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Multiobjective Duality for Convex Semidefinite Programming Problems

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Abstract. We treat some duality assertions regarding multiobjective convex semidefinite programming problems. Having a vector minimization problem with convex entries in the objective vector function, we establish a dual for it using the so-called conjugacy approach. In order to deal with the duality assertions between these problems we need to study the duality properties and the optimality conditions of the scalarized problem associated to the initial one. Using these results we present the weak, strong and converse duality assertions regarding the primal problem and the dual we obtained for it.

Keywords: *Multiobjective duality, semidefinite programming, convex optimization, Pareto efficiency*

AMS subject classification: Primary 49N15, secondary 90C22, 90C25, 90C29

1. Introduction

This paper presents some duality assertions regarding the multiobjective semidefinite programming problems. The duality model we are considering here has been introduced by W. Fenchel and R.T. Rockafellar and it consists in attaching to an optimization problem another problem, called its dual, by means of perturbation functions. This dual problem is important, because its solutions may reveal us in certain conditions the solutions of the initial problem. More on this subject may be found in [2, 6, 10, 11].

We deal further with semidefinite programming problems, namely, optimization problems with positive semidefinite constraints. The duality for the

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single objective linear case has already been presented in many papers, such as [1, 4, 5, 8, 9].

The word "multiobjective" appears in the title because we consider multiple vector valued objective functions. We treat the duality properties of these multiobjective functions considering the so-called Pareto efficiency. The approach we use to treat the multiobjective dual problems has been introduced in the articles [10, 11].

We begin with a problem of minimizing a vector function with convex entries subject to positive semidefinite inequality constraints. This vector minimization is considered using the so-called Pareto efficiency and proper efficiency, whose definitions are reminded here. Then we take the scalarized problem associated to it and we calculate its dual. From the duality assertions and optimality conditions, obtained further, we are able to extract duality properties regarding the primal multiobjective semidefinite optimization problem and its dual. Next we present the weak, strong and converse duality assertions regarding these problems. Finally, we derive as special cases the dual problems of the multiobjective semidefinite programming problem with linear objective function and of the multiobjective fractional programming problem with linear inequality constraints. The last one is presented also as a special case of the problem treated in [12].

2. Problem formulation

Let us consider the following multiobjective semidefinite programming problem with convex objective functions and convex constraints

(P) $v\text{-}\min_{x \in \mathcal{X}} f(x)$

where

$f = (f_1, ..., f_k)^T$

$\mathcal{X} = \left\{ x = (x_1, ..., x_m) \in \mathbb{R}^m : F(x) \geq_{\mathcal{S}^n_+} 0 \right\}$

$F(x) = F_0 + \sum_{i=1}^m x_i \cdot F_i.$

For each j, $f_j : \mathbb{R}^m \to \mathbb{R}$ is a real-valued convex function and also, for each i, $F_i \in \mathcal{S}^n$. Here we have denoted by $\mathcal{S}^n$ the linear subspace of the symmetric $n \times n$ matrices with real entries, i.e.

$$\mathcal{S}^n = \left\{ A \in \mathbb{R}^{n \times n} \mid A = A^T \right\},$$

and by S^n_+ the cone of the symmetric positive semidefinite $n \times n$ matrices with real entries, i.e.

$$\mathcal{S}^n_+ = \left\{ A \in \mathcal{S}^n \mid x^T \cdot A \cdot x \geq 0 \quad (x \in \mathbb{R}^n) \right\}$$

which introduces the so-called Löwner partial order $A \geq_{\mathcal{S}^n_+} B$ if and only if $A - B \in \mathcal{S}^n_+$ on $\mathcal{S}^n$. So our constraint $F(x) \geq_{\mathcal{S}^n_+} 0$ means actually that $F(x)$ is a symmetric positive semidefinite matrix. Also, on $\mathcal{S}^n_+$ we consider the scalar product from $\mathcal{S}^n$

$$\langle A, B \rangle = \sum_{i,j=1}^{n} A_{ij} \cdot B_{ij} = Tr\left(A^T \cdot B\right)$$

where $Tr(A)$ denotes the trace of the matrix A and "$\cdot$ " is the well-known product of matrices.

To deal with the dual properties of problem (P), using the method introduced in [10], we need to reformulate the feasible set by introducing a new function

$$g : \mathbb{R}^m \to \mathcal{S}^n, \qquad g(x) = -F_0 + \sum_{i=1}^{m} x_i \cdot (-F_i).$$

In this circumstance, the feasible set of problem (P) may be written as $\mathcal{X} = \left\{x \in \mathbb{R}^m : g(x) \leq_{\mathcal{S}^n_+} 0\right\}$.

There are several notions of solutions for this type of problems, but we use here so-called Pareto efficient and properly efficient solutions. Let us remind these notions.

Definition 1. With respect to problem (P), an element $\bar{x} \in \mathcal{X}$ is said to be

- *Pareto efficient* if $f(x) \leq_{\mathbb{R}^k_+} f(\bar{x})$ for $x \in \mathcal{X}$ implies $f(x) = f(\bar{x})$

- *properly efficient* if there exists $\lambda = (\lambda_1, \ldots, \lambda_k)^T \in \operatorname{int}(\mathbb{R}^k_+)$ such that $\sum_{i=1}^{k} \lambda_i f_i(\bar{x}) \leq \sum_{i=1}^{k} \lambda_i f_i(x)$ for all $x \in \mathcal{X}$.

Remark 1. We denote by "$\leq_{\mathbb{R}^k_+}$" the partial ordering induced by the non-negative orthant $\mathbb{R}^k_+ = \left\{x = (x_1, \ldots, x_k)^T \colon x_1, \ldots, x_k \geq 0\right\}$ on $\mathbb{R}^k$. Hence $\operatorname{int}(\mathbb{R}^k_+) = \left\{\lambda = (\lambda_1, ..., \lambda_k)^T \in \mathbb{R}^k_+ \colon \lambda_1, \ldots, \lambda_k > 0\right\}$.

Remark 2. A properly efficient element is also a Pareto efficient one, with respect to optimization problem (P).

