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Absolutely Continuous Functions of Several Variables and Quasiconformal Mappings

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Abstract. We prove that functions with bounded n-variation and n-absolutely continuous functions of n-variables in the sense of [4] are stable under quasiconformal mappings. The class of quasiconformal mappings is the best possible since every homeomorphism which induces a bounded operator between BV^n spaces is a quasiconformal mapping.

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1. Introduction

Absolutely continuous functions of one variable are admissible transformations for the change of variables in Lebesgue integral. Recently J. Malý [6] introduced a class of n-absolutely continuous functions giving an n-dimensional analogue of the notion of absolute continuity from this point of view. We study a modified class of n-absolutely continuous functions suggested by Zajíček which was introduced in [4]. Our aim is to find the largest class of transformations which preserves n-absolute continuity.

Suppose that $\Omega \subset \mathbb{R}^n$ is an open set and $0 < \lambda < 1$. We say that a function $f : \Omega \to \mathbb{R}^m$ is n, λ-*absolutely continuous* if for each $\varepsilon > 0$ there is a $\delta > 0$ such that, for each disjoint finite family $\{B_i(x_i, r_i)\}$ of balls in Ω,

$$\sum_i \mathcal{L}_n(B_i) < \delta \quad \Longrightarrow \quad \sum_i \left(\operatorname{osc}_{B_i(x_i,\lambda r_i)} f\right)^n < \varepsilon.$$

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Absolute continuity from [6] coincides with $n,1$-absolute continuity. It is proved in [4, 6] that n,λ-absolute continuity implies continuity, weak differentiability with gradient in L^n, differentiability almost everywhere and a formula on change of variables.

It was shown by Csörnyei [1] that there exists a $2,1$-absolutely continuous function with respect to balls, which is not a function of this type with respect to cubes, where the concept in question is defined by an obvious modification to the definition given above. On the contrary, n,λ-absolute continuity does not depend on the shape of the "ball" in the definition for $0<\lambda<1$ (see [4] for details). The class of absolutely continuous functions also does not depend on the precise value of λ if $0<\lambda<1$ (see Theorem 3.5 below). From this point of view it is more natural to work with the new definition (i.e. with $0<\lambda<1$).

Given a measurable set $A\subset\mathbb{R}^n$ and a function $f: A\to\mathbb{R}^m$, we define the n,λ-*variation* of f on A by

$$V_\lambda^n(f,A)=\sup\left\{\sum_i\left(\operatorname{osc}_{B(x_i,\lambda r_i)}f\right)^n:\ \begin{array}{l}\{B(x_i,r_i)\}\text{ is a disjoint}\\ \text{finite family of balls in } A\end{array}\right\}.$$

We denote by $BV_\lambda^n(\Omega)$ the class of all functions such that $V_\lambda^n(f,\Omega)<\infty$, define the space $AC_\lambda^n(\Omega)$ as the family of all n,λ-absolutely continuous functions in $BV_\lambda^n(\Omega)$ and write $AC_{\lambda,\mathrm{loc}}^n$ for the class of all functions f such that $f\in AC_\lambda^n(K)$ for every compact set $K\subset\Omega$.

We prove in Section 3 that if $\Omega\subset\mathbb{R}^n$ $(n\geq 2)$ is an open set, $0<\lambda<1$ and $F:\Omega\to\mathbb{R}^n$ is a quasiconformal mapping, then

(i) $f\in BV_\lambda^n(\Omega) \iff f\circ F^{-1}\in BV_\lambda^n(F(\Omega))$

(ii) $f\in AC_\lambda^n(\Omega) \iff f\circ F^{-1}\in AC_\lambda^n(F(\Omega))$.

This extends the result from [4] where F was a bi-Lipschitz mapping. Note that the class AC_1^n is not stable even under bi-Lipschitz mappings (see [5] for details).

