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Sequences of 0's and 1's Classes of Concrete 'big' Hahn Spaces

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Abstract. This paper continues the joint investigation by Bennett et al. (2001) and Zeltser et al. (2002) of the extent to which sequence spaces are determined by the sequences of 0's and 1's that they contain. Bennett et al. proved that each subspace E of ℓ^∞ containing the sequence $e = (1, 1, \ldots)$ and the linear space bs of all sequences with bounded partial sums is a Hahn space, that is, an FK-space F contains E whenever it contains (the linear hull $\chi(E)$ of) the sequences of 0's and 1's in E. In some sense these are 'big' subspaces of ℓ^∞. Theorem 2.6, one of the main results of this paper, tells us that this result remains true if we replace bs with suitably defined spaces $bs(N)$ which are subspaces of bs when N is a finite partition of $\mathbb{N}$. As an application of the main result, two large families of closed subspaces E of ℓ^∞ being Hahn spaces are presented: The bounded domain E of a weighted mean method (with positive weights) is a Hahn space if and only if the diagonal of the matrix defining the method is a null sequence; a similar result applies to the bounded domains of regular Nörlund methods. Since an FK-space E is a Hahn space if and only if $\chi(E)$ is a dense barrelled subspace of E, by these results, a large class of concrete closed subspaces E of ℓ^∞ such that $\chi(E)$ is a dense barrelled subspace can be identified by really simple conditions. A further application gives a negative answer to Problem 7.1 in the paper mentioned above.

Keywords: *Dense barrelled subspaces, Hahn space (property), separable Hahn property, matrix Hahn property, bounded domains, weighted mean method, Nörlund method*

AMS subject classification: 46A45, 46A35, 40C05, 40H05

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1. Introduction

We start with few preliminaries. (Otherwise, the terminology from the theory of locally convex spaces and summability is standard, we refer to Wilansky [9, 10] and Boos [3].)

ω denotes the space of all real-valued sequences, and any vector subspace of ω is called a *sequence space*.

Let χ be the set of all sequences of 0's and 1's and, if E is any sequence space, let $\chi(E)$ denote the linear hull of the sequences of 0's and 1's contained in E. It is always true that $\chi(E) \subset \chi(\ell^\infty) \cap E$, but equality fails in general.

An *FK-space* is a sequence space endowed with a complete, metrizable, locally convex topology under which the coordinate mappings $x \to x_k$ $(k \in \mathbb{N})$ are all continuous. A normable FK-space is called a *BK-space*.

The most natural FK-space is ω under the topology of coordinatewise convergence. Familiar examples of BK-spaces are ℓ^∞ (bounded sequences) with the supremum norm $\|\cdot\|_\infty$ and its closed subspaces c (convergent sequences) and c_0 (null sequences), ℓ^1 (absolutely summable sequences) with its usual norm, and $bs = \left\{x = (x_k) \in \omega \middle| \sup_n \left|\sum_{k=1}^n x_k\right| < \infty\right\} = \Sigma^{-1}(\ell^\infty)$ (bounded partial sums) with the norm $\|\cdot\|_{bs} = \|\cdot\|_\infty \circ \Sigma$ where Σ denotes the summation matrix.

A fundamental property of FK-spaces is that their topologies are monotonic: if $E \subset F$, then E is continuously embedded into F. This means that a sequence space can have at most one FK-topology, and we take advantage of this fact by not actually specifying the topology under consideration.

Let $B = (b_{nk})$ be an infinite matrix with real entries. The (*summability*) *domain* c_B of B is defined as

$$c_B = \left\{ x = (x_k) \in \omega \,\middle|\, Bx := \left(\sum_k b_{nk} x_k\right)_n \in c \right\}$$

where the definition of Bx implies the convergence of the series. The pair $(c_B, \lim_B)$ with $\lim_B : c_B \to \mathbb{R}$, $x \to \lim Bx$ is called a *matrix method* and we denote it also by B. By definition, the matrix (method) B is *coercive*, *conservative* and *regular*, if $\ell^\infty \subset c_B$, $c \subset c_B$, and $c \subset c_B$ with $\lim_B|_c = \lim$, respectively. Note that c_B is a separable FK-space and, if B is conservative, the *bounded domain* $\ell^\infty \cap c_B$ of B is a closed subspace of ℓ^∞.

In this paper, we continue the joint investigation of Bennett, Boos and Leiger in [1] and of Leiger and the authors in [11] of the extent to which sequence spaces are determined by the sequences of 0's and 1's that they contain. As in these papers we consider sequence spaces E with the property that

$$\chi(E) \subset F \implies E \subset F \tag{1.1}$$

whenever F is an arbitrary FK-space, a separable FK-space, and a matrix domain c_B, respectively. Then E is said to have the *Hahn property*, the *separable Hahn property*, and the *matrix Hahn property*, respectively. We call E a *Hahn space*, whenever E has the Hahn property. As well, we call a matrix *potent*, if its bounded domain has the matrix Hahn property.

A sequence space having any Hahn property is necessarily a subspace of ℓ^∞ (cf. [1: Theorem 5.1]). Obviously, the Hahn property implies the separable Hahn property, and the last one implies the matrix Hahn property. Each of the converse implications does not hold in general (cf. [1: Theorem 5.3] and [11: Theorem 1.3]).

The study of Hahn spaces is of functional analytical interest as, for instance, the following result shows.

Theorem 1.1 [2: Theorem 1]. *Let E be an FK-space. Then E has the Hahn property if and only if $\chi(E)$ is a dense, barrelled subspace of E.*

This result overlaps with results due to Drewnowski and Paúl (see, for instance, [5]) and pointed out in details by Stuart [8]. Namely, if the set $\mathcal{E} = \{\chi_{\{k\in\mathbb{N} \mid x_k\neq 0\}} \mid x \in \chi \cap E\}$ is a ring as a family of subsets of $\mathbb{N}$ and has the Nikodym property (cf. [5]), then $\chi(E)$ with $\|\cdot\|_\infty$ is barrelled; if in addition $\chi(E)$ is $\|\cdot\|_\infty$ dense in E and E is a closed subspace of $(\ell^\infty, \|\cdot\|_\infty)$, then E has the Hahn property. In the following investigations we consider sequence spaces E such that $\mathcal{E}$ is not necessarily a ring, so that the mentioned results do not apply in general.

It is well-known that, in general, it is complicated to identify dense, barrelled subspaces of Banach spaces. In this light we should see the following result which gives us a large and very useful class of 'big' Hahn spaces.

Theorem 1.2 [1: Theorem 3.4]. *If E is a sequence space satisfying $bs + \mathrm{Sp}\{e\} \subset E \subset \ell^\infty$, then E has the Hahn property.*

In particular, the bounded domains of strongly conservative matrices A (these are exactly those matrices with $bs + \mathrm{Sp}\{e\} \subset c_A$) have the Hahn property. As it is pointed out in [1: Problem 7.7], in the special case of conservative Hausdorff matrices the strongly conservative matrices are exactly the potent ones, so that in this case of bounded domains the matrix Hahn property and the Hahn property coincide. In the case of conservative weighted mean methods $A = (a_{nk})$ the situation is much more troublesome: On the one hand, as in the case of Hausdorff matrices, they are potent if and only if the diagonal (a_{nn}) converges to 0, and on the other hand, as simple examples prove, the bounded domain of a potent matrix does not necessarily contain $bs + \mathrm{Sp}\{e\}$. Thus Theorem 1.2 does not apply to this situation. In Section 2 we will essentially extend Theorem 1.2 by replacing bs by suitable subspaces $bs(N)$. In Section 3 we use this result for showing that also in the case of weighted mean methods the matrix Hahn property and the Hahn property coincide. In this way, we determine the class of all conservative Nörlund methods with the bounded domain having the Hahn property. These results may be seen as a further contribution to the investigation of the question *which bounded domains have the Hahn property* (cf. [1: Problem 7.7]). In Section 4 we show that the intersection of Hahn spaces is not necessarily a Hahn space which gives a negative answer to [1: Problem 7.1]. Moreover, we use some of the results in Section 2 to present a large class of such spaces.

2. General classes of big Hahn spaces

Let I be a non-empty subset of $\mathbb{N}$. For simplification, we assume $I = \mathbb{N}$ if I has infinite, and $I = \mathbb{N}_r$ if I has exactly r $(r \in \mathbb{N})$ elements. Furthermore, let $N_i = \{n_{ij} \in \mathbb{N} | \, j \in \mathbb{N}\}$ $(i \in I)$ be infinite ordered sets such that

$$\bigcup_{i\in I} N_i = \mathbb{N}, \qquad N_i \cap N_j = \emptyset \quad (i, j \in I, i \neq j) \tag{2.1}$$

that is, $N = (N_i | \, i \in I)$ is a partition of $\mathbb{N}$ (consisting of infinite sets). Then we define

$$bs(N) = \left\{ x \in \omega \,\middle|\, \|x\|_{bs(N)} = \sup_j \|(x_k)_{k\in N_j}\|_{bs} < \infty \right\}.$$

Generalizing the summation matrix map Σ, for any partition $N = (N_i | \, i \in I)$ satisfying (2.1) we consider the matrix map Σ_N defined by

$$[\Sigma_N x]_n = \sum_{j=1}^{k} x_{n_{ij}} \qquad \text{if } n = n_{ik} \quad (n, k \in \mathbb{N}, i \in I)$$

and its inverse map Σ_N^{-1}. Note, for every $y \in \omega$ we have

$$\left[\Sigma_N^{-1} y\right]_n = y_{n_{ik}} - y_{n_{i,k-1}} \quad \text{when } n = n_{ik} \quad (n, k \in \mathbb{N}, i \in I, y_{n_{i0}} := 0)$$

Remark 2.1.

(a) $bs(N) = \Sigma_N^{-1}(\ell^\infty)$, hence $bs(N)$ is a BK-space as the domain of a bijective matrix map with respect to the BK-space ℓ^∞.