3. The scalarized problem

In order to deal with the properly efficient solutions of problem (P) we consider the scalarized problem attached to it

(P$_\lambda$) $\inf_{x\in\mathcal{X}} \sum_{i=1}^{k} \lambda_i f_i(x)$

where $\lambda = (\lambda_1, \ldots, \lambda_k)^T \in \operatorname{int}(\mathbb{R}^k_+)$ and then study its duality properties according to the mentioned approach. For this, let us consider first the general semidefinite optimization problem

(P$_g$) $\inf_{x\in\mathcal{X}} \tilde{f}(x)$

where $\tilde{f} : \mathbb{R}^m \to \mathbb{R}$ is a convex function. To obtain the desired dual for problem (P$_g$) we use the method described in [10]. So let us consider the perturbation function

$$\Phi : \mathbb{R}^m \times \mathbb{R}^m \times \mathcal{S}^n \to \overline{\mathbb{R}}, \quad \Phi(x,p,Q) = \begin{cases} \tilde{f}(x+p) & \text{if } g(x) \leq_{\mathcal{S}^n_+} Q \\ +\infty & \text{otherwise.} \end{cases}$$

Its conjugate function Φ^* is given by

$$\begin{aligned} \Phi^*(x^*,p^*,Q^*) &= \sup_{\substack{x,p\in\mathbb{R}^m \\ g(x)-Q\leq_{\mathcal{S}^n_+} 0}} \Big\{ \langle x^*, x\rangle + \langle p^*, p\rangle + \langle Q^*, Q\rangle - \Phi(x,p,Q) \Big\} \\ &= \sup_{\substack{x,p\in\mathbb{R}^m \\ g(x)-Q\leq_{\mathcal{S}^n_+} 0}} \Big\{ \langle x^*, x\rangle + \langle p^*, p\rangle + \langle Q^*, Q\rangle - \tilde{f}(x+p) \Big\}. \end{aligned}$$

It is well-known that the space $\mathcal{S}^n$ is self-dual, i.e. $(\mathcal{S}^n)^* = \mathcal{S}^n$. In [13] there is proved that the cone $\mathcal{S}^n_+$ is also self-dual, i.e. $\left(\mathcal{S}^n_+\right)^* = \mathcal{S}^n_+$, a property will be used later.

The dual of problem (P$_g$) is obtained (cf. [2]) calculating the expression

(P$^*_g$) $\sup_{\substack{p^*\in\mathbb{R}^m \\ Q^*\in\mathcal{S}^n}} \big\{ -\Phi^*(0,p^*,Q^*) \big\}$.

To ease our calculation we introduce the new variables

$$\left.\begin{aligned} r &= x+p \\ S &= Q - g(x) \end{aligned}\right\}.$$

The expression of the conjugate function Φ^* of Φ becomes

$$\begin{aligned} \Phi^*(x^*,p^*,Q^*) &= \sup_{\substack{x,r\in\mathbb{R}^m \\ S\geq_{\mathcal{S}^n_+} 0}} \Big\{ \langle x^*, x\rangle + \langle p^*, r-x\rangle + \langle Q^*, S+g(x)\rangle - \tilde{f}(r) \Big\} \\ &= \sup_{S\geq_{\mathcal{S}^n_+} 0} \langle Q^*, S\rangle + \sup_{r\in\mathbb{R}^m} \big\{ \langle p^*, r\rangle - \tilde{f}(r) \big\} \\ &\quad + \sup_{x\in\mathbb{R}^m} \big\{ \langle x^*, x\rangle - \langle p^*, x\rangle + \langle Q^*, g(x)\rangle \big\}. \end{aligned} \tag{3.1}$$

As required above, taking $x^* = 0$ we get

$$\Phi^*(0, p^*, Q^*) = \sup_{S \geq_{\mathcal{S}^n_+} 0} \langle Q^*, S \rangle + \tilde{f}^*(p^*) + \sup_{x \in \mathbb{R}^m} \big\{ \langle Q^*, g(x) \rangle - \langle p^*, x \rangle \big\}$$

where $\tilde{f}^*(p^*) = \sup_{r \in \mathbb{R}^m} \{ \langle p^*, r \rangle - \tilde{f}(r) \}$ is the conjugate function of $\tilde{f}$ at p^*. It follows

$$\begin{aligned}
\Phi^*(0, p^*, Q^*) &= \sup_{S \geq_{\mathcal{S}^n_+} 0} \langle Q^*, S \rangle + \tilde{f}^*(p^*) \\
&\quad + \sup_{x \in \mathbb{R}^m} \left\{ - \sum_{i=1}^m x_i \langle Q^*, F_i \rangle - \langle Q^*, F_0 \rangle - \sum_{i=1}^m x_i p_i^* \right\} \\
&= \sup_{S \geq_{\mathcal{S}^n_+} 0} \langle Q^*, S \rangle + \tilde{f}^*(p^*) \\
&\quad + \sup_{x \in \mathbb{R}^m} \left\{ - \langle Q^*, F_0 \rangle - \sum_{i=1}^m x_i \left(\langle Q^*, F_i \rangle + p_i^* \right) \right\} \\
&= \sup_{S \geq_{\mathcal{S}^n_+} 0} \langle Q^*, S \rangle + \tilde{f}^*(p^*) - \langle Q^*, F_0 \rangle \\
&\quad + \sup_{x \in \mathbb{R}^m} \left\{ - \sum_{i=1}^m x_i \big(\langle Q^*, F_i \rangle + p_i^* \big) \right\}.
\end{aligned}$$

For the two suprema encountered above, we have

$$\sup_{S \geq_{\mathcal{S}^n_+} 0} \langle Q^*, S \rangle = \begin{cases} 0 & \text{if } \ Q^* \leq_{\mathcal{S}^n_+} 0 \\ +\infty & \text{otherwise} \end{cases}$$

and

$$\sup_{x \in \mathbb{R}^m} \left\{ - \sum_{i=1}^m x_i \big(\langle Q^*, F_i \rangle + p_i^* \big) \right\} = \begin{cases} 0 & \text{if } \langle Q^*, F_i \rangle + p_i^* = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

