Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 22 (2003), No. 4, 779–788

A System of Multi-Valued Generalized Order Complementarity Problems in Ordered Metric Spaces

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Abstract. Some existence results for a system of multi-valued generalized order complementarity problems are established in terms of fixed point theorems for multi-valued increasing-type mappings and decreasing mappings in ordered metric spaces.

Keywords: *Multi-valued generalized order complementarity problems, fixed point theorems, multi-valued increasing-type mappings, multi-valued decreasing mappings, ordered metric spaces*

AMS subject classification: 90C33, 54H25, 47H10, 47H07

1. Introduction

The complementarity problem, as an interesting and important subject of current mathematics, has been well studied by many authors. For details we can refer to [1, 3, 4, 6 - 11] and the references therein. The order complementarity problem, which is an extension of the classical complementarity problem, has obtained increasing attention (see, for instance, [1, 4, 6, 8 - 11]) for its potential applications to economics, mechanics and electric engineering, game theory, and optimization, etc. In 1991, a class of generalized order complementarity problems was introduced and studied by Isac and Kostreva [10] in ordered Banach spaces. The theoretic framework for the generalized order complementarity problem was established through fixed point theory.

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This work was supported by the National Natural Science Foundation of China and the Kyungnam University Research Fund 2003.

ISSN 0232-2064 / $ 2.50

In [11], Isac and Kostreva considered a class of multi-valued generalized order complementarity problems and proved the existence of solutions under some condition (K). Recently, Huang and Fang [6] further introduced a new system of multi-valued generalized order complementarity problems and presented some existence results of solutions by means of fixed point and coupled fixed point theorems, which extended and improved the corresponding results of Isac and Kostreva [10, 11].

The main objective of this paper is to present some further findings concerning some recent works (see [6, 10, 11]) in the area of solvability methods for a system of multi-valued generalized order complementarity problems. By using the concepts of multi-valued increasing-type mappings (see, for example, [6, 11, 12, 13]) and decreasing mappings, we first establish some new fixed point theorems in ordered metric spaces. In terms of these new fixed point theorems, some new existence results of solutions for the system of multi-valued generalized order complementarity problems are presented in the setting of ordered metric spaces.

2. Preliminaries

Let $(E, \leq)$ be a vector lattice, that is, for every pair $x, y \in E$, the supremum $x \vee y$ with respect to the partial order " $\leq$ " exists in E. In this case, for every $x, y, z \in E$, one has

$$(x+z) \vee (y+z) = z + x \vee y$$
$$(x+z) \wedge (y+z) = z + x \wedge y$$
$$\vee\{x \vee y, z\} = (x \vee y) \vee (y \vee z) = x \vee y \vee z$$

where $x \wedge y$ denotes the infimum.

Let (X, d) be a real complete metric space and $\phi: X \to \mathbb{R}$ a function. In 1976, Caristi [2] introduced a partial order " $\leq$ " in X by

$$x \leq y \iff d(x,y) \leq \phi(x) - \phi(y). \tag{2.1}$$

Clearly, $x \leq y$ implies $\phi(x) \geq \phi(y)$.

Definition 2.1. Let $(E, \leq)$ be a non-empty ordered set, A and B be two non-empty subsets of E. We say that

(1) $A \leq_1 B$ if, for each $a \in A$, there exists $b \in B$ such that $a \leq b$

(2) $A \leq_2 B$ if, for each $b \in B$, there exists $a \in A$ such that $a \leq b$

(3) $A \leq_3 B$ if $A \leq_1 B$ and $A \leq_2 B$.

If E is a set, then as usual we denote by 2^E the family of all non-empty subsets of E.

Definition 2.2. Let $(E, \leq)$ be a non-empty ordered set, $S: E \to 2^E$ and $T: E \times E \to 2^E$ be two multi-valued mappings. We say that

(1) S is $\leq_i$-*increasing* if $x, y \in E, x \leq y$ imply $Sx \leq_i Sy$ $(i = 1, 2, 3)$

(2) S is $\leq_i$-*decreasing* if $x, y \in E, x \leq y$ imply $Sy \leq_i Sx$ $(i = 1, 2, 3)$

(3) T is mixed $\leq_3$-*increasing* if $x_1, x_2, y_1, y_2 \in E, x_1 \leq x_2, y_2 \leq y_1$ imply $T(x_1, y_1) \leq_3 T(x_2, y_2)$.