(b) If N is infinite, then $bs(N) \not\subset bs$, since trivially $x \in bs(N) \setminus bs$ when x is chosen such that it is 1 exactly once on each set N_i, else 0. However, if N is finite, then we obviously have $bs(N) \subset bs$ and, moreover, $bs(N) \subsetneq bs$ if and only if N consists of more than one set. In the last case the codimension of $bs(N)$ in bs is 'uncountable infinite' as we now verify. It is sufficient to prove that it is not finite since the FK-space bs cannot be the union of an increasing sequence of FK-spaces. Now, if we suppose that $bs = bs(N) + X$ where $\dim X < \infty$, then $(X, \|\cdot\|_{bs})$ is a BK-space, hence $bs(N)$ is closed in bs by [10: Corollary 4.5.2]. On the other hand, we may easily find a sequence $(x^{(j)})$ in $bs(N)$ converging in bs to an $x \in bs \setminus bs(N)$. For instance, we may choose index sequences (μ_s) and (ν_s) with $n_{1\mu_s} < n_{2\nu_s} < n_{1\mu_{s+1}}$ $(s \in \mathbb{N})$ and consider the sequence $(x^{(j)})$ defined by

$$x^{(j)}_{n_{kl}} = \begin{cases} \frac{1}{s} & \text{if } k = 1 \text{ and } l = \mu_s \quad (s \in \mathbb{N}_j) \\ -\frac{1}{s} & \text{if } k = 2 \text{ and } l = \nu_s \quad (s \in \mathbb{N}_j) \\ 0 & \text{otherwise} \end{cases} \qquad (k, l \in \mathbb{N}).$$

(c) We get a decreasing chain of sequence spaces of the type $bs(N)$ if we proceed inductively, for instance, in the following way: We start with $\mathbb{N}$ and divide it into two infinite parts N_{i_1} $(i_1 = 1, 2)$, then we divide each N_{i_1} into two parts $N_{i_1 i_2}$ $(i_2 = 1, 2)$ and get in the k-th step a partition $N^{(k)}$ of $\mathbb{N}$ consisting of 2^k sets $N_{i_1 \cdots i_k}$, say N_{ki} $(i = 1, \ldots, 2^k)$. Obviously, we have

$$bs(N^{(k+1)}) \subsetneq bs(N^{(k)}) \subsetneq bs \qquad (k \in \mathbb{N}).$$

Further, similarly as in (b) we get that the codimension of $bs(N^{(k+1)})$ in $bs(N^{(k)})$ is 'uncountable infinite'.

(d) In the case that N is a finite partition the assumption in the definition of $bs(N)$ that the sets N_i have to be infinite is not essential since, whenever $N = (N_1, \ldots, N_s)$ is a partition of $\mathbb{N}$, we may define

$$\widehat{bs}(N) = \left\{ x \in \omega \,\middle|\, (x_k)_{k \in N_j} \in bs \text{ when } j \in \{1, \ldots, s\} \text{ and } |N_j| = \infty \right\}.$$

Obviously, there exists at least one infinite set N_{j^*}. Then we join this set with the finite sets among $N_1, \ldots, N_s$ and get a partition $M = (M_1, \ldots, M_r)$ of $\mathbb{N}$. Now, it is easy to check that $\widehat{bs}(N) = bs(M)$.

Proposition 2.2. *In general,* $\ell^1 \subset bs(N)$ *and*

$$\begin{aligned} \ell^1 &= \bigcap \left\{ bs(N) \,\middle|\, N = (N_1, N_2) \textit{ is a partition of } \mathbb{N} \textit{ satisfying } (2.1) \right\} \\ &= \bigcap \left\{ bs(N) \,\middle|\, N \textit{ is a partition of } \mathbb{N} \textit{ satisfying } (2.1) \right\}. \end{aligned}$$

Proof. Since $\ell^1 \subset bs(N)$ is obvious, the proof is done when for every $x \in bs \setminus \ell^1$ there exists a partition $N = (N_1, N_2)$ with $x \notin bs(N)$. For that, we fix $x \in bs \setminus \ell^1$, assume – without loss of generality – $(x_k^+) \notin \ell^1$, choose a subsequence (y_k) of (x_k^+) satisfying $(y_k) \notin bs$ and put $N_1 = \{k \in \mathbb{N} \mid y_k > 0\}$ and $N_2 = \mathbb{N} \setminus N_1$ ∎

In the following steps we verify that Theorem 1.2 remains true if we replace bs by any space $bs(N)$. The proofs are much the same as in the corresponding proofs in [1]. However, for the sake of completeness we adapt them to the case in hand.

Proposition 2.3. *If E is a sequence space satisfying $bs(N) + \mathrm{Sp}\{e\} \subset E \subset \ell^\infty$, then $E = bs(N) + \chi(E)$.*

Proof. Let $x \in E$ be given. Without loss of generality we may suppose that $0 \leq x_k < 1 \quad (k \in \mathbb{N})$. We set $y_{n_{i1}} = 0$, $z_{n_{ik}} = x_{n_{ik}} - y_{n_{ik}}$ and

$$y_{n_{i,k+1}} = \begin{cases} 1 & \text{if } \sum_{j=1}^k y_{n_{ij}} < \sum_{j=1}^k x_{n_{ij}} \\ 0 & \text{otherwise} \end{cases} \qquad (k \in \mathbb{N}, i \in I)$$

where $N_i = \{n_{ij} \mid j \in \mathbb{N}\}$ is the i-th partition set. Then $z \in bs(N)$ and $y = x - z \in E + bs(N) = E$, so $y \in \chi(E)$ ∎

Lemma 2.4. *Let E be a sequence space including $bs(N)$ for some partition $N = (N_i | i \in I)$. If $x \in E$ takes only the values $\{0, 1, \ldots, K\}$ in $\mathbb{N}$, then $x \in \chi(E)$.*

Proof. For each $j \in I$ we set $x_k^{(j)} = x_k$ for $k \in N_j$ and $x_k^{(j)} = 0$ otherwise $(k \in \mathbb{N})$. Further, let L be the least common multiple of $1, \ldots, K$. Following the proof of [1: Lemma 3.2], for every $j \in I$ we obtain subsets $V_1^j, \ldots, V_L^j$ of $\mathrm{supp}\, x^{(j)}$ such that $\chi_{V_k^j} - \frac{1}{L} x^{(j)} \in bs$ and $x^{(j)} = \sum_{k=1}^L \chi_{V_k^j}$. Since $\mathrm{supp}\,\big(\chi_{V_k^j} - \frac{1}{L} x^{(j)}\big) \subset N_j \quad (j \in I)$, then $\sum_{j \in I} \big(\chi_{V_k^j} - \frac{1}{L} x^{(j)}\big) \in bs(N)$ (coordinatewise sum). So $\chi_{V_k} \in E \cap \{0,1\}^{\mathbb{N}}$ where $V_k = \bigcup_{j \in I} V_k^j \quad (k = 1, \ldots, L)$. Hence

$$x = \sum_{j \in I} x^{(j)} = \sum_{j \in I} \sum_{k=1}^{L} \chi_{V_k^j} = \sum_{k=1}^{L} \chi_{V_k} \in \chi(E) \quad \text{(coordinatewise sum)}$$

and the lemma is proved ∎

Lemma 2.5. *We have $\chi(\ell^\infty) \subset \Sigma_N\big(\chi(bs(N) + \mathrm{Sp}\{e\})\big)$.*

Proof. Let w be an arbitrary sequence of 0's and 1's. The sequence $y = \Sigma_N^{-1} w$ belongs to $bs(N)$ and takes only the values $-1, 0, 1$. It follows from Lemma 2.4 that $y + e \in \chi(bs(N) + \mathrm{Sp}\{e\})$ and thus $y \in \chi(bs(N) + \mathrm{Sp}\{e\})$. Hence $w \in \Sigma_N\big(\chi(bs(N) + \mathrm{Sp}\{e\})\big)$ ∎

The following result generalizes Theorem 1.2.

Theorem 2.6. *If E is a sequence space satisfying $bs(N) + \mathrm{Sp}\{e\} \subset E \subset \ell^\infty$ for some partition N, then E has the Hahn property.*

Proof. Proposition 2.3 shows that $E = bs(N) + \mathrm{Sp}\{e\} + \chi(E)$. Since $\chi(E)$ certainly has the Hahn property, it suffices in view of [1: Proposition 2.1] to show that $bs(N) + \mathrm{Sp}\{e\}$ does, too. Suppose then that F is an FK-space containing $\chi(bs(N) + \mathrm{Sp}\{e\})$; we must show that F contains all of $bs(N) + \mathrm{Sp}\{e\}$. For this it is sufficient to check that $bs(N) \subset F$ or, what is equivalent, that $\ell^\infty \subset \Sigma_N(F)$. But this last assertion follows from [1: Corollary 1.2], it being plain from Lemma 2.5 that $\chi(\ell^\infty) \subset \Sigma_N(F)$ ∎

As an immediate consequence of Theorem 1.1 and Theorem 2.6 we get:

Corollary 2.7. *If E is any FK-space satisfying $bs(N)+\mathrm{Sp}\{e\} \subset E \subset \ell^\infty$ for some partition N, then $\chi(E)$ is both dense and barrelled in E.*

We apply the above results to more concrete sequence spaces than the spaces $bs(N)$ which are related to the bounded domains of weighted means. For that we replace in the definition of bs the summation matrix by more general triangles.

Let $p = (p_n)$ be any real sequence satisfying $p_n > 0 \quad (n \in \mathbb{N})$. Then we consider the matrices

$$\Sigma_p = \Sigma \,\mathrm{diag}\,(p_n) \qquad \text{and} \qquad \Sigma_p^{-1} = \mathrm{diag}\big(\tfrac{1}{p_n}\big)\Sigma^{-1}$$

and the domain

$$\ell^\infty_{\Sigma_p} = \Big\{ x = (x_k) \in \omega \Big| \sup_n \big| \sum_{k=1}^{n} p_k x_k \big| < \infty \Big\} = \Sigma_p^{-1}(\ell^\infty).$$

In particular, we have $\ell^\infty_{\Sigma_p} = \Sigma_p^{-1}(\ell^\infty) \subset \ell^\infty$ if and only if $\big(\frac{1}{p_n}\big) \in \ell^\infty$. Therefore, $\ell^\infty_{\Sigma_p} = \Sigma_p^{-1}(\ell^\infty)$ *does not have any Hahn property* when $\big(\frac{1}{p_n}\big) \notin \ell^\infty$ by [1: Theorem 5.1].