Using ideas from [2] we prove the following result in Section 4:

Let $0<\lambda\leq 1$ and $n\geq 2$. If a homeomorphism $F:\Omega\to\mathbb{R}^n$ induces a bounded operator from $BV_\lambda^n(F(\Omega))$ to $BV_\lambda^n(\Omega)$, then F is a quasiconformal mapping.

It follows that the results in Section 3 are sharp.

2. Preliminaries

Throughout the paper we consider an open set $\Omega \subset \mathbb{R}^n$ $(n > 1)$. We denote

- by $\mathcal{L}_n(A)$ or $|A|$ the n-dimensional Lebesgue measure of a set $A \subset \mathbb{R}^n$
- by λ a real number $0 < \lambda < 1$
- by $B(x, r)$ the n-dimensional Euclidean open ball with center x and diameter r (throughout the paper we use the letter B for balls only)
- by $\overline{B(x, r)}$ the corresponding closed ball
- $\lambda B = B(x, \lambda r)$ for a given ball $B = B(x, r)$
- by $S(x, r) = \{y \in \mathbb{R}^n : |x - y| = r\}$ a sphere
- by $\operatorname{osc}_A f$ the oscillation of $f : \Omega \to \mathbb{R}^m$ over the set $A \subset \Omega$, which is the diameter of $f(A)$
- by $F'(x)$ for a mapping $F : \Omega \to \mathbb{R}^n$ the Jacobi matrix of all partial derivatives of F at x
- by ∇F the weak (distributional) derivative
- by $J_F(x)$ the determinant of the Jacobi matrix of $F(x)$
- by $W^{1,p}(\Omega)$ and $W^{1,p}_{\mathrm{loc}}(\Omega)$ the Sobolev spaces.

A mapping $F : \Omega \to \mathbb{R}^n$ is called a *homeomorphism* if there exists its inverse F^{-1} and both F and F^{-1} are continuous. We write $f \circ F$ or $F^\star f$ for the composition of the functions $F : \Omega \to \mathbb{R}^n$ and $f : F(\Omega) \to \mathbb{R}^m$; that is $(f \circ F)(x) = (F^\star f)(x) = f(F(x))$ for every $x \in \Omega$. We say that a homeomorphism $F : \Omega \to \mathbb{R}^n$ induces a bounded operator $F^\star : BV^n_\lambda(F(\Omega)) \to BV^n_\lambda(\Omega)$ if there is a constant $C > 0$ such that $V^n_\lambda(F^\star f, \Omega) \le C\ V^n_\lambda(f, F(\Omega))$ for every $f \in BV^n_\lambda(F(\Omega))$.

We use the convention that C denotes a generic positive constant which may change from expression to expression.

3. Stability of AC^n_λ under quasiconformal mappings

In this section we will prove that classes of functions AC^n_λ and BV^n_λ are stable with respect to quasiconformal change of variables.

Definition 3.1. Let $1 \le K < \infty$. A mapping $F : \Omega \to \mathbb{R}^n$ is called K-*quasiconformal*, if it satisfies the following properties:

(i) F is a homeomorphism

(ii) $F \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$

(iii) $|\nabla F(x)|^n \le K|J_F(x)|$ for almost every $x \in \Omega$.

We say that a mapping F is *quasiconformal*, if there is $K < \infty$ such that f is K-quasiconformal.

For the history and basic properties of quasiconformal mappings we refer the reader to [8].

Definition 3.2. A function $F : \Omega \to \mathbb{R}^n$ is *η-quasisymmetric* if there is a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ such that, for every $a, b, x \in \Omega$ and $\rho \geq 0$,

$$|a - x| \leq \rho|b - x| \quad \Longrightarrow \quad |F(a) - F(x)| \leq \eta(\rho)|F(b) - F(x)|.$$

The following theorem [6: Theorem 2.4] states that quasiconformal mappings are locally quasisymmetric.