As the above infinite values are not relevant for our supremum problem (P_g^*), the dual problem becomes

$$(\mathbf{P}_g^*) \quad \sup_{\substack{Q^* \leq_{\mathcal{S}^n_+} 0,\, p^* = (p_1^*, \ldots, p_m^*) \in \mathbb{R}^m \\ p_i^* = -Tr(Q^* \cdot F_i) \ (1 \leq i \leq m)}} \big\{ - \tilde{f}^*(p^*) + \langle Q^*, F_0 \rangle \big\}$$

which, denoting $Q = -Q^*$, may be written after some transformations also as

$$(\mathbf{P}_g^*) \quad \sup_{Q \geq_{\mathcal{S}^n_+} 0} \Big\{ - \tilde{f}^* \big(Tr(Q \cdot F_1), \ldots, Tr(Q \cdot F_m) \big) - Tr(Q \cdot F_0) \Big\}.$$

Remark 3. (P_g^*) is the dual problem obtained using the conjugacy approach to our semidefinite optimization problem with general convex objective function (P_g). For $\tilde{f}(x) = \langle c, x\rangle \quad (x \in \mathbb{R}^m)$ it becomes

$$(\mathbf{P}_c^*) \qquad \sup_{\substack{Q \geq_{\mathcal{S}_+^n} 0 \\ Tr(Q\cdot F_i) = c_i \ (1\leq i\leq m)}} \big\{ -Tr(Q\cdot F_0)\big\}.$$

This is exactly the dual problem already obtained for the linear case in the literature (see [1, 8, 9, 13]).

So the dual to problem (P_λ) looks like

$$(\mathbf{P}_\lambda^*) \qquad \sup_{\substack{Q \geq_{\mathcal{S}_+^n} 0,\, p^* \in \mathbb{R}^m \\ p^* = (Tr(Q\cdot F_1),\ldots,Tr(Q\cdot F_m))}} \Big\{ -\big(\textstyle\sum_{i=1}^k \lambda_i f_i\big)^*(p^*) - Tr(Q\cdot F_0)\Big\}$$

which may be turned, using the formula (cf. [6])

$$\left(\sum_{i=1}^k \lambda_i f_i\right)^*(p^*) = \inf\left\{\sum_{i=1}^k (\lambda_i f_i)^*(\tilde{p}_i) \ : \ \sum_{i=1}^k \tilde{p}_i = p^*\right\},$$

into the problem

$$(\mathbf{P}_\lambda^*) \qquad \sup_{\substack{Q \geq_{\mathcal{S}_+^n} 0,\, \tilde{p}_i \in \mathbb{R}^m \\ \sum_{i=1}^k \tilde{p}_i = (Tr(Q\cdot F_1),\ldots,Tr(Q\cdot F_m))}} \Big\{ -\textstyle\sum_{i=1}^k (\lambda_i f_i)^*(\tilde{p}_i) - Tr(Q\cdot F_0)\Big\}.$$

Knowing that $(\lambda_i f_i)^*(\tilde{p}_i) = \lambda_i f_i^*\big(\frac{1}{\lambda_i}\tilde{p}_i\big)$ and denoting $p_i = \frac{1}{\lambda_i}\tilde{p}_i$, the dual problem may be simplified to

$$(\mathbf{P}_\lambda^*) \qquad \sup_{\substack{Q \geq_{\mathcal{S}_+^n} 0,\, p_i \in \mathbb{R}^m \\ \sum_{i=1}^k \lambda_i p_i = (Tr(Q\cdot F_1),\ldots,Tr(Q\cdot F_m))}} \Big\{ -\textstyle\sum_{i=1}^k \lambda_i f_i^*(p_i) - Tr(Q\cdot F_0)\Big\}.$$

4. Duality for the scalarized problem

The weak and strong duality assertions hold for the linear problem as it is proved in [9]. The scalarized problem we are currently treating is a natural extension of the linear problem, so similar duality properties are to be formulated for it. As the proof of the weak duality theorem is trivial we do not mention it here.

Theorem 1. *There holds weak duality between problems* (P_λ) *and* (P_λ^*)*, i.e.* $\inf(P_\lambda) \geq \sup(P_\lambda^*)$.

In order to prove the strong duality theorem we have to introduce the Slater Constraint Qualification

(SCQ) There exists $x' \in \mathbb{R}^m$ such that $F(x') >_{\mathcal{S}^n_+} 0$.

By " $>_{\mathcal{S}^n_+}$ " we have denoted the partial ordering on $\mathcal{S}^n$ introduced by the set of symmetric positive definite $n \times n$ matrices, which actually coincides with $\operatorname{int}(\mathcal{S}^n_+)$ (cf. [8]). We formulate the strong duality theorem for problem (P_g) from which we obtain the one regarding problem (P_λ).

Theorem 2. *Let be* $\inf(P_g) > -\infty$ *and condition* (SCQ) *be fulfilled. Then the dual problem* (P_g^*) *has an optimal solution and there is strong duality between problems* (P_g) *and* (P_g^*)*, i.e.* $\inf(P_g) = \max(P_g^*)$.

Proof. The convexity of f and g ensures the convexity of Φ. The constraint qualification (SCQ) being fulfilled, there exists $x' \in \mathbb{R}^m$ such that $F(x') \in \operatorname{int}(\mathcal{S}^n_+)$.

Next we prove that the function $\Phi(x', \cdot, \cdot)$ is continuous at $(0, 0)$. Proposition 2.3 and Theorem 4.1 in [2] imply in this case the existence of an optimal solution of problem (P_g^*) and state the equality of the optimal objective values of problems (P_g) and (P_g^*). Therefore let be $\varepsilon > 0$. The function $\tilde{f}$ being continuous over $\mathbb{R}^m$, there exists an open neighborhood V_1 of 0 in $\mathbb{R}^m$ such that, for all $p \in V_1$, $|\tilde{f}(x'+p) - \tilde{f}(x')| < \varepsilon$. Because $g(x') = -F(x') \in -\operatorname{int}(\mathcal{S}^n_+)$, there exists an open neighborhood $V_2 \subseteq \mathcal{S}^n$ of 0 such that, for all $Q \in V_2$,

$$g(x') \in Q - \mathcal{S}^n_+ \quad \Longleftrightarrow \quad g(x') \leq_{\mathcal{S}^n_+} Q.$$

Consider $V = V_1 \times V_2$, that is a neighborhood of $(0, 0)$ in $\mathbb{R}^m \times \mathcal{S}^n$. For all $(p, Q) \in V$ we have

$$\left|\Phi(x', p, Q) - \Phi(x', 0, 0)\right| = \left|\tilde{f}(x'+p) - \tilde{f}(x')\right| < \varepsilon$$

which actually means that $\Phi(x', \cdot, \cdot)$ is continuous at $(0, 0)$ ∎

Considering $\tilde{f} = \sum_{i=1}^{k} \lambda_i f_i$, we obtain the strong duality assertion for the scalarized problem.