Definition 2.3. Let $(E, \leq)$ be a vector lattice, $\Lambda: E \to E$ be a single-valued mapping, $S: E \to 2^E$ and $T: E \times E \to 2^E$ be two multi-valued mappings. We say that

(1) S is Λ-$\leq_i$-*increasing* if $(I + \Lambda)^{-1}$ is $\leq_i$-increasing and $S + \Lambda$ is $\leq_i$-increasing $(i = 1, 2)$

(2) T is *mixed* Λ-$\leq_3$-*increasing* if $(I + \Lambda)^{-1}$ is $\leq_3$-increasing and the mapping $(x, y) \mapsto T(x, y) + \Lambda(x)$ is mixed $\leq_3$-increasing.

Definition 2.4. Let E be a topological space, $S: E \to 2^E$ and $T: E \times E \to 2^E$ be two multi-valued mappings. We say that

(1) $x^* \in E$ is a *fixed point* of S if $x^* \in Sx^*$

(2) $(x^*, y^*) \in E \times E$ is a *periodic fixed point* of S if $x^* \in Sy^*$ and $y^* \in Sx^*$

(3) $(x^*, y^*) \in E \times E$ is a *coupled fixed point* of T if $x^* \in T(x^*, y^*)$ and $y^* \in T(y^*, x^*)$.

3. Some fixed point theorems

In this section, we establish some new fixed point theorems for multi-valued increasing-type mappings and decreasing mappings in ordered metric spaces.

Theorem 3.1. *Let (X, d) be a real complete metric space, $\phi: X \to \mathbb{R}$ a continuous and bounded from below function, and " $\leq$ " the partial order defined by* (2.1). *Further, suppose that $S: X \to 2^X$ is a multi-valued mapping satisfying the following conditions:*

(i) *For each $x \in X$, Sx is non-empty and compact.*

(ii) *S is $\leq_1$-increasing.*

(iii) *There exists a point $x_0 \in X$ such that $\{x_0\} \leq_1 Sx_0$.*

Then there exists $x^* \geq x_0$ such that $x^* \in S(x^*)$.

Proof. It is an easy consequence of [5: Theorem 2.1] ∎

Theorem 3.2. *Let (X, d) be a complete metric space, $\phi : X \to \mathbb{R}$ a continuous function with bounded range $\phi(X)$, and "$\leq$" the partial order defined by* (2.1)*. Further, suppose that $T : X \times X \to 2^X$ is a multi-valued mapping satisfying the following conditions:*

(i) *$T(x, y)$ is non-empty and compact for any $x, y \in X$.*

(ii) *T is mixed $\leq_3$-increasing.*

(iii) *There exist $x_0, y_0 \in X$ with $\{x_0\} \leq_1 T(x_0, y_0)$ and $T(y_0, x_0) \leq_2 \{y_0\}$.*

Then T has at least one coupled fixed point (x^*, y^*).

Proof. Let

$$C = \Big\{(x, y) \in X \times X : \{x\} \leq_1 T(x, y) \text{ and } T(y, x) \leq_2 \{y\}\Big\}.$$

Clearly, C is non-empty and $(x_0, y_0) \in C$. We define a partial order on C by

$$(x_1, y_1) \preceq (x_2, y_2) \quad \Longleftrightarrow \quad x_1 \leq x_2 \text{ and } y_2 \leq y_1$$