Theorem 3.3. *Suppose $n \geq 2$, $F : \Omega \to \mathbb{R}^n$ is a K-quasiconformal mapping and $x_0 \in \Omega, \alpha > 1, r > 0$ and $B(x_0, \alpha r) \subset \Omega$. Then $F|_{B(x_0,r)}$ is η-quasisymmetric where η depends only on n, K and α.*

Using this theorem for $\alpha = 2$ and a quasiconformal mapping $F : \Omega \to \mathbb{R}^n$, there is $0 < \rho_0 < 1$ such that, for a fixed $x \in \Omega$ and $r < \frac{\rho_0}{2}\operatorname{dist}(x, \partial\Omega)$,

$$\sup_{\{a:|x-a|\leq r\}} |F(x) - F(a)| \leq \frac{1}{4} \inf_{\{b:|x-b|=\frac{r}{\rho_0}\}} |F(x) - F(b)|. \tag{3.1}$$

Lemma 3.4. *Let $0 < \lambda \leq 1$, $f \in BV^n_\lambda(\Omega)$ and $f \in AC^n_{\lambda,\mathrm{loc}}(\Omega)$. Then $f \in AC^n_\lambda(\Omega)$.*

Proof. Fix $\varepsilon > 0$. It is not difficult to see from the definition of n, λ-variation that we can find a finite collection of pairwise disjoint balls $B(x_i, r_i)$ such that $\overline{B(x_i, r_i)} \subset \Omega$ and

$$\sum_i \left(\operatorname{osc}_{B(x_i,\lambda r_i)} f\right)^n > V^n_\lambda(f, \Omega) - \varepsilon.$$

Since Ω is open and $\overline{B(x_i, r_i)} \subset \Omega$, we can find $k \in \mathbb{N}$ such that for

$$\Omega_k = \left\{x \in \Omega : |x| < k \ \text{ and } \ \operatorname{dist}(x, \partial\Omega) > \frac{1}{k}\right\} \tag{3.2}$$

we have $B(x_i, r_i) \subset \Omega_k$ for each i and therefore $V^n_\lambda(f, \Omega_k) > V^n_\lambda(f, \Omega) - \varepsilon$. From this fact and $V^n_\lambda(\Omega_k) + V^n_\lambda(\Omega \setminus \Omega_k) \leq V^n_\lambda(\Omega)$ we obtain $V^n_\lambda(\Omega \setminus \Omega_k) < \varepsilon$.

For a given ε we can find δ_1 from the definition of $AC^n_\lambda(\Omega_{k+1})$ for f. Put

$$\delta = \min\left\{\delta_1, \mathcal{L}_n\Big(B\Big(0, \frac{1}{2k(k+1)}\Big)\Big)\right\}. \tag{3.3}$$

Fix pairwise disjoint balls $B_1, \ldots, B_l$ in Ω such that $\sum_{i=1}^l \mathcal{L}_n(B_i) < \delta$. From (3.3) we obtain $\operatorname{diam}(B_i) < \frac{1}{k} - \frac{1}{k+1}$ $(i \in \{1, \ldots, l\})$. Thus (3.2) gives that either $B_i \subset \Omega_{k+1}$ or $B_i \subset \Omega \setminus \Omega_k$ for every i. Hence we obtain from the definition of δ_1 and k that

$$\begin{aligned}\sum_i \operatorname{osc}^n_{\lambda B_i} f &\le \sum_{i:B_i\subset\Omega_{k+1}} \operatorname{osc}^n_{\lambda B_i} f + \sum_{i:B_i\subset\Omega\setminus\Omega_k} \operatorname{osc}^n_{\lambda B_i} f\\ &\le \sum_{i:B_i\subset\Omega_{k+1}} \operatorname{osc}^n_{\lambda B_i} f + V^n_\lambda(\Omega\setminus\Omega_k)\\ &\le \varepsilon+\varepsilon\\ &= 2\varepsilon\end{aligned}$$

and the proof is finished ∎

The following theorem [3: Theorem 3.1] gives us the opportunity to use any $\lambda \in (0,1)$ in the definition of the classes AC^n_λ and BV^n_λ. We will use this fact in the proof of Theorem 3.6.

