Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 22 (2003), No. 4, 757–765

Bounds for the Best Constant in an Improved Hardy-Sobolev Inequality

N. Chaudhuri

Abstract. In this note we show that the best constant C in the improved Hardy-Sobolev inequality of Adimurthi, Chaudhuri and Ramaswamy [1] for $2 \le p < n$ is bounded by $\frac{p-1}{p^2}\ \frac{n-p}{p}\ {}^{p-2} \le C \le \frac{p-1}{2}\ \frac{n-p}{p}\ {}^{p-2}$.

Keywords: *Hardy-Sobolev inequality, best constant in inequality*

AMS subject classification: 35P15, 35J20

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \ge 2)$ with $0 \in \Omega$. Adimurthi, Chaudhuri and Ramaswamy in [1] have obtained the following improved Hardy-Sobolev inequality. Let $1 < p < n$ and let $R \ge e^{\frac{2}{p}} \sup_\Omega |x|$. Then there exists a constant $C > 0$ such that

$$\int_\Omega |\nabla u|^p dx \ge \Big(\frac{n-p}{p}\Big)^p \int_\Omega \frac{|u|^p}{|x|^p}\, dx + C \int_\Omega \frac{|u|^p}{|x|^p} \Big(\log \frac{R}{|x|}\Big)^{-2} dx \qquad (1.1)$$

holds for all $u \in W_0^{1,p}(\Omega)$. In his book on *Sobolev Spaces* [14: Section 2.1.6] Maz'ja discovered that the classical multi-dimensional Hardy-type inequalities with sharp constant can be improved by adding different additional positive integrals. However, the above inequality have applications in proving existence, non-existence and regularity of solutions for differential equations involving the potential $\frac{1}{|x|^p}$ (see [1, 3, 10 - 12, 15]). Adimurthi and Esteban [2] extended the above inequality for $W^{1,p}$ functions and found interesting applications to the Schrödinger operator. However, finding the *best constant* in inequality (1.1) remains open. In this article we find interesting bounds for the best

N. Chaudhuri: Australian Nat. Univ., Centre for Math. and its Appl., Canberra, ACT 0200, Australia; chaudhur@maths.anu.edu.au

ISSN 0232-2064 / $ 2.50

constant $C(n,p,R,\Omega)$, defined in (1.4) below. In [1: Theorem 1.2] it has been shown that for $0<\mu<\left(\frac{n-p}{p}\right)^p$ the eigenvalue problem

$$\left.\begin{aligned} -\Big(\operatorname{div}(|\nabla u|^{p-2}\nabla u)+\frac{\mu}{|x|^p}|u|^{p-2}u\Big) &= \lambda\,\frac{|u|^{p-2}}{|x|^p\big(\log\frac{R}{|x|}\big)^2}\,u \quad \text{in } \Omega \\ u &= 0 \qquad\qquad \text{on } \partial\Omega \end{aligned}\right\} \tag{1.2}$$

admits a positive weak solution $u\in W_0^{1,p}(\Omega)$ corresponding to the eigenvalue $\lambda=\lambda_\mu^1>0$. Moreover, $\lambda_\mu^1\to C(n,p,R,\Omega)$ as $\mu\to\left(\frac{n-p}{p}\right)^p$. Thus the bounds on the best constant in inequality (1.1) gives bounds on the limiting behaviour of the first eigenvalue for the eigenvalue problem (1.2).

In [1], the following n-dimensional version of the Hardy-Sobolev inequality also has been established. For any bounded domain $\Omega\subset\mathbb{R}^n$ $(n\geq 2)$ with $0\in\Omega$,

$$\int_\Omega |\nabla u|^n dx \geq \Big(\frac{n-1}{n}\Big)^n \int_\Omega \frac{|u|^n}{|x|^n}\Big(\log\frac{R}{|x|}\Big)^{-n} dx \tag{1.3}$$

holds for every $u\in W_0^{1,n}(\Omega)$. Adimurthi and Sandeep [3] proved that the best constant herein is indeed $\left(\frac{n-1}{n}\right)^n$. For some interesting improvements of the classical Hardy-Sobolev inequality and their applications see [5 - 9, 13].