and prove that $(C, \preceq)$ has a maximal element (x^*, y^*). Without loss of generality, let $\{(x_\mu, y_\mu)\}_{\mu \in J} \subset (C, \preceq)$ be any totally ordered set, where J is a directed set with $\mu, \nu \in J$, and $\mu \preceq \nu$ if and only if $(x_\mu, y_\mu) \preceq (x_\nu, y_\nu)$. We now prove there exist $z, w \in X$ such that $x_\mu \to z$, $y_\mu \to w$ and $x_\mu \leq z$, $w \leq y_\mu$ for every $\mu \in J$. In fact, from (2.1), $\{\phi(x_\mu)\}_{\mu \in J}$ is a non-increasing net in $\mathbb{R}$. Since ϕ is bounded from below, $\{\phi(x_\mu)\}_{\mu \in J}$ is a Cauchy net in $\mathbb{R}$. It follows from (2.1) that $d(x_\mu, x_\nu) \leq \phi(x_\mu) - \phi(x_\nu)$ for all $\mu \prec \nu$. This implies that $\{x_\mu\}$ is a Cauchy net in X. Let $x_\mu \to z$. For any $\mu \in J$, one has

$$d(x_\mu, z) = \lim_\nu d(x_\mu, x_\nu) \leq \lim_\nu \big(\phi(x_\mu) - \phi(x_\nu)\big) = \phi(x_\mu) - \phi(z).$$

Hence $x_\mu \leq z$ for every $\mu \in J$. Similarly, $y_\mu \to w$ and $w \leq y_\mu$ for every $\mu \in J$. Since T is mixed $\leq_3$-increasing, $T(x_\mu, y_\mu) \leq_3 T(z, w)$ and $T(w, z) \leq_3 T(y_\mu, x_\mu)$. As follows from $\{x_\mu\} \leq_1 T(x_\mu, y_\mu)$ and $T(y_\mu, x_\mu) \leq_2 \{y_\mu\}$, there exist $z_\mu \in T(z, w)$ and $w_\mu \in T(w, z)$ such that $x_\mu \leq z_\mu$ and $w_\mu \leq y_\mu$ for every $\mu \in J$. From condition (i), there exist $\{z_\nu\} \subset \{z_\mu\}$ and $\{w_\nu\} \subset \{w_\mu\}$ such that $z_\nu \to \bar{z}$ and $w_\nu \to \bar{w}$. We claim that $z \leq \bar{z}$ and $\bar{w} \leq w$. Indeed, since ϕ is continuous,

$$\begin{aligned} d(z, \bar{z}) &\leq d(z, x_\nu) + d(x_\nu, z_\nu) + d(z_\nu, \bar{z}) \\ &\leq d(z, x_\nu) + d(z_\nu, \bar{z}) + \phi(x_\nu) - \phi(z_\nu) \\ &\to \phi(z) - \phi(\bar{z}) \end{aligned}$$

$$\begin{aligned} d(\bar{w}, w) &\leq d(w, y_\nu) + d(y_\nu, w_\nu) + d(w_\nu, \bar{w}) \\ &\leq d(w, y_\nu) + d(w_\nu, \bar{w}) + \phi(w_\nu) - \phi(y_\nu) \\ &\to \phi(\bar{w}) - \phi(w). \end{aligned}$$

Therefore, $z \leq \bar{z}$ and $\bar{w} \leq w$. Since $\bar{z} \in T(z,w)$ and $\bar{w} \in T(w,z)$, $(z,w) \in C$ and $(x_\mu, y_\mu) \preceq (z,w)$ for every $\mu \in J$. By Zorn lemma, $(C, \preceq)$ has a maximal element (x^*, y^*).

Now, we show that (x^*, y^*) is a coupled fixed point of T. Since $(x^*, y^*) \in C$, there exist $\hat{x} \in T(x^*, y^*)$ and $\hat{y} \in T(y^*, x^*)$ such that $x^* \leq \hat{x}$ and $\hat{y} \leq y^*$. Since T is mixed $\leq_3$-increasing, $T(x^*, y^*) \leq_3 T(\hat{x}, \hat{y})$ and $T(\hat{y}, \hat{x}) \leq_3 T(y^*, x^*)$. This implies that $(\hat{x}, \hat{y}) \in C$. Since (x^*, y^*) is the maximal element of $(C, \preceq)$, $\hat{x} = x^*$ and $\hat{y} = y^*$. So $x^* \in T(x^*, y^*)$ and $y^* \in T(y^*, x^*)$. The proof is complete ∎

If $T(x,y) = S(y)$ for all $x, y \in X$, then, from Theorem 3.2, we can obtain the following periodic fixed point theorem:

Theorem 3.3. *Let (X,d) be a complete metric space, $\phi : X \to \mathbb{R}$ a continuous function with bounded range $\phi(X)$ and " $\leq$ " the partial order defined by* (2.1). *Further, suppose that $S : X \to 2^X$ is a multi-valued mapping satisfying the following conditions:*

(i) *$S(x)$ is non-empty and compact for all $x \in X$.*

(ii) *S is $\leq_3$-decreasing.*

(iii) *There exist $x_0, y_0 \in X$ such that $\{x_0\} \leq_1 S(y_0)$ and $S(x_0) \leq_2 \{y_0\}$.*

Then there exist $x^*, y^* \in X$ such that $x^* \in S(y^*)$ and $y^* \in S(x^*)$.

Example 3.1. Let $X = \mathbb{R}^2$ and

$$\begin{aligned} d\big((x,y),(u,v)\big) &= |x-u| + |y-v| \\ \phi((x,y)) &= \frac{|x|+|y|}{1+|x|+|y|} \end{aligned} \qquad \big((x,y),(u,v) \in X\big).$$

Then (X,d) is a complete metric space and $\phi : X \to \mathbb{R}$ is a continuous function with bounded rang $\phi(X)$.

Remark 3.1. Some fixed theorems for single-valued increasing mappings and coupled fixed point theorems for single-valued mixed increasing mappings in ordered metric spaces were first established by Zhang [14].

4. A system of multi-valued generalized order complementarity problems

In this section we prove some new existence results of solutions for a system of multi-valued generalized order complementarity problems by virtue of the results presented in Section 3.

Let (X,d) be a real complete metric space, $\phi : X \to \mathbb{R}$ be function and "$\leq$" the partial order defined by (2.1) such that $(X,\leq)$ is a vector lattice. Further, let $K_0 = \{x \in X : x \geq 0\}$. Suppose that $f_i : X \times X \to 2^X$ $(i = 1,\ldots,m)$ are given multi-valued mappings of the form $f_i(x,y) = x - T_i(x,y)$, where $T_i : X \times X \to 2^X$ is a multi-valued mapping. We now consider the following problem associated with $\{f_i\}_{1\leq i\leq m}$ and K_0:

Find $(x^*,y^*) \in K_0 \times K_0, u_i^* \in T_i(x^*,y^*)$ and $v_i^* \in T_i(y^*,x^*)$ such that

$$\left.\begin{aligned} \inf\left\{x^*, x^* - u_1^*, \ldots, x^* - u_m^*\right\} = 0 \\ \inf\left\{y^*, y^* - v_1^*, \ldots, y^* - v_m^*\right\} = 0 \end{aligned}\right\} \tag{4.1}$$

which is called a system of multi-valued generalized order complementarity problems and which was considered already by Huang and Fang [6].

If $T_i(x,y) = T_i(y)$ for all $x,y \in X$, then problem (4.1) collapses to the following problem:

Find $(x^*,y^*) \in K_0 \times K_0, u_i^* \in T_i(y^*)$ and $v_i^* \in T_i(x^*)$ such that

$$\left.\begin{aligned} \inf\left\{x^*, x^* - u_1^*, \ldots, x^* - u_m^*\right\} = 0 \\ \inf\left\{y^*, y^* - v_1^*, \ldots, y^* - v_m^*\right\} = 0 \end{aligned}\right\}. \tag{4.2}$$

If $T_i(x,y) = T_i(x)$ for all $x,y \in X$, then problem (4.1) reduces to the following multi-valued generalized order complementarity problem:

Find $x^* \in K_0$ and $u_i^* \in T_i(x^*)$ such that

$$\inf\left\{x^*, x^* - u_1^*, \ldots, x^* - u_m^*\right\} = 0 \tag{4.3}$$

which was introduced and studied by Isac and Kostreva [11].