Corollary 1. *If* $\inf(P_\lambda) > -\infty$ *and condition* (SCQ) *holds, then the dual problem* (P_λ^*) *has an optimal solution and there is strong duality between problems* (P_λ) *and* (P_λ^*)*, i.e.* $\inf(P_\lambda) = \max(P_\lambda^*)$.

Further we need also optimality conditions regarding problem (P_λ) and its dual problem (P_λ^*). So we formulate and prove the following

Theorem 3.

(a) *Let condition* (SCQ) *be fulfilled and let* $\bar{x} \in \mathcal{X}$ *be a solution to problem* (P_λ). *Then there exists an optimal solution* $(\bar{p}_1, ..., \bar{p}_k, \bar{Q})$ *to problem* (P_λ^*) *satisfying the optimality conditions*

(i) $f_i(\bar{x}) + f_i^*(\bar{p}_i) = \langle \bar{p}_i, \bar{x} \rangle \quad (i = 1, ..., k)$.

(ii) $Tr(\bar{Q} \cdot F(\bar{x})) = 0$.

(b) *Let $\bar{x} \in \mathcal{X}$ and $\left(\bar{p}_1, ..., \bar{p}_k, \bar{Q}\right)$ feasible to problem (P_λ^*) satisfying conditions* (i) - (ii). *Then $\bar{x}$ turns out to be an optimal solution to problem (P_λ), $\left(\bar{p}_1, ..., \bar{p}_k, \bar{Q}\right)$ an optimal solution to problem (P_λ^*) and the strong duality between problems (P_λ) and (P_λ^*) is true,*

$$\sum_{i=1}^{k} \lambda_i f_i(\bar{x}) = -\sum_{i=1}^{k} \lambda_i f_i^*(\bar{p}_i) - Tr(\bar{Q} \cdot F_0).$$

Proof. (a) As condition (SCQ) is fulfilled, there exists an optimal solution $(\bar{p}_1, ..., \bar{p}_k, \bar{Q})$ to problem (P_λ^*) such that the equality above holds. It may be also written as

$$\sum_{i=1}^{k} \lambda_i \big(f_i^*(\bar{p}_i) + f_i(\bar{x}) \big) + Tr(\bar{Q} \cdot F_0) = 0.$$

Adding and subtracting in the left-hand side the term $\langle \sum_{i=1}^{k} \lambda_i \bar{p}_i, \bar{x} \rangle$ we get

$$\sum_{i=1}^{k} \lambda_i \big(f_i^*(\bar{p}_i) + f_i(\bar{x}) - \langle \bar{p}_i, \bar{x} \rangle \big) + \left\langle \sum_{i=1}^{k} \lambda_i \bar{p}_i, \bar{x} \right\rangle + Tr(\bar{Q} \cdot F_0) = 0.$$

As

$$\left\langle \sum_{i=1}^{k} \lambda_i \bar{p}_i, \bar{x} \right\rangle = \Big\langle (Tr(\bar{Q} \cdot F_1), ..., Tr(\bar{Q} \cdot F_m)), \bar{x} \Big\rangle = \sum_{i=1}^{m} \bar{x}_i Tr(\bar{Q} \cdot F_i)$$

the previous relation becomes

$$\sum_{i=1}^{k} \lambda_i \big(f_i^*(\bar{p}_i) + f_i(\bar{x}) - \langle \bar{p}_i, \bar{x} \rangle \big) + \sum_{i=1}^{m} \bar{x}_i Tr(\bar{Q} \cdot F_i) + Tr(\bar{Q} \cdot F_0) = 0$$

which is equivalent to

$$\sum_{i=1}^{k} \lambda_i \big(f_i^*(\bar{p}_i) + f_i(\bar{x}) - \langle \bar{p}_i, \bar{x} \rangle \big) + \langle \bar{Q}, F(\bar{x}) \rangle = 0. \tag{4.1}$$

As $\bar{x} \in \mathcal{X}$, there follows $F(\bar{x}) \geq_{\mathcal{S}_+^n} 0$. Also, knowing that $\bar{Q} \geq_{\mathcal{S}_+^n} 0$, we have

$$\langle \bar{Q}, F(\bar{x}) \rangle \geq 0. \tag{4.2}$$

From Young's inequality it stands

$$f_i^*(\bar{p}_i) + f_i(\bar{x}) - \langle p_i, \bar{x}\rangle \geq 0 \qquad (i = 1, ..., k) \tag{4.3}$$

and, as $\lambda_i > 0$,

$$\sum_{i=1}^{k} \lambda_i \big(f_i^*(\bar{p}_i) + f_i(\bar{x}) - \langle \bar{p}_i, \bar{x}\rangle\big) \geq 0.$$

Therefore

$$\sum_{i=1}^{k} \lambda_i \big(f_i^*(\bar{p}_i) + f_i(\bar{x}) - \langle \bar{p}_i, \bar{x}\rangle\big) + \langle \bar{Q}, F(\bar{x})\rangle \geq 0. \tag{4.4}$$

By (4.1) there follows that the inequalities encountered in (4.2) - (4.3) must be fulfilled as equalities. So optimality conditions (i) and (ii) are verified.