Before stating our theorem we define the *best constant* $C(n,p,R,\Omega)$ in inequality (1.1) by

$$C(n,p,R,\Omega)=\inf_{0\neq u\in W_0^{1,p}(\Omega)} Q_{\Omega,R}(u) \tag{1.4}$$

where

$$Q_{\Omega,R}(u)=\frac{\int_\Omega |\nabla u|^p dx-\left(\frac{n-p}{p}\right)^p\int_\Omega \frac{|u|^p}{|x|^p}dx}{\int_\Omega \frac{|u(x)|^p}{|x|^p}\left(\log\frac{R}{|x|}\right)^{-2}dx}.$$

It is also known (see [1]) that the best constant in $C(n,p,R,\Omega)$ is not achieved.

2. Result

In this article we prove the following

Theorem. *The constant* $C(n,p,R,\Omega)$ *defined by* (1.4) *is independent of the domain* Ω *and the choice of the constant* $R\geq e^{\frac{2}{p}}\sup_\Omega|x|$. *For* $2\leq p<n$,

$$\frac{p-1}{p^2}\Big(\frac{n-p}{p}\Big)^{p-2}\leq C(n,p)\leq \frac{p-1}{2}\Big(\frac{n-p}{p}\Big)^{p-2}.$$

It appears to the author that, for the case $2\leq p<n$, the constant $C(n,p)$ herein is indeed $\frac{p-1}{p^2}\left(\frac{n-p}{p}\right)^{p-2}$.

Proof of the theorem. We prove the independence and the bounds for the best constant through the following steps.

`Step 1.` We first prove that if B_i $(i = 1, 2)$ are concentric balls centered at origin of radii T_i, then $C(n, p, R_1, B_1) = C(n, p, R_2, B_2)$, where $R_i = \alpha T_i$ with $\alpha \geq e^{\frac{2}{p}}$. For this take $u \in W_0^{1,p}(B_2)$ and define $v(x) = u\big(\frac{T_2}{T_1}x\big)$ for $|x| < T_1$. Then

$$
\begin{aligned}
Q_{B_1,R_1}(v) &= \frac{\int_{B_1} |\nabla v|^p dx - (\frac{n-p}{p})^p \int_{B_1} \frac{|v|^p}{|x|^p} dx}{\int_{B_1} \frac{|v|^p}{|x|^p} \big(\log \frac{\alpha T_1}{|x|}\big)^{-2} dx} \\
&= \frac{\int_{B_2} |\nabla u|^p dx - (\frac{n-p}{p})^p \int_{B_2} \frac{|u|^p}{|x|^p} dx}{\int_{B_2} \frac{|u|^p}{|x|^p} \big(\log \frac{\alpha T_2}{|x|}\big)^{-2} dx} \\
&= Q_{B_2,R_2}(u)
\end{aligned}
$$

and hence $C(n, p, R_1, B_1) = C(n, p, R_2, B_2)$.

`Step 2.` Now we prove that $C(n, p, R, \Omega) = C(n, p, R, \Omega^*)$, where $\Omega^* = B(0, T)$ is the ball of radius $T = \big(\frac{|\Omega|}{|B(0,1)|}\big)^{1/n}$, $|\cdot|_n$ denoting the n-dimensional Lebesgue measure. Indeed, for any $u \in W_0^{1,p}(\Omega)$, $|u|^* \in W_0^{1,p}(\Omega^*)$, where $|u|^*$ is the symmetric decreasing rearrangement of $|u|$. By standard symmetrization arguments (see [4]) we conclude that, for any $u \in W_0^{1,p}(\Omega)$, $Q_{\Omega,R}(u) \geq Q_{\Omega^*,R}(u^*)$ and hence

$$
C(n, p, R, \Omega) \geq C(n, p, R, \Omega^*).
$$

To prove the other inequality, take $s > 0$ such that $B_s = B(0, s) \subseteq \Omega$. Then, clearly, $C(n, p, R, \Omega) \leq C(n, p, R, B_s)$ and hence, by Step 1, $C(n, p, R, \Omega) = C(n, p, R, \Omega^*)$.