Theorem 4.1. *Let $\phi : X \to \mathbb{R}$ be a continuous function with bounded range $\phi(X)$ and $\Lambda : X \to X$ a single-valued mapping. Further, let $f_i : X \times X \to 2^X$ $(i = 1,\ldots,m)$ have the form $f_i(x,y) = x - T_i(x,y)$, where $T_i : X \times X \to 2^X$ is multi-valued. Suppose the following:*

(i) *The set $\left\{\sup_{1\leq i\leq m} z_i : z_i \in T_i(x,y)\right\}$ is non-empty and compact for all $x,y \in X$.*

(ii) *T_i is mixed Λ-$\leq_3$-increasing for each i.*

(iii) Λ *is increasing, i.e.,* $x \leq y$ *implies* $\Lambda(x) \leq \Lambda(y)$.

(iv) *There exist* $x_0, y_0 \in X$ *such that* $\{x_0\} \leq_1 T_i(x_0, y_0)$ *and* $T_i(y_0, x_0) \leq_2 \{y_0\}$ *for some* i.

(v) $(I + \Lambda)^{-1}$ *is upper semi-continuous with compact values.*

Then problem (4.1) *is solvable.*

Proof. We define

$$T_0^+(x, y) = \big\{ \sup\{0, u_1, u_2, \ldots, u_m\} : u_i \in T_i(x, y) \big\} \tag{4.4}$$

$$T(x, y) = (I + \Lambda)^{-1}\big(T_0^+(x, y) + \Lambda(x)\big) \tag{4.5}$$

and show that T satisfies all the assumptions of Theorem 3.2.

(1) Conditions (i) and (v) imply that $T(x, y)$ is non-empty and compact for all $x, y \in X$.

(2) Next we prove that T is mixed $\leq_3$-increasing. For this, let $x_1 \leq x_2$, $y_2 \leq y_1$ and $u \in T(x_1, y_1)$. From (4.4) and (4.5), there exist $u_i^1 \in T_i(x_1, y_1)$ and $w^1 \in X$ such that

$$\begin{aligned} w^1 &= \sup\big\{0, u_1^1, u_2^1, \ldots, u_m^1\big\} \\ u &\in (I + \Lambda)^{-1}(w^1 + \Lambda(x_1)). \end{aligned} \tag{4.6}$$

Further,

$$w^1 + \Lambda(x_1) = \Lambda(x_1) \vee \big(\vee_{i=1}^m (u_i^1 + \Lambda(x_1))\big).$$

Since T_i is mixed Λ-$\leq_3$-increasing for each i and Λ is increasing, there exist $u_i^2 \in T_i(x_2, y_2)$ and $w^2 \in X$ such that

$$\left.\begin{aligned} &u_i^1 + \Lambda(x_1) \leq u_i^2 + \Lambda(x_2) \\ &w^2 = \sup\big\{0, u_1^2, u_2^2, \ldots, u_m^2\big\} \\ &w^1 + \Lambda(x_1) \leq w^2 + \Lambda(x_2) \end{aligned}\right\}. \tag{4.7}$$

Since $(I + \Lambda)^{-1}$ is $\leq_3$-increasing, there exists $v \in (I + \Lambda)^{-1}(w^2 + \Lambda(x_2))$ such that $u \leq v$. From (4.4), (4.5) and (4.7), $v \in T(x_2, y_2)$. Similarly, for any given $v' \in T(x_2, y_2)$, there exists $u' \in T(x_1, y_1)$ such that $u' \leq v'$.

(3) Condition (iv) and the definition of T imply that $\{x_0\} \leq_1 T(x_0, y_0)$ and $T(y_0, x_0) \leq_2 \{y_0\}$.

Now, by (1) - (3) and Theorem 3.2, there exist $x^*, y^* \in X$ such that $x^* \in T(x^*, y^*)$ and $y^* \in T(y^*, x^*)$, i.e.

$$\begin{aligned} x^* &\in (I + \Lambda)^{-1}\big(\sup\big\{0, u_1^*, u_2^*, \ldots, u_m^*\big\} + \Lambda(x^*)\big) \\ y^* &\in (I + \Lambda)^{-1}\big(\sup\big\{0, v_1^*, v_2^*, \ldots, v_m^*\big\} + \Lambda(y^*)\big). \end{aligned}$$