(b) All the calculations from part (a) may be carried out in the reverse direction ∎

5. The multiobjective dual problem

Now we are ready to introduce the multiobjective dual problem (D) to the primal problem (P)

$$\textbf{(D)} \quad v\text{-max}_{(p,Q,\lambda,t)\in\mathcal{Y}} \begin{pmatrix} -f_1^*(p_1) - \frac{1}{k\lambda_1} Tr(Q \cdot F_0) + t_1 \\ \vdots \\ -f_k^*(p_k) - \frac{1}{k\lambda_k} Tr(Q \cdot F_0) + t_k \end{pmatrix}$$

where

$p = (p_1, ..., p_k), p_1, \ldots, p_k \in \mathbb{R}^m, Q \in \mathcal{S}^n$

$\lambda = (\lambda_1, ..., \lambda_k)^T, t = (t_1, ..., t_k)^T \in \mathbb{R}^k$

and

$$\mathcal{Y} = \left\{ (p, Q, \lambda, t) \left| \begin{array}{l} \lambda \in \operatorname{int}(\mathbb{R}^k_+),\ \sum_{i=1}^k \lambda_i t_i = 0,\ Q \geq_{\mathcal{S}^n_+} 0 \\ \sum_{i=1}^k \lambda_i p_i = \big(Tr(Q \cdot F_1), ..., Tr(Q \cdot F_m)\big) \end{array} \right. \right\}.$$

As (D) is a maximum vector optimization problem, we have to specify that we consider here the so-called Pareto efficiency in the sense of maximum to distinguish its solutions. We recall its definition.

Definition 2. An element $(\bar{p}, \bar{Q}, \bar{\lambda}, \bar{t}) \in \mathcal{Y}$ is said to be *Pareto efficient* for problem (D) if, for each $j = 1, ..., k$ and $(p, Q, \lambda, t) \in \mathcal{Y}$, from

$$-f_j^*(\bar{p}_j) - \frac{1}{k\bar{\lambda}_j} Tr(\bar{Q} \cdot F_0) + \bar{t}_j \leq -f_j^*(p_j) - \frac{1}{k\lambda_j} Tr(Q \cdot F_0) + t_j$$

there follows equality therein.

Further, we formulate and prove the weak duality assertion for multiobjective problems (P) and (D).

Theorem 4. *There is no $x \in \mathcal{X}$ and no $(p, Q, \lambda, t) \in \mathcal{Y}$ such that*

$$f_i(x) \leq -f_i^*(p_i) - \frac{1}{k\lambda_i} Tr(Q \cdot F_0) + t_i \qquad (i = 1, ..., k)$$

and, for at least one $j \in \{1, ..., k\}$, we have therein strong inequality.

Proof. Let us assume the contrary, i.e. that there exist some x and (p, Q, λ, t) feasible to our problems fulfilling the conditions mentioned above. Then assembling the relations given in the hypothesis we obtain

$$\sum_{i=1}^{k} \lambda_i f_i(x) < \sum_{i=1}^{k} \lambda_i \Big(- f_i^*(p_i) - \frac{1}{k\lambda_i} Tr(Q \cdot F_0) + t_i \Big). \tag{5.1}$$

On the other hand,

$$\begin{aligned} &\sum_{i=1}^{k} \lambda_i \Big(- f_i^*(p_i) - \frac{1}{k\lambda_i} Tr(Q \cdot F_0) + t_i \Big) \\ &\qquad = - \sum_{i=1}^{k} \lambda_i f_i^*(p_i) - \sum_{i=1}^{k} \frac{\lambda_i}{k\lambda_i} Tr(Q \cdot F_0) + \sum_{i=1}^{k} \lambda_i t_i \\ &\qquad = - \sum_{i=1}^{k} \lambda_i f_i^*(p_i) - k\frac{1}{k} Tr(Q \cdot F_0) \\ &\qquad = - \sum_{i=1}^{k} \lambda_i f_i^*(p_i) - Tr(Q \cdot F_0). \end{aligned}$$

By Theorem 1 we know that

$$- \sum_{i=1}^{k} \lambda_i f_i^*(p_i) - Tr(Q \cdot F_0) \leq \sum_{i=1}^{k} \lambda_i f_i(x)$$

which implies the reverse inequality $\geq$ in (5.1) and our presumption is false ∎

Now we are ready to deal with strong duality between problems (P) and (D).

Theorem 5. *Let there exists an element* $x' \in \mathbb{R}^m$ *such that* $F(x') >_{\mathcal{S}^n_+} 0$. *If* $\bar{x}$ *is properly efficient to problem* (P)*, then there exists a Pareto-efficient solution* $(\bar{p}, \bar{Q}, \bar{\lambda}, \bar{t}) \in \mathcal{Y}$ *to problem* (D) *and strong duality between problems* (P) *and* (D) *is fulfilled, i.e.*

$$f_i(\bar{x}) = -f_i^*(\bar{p}_i) - \frac{1}{k\bar{\lambda}_i} Tr(\bar{Q} \cdot F_0) + \bar{t}_i \qquad (i = 1, ..., k).$$

Proof. The element $\bar{x}$ being properly efficient to problem (P) implies the existence of a $\bar{\lambda} \in \operatorname{int}(\mathbb{R}^m_+)$ such that $\bar{x}$ solves problem ($P_{\bar{\lambda}}$), i.e.

$$\sum_{i=1}^k \bar{\lambda}_i f_i(\bar{x}) = \min_{x \in \mathcal{X}} \sum_{i=1}^k \bar{\lambda}_i f_i(x).$$

Condition (SCQ) is fulfilled, so there exists an optimal solution to problem ($P^*_{\bar{\lambda}}$) satisfying the optimality conditions in Theorem 3. Let us denote it by $(\bar{p}_1, ..., \bar{p}_k, \bar{Q})$ and define

$$\bar{t}_j = \bar{p}_j^T \bar{x} + \frac{1}{k\bar{\lambda}_j} Tr(\bar{Q} \cdot F_0) \in \mathbb{R} \qquad (j = 1, ..., k).$$

It follows

$$\begin{aligned}
\sum_{j=1}^k \bar{\lambda}_j \bar{t}_j &= \sum_{j=1}^k \bar{\lambda}_j \bar{p}_j^T \bar{x} + \sum_{j=1}^k \bar{\lambda}_j \frac{1}{k\bar{\lambda}_j} Tr(\bar{Q} \cdot F_0) \\
&= \left\langle \sum_{j=1}^k \bar{\lambda}_j \bar{p}_j, \bar{x} \right\rangle + Tr(\bar{Q} \cdot F_0) \\
&= \sum_{j=1}^k \langle \bar{Q}, F_j \rangle \bar{x}_j + \langle \bar{Q}, F_0 \rangle \\
&= \langle \bar{Q}, F(\bar{x}) \rangle \\
&= 0.
\end{aligned}$$