Now if Ω_i $(i = 1, 2)$ are two bounded domains with $R_i \geq e^{\frac{2}{p}} \sup_{\Omega_i} |x|$, by Steps 1 and 2, $C(n, p, R_1, \Omega_1) = C(n, p, R_2, \Omega_2)$ and hence the constant is independent of the domain and the choice of R. We shall denote this constant simply by $C(n, p)$.

`Step 3.` `Lower Bound:` The lower bound for the best constant $C(n, p)$ essentially follows from [1: Proof of Theorem 1.1], but for sake of completeness we include a proof. Since $C(n, p)$ is independent of the domain, without loss of generality we assume Ω to be the unit ball $B = B(0, 1)$. Let $R \geq e^{\frac{2}{p}}$. For $0 < u \in C_0^2(B)$ radially non-increasing we define

$$
v(r) = u(r)\, r^{\frac{n-p}{p}} \qquad (r = |x|). \tag{2.1}
$$

Here without loss of generality we as well assume $u'(r) < 0$ (replacing u by $u + \varepsilon(1-r)$ for $\varepsilon > 0$ sufficiently small). Now we observe that

$$\begin{aligned}
&\int_B |\nabla u|^p dx - \Big(\frac{n-p}{p}\Big)^p \int_B \frac{|u(x)|^p}{|x|^p} dx \\
&\quad = \omega_n \int_0^1 \Big|\frac{n-p}{p} r^{-\frac{n}{p}} v(r) - r^{1-\frac{n}{p}} v'(r)\Big|^p r^{n-1} dr \\
&\quad\quad - \Big(\frac{n-p}{p}\Big)^p \omega_n \int_0^1 \frac{v^p(r)}{r}\, dr \\
&\quad = \omega_n \Big(\frac{n-p}{p}\Big)^p \int_0^1 v^p(r) \Big\{\Big|1 - \frac{pv'(r)r}{(n-p)v(r)}\Big|^p - 1\Big\} \frac{dr}{r}
\end{aligned}$$

where ω_n is the volume of the $(n-1)$-dimensional sphere. Since u is a decreasing function, from (2.1) we have $v'(r) - \frac{(n-p)v(r)}{pr} < 0$. We set $x(r) = -\frac{pv'(r)r}{(n-p)v(r)}$ so that $x(r) > -1$. By using the inequality $(1+x)^p \geq 1 + px + (p-1)x^2$ for all $x \geq -1$ and for all $p \geq 2$ we obtain

$$\begin{aligned}
&\int_B |\nabla u|^p - \Big(\frac{n-p}{p}\Big)^p \int_B \frac{|u(x)|^p}{|x|^p} \\
&\quad \geq \omega_n (p-1) \Big(\frac{n-p}{p}\Big)^{p-2} \int_0^1 v^{p-2}(r) |v'(r)|^2 r dr \\
&\quad\quad - \omega_n p \Big(\frac{n-p}{p}\Big)^{p-1} \int_0^1 v^{p-1}(r) v'(r) dr \\
&\quad = \frac{4\omega_n (p-1)}{p^2} \Big(\frac{n-p}{p}\Big)^{p-2} \int_0^1 \big|(v^{p/2}(r))'\big|^2 r\, dr
\end{aligned}$$

since $v \in C_0^1(0,T)$. By applying the n-dimensional Hardy inequality (1.3) with $n = 2$ for the function $v^{\frac{p}{2}}$, we obtain