It follows that

$$\left.\begin{aligned} x^* &= \sup\left\{0, u_1^*, u_2^*, \ldots, u_m^*\right\} \\ y^* &= \sup\left\{0, v_1^*, v_2^*, \ldots, v_m^*\right\} \end{aligned}\right\}. \tag{4.8}$$

From (4.1), one has

$$\inf\left\{x^*, x^* - u_1^*, x^* - u_2^*, \ldots, x^* - u_m^*\right\} = x^* - \sup\left\{0, u_1^*, u_2^*, \ldots, u_m^*\right\} = 0$$
$$\inf\left\{y^*, y^* - v_1^*, y^* - v_2^*, \ldots, y^* - v_m^*\right\} = y^* - \sup\left\{0, v_1^*, v_2^*, \ldots, v_m^*\right\} = 0.$$

Furthermore, from (4.8), $(x^*, y^*) \in K_0 \times K_0$. The proof is complete ∎

By Theorem 4.1, we can obtain an existence result for problem (4.2) as follows:

Theorem 4.2. *Let $\phi : X \to \mathbb{R}$ be a continuous function with bounded range $\phi(X)$ and $\Lambda : X \to X$ a single-valued mapping. Further, let $f_i : X \to 2^X$ $(i = 1, \ldots, m)$ have the form $f_i(x, y) = x - T_i(y)$, where $T_i : X \to 2^X$ is multi-valued. Suppose the following:*

(i) *The set $\left\{\sup_{1\le i\le m} z_i : z_i \in T_i(x)\right\}$ is non-empty and compact for all $x \in X$.*

(ii) *T_i is Λ-$\le_3$-decreasing for each i.*

(iii) *Λ is increasing.*

(iv) *There exist $x_0, y_0 \in X$ such that $\{x_0\} \le_1 T_i(y_0)$ and $T_i(x_0) \le_2 \{y_0\}$ for some i.*

(v) *$(I + \Lambda)^{-1}$ is upper semi-continuous with compact values.*

Then problem (4.2) is solvable.

The proof is similarly to that of Theorem 4.1 and thus omitted.

Theorem 4.3. *Let $\phi : X \to \mathbb{R}$ be a continuous bounded from below function and $\Lambda : X \to X$ a single-valued mapping. Further, let $f_i : X \to 2^X$ $(i = 1, \ldots, m)$ have the form $f_i(x, y) = x - T_i(x)$, where $T_i : X \to 2^X$ is multi-valued. Suppose the following:*

(i) *The set $\left\{\sup_{1\le i\le m} z_i : z_i \in T_i(x)\right\}$ is non-empty and compact for all $x \in X$.*

(ii) *T_i is Λ-$\le_1$-increasing for each i.*

(iii) *There exists $x_0 \in K_0$ such that $\{x_0\} \le_1 T_i(x_0)$ for some i.*

(iv) *$(I + \Lambda)^{-1}$ is upper semi-continuous with compact values.*

Then problem (4.3) is solvable.

Proof. We define

$$T_0^+(x) = \left\{\sup_{1\le i\le m} u_i : u_i \in T_i(x)\right\} \tag{4.9}$$

$$T(x) = (I + \Lambda)^{-1}\big(T_0^+(x) + \Lambda(x)\big) \tag{4.10}$$

and show that T satisfies all the assumptions of Theorem 3.1.

(1) First, conditions (i) and (iv) imply that $T(x)$ is non-empty and compact for all $x \in X$.

(2) Next we prove that T is $\leq_1$-increasing. For this, let $x \leq y$ and $u \in T(x)$. From (4.9) and (4.10), there exist $u_i \in T(x)$ and $w \in X$ such that

$$w = \sup_{1 \leq i \leq m} u_i \qquad \text{and} \qquad u \in (I + \Lambda)^{-1}(w + \Lambda(x)). \tag{4.11}$$

Since

$$w + \Lambda(x) = \vee\big\{u_i + \Lambda(x) : u_i \in T_i(x)\big\}$$

and T_i is Λ-$\leq_1$-increasing, there exists $v_i \in T_i(y)$ such that $u_i + \Lambda(x) \leq v_i + \Lambda(y)$. Let $\omega = \vee_{i=1}^m(\Lambda(y) + v_i)$. It follows that

$$w + \Lambda(x) = \vee_{i=1}^m(u_i + \Lambda(x)) \leq \vee_{i=1}^m(v_i + \Lambda(y)) = \omega + \Lambda(y).$$

Since $(I + \Lambda)^{-1}$ is $\leq_1$-increasing, there exists $v \in (I + \Lambda)^{-1}(\omega + \Lambda(y))$ such that $u \leq v$. By the definition of T, $v \in T(y)$.