So far we have proved that the element $(\bar{p}, \bar{Q}, \bar{\lambda}, \bar{t})$ belongs to the set $\mathcal{Y}$. Let us show the remaining requirement, namely that, for all $i = 1, ..., k$,

$$f_i(\bar{x}) = -f_i^*(\bar{p}_i) - \frac{1}{k\bar{\lambda}_i} Tr(\bar{Q} \cdot F_0) + \bar{t}_i.$$

According to Theorem 3/(i) we have

$$\begin{aligned}
&- f_i^*(\bar{p}_i) - \frac{1}{k\bar{\lambda}_i} Tr(\bar{Q}\cdot F_0) + \bar{t}_i \\
&\quad = -f_i^*(\bar{p}_i) - \frac{1}{k\bar{\lambda}_i} Tr(\bar{Q}\cdot F_0)\bar{p}_i^T\bar{x} + \frac{1}{k\bar{\lambda}_i} Tr(\bar{Q}\cdot F_0) \\
&\quad = -f_i^*(\bar{p}_i) + \bar{p}_i^T\bar{x} \\
&\quad = f_i(\bar{x}).
\end{aligned}$$

With the weak duality (cf. Theorem 4) there follows that $(\bar{p}, \bar{Q}, \bar{\lambda}, \bar{t})$ is Pareto efficient to problem (D) ∎

We can also formulate the converse duality theorem, whose proof is not mentioned here (cf. [10]).

Theorem 6. Assume that condition (SCQ) is fulfilled and that for each $\lambda \in int(\mathbb{R}^n_+)$ the property

$$\textbf{(C)}\quad \inf_{x\in\mathcal{X}} \sum_{i=1}^k \lambda_i f_i(x) > -\infty \implies \inf_{x\in\mathcal{X}} \sum_{i=1}^k \lambda_i f_i(x) = \sum_{i=1}^k \lambda_i f_i(x_\lambda) \quad \text{for some } x_\lambda \in \mathcal{X}$$

holds. Then:

(a) For any Pareto-efficient solution $(\bar{p}, \bar{Q}, \bar{\lambda}, \bar{t})$ of problem (D) we have

$$\begin{pmatrix} -f_1^*(\bar{p}_1) - \frac{1}{k\bar{\lambda}_1} Tr(\bar{Q}\cdot F_0) + \bar{t}_1 \\ \vdots \\ -f_k^*(\bar{p}_k) - \frac{1}{k\bar{\lambda}_k} Tr(\bar{Q}\cdot F_0) + \bar{t}_k \end{pmatrix} \in \mathrm{cl}\big(f(\mathcal{X}) + \mathbb{R}^k_+\big)$$

and there exists a properly efficient solution $\bar{x}_{\bar{\lambda}} \in \mathcal{X}$ to problem (P) such that

$$\sum_{i=1}^k \bar{\lambda}_i \Big[f_i(\bar{x}_{\bar{\lambda}}) + f_i^*(\bar{p}_i) + \frac{1}{k\bar{\lambda}_i} Tr(\bar{Q}\cdot F_0) - \bar{t}_i \Big] = 0.$$

(b) If, additionally, $f(\mathcal{X})$ is $\mathbb{R}^k_+$-closed (i.e. $f(\mathcal{X}) + \mathbb{R}^k_+$ is closed), then there exists an $\bar{x} \in \mathcal{X}$ properly efficient to problem (P) such that

$$\begin{aligned}
&\sum_{i=1}^k \bar{\lambda}_i f_i(\bar{x}_{\bar{\lambda}}) = \sum_{i=1}^k \bar{\lambda}_i f_i(\bar{x}) \\
&\qquad f_i(\bar{x}) = -f_i^*(\bar{p}_i) - \frac{1}{k\bar{\lambda}_i} Tr(\bar{Q}\cdot F_0) + \bar{t}_i.
\end{aligned}$$

6. Special cases

6.1 Special case I. Let us consider the initial vector minimization problem with the linear objective function $f = \big(\langle c_1, x\rangle, ..., \langle c_k, x\rangle\big)^T$. We have

(P$_l$) $v\text{-}\min_{x\in\mathcal{X}} \big(\langle c_1, x\rangle, \ldots, \langle c_k, x\rangle\big)^T.$

To be able to calculate the problem dual to problem (P) using the method presented before, we have to determine the conjugate function f_i^* of each of the linear functions $f_i(\cdot) = \langle c_i, \cdot\rangle$,

$$\begin{aligned} f_i^*(p_i) &= \sup_{x\in\mathbb{R}^m} \big\{\langle p_i, x\rangle - \langle c_i, x\rangle\big\} \\ &= \sup_{x\in\mathbb{R}^m} \big\{\langle p_i - c_i, x\rangle\big\} \\ &= \begin{cases} 0 & \text{if } p_i = c_i \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

By this, the dual of problem (P$_l$) looks like

$$\textbf{(D}_l\textbf{)} \quad v\text{-}\max_{(Q,\lambda,t)\in\mathcal{Y}_l} \begin{pmatrix} -\frac{1}{k\lambda_1} Tr(Q\cdot F_0) + t_1 \\ \vdots \\ -\frac{1}{k\lambda_k} Tr(Q\cdot F_0) + t_k \end{pmatrix}$$

where

$$\mathcal{Y}_l = \left\{ (Q,\lambda,t) \,\middle|\, \begin{array}{l} \lambda\in \operatorname{int}(\mathbb{R}^k_+),\ \sum_{i=1}^k \lambda_i t_i = 0,\ Q \geq_{\mathcal{S}^n_+} 0 \\ \sum_{i=1}^k \lambda_i c_i = \big(Tr(Q\cdot F_1), ..., Tr(Q\cdot F_m)\big) \end{array} \right\}.$$

Let us denote $d_i = t_i - \frac{1}{k\lambda_i} Tr(Q\cdot F_0)$. The condition $\sum_{i=1}^k \lambda_i t_i = 0$ becomes $\sum_{i=1}^k \lambda_i\big(d_i + \frac{1}{k\lambda_i} Tr(Q\cdot F_0)\big) = 0$, which implies

$$\sum_{i=1}^k \lambda_i d_i = -k\frac{1}{k} Tr(Q\cdot F_0) = -Tr(Q\cdot F_0).$$

So the dual of problem (P$_l$) is

(D$_l$) $v\text{-}\max_{(Q,\lambda,d)\in\mathcal{Y}_l} \big(d_1, \ldots, d_k\big)^T$

where

$$\mathcal{Y}_l = \left\{ (Q,\lambda,d) \,\middle|\, \begin{array}{l} \lambda\in \operatorname{int}(\mathbb{R}^k_+),\ \sum_{i=1}^k \lambda_i d_i = -Tr(Q\cdot F_0),\ Q \geq_{\mathcal{S}^n_+} 0 \\ \sum_{i=1}^k \lambda_i c_i = \big(Tr(Q\cdot F_1), ..., Tr(Q\cdot F_m)\big) \end{array} \right\}.$$

