\mathbb{S}^{n-2}} \int_0^{d_i} |\hat{u}_\omega(x)|^2 \rho^{n-1}(x,\omega)\, dxd\omega \\
&\quad = \frac{1}{n-1} \int_{\mathbb{S}^{n-2}} \int_0^{d_i} \hat{u}_\omega(x) u(x,0) \rho^{n-1}(x,\omega)\, dxd\omega \\
&\quad \ge \int_0^{d_i} |u(x,0)|^2 R(x)\, dx - M \left| \int_0^{d_i} u(x,0)R(x)\, dx \right|
\end{aligned}$$

where $M = \max_{\omega \in \mathbb{S}^{n-2}} |\overline{u}_\omega|$. By $\int_{\Omega_i} u(x,y)\, dxdy = 0$, (3.2) and (3.3) we are lead to

$$\begin{aligned}
\int_0^{d_i} |u(x,0)|^2 R(x)\, dx &\le \frac{d_i^2}{\pi^2} \int_0^{d_i} \left| \frac{\partial u}{\partial x}(x,0) \right|^2 R(x)\, dx + c_1 |\Omega_i| M \delta \\
&\le \frac{d_i^2}{\pi^2} \int_{\Omega_i} |\nabla u(x,y)|^2 dxdy + \Big(c_1 M + c_2 \frac{d_i^2}{\pi^2} \Big) |\Omega_i| \delta.
\end{aligned}$$

From (3.4) and the summation over i we obtain

$$\int_\Omega |u(x,y)|^2 dxdy \leq \frac{d^2}{\pi^2}\int_\Omega |\nabla u(x,y)|^2 dxdy + \Big(c_1 M + c_2\frac{d^2}{\pi^2} + c_3\Big)|\Omega|\delta$$

and, since $\delta > 0$ is arbitrary, the desired estimate is proven. The assertion follows from the density of $C^\infty(\overline{\Omega})$ in $H^1(\Omega)$ ∎

Remark 3.3. In [5] it is claimed that R from (3.1) is a concave function, which is not true for $n \geq 3$ as can be seen from the following example. Let

$$\Omega = \Big\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \sqrt{x_2^2 + x_3^2} \leq \frac{1}{2}(x_1 + 1),\, 0 \leq x_1 \leq 1\Big\}.$$

Then Ω is convex, $\operatorname{vol} R(0) = \frac{\pi}{4}$ and $\operatorname{vol} R(1) = \pi$. But

$$\operatorname{vol} R\Big(\frac{1}{2}\Big) = \frac{9}{16}\pi < \frac{5}{8}\pi = \frac{1}{2}\operatorname{vol} R(0) + \frac{1}{2}\operatorname{vol} R(1).$$

Hence, R is not concave.

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