Theorem 3.5. *Let* $0 < \lambda_1 < \lambda_2 < 1$ *and* $f : \Omega \to \mathbb{R}^m$. *Then* $BV^n_{\lambda_1}(\Omega) = BV^n_{\lambda_2}(\Omega)$ *and* $AC^n_{\lambda_1}(\Omega) = AC^n_{\lambda_2}(\Omega)$.

Now we can prove the main result of this section.

Theorem 3.6. *Let* $n \ge 2$ *and* $0 < \lambda < 1$. *Suppose that the mapping* $F : \Omega \to \mathbb{R}^n$ *is* K*-quasiconformal and* $f : \Omega \to \mathbb{R}$. *Then:*

(i) $f \circ F^{-1} \in BV^n_\lambda(F(\Omega)) \implies f \in BV^n_\lambda(\Omega)$

(ii) $f \circ F^{-1} \in AC^n_\lambda(F(\Omega)) \implies f \in AC^n_\lambda(\Omega)$.

Proof. Let us first suppose that $f \circ F^{-1} \in BV^n_\lambda(F(\Omega))$. Thanks to Theorem 3.5 we can suppose that $\lambda = \frac{1}{2}$. We will prove that $f \in BV^n_{\frac{\rho_0}{2}}(\Omega)$. Recall that the constant $0 < \rho_0 < 1$ comes from (3.1).

Suppose that $B_i = B(x_i, r_i) \subset \Omega$ are pairwise disjoint balls. Clearly,

$$F\left(B\left(x_i, \frac{\rho_0}{2} r_i\right)\right) \subset B\left(F(x_i), \operatorname{osc}_{B(x_i,\frac{\rho_0}{2}r_i)} F\right). \tag{3.4}$$

Thanks to (3.1), for $r = \frac{\rho_0}{2} r_i$ and $x = x_i$ we have

$$\begin{aligned}&B\left(F(x_i), 2\operatorname{osc}_{B(x_i,\frac{\rho_0}{2}r_i)} F\right)\\ &\quad\subset B\Bigg(F(x_i), 4 \sup_{\{a:|x_i-a|\le\frac{\rho_0}{2}r_i\}} |F(x_i)-F(a)|\Bigg)\\ &\quad\subset B\Bigg(F(x_i), \inf_{\{b:|x_i-b|=\frac{1}{\rho_0}\frac{\rho_0}{2}r_i\}} |F(x_i)-F(b)|\Bigg)\\ &\quad\subset F\Big(B\Big(x_i, \frac{1}{2}r_i\Big)\Big).\end{aligned} \tag{3.5}$$

Hence the balls $\widetilde{B}_i = B\big(F(x_i), 2\mathrm{osc}_{B(x_i,\frac{\rho_0}{2}r_i)}F\big)$ are pairwise disjoint in $F(\Omega)$. Thus (3.4) gives us

$$\begin{aligned}\sum_i \mathrm{osc}^n_{B(x_i,\frac{\rho_0}{2}r_i)} f &= \sum_i \mathrm{osc}^n_{F(B(x_i,\frac{\rho_0}{2}r_i))} f\circ F^{-1}\\ &\le \sum_i \mathrm{osc}^n_{B(F(x_i),\mathrm{osc}_{B(x_i,\frac{\rho_0}{2}r_i)}F)} f\circ F^{-1}\\ &= \sum_i \mathrm{osc}^n_{\frac{1}{2}\widetilde{B}_i} f\circ F^{-1}\\ &\le V^n_{\frac{1}{2}}\big(f\circ F^{-1}, F(\Omega)\big).\end{aligned} \tag{3.6}$$

It follows that $V^n_{\frac{\rho_0}{2}}(f,\Omega) \le V^n_{\frac{1}{2}}(f\circ F^{-1}, F(\Omega)) < \infty$.