6.2 Special case II. The next optimization problem we treat is known as multiobjective fractional program with linear inequality constraints. A generalized case is presented in [12], while its single objective scalar case is mentioned in [9], where it is treated by means of semidefinite programming. For

$$(\mathbf{P}_f) \quad v\text{-}\min_{A\cdot x \leqq_{\mathbb{R}^p_+} b} \left(\frac{\langle c^1, x\rangle^2}{\langle d^1, x\rangle}, \ldots, \frac{\langle c^k, x\rangle^2}{\langle d^k, x\rangle} \right)^T$$

with $A = (a_{ij}) \in \mathbb{R}^{p\times m}, b \in \mathbb{R}^p$ and $c^j, d^j \in \mathbb{R}^m \quad (j = 1, ..., k)$ we assume that, for each feasible x, $\langle d^j, x\rangle > 0$.

To be able to deal with problem (P_f) within the framework of the present paper we reformulate it as a semidefinite programming problem. First, it is obvious that (P_f) may be written also as

$$(\mathbf{P}_f) \quad v\text{-} \min_{\substack{A\cdot x \leqq_{\mathbb{R}^p_+} b \\ \frac{\langle c^j, x\rangle^2}{\langle d^j, x\rangle} \leqq y_j \ (1\leqq j\leqq k)}} \left(y_1, \ldots, y_k\right)^T.$$

The system of constraints above is equivalent (cf. [3, 9, 13]) to the semidefiniteness of the matrix

$$F(x,y) = \left(\begin{array}{c|c|c|c} \mathrm{diag}(b - A\cdot x) & 0 & 0 & 0 \\ \hline 0 & H_1 & 0 & 0 \\ \hline 0 & 0 & \ddots & 0 \\ \hline 0 & 0 & 0 & H_k \end{array} \right)$$

with $H_j = \begin{pmatrix} y_j & \langle c^j, x\rangle \\ \langle c^j, x\rangle & \langle d^j, x\rangle \end{pmatrix} \quad (j = 1, ..., k)$. As $A\cdot x = \left(\sum_{i=1}^p a_{1i}x_i, \ldots, \sum_{i=1}^p a_{mi}x_i\right)^T$, $c^j = (c_1^j, ..., c_m^j)$ and $d^j = (d_1^j, ..., d_m^j)$, $F(x,y)$ may be written as sum of symmetric matrices

$$F(x,y) = \sum_{i=1}^m x_i \left(\begin{array}{c|cc|c|cc} \mathrm{diag}(-a_{1i}, ..., -a_{pi}) & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & c_i^1 & 0 & 0 & 0 \\ 0 & c_i^1 & d_i^1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \ddots & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & c_i^k \\ 0 & 0 & 0 & 0 & c_i^k & d_i^k \end{array} \right)$$

$$+\sum_{j=1}^{k} y_j \left(\begin{array}{c|c|c|c|c} 0 & & & & \\ \hline & \ddots & & & \\ \hline & & 1 & & \\ \hline & & & \ddots & \\ \hline & & & & 0 \end{array}\right) + \begin{pmatrix} \operatorname{diag}(b) & 0 \\ 0 & 0 \end{pmatrix} \geq_{\mathcal{S}_+^{p+2k}} 0.$$

Let us denote

- by F_i the matrix multiplied above by x_i $(i = 1, ..., m)$
- by F_{m+j} $(1 \leq j \leq k)$ the one multiplied above by y_j, namely, the $(p+2k) \times (p+2k)$ matrix with all entries equal to 0, but the one in the position $(p+2j-1, p+2j-1)$ whose value is 1, and
- by F_0 the last matrix.

One may notice that all the matrices encountered above are symmetric and problem (P_f) has been written in the same form as primal problem (P). In order to determine the dual of problem (P_f) we need to calculate the conjugates of the entries of the vectorial objective function. For the functions $f_j(x,y) = y_j$ $(j = 1, ..., k)$ the conjugates are

$$\begin{aligned} f_j^*(u,v) &= \sup_{x \in \mathbb{R}^m, y \in \mathbb{R}^k} \left\{ \langle u, x \rangle + \langle v, y \rangle - f_j(x,y) \right\} \\ &= \sup_{x \in \mathbb{R}^m, y \in \mathbb{R}^k} \left\{ \sum_{i=1}^{m} u_i x_i + \sum_{l=1}^{k} v_l y_l - y_j \right\} \\ &= \begin{cases} 0 & \text{if } u = 0, v_j = 1, v_l = 0, l \neq j \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

The previous results lead us to the dual to problem (P_f)

$$\textbf{(D}_f\textbf{)} \quad v\text{-}\max_{(Q,\lambda,t) \in \mathcal{Y}_f} \begin{pmatrix} -\frac{1}{k\lambda_1} Tr(Q \cdot F_0) + t_1 \\ \vdots \\ -\frac{1}{k\lambda_k} Tr(Q \cdot F_0) + t_k \end{pmatrix}$$

where

$$\mathcal{Y}_f = \left\{ (Q, \lambda, t) \left| \begin{array}{l} \lambda \in \operatorname{int}(\mathbb{R}_+^k),\ \sum_{i=1}^{k} \lambda_i t_i = 0,\ Q \geq_{\mathcal{S}_+^{p+2k}} 0,\ Tr(Q \cdot F_i) = 0 \\ (1 \leq i \leq m),\ Tr(Q \cdot F_{m+j}) = \lambda_j\ (1 \leq j \leq k) \end{array} \right. \right\}.$$