Proposition 2.8. *Let $r \in \mathbb{N}$ and p, q be sequences in $\mathbb{N}_r$ and, moreover, let $N_{r(i-1)+j} = \big\{k \in \mathbb{N} | \, p_k = i, \ q_k = j\big\} \quad (i, j = 1, \dots, r)$. Then $bs(N) \subset \ell^\infty_{\Sigma_p} \cap \ell^\infty_{\Sigma_q}$ $(\subset \ell^\infty)$ holds where $N = \{N_{r(i-1)+j} \,|\, i, j = 1, \dots, r\}$. Consequently (cf. Theorem 2.6), every linear subspace of ℓ^∞ containing $(\ell^\infty_{\Sigma_p} \cap \ell^\infty_{\Sigma_q}) \oplus \mathrm{Sp}\{e\}$ has the Hahn property. In particular, $\ell^\infty_{\Sigma_p} \oplus \mathrm{Sp}\{e\}$, $(\ell^\infty_{\Sigma_p} \cap \ell^\infty_{\Sigma_q}) \oplus \mathrm{Sp}\{e\}$, $(\ell^\infty_{\Sigma_p} \oplus \mathrm{Sp}\{e\}) \cap (\ell^\infty_{\Sigma_q} \oplus \mathrm{Sp}\{e\})$ have the Hahn property.*

Proof. Note first that some of the sets N_i might be empty or finite. For that see Remark 2.1(d). Since

$$(\ell^\infty_{\Sigma_p} \cap \ell^\infty_{\Sigma_q}) \oplus \mathrm{Sp}\{e\} \subset (\ell^\infty_{\Sigma_p} \oplus \mathrm{Sp}\{e\}) \cap (\ell^\infty_{\Sigma_q} \oplus \mathrm{Sp}\{e\}),$$

by Proposition 2.6 it suffices to verify that $bs(N) \subset \ell^\infty_{\Sigma_p} \cap \ell^\infty_{\Sigma_q}$ for the partition N fixed in the proposition. Let $x \in bs(N)$ be arbitrarily given. Then

$$\begin{aligned}
\sup_n \left| \sum_{k=1}^n p_k x_k \right| &= \sup_n \left| \sum_{i=1}^r i \sum_{\substack{k=1 \\ p_k = i}}^n x_k \right| \\
&\leq \sum_{i=1}^r i \sum_{j=1}^r \sup_n \left| \sum_{k \in N_{r(i-1)+j} \cap \mathbb{N}_n} x_k \right| \\
&< \infty.
\end{aligned}$$

Hence $x \in \ell^\infty_{\Sigma_p}$. Analogously we may prove $x \in \ell^\infty_{\Sigma_q}$ ∎

The following example shows that the statement in Proposition 2.8 fails in general if p is not bounded.

Example 2.9. Let $p = (n)$. Then $\ell^\infty_{\Sigma_p} \oplus \mathrm{Sp}\{e\}$ does not enjoy any of the Hahn properties. (However, compare this result with that in Example 3.5.)

For a proof note that

$$\varphi \subsetneq \ell^\infty_{\Sigma_p} \subset \left\{ x \in \omega : x_n = \mathfrak{O}\left(\frac{1}{p_n}\right) \right\} \subsetneq c_0$$

and

$$\chi(\ell^\infty_{\Sigma_p} \oplus \mathrm{Sp}\{e\}) = \chi(\varphi \oplus \mathrm{Sp}\{e\}) = \chi(c) = \varphi \oplus \mathrm{Sp}\{e\}.$$

Furthermore, $x = \left((-1)^k \frac{1}{k}\right) \in \ell^\infty_{\Sigma_p}$, $x \notin c_A$ where $A = (a_{nk})$ is the matrix with $a_{nk} = (-1)^k$ when $k \leq 2n$ and $a_{nk} = 0$ otherwise, and obviously $\varphi \oplus \mathrm{Sp}\{e\} \subset c_A$. So $\ell^\infty_{\Sigma_p} \oplus \mathrm{Sp}\{e\}$ does not enjoy the matrix Hahn property.

3. Bounded domains of Riesz and Nörlund means

In connection with Riesz and Nörlund means we consider exclusively **real** sequences (p_k) with

$$p_1 > 0, \quad p_k \geq 0 \ \ (k \in \mathbb{N}), \quad P_n := \sum_{k=1}^n p_k \ \ (n \in \mathbb{N}). \tag{3.1}$$

The *Riesz matrix* $R_p = (r_{nk})$ and the *Nörlund matrix* $N_p = (p_{nk})$ (associated with p) are defined by

$$r_{nk} = \begin{cases} \frac{p_k}{P_n} & \text{if } k \leq n \\ 0 & \text{otherwise} \end{cases} \qquad (n, k \in \mathbb{N})$$

and

$$p_{nk} = \begin{cases} \frac{p_{n-k+1}}{P_n} & \text{if } k \leq n \\ 0 & \text{otherwise} \end{cases} \qquad (n, k \in \mathbb{N}),$$

respectively. The summability method corresponding R_p is called *Riesz method*, *Riesz mean* or *weighted mean method* whereas the summability method corresponding N_p is called *Nörlund method* or *Nörlund mean*.

Note that $R_p = \mathrm{diag}\big(\frac{1}{P_n}\big)\Sigma_p$ and, if $P_n \to \infty$, that obviously $\ell^\infty_{\Sigma_p} \subset c_{0R_p}$.

We recall some basic properties of Riesz and Nörlund matrices (methods) (cf. [3: Sections 3.2 and 3.3], [4] and [6]).

Remarks 3.1.

(a) R_p is conservative and it is either regular (being equivalent to $P_n \to \infty$) or coercive (cf. also [7]).

(b) If $p_k > 0$ for all $k \in \mathbb{N}$, then $c_{R_p} = c, \ell^\infty \cap c_{R_p} = c$ and $\big(\frac{P_n}{p_n}\big) \in \ell^\infty$ are equivalent. In particular, in this case R_p is not potent.

(c) If (P_n) is bounded, then R_p is potent since $\ell^\infty \cap c_{R_p} = \ell^\infty$ (cf. assertion (a)). If (P_n) is unbounded, that is $P_n \to \infty$, then the potency of R_p is equivalent to $\big(\frac{p_n}{P_n}\big) \in c_0$ (which is obviously equivalent to $\big(\frac{\max_{i \in \mathbb{N}_n} p_i}{P_n}\big)_n \in c_0$). Note that this equivalence is proven in [3: Sections 3.2 and 3.3] (cf. also [4, 6]) under the assumption $p_k > 0$ for all $k \in \mathbb{N}$, but the proofs work also without this assumption.

(d) If (p_k) is monotonically decreasing, then $c_{R_p} \supset c_{C_1}$ with consistency.

(e) If (p_k) is monotonically increasing, then $c_{R_p} \subset c_{C_1}$ with consistency.

(f) If (p_k) is monotonically increasing, then $c_{R_p} = c_{C_1}$ with consistency if and only if $\big(\frac{np_n}{P_n}\big) \in \ell^\infty$.

Remarks 3.2.

(a) The Nörlund method N_p is conservative if and only if $\big(\frac{p_n}{P_n}\big) \in c$, and regular if and only if $\big(\frac{p_n}{P_n}\big) \in c_0$.

(b) If N_p is conservative and non-regular, then it is coercive (cf. [7: Theorem 2]).

By Remark 3.1(c) we have a very simple as well as satisfactory characterization of those Riesz means R_p satisfying (3.1) with $\ell^\infty \cap c_{R_p}$ having the matrix Hahn property. This is quite different in the case of the (regular) Nörlund means: Up to now, there was not known a similar satisfactory necessary or sufficient condition for the potency of regular Nörlund methods.

As an application of Theorem 2.6 we will prove in this section the strong result that the bounded domain of a regular potent Riesz matrix R_p (p as in (3.1), $P_n \to \infty$ and $\big(\frac{p_n}{P_n}\big) \in c_0$) has the Hahn property and (this is like a nice additional gift that the bounded domain of the corresponding Nörlund matrix N_p has also the Hahn property) in particular that N_p is potent.

Before we start this program, we consider N_p in the much easier open case that $p \in \ell^1$.

Theorem 3.3. *If* (*in addition*) $p \in \ell^1$, *then the Nörlund method* N_p *is* (*obviously regular and*) *not potent.*

Proof. Let $k_1 \in \mathbb{N}$ be chosen such that $\sum_{k=k_1+1}^{\infty} p_k < \frac{p_1}{8}$ and $p_{k_1} > 0$. Let X be the set of all elements in $\chi \cap c_{N_p}$ such that

$$\left| \sum_{k=1}^{k_1} p_k x_{n+1-k} - \sum_{k=1}^{k_1} p_k x_{n-k} \right| < \frac{p_1}{2}$$

for every $n > k_1$. Obviously, $X \neq \emptyset$ since $e \in X$. If x_{n-k} $(k = 1, \ldots, k_1)$ are fixed $(n > k_1)$, then there exists at most one $x_n \in \{0, 1\}$ such that

$$\left| \sum_{k=2}^{k_1} p_k x_{n+1-k} + p_1 x_n - \sum_{k=1}^{k_1} p_k x_{n-k} \right| < \frac{p_1}{2}.$$

So each element $x \in X$ is completely defined by the first k_1 coordinates $x_1, \ldots, x_{k_1}$. Evidently, $n_0 := |X| \leq 2^{k_1}$. Since there exist exactly 2^{k_1} different combinations $x_{n-k} \in \{0, 1\}$ $(k = 1, \ldots, k_1)$ and x_n is uniquely defined by x_{n-k} $(k = 1, \ldots, k_1)$, it follows that every element $x \in X$ is periodic starting with some place. Now, if $x \in \chi \cap c_{N_p}$ and $M = \sum_k p_k$, then there exists $N \in \mathbb{N}$, $N > k_1$, such that

$$\big| [N_p x]_m - [N_p x]_n \big| < \frac{p_1}{8M} \quad \text{and} \quad p_n |[N_p x]_{n-1}| < \frac{p_1}{8}$$

for all $m, n \geq N$. Hence

$$\begin{aligned} &\left| \sum_{k=1}^{k_1} p_k x_{n+1-k} - \sum_{k=1}^{k_1} p_k x_{n-k} \right| \\ &\qquad \leq P_n \big| [N_p x]_n - [N_p x]_{n-1} \big| + p_n |[N_p x]_{n-1}| + 2 \sum_{k=k_1+1}^{\infty} p_k < \frac{p_1}{2} \end{aligned}$$

for each $n > N$. So $\chi \cap c_{N_p} \subset \varphi + X$.