Now let us suppose that $f\circ F^{-1} \in AC^n_\lambda(F(\Omega))$. As before we can assume that $\lambda = \frac{1}{2}$. From the conclusions above we obtain $f \in BV^n_{\frac{\rho_0}{2}}(\Omega)$. Thanks to Lemma 3.4 and Theorem 3.5 it is enough to prove that $f \in AC^n_{\frac{\rho_0}{2},\mathrm{loc}}(\Omega)$.

Fix $\varepsilon > 0$ and $\Omega' \subset \Omega$ such that $\overline{\Omega'} \subset \Omega$. Choose δ_1 from the definition of $AC^n_{\frac{1}{2}}(\Omega)$ for function $f\circ F^{-1}$. By [4: Theorem 4.3], quasiconformal mappings are locally absolutely continuous and therefore $F \in AC^n_\lambda(\Omega')$. Hence for a given $\varepsilon_1 = \frac{\delta_1}{2^n}$ we can choose δ_2 from the definition of $AC^n_{\frac{\rho_0}{2}}(\Omega')$ for the function F.

Suppose that the balls $B_i = B(x_i, r_i) \subset \Omega'$ are pairwise disjoint and $\sum_i \mathcal{L}_n(B_i) < \delta_2$. As before we obtain (3.4) and (3.5). Therefore the balls

$$\widetilde{B}_i = B\left(F(x_i), 2\mathrm{osc}_{B\left(x_i,\frac{\rho_0}{2}r_i\right)}F\right)$$

are pairwise disjoint in $F(\Omega')$. Further, $\sum_i \mathcal{L}_n(B_i) < \delta_2$ and the definition of δ_2 give us

$$\sum_i \mathcal{L}_n(\widetilde{B}_i) = 2^n \sum_i \mathrm{osc}^n_{B\left(x_i,\frac{\rho_0}{2}r_i\right)} F \le 2^n\varepsilon_1 = 2^n\frac{\delta_1}{2^n} = \delta_1.$$

Analogously to (3.6) we obtain from the definition of δ_1 that

$$\sum_i \mathrm{osc}^n_{B(x_i,\frac{\rho_0}{2}r_i)} f \le \sum_i \mathrm{osc}^n_{\frac{1}{2}\widetilde{B}_i} f\circ F^{-1} < \varepsilon$$

and the proof is finished ∎

The inverse mapping to a quasiconformal mapping is also quasiconformal [7: Corollary 13.3] and hence we have the following

Corollary 3.7. *Let $0 < \lambda < 1$, $n \geq 2$ and let $f : \Omega \to \mathbb{R}^m$. Suppose that $F : \Omega \to \mathbb{R}^n$ is a quasiconformal mapping. Then:*

(i) $f \in BV^n_\lambda(\Omega) \iff f \circ F^{-1} \in BV^n_\lambda(F(\Omega))$

(ii) $f \in AC^n_\lambda(\Omega) \iff f \circ F^{-1} \in AC^n_\lambda(F(\Omega))$.

The following elementary example shows that the assumption $f \in BV^n_\lambda$ from the definition of the class AC^n_λ is important in Theorem 3.6.

Example 3.8. Let $0 < \lambda < 1$. There exists a domain $\Omega \subset \mathbb{R}^2$ and a 1-quasiconformal mapping $F : \Omega \to \mathbb{R}^2$ such that $f \circ F^{-1}$ is $2, \lambda$-absolutely continuous on $F(\Omega)$ but f is not $2, \lambda$-absolutely continuous on Ω.

Indeed, set $\Omega = \{[x, y] : x > 0\}$ and $F(x, y) = \left[\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right]$. In other words, for $z \in \mathbb{C}$ we define $F(z) = \frac{1}{z}$ (thus also $F^{-1}(z) = \frac{1}{z}$). It is well known that the mapping $\frac{1}{z}$ is conformal and hence also 1-quasiconformal [7: Theorem 8.1]. Plainly, $F(\Omega) = \{[x, y] : x > 0\}$. Put

$$\tilde{f}(x) = \sum_{k=1}^{\infty} \max\{0, 1 - \operatorname{dist}(x, [2k, 0])\}.$$

Clearly, $\tilde{f}$ is a Lipschitz function with Lipschitz constant 1 on $F(\Omega)$ and hence also $2, \lambda$-absolutely continuous.