$$\begin{aligned}
\int_0^1 \big|(v^{p/2}(r))'\big|^2 r\, dr &\geq \frac{1}{4} \int_0^1 \Big(\frac{v^{p/2}(r)}{r \log \frac{R}{r}}\Big)^2 r\, dr \\
&= \frac{1}{4} \int_0^1 \frac{u^p(r)}{r^p} \Big(\log \frac{R}{r}\Big)^{-2} r^{n-1} dr \\
&= \frac{1}{4\omega_n} \int_B \frac{|u(x)|^p}{|x|^p} \Big(\log \frac{R}{|x|}\Big)^{-2} dx.
\end{aligned}$$

Hence for all radially non-increasing functions $0 < u \in C_0^2(B)$ we have

$$\begin{aligned}
&\int_B |\nabla u|^p - \Big(\frac{n-p}{p}\Big)^p \int_B \frac{|u(x)|^p}{|x|^p} \\
&\quad \geq \frac{p-1}{p^2} \Big(\frac{n-p}{p}\Big)^{p-2} \int_B \frac{|u(x)|^p}{|x|^p} \Big(\log \frac{R}{|x|}\Big)^{-2} dx.
\end{aligned}$$

Now by standard approximation and symmetrization this inequality holds for all $u \in W_0^{1,p}(B)$ and hence $C(n,p) \geq \frac{p-1}{p^2}\left(\frac{n-p}{p}\right)^{p-2}$.

`Step 3.` `Upper Bound:` Here our idea is to construct a family of functions $\{u_{\varepsilon,k}\}_{0<\varepsilon<1}$ in $W_0^{1,p}(B)$, where $B = B(0,1)$ is the unit ball, and then to estimate $Q_{B,R}$ for this family. Similar to the family found in [1], for any $0 < \varepsilon < 1$ and for $2 \leq k \in \mathbb{N}$ we define

$$u_{\varepsilon,k}(r) = \begin{cases} 0 & \text{for } r \leq \varepsilon^k \\ \dfrac{\log \frac{r}{\varepsilon^k}}{(k-1)r^{\frac{n-p}{p}} \log \frac{1}{\varepsilon}} & \text{for } \varepsilon^k \leq r \leq \varepsilon \\ \dfrac{\log \frac{1}{r}}{r^{\frac{n-p}{p}} \log \frac{1}{\varepsilon}} & \text{for } \varepsilon \leq r \leq 1. \end{cases}$$

Clearly, $u_{\varepsilon,k} \in W_0^{1,p}(B)$ is continuous and differentiable a.e., and its derivative is given by

$$u'_{\varepsilon,k}(r) = \begin{cases} 0 & \text{for } 0 \leq r \leq \varepsilon^k \\ \dfrac{1}{(k-1)r^{\frac{n}{p}} \log \frac{1}{\varepsilon}} \Big[1 - \dfrac{n-p}{p} \log \dfrac{r}{\varepsilon^k}\Big] & \text{for } \varepsilon^k \leq r \leq \varepsilon \\ -\dfrac{1}{r^{\frac{n}{p}} \log \frac{1}{\varepsilon}} \Big[1 + \dfrac{n-p}{p} \log \dfrac{1}{r}\Big] & \text{for } \varepsilon \leq r \leq 1. \end{cases}$$

Since $\varepsilon > 0$ is sufficiently small, after a change of variables and the use of Neumann series we have the estimates