Let $x^{(1)}, \ldots, x^{(n_0)}$ be elements of X and let $\nu_0 \in \mathbb{N}$, $\nu_0 \leq n_0$, be the maximal integer such that $N := \cap \big\{ \operatorname{supp} x^{(i)} \,|\, i = 1, \ldots, \nu_0 \big\}$ is infinite. Let (k_i) be the index sequence of all elements in N. We consider the matrix $A = (a_{nk})$ with

$$a_{nk} = \begin{cases} n & \text{if } k = k_{2n} \\ -n & \text{if } k = k_{2n-1} \\ 0 & \text{otherwise} \end{cases} \qquad (n, k \in \mathbb{N}).$$

It is easy to see that $\varphi + X \subset c_A$. So $\chi \cap c_{N_p} \subset c_A$. On the other hand, A is not conservative, so that $\ell^\infty \cap c_{N_p} \not\subset c_A$. Thus N_p is not potent ∎

Now, for instance, $bs \subset \ell^\infty \cap c_{R_p}$ implies that $\ell^\infty \cap c_{R_p}$ has the Hahn property. So in a first step on the way to the goal aimed at we give a characterization of those **potent** regular Riesz matrices fulfilling this sufficient condition.

Proposition 3.4. *Let (in addition) $p = (p_k) \notin \ell^1$ and $\left(\frac{p_n}{P_n}\right) \in c_0$. Then the following statements are equivalent:*

(a) $bs \subset c_{R_p}$

(b) $bs \subset c_{N_p}$

(c) $bs \subset c_{0R_p} \cap c_{0N_p} q$

(d) $\lim_n \frac{1}{P_n} \sum_{k=1}^{n-1} |p_k - p_{k+1}| = 0.$

Proof. This is an obvious consequence of [3: Exercise 2.4.19] ∎

Examples 3.5.

(a) Let $p = (p_n) := (n)$. Obviously, (p_n) is monotonically increasing, $\left(\frac{P_n}{p_n}\right) \notin \ell^\infty$, $\left(\frac{p_n}{P_n}\right) \in c_0$ and $\left(\frac{n\,p_n}{P_n}\right) \in \ell^\infty$. Therefore, $c \subsetneq \ell^\infty \cap c_{R_p} = \ell^\infty \cap c_{C_1} \subsetneq c_{R_p} = c_{C_1}$ and R_p is potent (cf. Remark 3.1(b) and (c)). In particular, $\ell^\infty \cap c_{R_p}$ has the Hahn property since $\ell^\infty \cap c_{C_1}$ has (because $bs \oplus \mathrm{Sp}\,\{e\} \subset \ell^\infty \cap c_{C_1}$, cf. [1: Theorem 3.4]).

We should note that $\ell^\infty_{\Sigma_p} \oplus \mathrm{Sp}\,\{e\} \subset \ell^\infty \cap c_{R_p} \supset bs \oplus \mathrm{Sp}\,\{e\}$ and that $\ell^\infty_{\Sigma_p} \oplus \mathrm{Sp}\,\{e\}$ does not have the matrix Hahn property (cf. Example 2.9) whereas both $\ell^\infty \cap c_{R_p}$ and $bs \oplus \mathrm{Sp}\,\{e\}$ have the Hahn property.

(b) If $p \in c \backslash c_0$, then the assumptions and assertion (d) of Proposition 3.4 are fulfilled, thus $bs \subset c_{0R_p} \cap c_{0N_p}$, and consequently both $\ell^\infty \cap c_{R_p}$ and $\ell^\infty \cap c_{N_p}$ have the Hahn property.

(c) If $p_n \in \mathbb{N}_r \;\; (n \in \mathbb{N})$ for some $r \in \mathbb{N}$, then $\ell^\infty \cap c_{R_p}$ has the Hahn property by Proposition 2.8 since $\ell^\infty_{\Sigma_p} \oplus \mathrm{Sp}\{e\} \subset \ell^\infty \cap c_{R_p}$. Note, in this case we have $p \notin \ell^1$ and $\left(\frac{p_n}{P_n}\right) \in c_0$, but p fails in general the condition of Proposition 3.4(d) as, for instance, $p = (1, 2, 1, 2, \ldots)$ shows.

For the proof of the promised main result we need two lemmas.

Lemma 3.6. *Let $n \in \mathbb{N}$ and let $p_1, \ldots, p_n$ be a (finite) sequence of numbers. Then there exists a partition $(M_1, \ldots, M_t)$ of $\{1, \ldots, n\}$ such that $t \leq 2\sqrt{n}$ and $(p_i)_{i \in M_j}$ $(j \in \mathbb{N}_t)$ is monotone.*

Proof. A non-decreasing subsequence of $(p_i)_{i=1}^n$ is temporarily called a chain and an r-chain when its length is r. Let $s \in \mathbb{N}$ be the maximal length of all chains of $(p_i)_{i=1}^n$.

First we suppose $s \leq \sqrt{n}$. We choose all possible indices $n_1^1, \ldots, n_{k_1}^1$ with $n_1^1 < n_2^1 < \ldots < n_{k_1}^1$ such that $p_{n_j^1}$ is the beginning of some s-chain $(j = 1, \ldots, k_1)$. Note that for every $j = 1, \ldots, k_1 - 1$ we have $p_{n_j^1} > p_{n_{j+1}^1}$.

(Otherwise, $p_{n_j^1}, p_{n_{j+1}^1}$ would be the beginning of an $(s+1)$-chain.) So $(p_{n_j^1})_{j=1}^{k_1}$ is decreasing.

Now, we exclude all the elements $p_{n_j^1}$ $(j \in \mathbb{N}_{k_1})$ from the sequence $(p_i)_{i=1}^n$ and consider all the remaining elements in the original order with the original indices. The maximal length of all sorts of chains is now $s-1$. We pick out all possible indices $n_1^2, \ldots, n_{k_2}^2$ with $n_1^2 < n_2^2 < \ldots < n_{k_2}^2$ such that $p_{n_j^2}$ is the beginning of some $(s-1)$-chain $(j = 1, \ldots, k_2)$. Again, the sequence $(p_{n_j^2})_{j=1}^{k_2}$ is decreasing.

We now exclude from the (already reduced) sequence $(p_i)_{i=1}^n$ also all the elements $p_{n_j^2}$ $(j \in \mathbb{N}_{k_2})$ and consider the remaining sequence where the elements are listed in the original order with the original indices.

Continuing this procedure, we get $s-2$ decreasing subsequences $(p_{n_j^i})_{j=1}^{k_i}$ $(i \in \mathbb{N}_{s-2})$ of $(p_i)_{i=1}^n$. We now exclude from the sequence $(p_i)_{i=1}^n$ all the elements $p_{n_j^i}$ $(i \in \mathbb{N}_{s-2},\ j \in \mathbb{N}_{k_i})$ and consider the remaining sequence where the elements are listed in the original order with original indices. Let $t_1, \ldots, t_\nu$ (with $t_1 < t_2 < \ldots < t_\nu$) be the indices of it. The maximal length of its chains is obviously 2. Let l_1 be the minimal index such that $p_{t_{l_1}}$ is the beginning of some 2-chain and let $m_1 > l_1$ be the minimal index such that $p_{t_{l_1}} \le p_{t_{m_1}}$. Now we choose the minimal index l_2 with $l_2 > m_1$ such that $p_{t_{l_2}}$ is the beginning of some 2-chain and pick out the minimal index m_2 with $m_2 > l_2$ such that $p_{t_{l_2}} \le p_{t_{m_2}}$. Continuing this procedure we obtain the minimal index $l_{k_{s-1}}$ with $l_{k_{s-1}} > m_{k_{s-1}-1}$ such that $p_{t_{l_{k_{s-1}}}}$ is the beginning of some 2-chain and we may pick out the minimal index $m_{k_{s-1}}$ with $m_{k_{s-1}} > l_{k_{s-1}}$ such that $p_{t_{l_{k_{s-1}}}} \le p_{t_{m_{k_{s-1}}}}$.

Note that $p_{t_{l_j-1}} > p_{t_{m_j}}$: Namely, if $l_j - 1 > m_{j-1}$ $(m_0 := 0)$, this statement follows since l_j is the minimal index greater than m_{j-1} such that $p_{t_{l_j}}$ is the beginning of some 2-chain, and if $l_j - 1 = m_{j-1}$ it follows since otherwise we would obtain the 3-chains $p_{t_{l_{j-1}}} \le p_{t_{m_{j-1}}} \le p_{t_{m_j}}$ $(j \in \mathbb{N}_{k_{s-1}})$. We also have $p_{t_{m_{j-1}-1}} > p_{t_{l_j}}$ because otherwise $p_{t_{m_{j-1}-1}} \le p_{t_{l_j}} \le p_{t_{m_j}}$ $(j = 2, \ldots, k_{s-1})$ would be 3-chains.