(3) Condition (iii) and the definition of T imply that $\{x_0\} \leq_1 T(x_0)$.

Now, by Theorem 3.1, there exists $x^* \geq x_0$ such that $x^* \in T(x^*)$, i.e.

$$x^* \in (I + \Lambda)^{-1}\Big(\sup_{1 \leq i \leq m} u_i^* + \Lambda(x^*)\Big).$$

It follows that $x^* = \sup_{1 \leq i \leq m} u_i^*$. Then from (4.1) one get

$$\begin{aligned}\inf\big\{x^*, x^* - u_1^*, x^* - u_2^*, \ldots, x^* - u_m^*\big\} \\ &= x^* \wedge \big(x^* - \sup\{u_1^*, u_2^*, \ldots, u_m^*\}\big) \\ &= x^* \wedge 0.\end{aligned}$$

Since $x^* \geq x_0$, it follows that $x^* \in K_0$ and

$$\inf\big\{x^*, x^* - u_1^*, x^* - u_2^*, \ldots, x^* - u_m^*\big\} = 0.$$

The proof is complete ∎

Acknowledgment. The authors would like to express their deep gratitude to the referees for their helpful comments and suggestions.

References

[1] Borwein, J. M. and M. A. H. Dempster: *The linear order complementarity problem.* Math. Oper. Res. 14 (1989), 534 – 558.

[2] Caristi, J.: *Fixed point theorems for mappings satisfying inwardness conditions.* Trans. Amer. Math. Soc. 215 (1976), 241 – 251.

[3] Cottle, R. W., Pang, J. S. and R. E. Stone: *The Linear Complementarity Problems.* New York: Acad. Press 1992.

[4] Gowda, M. S. and R. Sznajder: *The generalized ordered complementarity problem.* SIAM J. Matrix Anal. Appl. 15 (1994), 779 – 795.

[5] Heikkilä, S. and S. Hu: *On fixed points of multifunctions in ordered spaces.* Appl. Anal. 51 (1993), 115 – 127.

[6] Huang, N. J. and Y. P. Fang: *Fixed point theorems and a new system of multivalued generalized order complementarity problems.* Positivity (to appear).

[7] Huang, N. J., Gao, C. J. and X. P. Huang: *Exceptional family of elements and feasibility for nonlinear complementarity problems.* J. Global Optim. 25 (2003), 337 – 344.

[8] Isac, G.: *Topological Methods in Complementarity Theory.* Dordrecht: Kluwer Acad. Publ. 2000.

[9] Isac, G. and D. Goeleven: *The implicit general order complementarity problem, models and iterative methods.* Ann. Oper. Res. 44 (1993), 63 – 92.

[10] Isac, G. and M. Kosetreva: *The generalized order complementarity problem.* J. Optim. Theory Appl. 3 (1991), 517 – 534.

[11] Isac, G. and M. Kosetreva: *Kneser's theorem and the multivalued generalized order complementarity problem.* Appl. Math. Lett. 4 (1991)6, 81 – 85.

[12] Nguyen, B. Y. and H. K. Nguyen: *Fixed point for multivalued increasing operators.* J. Math. Anal. Appl. 250 (2000), 368 – 371.

[13] Nishnianidze, Z. G.: *Fixed points of monotonic multiple-valued operators.* Bull. Acad. Sci. Georgian SSR 114 (1984), 489 – 491.

[14] Zhang, X.: *Fixed point theorems of monotone mappings and coupled fixed point theorems of mixed monotone mappings in ordered metric spaces.* Acta Math. Sinica 44 (2001), 641 – 646.

Received 20.03.2003; in revised form 02.06.2003