Set $f = \tilde{f} \circ F$ (hence $\tilde{f} = f \circ F^{-1}$) and $B_k = B([2k, 0], 1)$. Properties of inversion and easy computation gives us

$$\widetilde{B}_k := F^{-1}(B_k) = B\left(\left[\frac{\frac{1}{2k+1} + \frac{1}{2k-1}}{2}, 0\right], \frac{\frac{1}{2k-1} - \frac{1}{2k+1}}{2}\right)$$

for every $k \in \mathbb{N}$. From $\operatorname{osc}_{\widetilde{B}_k} f \geq 1$ and $\operatorname{diam} \widetilde{B}_k \to 0$ we obtain that f is not $2, \lambda$-absolutely continuous.

It is not difficult to prove that the condition $\mathcal{L}_n(\Omega) < \infty$ guarantees that any n, λ-absolutely continuous function f on Ω satisfies $f \in BV^n_\lambda(\Omega)$. Hence such an example can exist only if $\mathcal{L}_n(F(\Omega)) = \infty$ in view of Theorem 3.6.

4. Continuous homeomorphisms $F : BV^n_\lambda \to BV^n_\lambda$

In this section we will use ideas of Gold'stein, Gurov and Romanov [2]. They proved that a homeomorphism $F : \Omega \to \mathbb{R}^n$ which induces a bounded operator from $W^{1,n}(F(\Omega))$ to $W^{1,n}(\Omega)$ is a quasiconformal mapping (see [2] for details and [3] for the history of similar problems).

Let us denote $F'_v(x) = \lim_{r\to 0} \frac{|F(B(x,r))|}{|B(x,r)|}$. We shall need the following connection between F'_v and the Jacobian of F [7: Theorems 24.2 and 24.4].

Theorem 4.1. *Let $F : \Omega \to \mathbb{R}^n$ be a homeomorphism. Then:*

(i) *$F'_v < \infty$ almost everywhere.*

(ii) *F'_v is a measurable function.*

(iii) *For each measurable set $A \subset \Omega$, $|F(A)| \ge \int_A F'_v(x)\,dx$.*

(iv) *If F is differentiable at x and $J_F(x)$ is the Jacobi matrix of F at x, then $F'_v = |J_F(x)|$.*

Lemma 4.2. *If a homeomorphism $F : \Omega \to \mathbb{R}^n$ induces the bounded operator $F^\star : BV^n_\lambda(F(\Omega)) \to BV^n_\lambda(\Omega)$, then F is differentiable almost everywhere on Ω.*

Proof. Fix $R > 0$. The mapping F is a homeomorphism and therefore the set

$$A_R = \{x \in \Omega : F(x) \in B(0,R)\} = F^{-1}(B(0,R))$$

is open. Fix $1 \le i \le n$. Plainly, there is a Lipschitz function $f : F(\Omega) \to \mathbb{R}$ such that

$$f(x) = \begin{cases} x_i & \text{for } x \in F(\Omega), |x| < R \\ 0 & \text{for } x \in F(\Omega), |x| > R+1. \end{cases}$$

Hence $f \in BV^n_\lambda(F(\Omega))$ implies $F^\star f = f \circ F \in BV^n_\lambda(\Omega)$. If $|F(x)| < R$, then $f \circ F = F_i(x)$. Thus $F_i(x) \in BV^n_\lambda(A_R)$. Functions from $BV^n_\lambda(A)$ are differentiable almost everywhere on A for every open set A (see [6: Theorem 3.3] and [4: Theorem 3.4] for details) and hence F_i is differentiable almost everywhere on A_R. Since $A_R \to \Omega$ as $R \to \infty$ we obtain that F_i is differentiable almost everywhere on Ω ∎