Evidently, $p_{t_\alpha} > p_{t_{\alpha+1}}$ for every $j \in \{1, \ldots, k_{s-1} - 1\}$, $\alpha \in \{m_{j-1} + 1, \ldots, l_j - 2\}$ and $\alpha \in \{m_{k_{s-1}}, \ldots, \nu\}$. Moreover, we have in fact $p_{t_\alpha} > p_{t_{\alpha+1}}$ for every $j \in \mathbb{N}_{k_{s-1}}$ and $\alpha \in \{l_j + 1, \ldots, m_j - 2\}$. Namely, if $j \in \mathbb{N}_{k_{s-1}}$ and $\alpha \in \{l_j + 1, \ldots, m_j - 2\}$ is the minimal integer such that $p_{t_\alpha} \le p_{t_{\alpha+1}}$, then $p_{t_{\alpha+1}} > p_{t_{m_j}}$ (otherwise $p_{t_\alpha} \le p_{t_{\alpha+1}} \le p_{t_{m_j}}$ would be a 3-chain); hence $p_{t_{l_j}} > p_{t_{\alpha+1}} > p_{t_{m_j}}$, yielding a contradiction.

Summarizing, we constructed two increasing sequences

$$\begin{aligned} &p_{t_{l_1}} > p_{t_{l_1+1}} > \ldots > p_{t_{m_1-1}} > p_{t_{l_2}} > \\ &\qquad \ldots > p_{t_{m_2-1}} > \ldots > p_{t_{l_{k_{s-1}}}} > \ldots > p_{t_{m_{k_{s-1}}-1}} \\ &p_{t_1} > p_{t_2} > \ldots > p_{t_{l_1-1}} > p_{t_{m_1}} > \\ &\qquad \ldots > p_{t_{l_2-1}} > \ldots > p_{t_{m_{k_{s-1}}}} > \ldots > p_{t_\nu}. \end{aligned}$$

Altogether we divided the sequence $(p_i)_{i=1}^n$ into s, $s \le \sqrt{n}$, monotone (decreasing) subsequences.

Now, we suppose $s > \sqrt{n}$. We extract from $(p_i)_{i=1}^n$ some s-chain (excluding the referring elements from $(p_i)_{i=1}^n$). If the maximal length s_1 of the new sequence is greater than $\sqrt{n}$, we extract from $(p_i)_{i=1}^n$ some s_1-chain. We continue this procedure until the maximal length s_k of the chain is less than or equal to $\sqrt{n}$. The number k of extracted chains is less than or equal to $\sqrt{n}$. Now in the same way as in the first part of the proof we divide the remaining sequence into s_k, $s_k \le \sqrt{n}$, decreasing subsequences. Altogether, we divided the sequence $(p_i)_{i=1}^n$ into $k+s_k$, $k+s_k \le 2\sqrt{n}$ monotone subsequences ∎

Lemma 3.7. *If*

$$\frac{1}{P_n} \sum_{\substack{s\in I \\ N_s\cap[1,n]\neq\emptyset}} \left(\sum_{\substack{j\in\mathbb{N} \\ n_{s,j+1}\le n}} |p_{n_{sj}} - p_{n_{s,j+1}}| + \sum_{\substack{j\in\mathbb{N} \\ n_{sj}\le n<n_{s,j+1}}} p_{n_{sj}} \right) \to 0 \qquad (3.2)$$

as $n \to \infty$ holds for some partition $N = (N_i|\, i \in I)$, then $bs(N) \subset c_{0R_p} \cap c_{0N_p} \subset c_{R_p} \cap c_{N_p}$.

Proof. Let $x \in bs(N)$ be fixed. For any fixed $n \in \mathbb{N}$ we get

$$\begin{aligned} &\frac{1}{P_n}\left|\sum_{k=1}^n p_k x_k\right| \\ &\quad = \frac{1}{P_n}\left| \sum_{\substack{s\in I \\ N_s\cap[1,n]\neq\emptyset}} \sum_{\substack{j\in\mathbb{N} \\ n_{sj}\le n}} p_{n_{sj}} x_{n_{sj}} \right| \\ &\quad \le \frac{1}{P_n} \sum_{\substack{s\in I \\ N_s\cap[1,n]\neq\emptyset}} \|(x_{n_{sj}})_j\|_{bs} \left(\sum_{\substack{j\in\mathbb{N} \\ n_{s,j+1}\le n}} |p_{n_{sj}} - p_{n_{s,j+1}}| + \sum_{\substack{j\in\mathbb{N} \\ n_{sj}\le n<n_{s,j+1}}} p_{n_{sj}} \right) \\ &\quad \le \frac{\|x\|_{bs(N)}}{P_n} \sum_{\substack{s\in I \\ N_s\cap[1,n]\neq\emptyset}} \left(\sum_{\substack{j\in\mathbb{N} \\ n_{s,j+1}\le n}} |p_{n_{sj}} - p_{n_{s,j+1}}| + \sum_{\substack{j\in\mathbb{N} \\ n_{sj}\le n<n_{s,j+1}}} p_{n_{sj}} \right). \end{aligned}$$

Hence $R_p x \in c_0$. Considering the Nörlund method for a fixed $n \in \mathbb{N}$ we get

$$\begin{aligned}
&\frac{1}{P_n}\left|\sum_{k=1}^{n} p_{n-k+1} x_k\right| \\
&\quad = \frac{1}{P_n}\left|\sum_{\substack{s\in I\\ N_s\cap[1,n]\neq\emptyset}} \sum_{\substack{j\in\mathbb{N}\\ n_{sj}\le n}} p_{n-n_{sj}+1} x_{n_{sj}}\right| \\
&\quad = \frac{1}{P_n}\left|\sum_{\substack{s\in I\\ N_s\cap[1,n]\neq\emptyset}} \sum_{\substack{j\in\mathbb{N}\\ n_{sj}\le n}} p_{n_{sj}} x_{n-n_{sj}+1}\right| \\
&\quad \le \frac{1}{P_n} \sum_{\substack{s\in I\\ N_s\cap[1,n]\neq\emptyset}} \sup_j \left|\sum_{k=1}^{j} x_{n-n_{sk}+1}\right| \\
&\qquad \times \left(\sum_{\substack{j\in\mathbb{N}\\ n_{s,j+1}\le n}} |p_{n_{sj}} - p_{n_{s,j+1}}| + \sum_{\substack{j\in\mathbb{N}\\ n_{sj}\le n< n_{s,j+1}}} p_{n_{sj}} \right) \\
&\quad \le \frac{1}{P_n} \sum_{\substack{s\in I\\ N_s\cap[1,n]\neq\emptyset}} 2\,\|(x_{n_{sj}})_j\|_{bs} \\
&\qquad \times \left(\sum_{\substack{j\in\mathbb{N}\\ n_{s,j+1}\le n}} |p_{n_{sj}} - p_{n_{s,j+1}}| + \sum_{\substack{j\in\mathbb{N}\\ n_{sj}\le n< n_{s,j+1}}} p_{n_{sj}} \right) \\
&\quad \le \frac{2\|x\|_{bs(N)}}{P_n} \\
&\qquad \times \sum_{\substack{s\in I\\ N_s\cap[1,n]\neq\emptyset}} \left(\sum_{\substack{j\in\mathbb{N}\\ n_{s,j+1}\le n}} |p_{n_{sj}} - p_{n_{s,j+1}}| + \sum_{\substack{j\in\mathbb{N}\\ n_{sj}\le n< n_{s,j+1}}} p_{n_{sj}} \right).
\end{aligned}$$

Thus $N_p x \in c_0$ ∎

Theorem 3.8. *If $p \notin \ell^1$ and R_p is a potent matrix, then there exists a partition $N = (N_i|\, i \in I)$ of $\mathbb{N}$ satisfying (2.1) and $bs(N) \subset \ell^\infty \cap c_{R_p} \cap c_{N_p}$. In particular, by Theorem 2.6, $\ell^\infty \cap c_{R_p} \cap c_{N_p}, \ell^\infty \cap c_{R_p}$ and $\ell^\infty \cap c_{N_p}$ have the Hahn property.*

As an immediate consequence, we get the following extension of the characterization of the potency of weighted means due to Kuttner and Parameswaran (cf. [6] and also [3, 4]).

Corollary 3.9.

(i) *If R_p is conservative and non-regular, that is $p \in \ell^1$, then $\ell^\infty \cap c_{R_p}$ has the Hahn property (and R_p is potent).*

(ii) *If R_p is regular, that is $p \notin \ell^1$, then the following statements are equivalent:*

(a) *R_p is potent, that is $\ell^\infty \cap c_{R_p}$ has the matrix Hahn property.*

(b) $\left(\frac{p_n}{P_n}\right) \in c_0$.

(c) $\left(\frac{1}{P_n} \max_{k \le n} p_k\right) \in c_0$.

(d) *$R_p \in \mathrm{KG}$, that is each matrix A with $\chi(\ell^\infty \cap c_{R_p}) \subset c_A$ is conservative.*

(e) *$\ell^\infty \cap c_{R_p}$ has the separable Hahn property.*

(f) *$\ell^\infty \cap c_{R_p}$ has the Hahn property.*

By Remark 3.2, Theorem 3.3 and Theorem 3.8 we also have a full characterization of the potent conservative Nörlund matrices.

Corollary 3.10.

(i) *If N_p is conservative and non-regular, that is $(\frac{p_n}{P_n}) \in c \setminus c_0$, then $\ell^\infty \cap c_{N_p}$ has the Hahn property (and N_p is potent).*

(ii) *If N_p is regular, that is $(\frac{p_n}{P_n}) \in c_0$, then the following statements are equivalent:*

(a) *N_p is potent, that is $\ell^\infty \cap c_{N_p}$ has the matrix Hahn property.*

(b) $p \notin \ell^1$.

(c) *$\ell^\infty \cap c_{N_p}$ has the separable Hahn property.*

(d) *$\ell^\infty \cap c_{N_p}$ has the Hahn property.*

Proof of Theorem 3.8. Aiming to a partition $N = (N_i | i \in I)$ of $\mathbb{N}$ satisfying (2.1) and $bs(N) \subset c_{R_p} \cap c_{N_p}$, it is sufficient to find a partition which fulfils condition (3.2) in Lemma 3.7.