$$\begin{aligned} \int_B |\nabla u_{\varepsilon,k}|^p dx &= \frac{\omega_n}{\left(\log \frac{1}{\varepsilon}\right)^p} \Bigg[\frac{1}{(k-1)^p} \int_{\varepsilon^k}^{\varepsilon} \Big| \frac{n-p}{p} \log \frac{r}{\varepsilon^k} - 1 \Big|^p \frac{dr}{r} \\ &\quad + \int_{\varepsilon}^{1} \Big| 1 + \frac{n-p}{p} \log \frac{1}{r} \Big|^p \frac{dr}{r} \Bigg] \\ &= \frac{\lambda_{n,p}\omega_n}{p+1} \log \frac{1}{\varepsilon} \Bigg[(k-1)\Big(1 - \frac{p}{(k-1)(n-p)\log \frac{1}{\varepsilon}}\Big)^{p+1} \\ &\quad + \Big(1 + \frac{p}{(n-p)\log \frac{1}{\varepsilon}}\Big)^{p+1} \Bigg] \\ &= \frac{\lambda_{n,p}\omega_n}{p+1} \log \frac{1}{\varepsilon} \Bigg[(k-1) - \frac{p(p+1)}{(n-p)\log \frac{1}{\varepsilon}} \\ &\quad + \frac{p(p+1)}{2(k-1)} \Big(\frac{p}{(n-p)\log \frac{1}{\varepsilon}} \Big)^2 \end{aligned}$$

$$+ O\Big(\frac{1}{(k-1)^2\big(\log\frac{1}{\varepsilon}\big)^3}\Big) + 1 + \frac{p(p+1)}{(n-p)\log\frac{1}{\varepsilon}}$$
$$+ \frac{p(p+1)}{2}\Big(\frac{p}{(n-p)\log\frac{1}{\varepsilon}}\Big)^2 + O\Big(\frac{1}{\log\frac{1}{\varepsilon}}\Big)^3\Big]$$
$$= \frac{k\lambda_{n,p}\omega_n}{p+1}\log\frac{1}{\varepsilon} + \frac{kp\omega_n}{2(k-1)}\Big(\frac{n-p}{p}\Big)^{p-2}\Big(\log\frac{1}{\varepsilon}\Big)^{-1} \tag{2.2}$$
$$+ O\Big(\frac{1}{(k-1)\log\frac{1}{\varepsilon}}\Big)^2 + O\Big(\frac{1}{\log\frac{1}{\varepsilon}}\Big)^2.$$

Then we have

$$\begin{aligned}\int_B \frac{|u_{\varepsilon,k}|^p}{|x|^p}dx &= \frac{\omega_n}{\big(\log\frac{1}{\varepsilon}\big)^p}\Big[\frac{1}{(k-1)^p}\int_{\varepsilon^k}^{\varepsilon}\Big(\log\frac{r}{\varepsilon^k}\Big)^p\frac{dr}{r} + \int_{\varepsilon}^{1}\Big(\log\frac{1}{r}\Big)^p\frac{dr}{r}\Big] \\ &= \frac{\omega_n}{(p+1)\big(\log\frac{1}{\varepsilon}\big)^p}\Big[\frac{1}{(k-1)^p}\int_{\varepsilon^k}^{\varepsilon}\frac{d}{dr}\Big(\log\frac{r}{\varepsilon^k}\Big)^{p+1}dr \\ &\quad - \int_{\varepsilon}^{1}\frac{d}{dr}\Big(\log\frac{1}{r}\Big)^{p+1}dr\Big] \\ &= \frac{k\omega_n}{(p+1)}\log\frac{1}{\varepsilon}.\end{aligned} \tag{2.3}$$

Thus (2.2) - (2.3) yield

$$\begin{aligned}&\int_B|\nabla u_{\varepsilon,k}|^p - \Big(\frac{n-p}{p}\Big)^p\int_B\frac{|u_{\varepsilon,k}|^p}{|x|^p} \\ &\quad = \frac{kp\omega_n}{2(k-1)}\Big(\frac{n-p}{p}\Big)^{p-2}\Big(\log\frac{1}{\varepsilon}\Big)^{-1} + O\Big(\frac{1}{\log\frac{1}{\varepsilon}}\Big)^2.\end{aligned} \tag{2.4}$$