In the proof of Theorem 4.4 below we will need the following elementary lemma [2: Lemma 3.5]:

Lemma 4.3. *Let $F : \Omega \to \mathbb{R}^n$ be a continuous mapping and $G \subset \mathbb{R}^k$. Suppose that $\{K_y\}_{y\in G}$ is a family of pairwise disjoint compact sets such that $K_y \subset F(\Omega)$. Then $\mathcal{L}_n(F^{-1}(K_y)) = 0$ for all $y \in G$ except possibly a countable subset of G.*

Theorem 4.4. *Let $0 < \lambda \le 1$ and $n \ge 2$. If a homeomorphism $F : \Omega \to \mathbb{R}^n$ induces the bounded operator $F^\star : BV^n_\lambda(F(\Omega)) \to BV^n_\lambda(\Omega)$, then $F \in W^{1,n}_{\mathrm{loc}}(\Omega)$ and there is a number K such that*

$$|\nabla F_i|^n \le K F'_v(x)$$

for almost all $x \in \Omega$ and for all $i = 1, 2, \dots, n$.

Proof. In this proof we will follow the ideas from [2: Theorem 3.6]. By Theorem 4.1, $F'_v(x) < \infty$ a.e. Fix $\varepsilon > 0$ and a point $x_0 \in \Omega$ such that

$F'_v(x_0) < \infty$. There is r_0 such that for all $r \in (0, r_0)$ we have

$$|F(B(x_0,2r))| \leq (F'_v(x_0)+\varepsilon)|B(x_0,2r)| \\ = (F'_v(x_0)+\varepsilon)2^n|B(x_0,r)|. \tag{4.1}$$

Set $M = (F'_v(x_0)+\varepsilon)2^n$. Let us call a cube Q *h-regular* if all its edges are parallel to the coordinate axes, the length of the edge is h and every vertex has the form $[k_1h,\dots,k_nh]$ where $k_1,\dots,k_n$ are integers. Fix $r < r_0$ and choose $h > 0$ such that

$$h < \frac{1}{2\sqrt{n}}\operatorname{dist}\Big(F(S(x_0,2r)), F(S(x_0,r))\Big).$$

Let A be the union of all h-regular cubes Q such that $Q \cap F(B(x_0,r)) \neq \emptyset$. It is evident that

$$F(B(x_0,r)) \subset A \subset F(B(x_0,2r)).$$

Fix $j \in \{1,\dots,n\}$ and let us focus on the j-th coordinate. Denote the hyperplanes $x_j = th$ by L_t. The hyperplanes L_m (m an integer) divide $\mathbb{R}^n$ into layers

$$Z_m = \big\{x \in \mathbb{R}^n : mh < x_j < (m+1)h\big\}.$$

Put $A_m = Z_m \cap A$.

For every A_m we construct three functions

$$\psi_{m,1} = x_j - mh$$
$$\psi_{m,2} = (m+1)h - x_j$$
$$\psi_{m,3} = \frac{h}{2} - \operatorname{dist}\big(P_j(x), P_j(A_m)\big).$$

Here $P_j : \mathbb{R}^n \to \mathbb{R}^{n-1}_j$ is the orthogonal projection of $\mathbb{R}^n$ onto $\mathbb{R}^{n-1}_j$. Consider the functions

$$\psi_m = \max\big\{0, \min\{\psi_{m,1},\psi_{m,2},\psi_{m,3}\}\big\} \quad \text{and} \quad \psi = \sum_m \psi_m.$$

Put

$$E = \big\{x \in G : \psi(x) \ \text{ is not differentiable at } x\big\}.$$

It follows from the definition of ψ that:

(1) $\operatorname{supp}(\psi) \subset F(B(x_0,2r))$
(2) ψ is Lipschitz with constant 1
(3) $\psi \in BV^n_\lambda(F(\Omega))$
(4) ψ is differentiable almost everywhere

(5) $\psi(x) = \pm x_j + \text{const}$ in all components of the set $F(B(x_0,r)) \setminus E$.