Since $(\frac{p_n}{P_n}) \in c_0$ (see Remark 3.1(c)), we may choose $m_0 \in \mathbb{N}$ such that $p_n < \frac{1}{2}P_n = \frac{1}{2}P_{n-1} + \frac{1}{2}p_n$ or, equivalently, $p_n < P_{n-1}$, for each $n \ge m_0$. Let $i_0 \in \mathbb{N}$ be such that $4^{i_0 - 1} \le P_{m_0} < 4^{i_0}$. For every $i \ge i_0$ we denote by m_i the minimal integer n such that $P_n \ge 4^i$. Then $P_{m_i} = P_{m_i - 1} + p_{m_i} < 2P_{m_i - 1} < 2 \cdot 4^i$, so $P_{m_i} > 8^{-1} P_{m_{i+1}}$ and $P_{m_i} < 2^{-1} P_{m_{i+1}}$ $(i \ge i_0)$. Since R_p is potent, $\varepsilon_i := \frac{\max\{p_k | k \le m_i\}}{P_{m_i}} \to 0$ $(i \to \infty)$ by Remark 3.1(c). Without loss of generality, we may assume that $\varepsilon_i < 1$ $(i \ge i_0)$. (Otherwise, we may choose a bigger m_0.) For every $i \ge i_0$ we choose the minimal integer $j_i \in \mathbb{N}$ such that $m_i^{1/j_i} < \varepsilon_i^{-1/2}$. Since $\varepsilon_i = \frac{\max\{p_k | k \le m_i\}}{P_{m_i}} \ge \frac{1}{m_i}$, we have $m_i \ge \varepsilon_i^{-1}$ $(i \ge i_0)$. Hence $j_i > 2$ $(i \ge i_0)$. Note that $m_i^{1/j_i} \ge \varepsilon_i^{-1/4}$ $(i \ge i_0)$. [Otherwise $m_i^{1/[(j_i+2)/2]} \le m_i^{2/j_i} < \varepsilon_i^{-1/2}$, so $\frac{j_i}{2} + 1 \ge \left[\frac{j_i}{2}\right] + 1 \ge j_i$, which is equivalent to $j_i \le 2$ and contradicts $j_i > 2$.]

Set $\alpha_i = \frac{P_{m_{i+1}}}{m_{i+1}}$ and $\beta_i = \max\{p_k | k \le m_{i+1}\}$ $(i \in \mathbb{N})$. For every $i \in \mathbb{N}$

and $s = 2, \dots, j_{i+1} - 2$ we use the notations

$$\begin{aligned} M_{i1} &= \Big\{ m_i < k \le m_{i+1} \,|\, p_k \le m_{i+1}^{1/j_{i+1}} \alpha_i \Big\} \\ M_{is} &= \Big\{ m_i < k \le m_{i+1} \,|\, m_{i+1}^{(s-1)/j_{i+1}} \alpha_i < p_k \le m_{i+1}^{s/j_{i+1}} \alpha_i \Big\} \\ M_{i,j_{i+1}-1} &= \Big\{ m_i < k \le m_{i+1} \,|\, m_{i+1}^{(j_{i+1}-2)/j_{i+1}} \alpha_i < p_k \Big\}. \end{aligned}$$

Set $\nu_{is} = |M_{is}| \;\; (i \in \mathbb{N}, s = 1, \dots, j_{i+1} - 1)$. Note,

$$\nu_{is} \le \frac{m_{i+1}}{m_{i+1}^{(s-1)/j_{i+1}}} = m_{i+1}^{(j_{i+1}+1-s)/j_{i+1}}$$

since otherwise

$$\begin{aligned} P_{m_{i+1}} &\ge \nu_{is} m_{i+1}^{(s-1)/j_{i+1}} \alpha_i \\ &> m_{i+1}^{(j_{i+1}+1-s)/j_{i+1}} m_{i+1}^{(s-1)/j_{i+1}} \alpha_i \\ &= m_{i+1} \alpha_i \\ &= P_{m_{i+1}}. \end{aligned}$$

By Lemma 3.6, for every $i \in \mathbb{N}$ and $s = 1, \dots, j_{i+1} - 1$ we may choose a partition $(S_{is1}, \dots, S_{isk_{is}})$ of M_{is} with $k_{is} \le 2\sqrt{\nu_{is}}$ such that $(p_\nu)_{\nu \in S_{ist}}$ is monotone $(t = 1, \dots, k_{is})$. Let $e_{isk} := |S_{isk}|$ and let $(\xi_{iskl})_{l=1}^{e_{isk}}$ be the finite sequence of all elements of S_{isk} arranged in ascending ordering ($i \in \mathbb{N}$, $s = 1, \dots, j_{i+1} - 1$ and $k = 1, \dots, k_{is}$). Then

$$\begin{aligned} &\sum_{k=1}^{k_{is}} \left(p_{\xi_{isk1}} + \sum_{l=1}^{e_{isk}-1} |p_{\xi_{iskl}} - p_{\xi_{i,s,k,l+1}}| + p_{\xi_{iske_{isk}}} \right) \\ &\qquad = \sum_{k=1}^{k_{is}} 2 \max \left\{ p_{\xi_{isk1}}, p_{\xi_{iske_{isk}}} \right\} \\ &\qquad \le 2 k_{is} m_{i+1}^{s/j_{i+1}} \alpha_i \\ &\qquad \le 4 \sqrt{\nu_{is}} m_{i+1}^{s/j_{i+1}} \alpha_i \\ &\qquad \le 4 m_{i+1}^{(j_{i+1}+1-s)/2j_{i+1}} m_{i+1}^{s/j_{i+1}} \alpha_i \\ &\qquad = 4 m_{i+1}^{(j_{i+1}+1+s)/2j_{i+1}} \alpha_i \end{aligned}$$

for $s = 1, \dots j_{i+1} - 2$ and

$$\begin{aligned}\sum_{k=1}^{k_{is}} \Big(p_{\xi_{isk1}} + \sum_{l=1}^{e_{isk}-1} |p_{\xi_{iskl}} - p_{\xi_{i,s,k,l+1}}| + p_{\xi_{iske_{isk}}} \Big) \\ &= \sum_{k=1}^{k_{is}} 2 \max \{ p_{\xi_{isk1}}, p_{\xi_{iske_{isk}}} \} \\ &\le 4\sqrt{\nu_{is}}\beta_i \\ &\le 4 m_{i+1}^{1/j_{i+1}} \beta_i\end{aligned}$$

for $s = j_{i+1} - 1$ $(i \in \mathbb{N})$. Hence for every $i \ge i_0$ we get

$$\begin{aligned}\frac{1}{P_{m_{i+1}}} \sum_{s=1}^{j_{i+1}-1} \sum_{k=1}^{k_{is}} \Big(p_{\xi_{isk1}} + \sum_{l=1}^{e_{isk}-1} |p_{\xi_{iskl}} - p_{\xi_{i,s,k,l+1}}| + p_{\xi_{iske_{isk}}} \Big) \\ &\le \frac{1}{P_{m_{i+1}}} \Big(\sum_{s=1}^{j_{i+1}-2} 4 m_{i+1}^{(j_{i+1}+1+s)/2j_{i+1}} \alpha_i + 4 m_{i+1}^{1/j_{i+1}} \beta_i \Big) \\ &= 4 \frac{m_{i+1}^{(j_{i+1}+1)/2j_{i+1}} \alpha_i}{m_{i+1}\alpha_i} \left(\frac{m_{i+1}^{(j_{i+1}-1)/2j_{i+1}} - 1}{m_{i+1}^{1/2j_{i+1}} - 1} - 1 \right) + 4 \frac{\beta_i m_{i+1}^{1/j_{i+1}}}{P_{m_{i+1}}} \\ &\le 4 m_{i+1}^{(-j_{i+1}+1)/2j_{i+1}} \frac{m_{i+1}^{(j_{i+1}-1)/2j_{i+1}}}{4^{-1} m_{i+1}^{1/2j_{i+1}}} + 4\varepsilon_{i+1}\varepsilon_{i+1}^{-1/2} \\ &= 16 m_{i+1}^{-1/2j_{i+1}} + 4\varepsilon_{i+1}^{1/2} \\ &\le 16 \varepsilon_{i+1}^{1/8} + 4\varepsilon_{i+1}^{1/2}.\end{aligned}$$

Aiming to a definition of N_i $(i \in \mathbb{N})$, for any subset $N \subset \mathbb{N}$ and $n \in \mathbb{N}$ we use the notation

$$N|_n = \{ k \in N | \, k \le n \}.$$

Set

$$\begin{aligned} N_1|_{m_{i_0}} &= \{1, \dots, m_{i_0}\} \\ N_j|_{m_{i_0}} &= \emptyset \quad (j > 1). \end{aligned}$$

Let $s_1^0, \dots, s_{t_0}^0$ be all indices such that $M_{i_0 s_t^0} \neq \emptyset$ $(t = 1, \dots, t_0)$. We set

$$N_j|_{m_{i_0+1}} = \begin{cases} N_j|_{m_{i_0}} \cup S_{i_0 s_t^0 k} & \text{for } j = \sum_{\tau=1}^{t-1} k_{i_0 s_\tau^0} + k \\ & \quad (t = 1, \dots, t_0; k = 1, \dots, k_{i_0 s_t^0}) \\ N_j|_{m_{i_0}} & \text{for } j > \sum_{\tau=1}^{t_0} k_{i_0 s_\tau^0}. \end{cases}$$

Continuing inductively for $i > i_0$, let $s_1^{i-i_0}, \dots, s_{t_{i-i_0}}^{i-i_0}$ be all indices such that $M_{is_t^{i-i_0}} \neq \emptyset$ $(t = 1, \dots, t_{i-i_0})$. Then we set

$$N_j|_{m_{i+1}} = \begin{cases} N_j|_{m_i} \cup S_{is_t^{i-i_0}k} & \text{for } j = \sum_{\tau=1}^{t-1} k_{is_\tau^{i-i_0}} + k \\ & \qquad (t = 1, \dots, t_{i-i_0}; k = 1, \dots, k_{is_t^{i-i_0}} \\ N_j|_{m_i} & \text{for } j > \sum_{\tau=1}^{t_{i-i_0}} k_{is_\tau^{i-i_0}}. \end{cases}$$