Finally, let us try to find a “good” estimate of the integral

$$\begin{aligned}I_p &= \int_B\frac{|u_{\varepsilon,k}|^p}{|x|^p}\Big(\log\frac{R}{|x|}\Big)^{-2}dx \\ &= \frac{\omega_n}{\big(\log\frac{1}{\varepsilon}\big)^p}\Big[\frac{1}{(k-1)^p}\int_{\varepsilon^k}^{\varepsilon}\frac{\big(\log\frac{r}{\varepsilon^k}\big)^p}{r\big(\log\frac{R}{r}\big)^2}dr + \int_{\varepsilon}^{1}\frac{\big(\log\frac{1}{r}\big)^p}{r\big(\log\frac{R}{r}\big)^2}dr\Big].\end{aligned}$$

By change of variable $r \mapsto \log\frac{R}{r}$ and denoting $a_\varepsilon = \log\frac{R}{\varepsilon}$, $b_\varepsilon = \log\frac{R}{\varepsilon^k}$ and $c = \log R$ we get

$$\begin{aligned}I_p &= \frac{\omega_n}{\big((k-1)\log\frac{1}{\varepsilon}\big)^p}\int_{a_\varepsilon}^{b_\varepsilon}\frac{\big(\log\frac{Re^{-r}}{\varepsilon^k}\big)^p}{r^2}dr \\ &\quad + \frac{\omega_n}{\big(\log\frac{1}{\varepsilon}\big)^p}\int_{c}^{a_\varepsilon}\frac{\big(\log\frac{e^r}{R}\big)^p}{r^2}dr \\ &=: I_p^1 + I_p^2.\end{aligned}$$

For the integrals I_p^1 and I_p^2 we get the estimations

$$\begin{aligned}
I_p^1 &= \int_{a_\varepsilon}^{b_\varepsilon} \Big(\log \frac{R}{\varepsilon^k} - r\Big)^p \frac{dr}{r^2} \\
&= b_\varepsilon^p \int_{a_\varepsilon}^{b_\varepsilon} \Big(1 - \frac{r}{b_\varepsilon}\Big)^p \frac{dr}{r^2} \\
&\geq b_\varepsilon^p \int_{a_\varepsilon}^{b_\varepsilon} \Big(1 - \frac{pr}{b_\varepsilon} + \frac{(p-1)r^2}{b_\varepsilon^2}\Big) \frac{dr}{r^2} \\
&= \frac{b_\varepsilon^p}{a_\varepsilon} \bigg[\Big(1 - \frac{a_\varepsilon}{b_\varepsilon}\Big)\Big(1 + (p-1)\frac{a_\varepsilon}{b_\varepsilon}\Big) - \frac{pa_\varepsilon}{b_\varepsilon} \log \frac{b_\varepsilon}{a_\varepsilon}\bigg]
\end{aligned}$$

and

$$\begin{aligned}
I_p^2 &= \int_c^{a_\varepsilon} (r - \log R)^p \frac{dr}{r^2} \\
&= \int_c^{a_\varepsilon} r^{p-2} \Big(1 - \frac{c}{r}\Big)^p dr \\
&\geq \int_c^{a_\varepsilon} r^{p-2} \Big(1 - \frac{pc}{r} + \frac{(p-1)c^2}{r^2}\Big) dr \\
&= \begin{cases}
a_\varepsilon \Big[\Big(1 - \frac{c}{a_\varepsilon}\Big) - 2\frac{c}{a_\varepsilon} \log \frac{a_\varepsilon}{c} + o(1)\Big] & \text{for } p = 2 \\
a_\varepsilon^2 \Big[\frac{1}{2}\Big(1 - \Big(\frac{c}{a_\varepsilon}\Big)^2\Big) + 2\Big(\frac{c}{a_\varepsilon}\Big)^2 \log \frac{a_\varepsilon}{c} + o(1)\Big] & \text{for } p = 3 \\
a_\varepsilon^{p-1} \Big[\frac{1}{p-1}\Big(1 - \Big(\frac{c}{a_\varepsilon}\Big)^{p-1}\Big) + o(1)\Big] & \text{for } p \neq 2, p \neq 3
\end{cases} \\
&= a_\varepsilon^{p-1} \Big[\frac{1}{p-1} + o(1)\Big]
\end{aligned}$$