As the matrices F_i $(i = 0, ..., m+k)$ are known and Q's entries may be denoted by (q_{ij}) $(i, j = 1, ..., p+2k)$, we can develop a simpler shape of the dual problem. So let us calculate the values of the scalar products between Q and F_i $(i = 0, ..., m + k)$:

$$
\begin{aligned}
Tr(Q \cdot F_0) &= \sum_{i=1}^{p} q_{ii} b_i \\
Tr(Q \cdot F_i) &= \sum_{i=1}^{p} -a_{li} q_{ll} + 2 \sum_{j=1}^{k} q_{p+2j-1,p+2j} c_i^j + \sum_{i=1}^{p} q_{p+2j,p+2j} d_i^j \\
&\quad (i = 1, \ldots, m) \\
Tr(Q \cdot F_{m+j}) &= q_{p+2j-1,p+2j-1} \ (j = 1, ..., k).
\end{aligned}
$$

Because $Tr(Q \cdot F_{m+j}) = \lambda_j$, one has $\lambda_j = q_{p+2j-1,p+2j-1}$ $(j = 1, ..., k)$. So the variables λ_j may be eliminated from the dual problem whose form becomes

$(\mathbf{D}_f)$ $v\text{-max}_{(Q,t) \in \mathcal{Y}_f} \left(h_1(Q,t), \ldots, h_k(Q,t) \right)^T$

where

$$
h_j(Q,t) = -\frac{1}{k q_{p+2j-1,p+2j-1}} \sum_{i=1}^{p} q_{ii} b_i + t_j \qquad (j = 1, ..., k)
$$

and

$$
\mathcal{Y}_f = \left\{ (Q,t) \;\middle|\; \begin{array}{l} Q = (q_{ij}) \geq_{\mathcal{S}_+^{p+2k}} 0, \ \displaystyle\sum_{j=1}^{k} q_{p+2j-1,p+2j-1} t_j = 0 \\ q_{p+2j-1,p+2j-1} > 0 \ (j = 1, ..., k) \\ A^T \cdot (q_{11}, ..., q_{pp})^T = \\ 2 \displaystyle\sum_{j=1}^{k} q_{p+2j-1,p+2j} c^j + \sum_{i=1}^{k} q_{p+2j,p+2j} d^j \end{array} \right\}.
$$

In [12] there is obtained the dual to problem (P_f)

$(\mathbf{D}'_f)$ $v\text{-max}_{(\lambda,\delta,q^s,q^t) \in \mathcal{Y}'_f} h'(\lambda, \delta, q^s, q^t)$

where $h'(\lambda, \delta, q^s, q^t) = \big(-\langle \delta_1, b \rangle, \ldots, -\langle \delta_k, b \rangle \big)^T$ and

$$
\mathcal{Y}'_f = \left\{ (\lambda, \delta, q^s, q^t) \;\middle|\; \begin{array}{l} \lambda \in \operatorname{int}(\mathbb{R}_k^+), \ q^s, q^t \in \mathbb{R}_+^k, \ \delta = (\delta_1, ..., \delta_k), \ \delta_j \in \mathbb{R}^k \\ (q_j^s)^2 \leq 4 q_j^t \ (j = 1, ..., k), \ \displaystyle\sum_{j=1}^{k} \lambda_j \delta_j \geq_{\mathbb{R}_+^p} 0 \\ A^T \cdot \left(\displaystyle\sum_{j=1}^{k} \lambda_j \delta_j \right) + \displaystyle\sum_{j=1}^{k} \lambda_j (q_j^s c^j - q_j^t d^j) = 0 \end{array} \right\}.
$$

In order to find some connections between these two multiobjective dual problems we study the relation of inclusion between the image sets of their objective functions over the corresponding feasible sets. Therefore, let be $d \in h'(\mathcal{Y}'_f)$. So there exists a tuple $(\lambda, \delta, q^s, q^t) \in \mathcal{Y}'_f$ such that $d = h'(\lambda, \delta, q^s, q^t)$. Let us consider

$$q_{uv} = \begin{cases} (\sum_{j=1}^k \lambda_j \delta_j)_u & \text{if } u = v = 1, ..., p \\ \lambda_j & \text{if } u = v = p + 2j - 1 \\ \lambda_j q_j^t & \text{if } u = v = p + 2j \\ -\frac{\lambda_j q_j^s}{2} & \text{if } (u, v) = \begin{cases} (p + 2j, p + 2j - 1) \\ \text{or} \\ (p + 2j - 1, p + 2j) \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

where by $(a)_u$ we have denoted the u-th entry of the vector a. Also, let us introduce $t_j = -\langle \delta_j, b\rangle + \frac{1}{k\lambda_j}\langle \sum_{j=1}^k \lambda_j \delta_j, b\rangle$.

Using properties of positive semidefinite matrices (cf. [3, 9, 13]) one may notice that $Q = (q_{uv})_{u,v=1}^{p+2k} \in \mathcal{S}_+^{p+2k}$. On the other hand, for each $j = 1, ..., k$ we have $q_{p+2j-1,p+2j-1} > 0$. Simple calculations give the relations

$$\sum_{j=1}^k q_{p+2j-1,p+2j-1} t_j = 0$$

$$A^T \cdot (q_{11}, ..., q_{pp})^T - 2\sum_{j=1}^k q_{p+2j-1,p+2j} c^j - \sum_{i=1}^k q_{p+2j,p+2j} d^j = 0.$$

By these, $(Q, t) \in \mathcal{Y}_f$. For each component of the objective function $h(Q, t)$, we have

$$\begin{aligned} h_j(Q, t) &= -\frac{1}{kq_{p+2j-1,p+2j-1}} \langle (q_{11}, ..., q_{pp})^T, b\rangle + t_j \\ &= -\frac{1}{k\lambda_j}\left\langle \sum_{j=1}^k \lambda_j \delta_j, b \right\rangle - \langle \delta_j, b\rangle + \frac{1}{k\lambda_j}\left\langle \sum_{j=1}^k \lambda_j \delta_j, b \right\rangle \\ &= -\langle \delta_j, b\rangle. \end{aligned}$$

Hence $d = h(Q, t) \in h(\mathcal{Y}_f)$, which means that $h'(\mathcal{Y}'_f) \subseteq h_f(\mathcal{Y}_f)$. One may notice that the reverse inclusion does not hold.

A detailed analysis of the relations between different duals introduced in the literature to a general convex multiobjective problem will be given in a forthcoming paper.

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