Let I be the set of all indices $i \in \mathbb{N}$ such that $N_i \neq \emptyset$. If I is infinite, then, by our construction, every N_i is infinite. If I is finite, then without loss of generality (cf. Remark 2.1(d)) we may assume that N_i is infinite $(i \in I)$. Let $n \in \mathbb{N}$ with $m_i < n \leq m_{i+1}$ and $i \geq i_0$ be fixed. Then

$$\begin{aligned} B_n :&= \frac{1}{P_n} \sum_{\substack{s \in I \\ N_s \cap [1,n] \neq \emptyset}} \left(\sum_{\substack{j \in \mathbb{N} \\ n_{s,j+1} \leq n}} |p_{n_{sj}} - p_{n_{s,j+1}}| + \sum_{\substack{j \in \mathbb{N} \\ n_{sj} \leq n < n_{s,j+1}}} p_{n_{sj}} \right) \\ &\leq \frac{1}{P_{m_i}} \Bigg(2 \sum_{k=1}^{m_{i_0}} p_k \\ &\quad + \sum_{\tau=i_0}^{i} \sum_{s=1}^{j_{\tau+1}-1} \sum_{k=1}^{k_{\tau s}} \Big(p_{\xi_{isk1}} + \sum_{l=1}^{e_{\tau sk}-1} |p_{\xi_{\tau skl}} - p_{\xi_{\tau,s,k,l+1}}| + p_{\xi_{iske_{isk}}} \Big) \Bigg) \\ &\leq \frac{8}{P_{m_{i+1}}} \left(2P_{m_{i_0}} + \sum_{\tau=i_0}^{i} \left(16\varepsilon_{\tau+1}^{1/8} + 4\varepsilon_{\tau+1}^{1/2} \right) P_{m_{\tau+1}} \right) \\ &\leq \frac{16}{2^{i-i_0+1}} + 32 \sum_{\tau=i_0}^{i} \frac{1}{2^{i-\tau}} \left(4\varepsilon_{\tau+1}^{1/8} + \varepsilon_{\tau+1}^{1/2} \right). \end{aligned}$$

Consider the matrix $A = (a_{i\tau})$ with $a_{i\tau} = 2^{\tau-i}$ for $\tau \leq i$ and $a_{i\tau} = 0$ otherwise. Evidently, A is regular for null sequences, so it sums the null sequence $\left(4\varepsilon_{\tau+i_0}^{1/8} + \varepsilon_{\tau+i_0}^{1/2}\right)_\tau$ to zero. Thus $B_n \to 0$ $(n \to \infty)$. Now, applying Lemma 3.7, we get the desired inclusion $bs(N) \subset c_{R_p} \cap c_{N_p}$ ∎

The following example shows that there exists a potent Riesz method R_p such that $bs(N) \not\subset \ell^\infty \cap c_{R_p}$ for each finite partition $N = (N_1, \dots, N_s)$ of $\mathbb{N}$. Since R_p is potent, $\ell^\infty \cap c_{R_p}$ has the Hahn property by Theorem 3.8 and, by the proof of this theorem, we have $bs(M) \subset \ell^\infty \cap c_{R_p}$ for a suitable partition $M = (M_i | i \in I)$ of $\mathbb{N}$. This shows that, in the proof of Theorem 3.8, we applied Theorem 2.6 in its full generality.

Example 3.11. Let $k_1 = 1$ and $k_{i+1} = k_i + (i+1)$ $(i \in \mathbb{N})$. We consider the Riesz method R_p with

$$p_1 = p_2 = 1, \quad p_{k_i} = p_{k_i+1} = \frac{1}{2^i i}$$
$$p_{k_i+j} = \sum_{l=0}^{j-1} p_{k_i+l} = \frac{2^{j-1}}{2^i i} = \frac{2^{j-i-1}}{i}$$

for $i, j \in \mathbb{N},\ i > 1,\ j = 2, \ldots, i$. Evidently, $(p_n) \in c_0$. Furthermore,

$$P_{k_l-1} = \sum_{i=1}^{l-1} \sum_{j=k_i}^{k_{i+1}-1} p_j = \sum_{i=1}^{l-1} \frac{1}{2^i i} 2^i = \sum_{i=1}^{l-1} \frac{1}{i} \to \infty \qquad (l \to \infty),$$

hence $p \notin \ell^1$. Consequently, R_p is regular and potent, and $\ell^\infty \cap c_{R_p}$ has the Hahn property by Theorem 3.8.

Now, we verify that *the inclusion $bs(N) \subset \ell^\infty \cap c_{R_p}$ fails for all finite partitions $N = (N_1, \ldots, N_s)$ of $\mathbb{N}$.* Since $\ell^\infty_{\Sigma_p} \subset c_{R_p}$, we get as an immediate consequence that *$bs(N) \not\subset \ell^\infty_{\Sigma_p}$ for all $s \in \mathbb{N}$ and partitions $N = (N_1, \ldots, N_s)$ of $\mathbb{N}$.*

Suppose, by contrast, that $bs(N) \subset \ell^\infty \cap c_{R_p}$ for some $s \in \mathbb{N}$ and partition $N = (N_1, \ldots, N_s)$ of $\mathbb{N}$. Then, on the one hand, by Proposition 3.4 we have

$$A_l := \frac{1}{P_{k_l-1}} \sum_{i=1}^{s} \sum_{\{j \mid n_{i,j+1} < k_l - 1\}} |p_{n_{ij}} - p_{n_{i,j+1}}| \to 0 \qquad (l \to \infty).$$

On the other hand, we get

$$\begin{aligned} A_l &\geq \frac{1}{P_{k_l-1}} \sum_{\nu=1}^{l-1} \sum_{i=1}^{s} \sum_{\{j \mid k_\nu \leq n_{ij} < n_{i,j+1} < k_{\nu+1}\}} |p_{n_{ij}} - p_{n_{i,j+1}}| \\ &\geq \frac{1}{P_{k_l-1}} \sum_{\nu=1}^{l-1} \sum_{i=1}^{s} \sum_{\{j \mid k_\nu \leq n_{ij} < n_{i,j+1} < k_{\nu+1}\}} |p_{n_{ij}} - p_{n_{ij}+1}| \end{aligned}$$

and, since

$$\left.\begin{aligned} &\left|\{(i,j) \mid k_\nu \leq n_{ij} < k_{\nu+1} \leq n_{i,j+1}\}\right| \leq s \\ &p_{k+1} - p_k > p_k - p_{k-1} > 0 \ \text{ for } k \in [k_\nu + 1, k_{\nu+1} - 2] \end{aligned}\right. \qquad (\nu \in \mathbb{N}),$$

the estimation

$$\begin{aligned}
A_l &\geq \frac{1}{P_{k_l-1}} \sum_{\nu=1}^{l-1} \sum_{j=k_\nu}^{k_{\nu+1}-2-s} (p_{j+1} - p_j) \\
&= \frac{1}{P_{k_l-1}} \sum_{\nu=1}^{l-1} (p_{k_{\nu+1}-1-s} - p_{k_\nu}) \\
&= \frac{1}{\sum_{\nu=1}^{l-1} \frac{1}{\nu}} \sum_{\nu=1}^{l-1} \frac{1}{\nu} \Big(\frac{1}{2^{1+s}} - \frac{1}{2^\nu} \Big) \\
&= \frac{1}{2^{1+s}} - \frac{1}{\sum_{\nu=1}^{l-1} \frac{1}{\nu}} \sum_{\nu=1}^{l-1} \frac{1}{\nu 2^\nu} \\
&\geq \frac{1}{2^{2+s}}
\end{aligned}$$

for l sufficiently large, contradicting $(A_l) \in c_0$.

The following example shows that there exist non-potent Riesz means R_p fulfilling $c \subsetneqq \ell^\infty \cap c_{R_p}$. In particular, the FK-space $(\ell^\infty \cap c_{R_p}, \|\cdot\|_\infty)$ is non-separable and does not have the matrix Hahn property (see [1: Theorems 2.5 and 5.1]), hence not any Hahn property.

Example 3.12. For any index sequence (n_ν) with $n_1 = 1$ and $n_\nu + 1 < n_{\nu+1}$ $(\nu \in \mathbb{N})$ we define $p = (p_n)$ inductively by

$$p_n = \begin{cases} 1 & \text{if } n < n_2 \\ P_{n-1} = \sum_{k=1}^{n-1} p_k & \text{if } n = n_\nu \\ p_{n_\nu} & \text{if } n_\nu < n < n_{\nu+1} \end{cases} \qquad (n \in \mathbb{N}, \nu \geq 2).$$

Using this definition of (p_n) we finally define (n_ν): Having fixed n_ν for a $\nu \in \mathbb{N}$ we choose an $n_{\nu+1}$ with $n_\nu + 1 < n_{\nu+1}$ such that $\frac{P_{n_{\nu+1}-1}}{p_{n_{\nu+1}-1}} > \nu$. Obviously, (p_n) is monotonically increasing, $(\frac{P_n}{p_n}) \notin \ell^\infty$ and $(\frac{p_n}{P_n}) \notin c_0$. Therefore, $c \subsetneqq \ell^\infty \cap c_{R_p} \subsetneqq c_{R_p} \subset c_{C_1}$ and R_p is not potent.

Note, $\ell^\infty \cap c_{R_p}$ does not contain any space $bs(N)$ since otherwise it would have the Hahn property by Theorem 2.6.

Now, we give an example showing that, in general, the bounded domains of a potent R_p and of the corresponding N_p are different.

Example 3.13.

(a) Let $p = (p_n) := (1, 2, 1, 2, \ldots)$. Then, by Theorem 3.8, both $\ell^\infty \cap c_{N_p}$ and $\ell^\infty \cap c_{R_p}$ have the Hahn property. Note that $\ell^\infty \cap c_{0N_p} \subsetneqq \ell^\infty \cap c_{0R_p}$ since $x = (2, -1, 2, -1, \ldots) \in \ell^\infty \cap c_{0R_p}$, but $x \notin \ell^\infty \cap c_{0N_p}$, and since $\ell^\infty \cap$

$c_{0N_p} \subset \ell^\infty \cap c_{0R_p}$ as we may easily check (note, $[R_p x]_{2n-1} = [N_p x]_{2n-1}$ and $\frac{p_{2n}}{P_{2n}} x_{2n} \to 0 \ \ (n \to \infty)$ for every $x \in \ell^\infty$).