where $o(1) \to 0$ as $\varepsilon \to 0$. From these estimations for I_p^1 and I_p^2 we obtain

$$\begin{aligned}
I_p &\geq J_{k,\varepsilon} \\
&:= \frac{\omega_n}{\big((k-1)\log \frac{1}{\varepsilon}\big)^p} \frac{b_\varepsilon^p}{a_\varepsilon} \bigg[\Big(1 - \frac{a_\varepsilon}{b_\varepsilon}\Big)\Big(1 + (p-1)\frac{a_\varepsilon}{b_\varepsilon}\Big) - \frac{pa_\varepsilon}{b_\varepsilon} \log \frac{b_\varepsilon}{a_\varepsilon}\bigg] \\
&\quad + \frac{\omega_n}{\big(\log \frac{1}{\varepsilon}\big)^p} a_\varepsilon^{p-1} \Big[\frac{1}{p-1} + o(1)\Big].
\end{aligned}$$

Hence from (2.4) we obtain

$$\begin{aligned}
Q_{B,R}(u_{\varepsilon,k}) \leq &\frac{pk}{2(k-1)} \Big(\frac{n-p}{p}\Big)^{p-2} \Big(\log \frac{1}{\varepsilon}\Big)^{p-1} \\
&\times \bigg[\frac{b_\varepsilon^p}{(k-1)^p a_\varepsilon} \Big\{\Big(1 - \frac{a_\varepsilon}{b_\varepsilon}\Big)\Big(1 + (p-1)\frac{a_\varepsilon}{b_\varepsilon}\Big)\Big\}
\end{aligned}$$

$$
\begin{aligned}
&+ a_\varepsilon^{p-1}\Big\{\frac{1}{p-1}+o(1)\Big\}\Bigg]^{-1} + J_{k,\varepsilon}^{-1}\Big[O\Big(\frac{1}{\log\frac{1}{\varepsilon}}\Big)^2\Big]\\
&= \frac{pk}{2(k-1)}\Big(\frac{n-p}{p}\Big)^{p-2}\\
&\quad\times\Bigg[\frac{(k-1)^{-p}b_\varepsilon^p}{a_\varepsilon\big(\log\frac{1}{\varepsilon}\big)^{p-1}}\Big\{\Big(1-\frac{a_\varepsilon}{b_\varepsilon}\Big)\Big(1+(1-p)\frac{a_\varepsilon}{b_\varepsilon}\Big)\Big\}\\
&\quad+\Big(\frac{a_\varepsilon}{\log\frac{1}{\varepsilon}}\Big)^{p-1}\Big\{\frac{1}{p-1}+o(1)\Big\}\Bigg]^{-1} + J_{k,\varepsilon}^{-1}\Big[O\Big(\frac{1}{\log\frac{1}{\varepsilon}}\Big)^2\Big].
\end{aligned}
$$

Here we note that $\frac{b_\varepsilon^p}{a_\varepsilon}\big(\log\frac{1}{\varepsilon}\big)^{p-1} \to k^p$ as $\varepsilon \to 0$ and hence $J_{k,\varepsilon}^{-1}\big[O\big(\frac{1}{\log\frac{1}{\varepsilon}}\big)^2\big] \to 0$ as either $\varepsilon \to 0$ or $k \to \infty$. Thus