The set $E \cap F(B(x_0,r))$ belongs to a finite union of hyperplanes $L_{t_1}, \dots, L_{t_s}$ where $2t_i$ is an integer. By Lemma 4.3, for almost all small translations τ_y parallel to the axis x_j we have

$$\left| F^{-1}\left(\tau_y \left(\bigcup_{i=-\infty}^{\infty} L_{\frac{i}{2}} \right) \cap F(\overline{B(x_0,r)}) \right) \right| = 0.$$

Thus we can assume without loss of generality that

$$\left| F^{-1}\big(E \cap F(B(x_0,r))\big) \right| = 0. \tag{4.2}$$

Otherwise it is possible to change the j-th coordinate of the point $[0,\dots,0]$ at the beginning of the construction of ψ.

By the assumption of the theorem, $F^\star \psi = \psi \circ F \in BV^n_\lambda(\Omega)$. It follows from (5) and (4.2) that $(\psi \circ F)(x) = \pm F_j(x) + \text{const}$ for almost all $x \in B(x_0,r)$. It is easy to see from the proof of [5: Theorem 3.2] that $BV^n_\lambda(\Omega)$ is continuously embedded into $W^{1,n}(\Omega)$. These two facts give us

$$\begin{aligned} \int_{B(x_0,r)} |\nabla(F_j(x))|^n dx &\le \int_\Omega |\nabla(\psi \circ F)|^n dx \\ &\le C V^n_\lambda(\psi \circ F, \Omega) \\ &= C V^n_\lambda(F^\star(\psi), \Omega). \end{aligned} \tag{4.3}$$

Since $F^\star$ is continuous we have

$$V^n_\lambda(F^\star(\psi), \Omega) \le C V^n_\lambda(\psi, \Omega). \tag{4.4}$$

The function ψ is Lipschitz with constant 1 and hence

$$\operatorname{osc}^n_{B(x,s)} \psi \le (2s)^n = C|B(x,s)| \tag{4.5}$$

for each x and every s. Thanks to (4.5), the continuity of ψ and $\operatorname{supp}(\psi) \subset F(B(x_0,2r))$ we have

$$V^n_\lambda(\psi, \Omega) \le C|F(B(x_0,2r))|. \tag{4.6}$$

From (4.3), (4.4) and (4.6) it follows that

$$\int_{B(x_0,r)} |\nabla(F_j(x))|^n dx \le C|F(B(x_0,2r))|.$$

By (4.1),

$$\int_{B(x_0,r)} |\nabla(F_j(x))|^n dx \le CM|B(x_0,r)|.$$

Hence

$$\limsup_{r\to 0} \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} |\nabla(F_j(x))|^n dx \le CM.$$

The Lebesgue Theorem gives us $|\nabla F_j(x_0)|^n \le C(F_v'(x_0)+\varepsilon)$ for almost all $x_0 \in \Omega$. Letting $\varepsilon \to 0$ we obtain

$$|\nabla F_j(x_0)|^n \le CF_v'(x_0) \tag{4.7}$$

for almost all $x_0 \in \Omega$. For every compact set $K \subset \Omega$ we obtain from Theorem 4.1 and (4.7) that

$$\int_K |\nabla F_j(x)|^n dx \le C \int_K F_v'(x)\,dx \le C|F(K)| < \infty.$$

Thus $F \in W^{1,n}_{\rm loc}(\Omega)$ ∎

Thanks to Lemma 4.2 and Theorem 4.1/(iv) we obtain the following

Corollary 4.5. *Let* $0 < \lambda \le 1$ *and* $\Omega \subset \mathbb{R}^n$ $(n \ge 2)$. *Each homeomorphism* $F : \Omega \to \mathbb{R}^n$ *that induces a bounded operator from* $BV^n_\lambda(F(\Omega))$ *to* $BV^n_\lambda(\Omega)$ *is quasiconformal.*

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