(b) Let $p = (p_n) = (2^{n-1})$. Then $c_{R_p} = c$ since $(\frac{P_n}{p_n}) \in \ell^\infty$ (cf. Remark 3.1(b)). In particular, R_p is not potent. By contrast, N_p is coercive by the Schur Theorem (cf. [3: Theorem 2.4.1]), so that $\ell^\infty \cap c_{N_p}$ has obviously the Hahn property. More generally, we may assume that p satisfies (3.1) and $(\frac{p_n}{P_n}) \in c \setminus c_0$. Then R_p is not potent, but N_p is coercive (cf. Remark 3.2).

4. Intersection of Hahn spaces

In [1: Problem 7.1] the question is posed whether the intersection of Hahn spaces is also a Hahn space. In the first step we give positive results by presenting big classes of Hahn spaces such that the finite intersection of some of their members is also a Hahn space. These results prove the power of Theorem 2.6.

Remark 4.1.

(a) If E and F are spaces lying between $bs(N) \oplus \mathrm{Sp}\{e\}$ and ℓ^∞ for some partition N of $\mathbb{N}$, then E, F and also $E \cap F$ have the Hahn property by Theorem 2.6.

(b) If (p_n) satisfies the conditions in Theorem 3.8, then the bounded domains $\ell^\infty \cap c_{R_p}$ and $\ell^\infty \cap c_{N_p}$ and their intersection $\ell^\infty \cap c_{R_p} \cap c_{N_p}$ have the Hahn property.

In the following theorem we give a negative answer to [1: Problem 7.1] by presenting a big class of such examples. For that, we consider the set $\mathcal{T}$ of all thin sequences: A sequence is called *thin* if there exists an index sequence (k_ν) with $k_{\nu+1} - k_\nu \to \infty$ such that $x_k = 1$ if $k = k_\nu$ and $x_k = 0$ otherwise $(k, \nu \in \mathbb{N})$.

Theorem 4.2. *The intersection of Hahn spaces does not necessarily have any of the Hahn properties. In particular, we have the following results:*

(a) $(bs \oplus \mathrm{Sp}\{e\}) \cap (\ell^\infty \cdot \mathrm{Sp}\mathcal{T})$ *does not have the matrix Hahn property* (*which is necessary for both the separable Hahn property and the Hahn property*).

(b) *If* $N = (N_i | \, i \in I)$ *is a finite partition of* $\mathbb{N}$ *satisfying* (2.1), *then* $(bs(N) \oplus \mathrm{Sp}\,\{e\}) \cap (\ell^\infty \cdot \mathrm{Sp}\mathcal{T})$ *does not have any Hahn property.*

Proof.

(a) $bs \oplus \mathrm{Sp}\,\{e\}$ and $\ell^\infty \cdot \mathrm{Sp}\mathcal{T}$ have the Hahn property (cf. [1: Theorem 3.4] and [1: (4.6) in the proof of Theorem 4.1], respectively).

Now, we set $G = (bs \oplus \mathrm{Sp}\,\{e\}) \cap (\ell^\infty \cdot \mathrm{Sp}\,\mathcal{T})$ and prove $\chi(G) = \varphi$. Then we can conclude that G does not enjoy the matrix Hahn property: If I denotes the identity matrix, then obviously $\chi(G) = \varphi \subset c = c_I$, but $y = (y_k) \in G \setminus c_I$ where

$$y_k = \begin{cases} 1 & \text{if } k = 2^n \quad (n \in \mathbb{N}) \\ -1 & \text{if } k = 2^n + 1 \quad (n \in \mathbb{N}) \\ 0 & \text{otherwise.} \end{cases}$$

Note, $y \in \operatorname{Sp} \mathcal{T}$ since y takes only finitely many positive values and the length of the 'zero-gaps' of y tends monotonically to infinity.

The inclusion $\varphi \subset \chi(G)$ is obvious since $\varphi \subset G$. For a proof of the contrary inclusion we consider an arbitrary $x \in \chi \cap (G \setminus \varphi)$. Then

$$x = y \cdot \sum_{i=1}^{N} \alpha_i x^{(i)} \qquad \text{with} \begin{cases} N \in \mathbb{N} \\ \alpha_i \neq 0, \; x^{(i)} \in \mathcal{T} \quad (i = 1, \ldots, N) \\ y \in \ell^\infty. \end{cases}$$

First we estimate the least number of zeros and the utmost number of ones which x_ν takes for $\nu \leq k \quad (k \in \mathbb{N})$. By S_x, S_y and $S_{x^{(i)}}$ we denote the support of x, y and $x^{(i)}$, respectively. Thus $S_x \subset S_y \cap \bigcup_{i=1}^N S_{x^{(i)}} =: S_y \cap S$. So, x takes the value 1 at most for $k \in S$ (and the value 0 outside of S). Let $\alpha \in (0,1)$ be given and $j \in \mathbb{N}$, $j > 2$, be chosen such that

$$\frac{1}{j} < \alpha, \qquad \text{that is } (1-\alpha) - \alpha(j-1) < 0. \tag{4.1}$$

Because $x^{(i)} \in \mathcal{T} \quad (i = 1, \ldots, N)$, for $M := jN$ there exists a $k_0 \in \mathbb{N}$ such that for every $i \in \mathbb{N}_N$ the sequence $x^{(i)}$ takes at most once the value 1 in each interval $[k, k+M) \quad (k \geq k_0)$. Thus x takes the value 1 at most N times in each such interval. Let $k_\nu := k_0 + \nu M \quad (\nu \in \mathbb{N})$. Now, for $k \in \mathbb{N}$ we set

$$\begin{aligned} g(k) &= \big|\{\nu \leq k | \, x_\nu = 1\}\big| \\ f(k) &= k - g(k) = \big|\{\nu \leq k | \, x_\nu = 0\}\big| \\ a &= g(k_0) \\ b &= f(k_0) = k_0 - a. \end{aligned}$$

With that, for $\nu \in \mathbb{N}$ we get $g(k_\nu) \leq a + \nu N$ and therefore

$$f(k_\nu) = k_\nu - g(k_\nu) \geq k_0 + j\nu N - a - \nu N = k_0 - a + (j-1)\nu N.$$

Aiming to a contradiction we use that x is assumed to be a member of $bs \oplus \operatorname{Sp}\{e\}$, that is $x = y + \alpha e$ where $y \in bs$ and $\alpha \in \mathbb{R}$ are suitably chosen. Considering the special cases $\alpha = 0, \alpha = 1, \alpha > 1, \alpha < 0$ and $0 < \alpha < 1$ we get the contradiction $y \notin bs$ as follows:

The case $\alpha = 0$: $y_k = x_k - \alpha = x_k$, thus y is a 0-1-sequence which takes the value 1 at infinitely many positions since $x \notin \varphi$.

The case $\alpha = 1$: Then we have

$$y_k = x_k - 1 = \begin{cases} 0 & \text{if } x_k = 1 \\ -1 & \text{if } x_k = 0 \end{cases} \qquad (k \in \mathbb{N}),$$

thus y is a -1-0-sequence which takes the value -1 at infinitely many positions by the foregoing considerations.

The case $\alpha > 1$: Here, y takes only the negative values $1 - \alpha$ and $-\alpha$.

The case $\alpha < 0$: In this case y takes only the positive values $1 - \alpha$ and $-\alpha$.

The case $0 < \alpha < 1$: We have

$$y_k = x_k - \alpha = \begin{cases} 1 - \alpha > 0 & \text{if } x_k = 1 \\ -\alpha < 0 & \text{if } x_k = 0. \end{cases}$$

Using the notation in the foregoing (preparing) considerations we get

$$\begin{aligned} \sum_{k=1}^{k_\nu} y_k &\leq (1-\alpha)(a + \nu N) - \alpha(k_0 - a + (j-1)\nu N) \\ &= (1-\alpha)a - \alpha(k_0 - a) + \big((1-\alpha) - \alpha(j-1)\big)\nu N \\ &\to -\infty \quad (\nu \to \infty) \end{aligned}$$

(see (4.1)). Therefore $y \notin bs$ which contradicts $x \in bs \oplus \mathrm{Sp}\,\{e\}$.

(b) We set

$$\begin{aligned} F &= (bs(N) \oplus \mathrm{Sp}\,\{e\}) \cap (\ell^\infty \cdot \mathrm{Sp}\,\mathcal{T}) \\ G &= (bs \oplus \mathrm{Sp}\,\{e\}) \cap (\ell^\infty \cdot \mathrm{Sp}\,\mathcal{T}). \end{aligned}$$

Obviously, we have $F \subset G$ (since I is assumed to be finite), thus $\chi(F) \subset \chi(G) = \varphi$ where the last identity has been proved in part (a). Now we proceed analogously to the beginning of the proof of assertion (a): We consider the identity matrix I and get obviously $\chi(F) = \varphi \subset c = c_I$, but $y = (y_k) \in F \setminus c_I$ where

$$y_k = \begin{cases} 1 & \text{if } k = k_{1,2^\nu} \quad (\nu \in \mathbb{N}) \\ -1 & \text{if } k = k_{1,2^\nu+1} \quad (\nu \in \mathbb{N}) \\ 0 & \text{otherwise} \end{cases} \qquad (k \in \mathbb{N})$$

and $(k_{1\nu})$ is the sequence of all members of N_1 arranged in the ascending order. Note that here $y \in bs(N)$ is obvious and $y \in \mathrm{Sp}\,\mathcal{T}$ since y may easily be represented as difference of thin sequences ∎

Problems. In addition to the problems in [1: Section 7] we pose the following:

1. Still we do not know whether there exists a bounded domain of a matrix having the matrix Hahn property but not the (separable) Hahn property.

2. Find a bounded domain $\ell^\infty \cap c_A$ having any Hahn property and a sequence space E with $\ell^\infty \cap c_A \subset E \subset \ell^\infty$ and having not this Hahn property. (Note, in this situation we have necessarily $E \neq \ell^\infty \cap c_A + \chi(E)$.)

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