$$
\begin{aligned}
Q_{B,R}(u_{\varepsilon,k}) &\to \frac{pk}{2(k-1)}\Big(\frac{n-p}{p}\Big)^{p-2}\\
&\quad\times\Bigg[\Big(\frac{k}{k-1}\Big)^p\Big\{\Big(1-\frac{1}{k}\Big)\Big(1+\frac{p-1}{k}\Big)\\
&\quad+\frac{p}{k}\log\frac{1}{k}\Big\}+\frac{1}{p-1}\Bigg]^{-1}\quad(\varepsilon\to 0)\\
&\to \frac{p}{2}\Big(\frac{n-p}{p}\Big)^{p-2}\Big[1+\frac{1}{p-1}\Big]^{-1}\quad(k\to\infty)\\
&= \frac{p-1}{2}\Big(\frac{n-p}{p}\Big)^{p-2}.
\end{aligned}
$$

Since $C(n,p) \le Q_{B,R}(u_{\varepsilon,k})$ for all $k \ge 2$ and for any sufficiently small $\varepsilon > 0$, by passing through the limits as $\varepsilon \to 0$ and $k \to \infty$ we get $C(n,p) \le \frac{p-1}{2}\big(\frac{n-p}{p}\big)^{p-2}$ and hence the theorem is proved ∎

Acknowledgements. I would like to thank the referee for pointing out the work of V. Maz'ja [14] on improved Hardy-Sobolev inequality.

References

[1] Adimurthi, Chaudhuri, N. and M. Ramaswamy: *An improved Hardy-Sobolev inequality and its application.* Proc. Amer. Math. Soc. 130 (2002), 489 – 505.

[2] Adimurthi and M. J. Esteban: *An improved Hardy-Sobolev inequality in $W^{1,p}$ and its applications to Schrödinger operator.* Preprint 2002.

[3] Adimurthi and K. Sandeep: *Existence and non-existence of the first eigenvalue of the perturbed Hardy-Sobolev operator.* Proc. Roy. Soc. Edinburgh (Sect. A) 132 (2002), 1021 – 1043.

[4] Bandle, C.: *Isoperimetric Inequalities and Applications.* London: Pitman 1980.

[5] Brézis, H. and M. Marcus: *Hardy's inequality revisited.* Ann. Scuola. Norm. Sup. Pisa 25 (1997), 217 – 237.

[6] Brézis, H., Marcus, M. and I. Shafrir: *Extremal functions for Hardy's inequality with weight.* J. Funct. Anal. 171 (2000), 177 – 191.

[7] Brézis, H. and J. L. Vázquez: *Blowup solutions of some nonlinear elliptic problems.* Revista Mat. Univ. Complutense Madrid 10 (1997), 443 – 469.

[8] Cabré, X. and Y. Martel: *Weak eigenfunctions for the linearization of extremal elliptic problems.* J. Funct. Anal. 156 (1998), 30 – 56.

[9] Cabré, X. and Y. Martel: *Existence versus instantaneous blowup for linear heat equations with singular potentials.* C.R. Acad. Sci. Paris 329 (1999), 973 – 978.

[10] Chaudhuri, N.: *On improved Hardy-Sobolev inequality and its application to certain singular problems.* Ph.D. Thesis. Bangalore (India): Indian Inst. Sci./Dept. Math. 2001.

[11] Chaudhuri, N. and Ramaswamy: *Existence of positive solutions of some semilinear elliptic equations with singular coefficients.* Proc. Roy. Soc. Edinburgh (Sect. A) 131 (2001), 1275 – 1295.

[12] Chaudhuri, N. and K. Sandeep: *On a heat problem involving perturbed Hardy-Sobolev operator.* Preprint 2001.

[13] Filippas, S. and A. Tertikas: *Optimizing improved Hardy inequalities.* J. Funct. Anal. 192 (2002), 186 – 233.

[14] Maz'ja, V. G.: *Sobolev Spaces.* Berlin et al.: Springer-Verlag 1985.

[15] Sandeep, K.: *On the first eigenfunction of a perturbed Hardy-Sobolev operator.* Preprint 2001.

Received 25.03.2003; in revised form 25.08.2003