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The Dirichlet Energy Integral on Intervals in Variable Exponent Sobolev Spaces

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Abstract. In this article we consider Dirichlet energy integral minimizers in variable exponent Sobolev spaces defined on intervals of the real line. We illustrate by examples that the minimizing question is interesting even in this case that is trivial in the classical fixed exponent space. We give an explicit formula for the minimizer, and some simple conditions for when it is convex, concave or Lipschitz continuous. The most surprising conclusion is that there does not exist a minimizer even for every smooth exponent.

Keywords: *Variable exponent Sobolev space, zero boundary values, Sobolev capacity, Dirichlet energy integral, minimizing problem*

AMS subject classification: 46E35, 31C45, 35J65

1. Introduction

An intensive study of variable exponent Lebesgue and Sobolev spaces has been undertaken during the last couple of years by several authors, inspired primarily by the article [12] of Kováčik and Rákosník from 1991. These spaces have turned up in the modeling of non-homogeneous fluids, see the monograph by Růžička [15] and the article [2] by Acerbi and Mingione for newer results. However, the special case of variable Lebesgue spaces on the real line had been studied already by Sharapudinov [17] in the late 70's. Some questions are now fairly well understood; as an example we mention the boundedness of the Hardy-Littlewood maximal operator, thanks to the investigations of Pick and Růžička [14], Diening [5], Nekvinda [13] and Cruz-Uribe, Fiorenza and Neugebauer [4]. Other questions, like Dirichlet energy integral minimizers, are only now beginning to be investigated.

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In a recent article Harjulehto, Hästö, Koskenoja and Varonen [10] showed that the Dirichlet energy integral, with boundary values given in the Sobolev sense, has a minimizer provided the variable exponent satisfies a certain jump condition. The energy integral had previously been considered also by Coscia and Minginone [3] and by Acerbi and Mingione [1], but their condition on the exponent was much stricter, excluding for instance all exponents with a discontinuity.

In the present article we will consider the Dirichlet energy integral on an interval of the real line. Our motivation for this research was three-fold:

– We believe that studying this integral on the real line will provide us with hints as to how it behaves in more general settings.

– Moreover, it is clear that a one-dimensional minimization problem can be extended to higher dimensions, simply by choosing the exponent to depend on one coordinate only. Therefore, we get several necessary conditions also for the higher-dimensional case. For instance, we will show that the assumptions used by Coscia and Mingione [3] in their study of energy integral minimizers are in some sense necessary.

– Also, we think that this question is of interest on its own right, since it turns out that even one-dimensional problems are often difficult in the variable exponent setting. For instance Edmunds and Meskhi [6] have recently studied potential-type operators in variable exponent spaces on the real line.

In the next section we briefly review the definition and basic properties of variable exponent spaces. Since the energy integral problem on an interval is trivial in fixed exponent Sobolev spaces, we start Section 3 by giving an example which shows that the question merits study in variable exponent spaces. We then give the explicit closed form of the solution of the Dirichlet energy integral problem. The most important and striking conclusion is that even for very smooth exponents no minimizer need exist. In particular this renders support to the intuition of previous researchers that some restrictions on the exponent are necessary in order to get minimizers. In Section 4 we use the explicit formula to study convexity of minimizer and in Section 5 we study its Lipschitz and Hölder continuity.

2. Variable exponent Sobolev spaces on the real line

In this section we review the standard theory of variable exponent Lebesgue and Sobolev spaces as it pertains to the one-dimensional case.

Let $\Omega \subset \mathbb{R}$ be an open set and let $p\colon \Omega \to [1,\infty)$ be a measurable function (called the *variable exponent* on Ω). Throughout this paper the function p always denotes a variable exponent; also, we define $p^+ = \operatorname{ess\,sup}_{x\in\Omega} p(x)$ and $p^- = \operatorname{ess\,inf}_{x\in\Omega} p(x)$. We define the *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$

to consist of all measurable functions $u : \Omega \to \mathbb{R}$ such that $\varrho_{p(\cdot)}(\lambda u) = \int_\Omega |\lambda u(x)|^{p(x)} dx < \infty$ for some $\lambda > 0$. The function $\varrho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to [0,\infty)$ is called the *modular* of the space $L^{p(\cdot)}(\Omega)$. We define a norm, the so-called *Luxemburg norm*, on this space by the formula $\|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \varrho_{p(\cdot)}\big(\frac{u}{\lambda}\big) \le 1\}$. The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ is the space of measurable functions $u : \Omega \to \mathbb{R}$ such that u and the distributional derivative u' are in $L^{p(\cdot)}(\Omega)$. The function

$$\varrho_{1,p(\cdot)} : W^{1,p(\cdot)}(\Omega) \to [0,\infty)$$

is defined by

$$\varrho_{1,p(\cdot)}(u) = \varrho_{p(\cdot)}(u) + \varrho_{p(\cdot)}(u').$$

The norm $\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|u'\|_{p(\cdot)}$ makes $W^{1,p(\cdot)}(\mathbb{R})$ a Banach space. For more details on variable exponent spaces see [12].

In [9] we introduced a Sobolev capacity $C_{p(\cdot)}$ in variable exponent spaces and in [10] we used it to define Sobolev spaces with zero boundary values, following the ideas of Kilpeläinen, Kinnunen and Martio [11] in metric spaces. In the one-dimensional case it is possible to dispense with much of the fancy stuff, claims holding only quasi-everywhere etc., and to give simpler definitions, which is what we do next. For the complete definitions, valid also in higher dimensions, the reader is referred to the above mentioned papers.

Let again $\Omega \subset \mathbb{R}$ be an open set. Since every element in the space $W^{1,p(\cdot)}(\Omega)$ has a continuous representative, we will assume throughout this paper that every function in a Sobolev space is continuous. We denote $u \in W_0^{1,p(\cdot)}(\Omega)$ and say that u belongs to the *variable exponent Sobolev space with zero boundary values* if it can be continuously continued by 0 outside Ω (the extension is again denoted by u). The space $W_0^{1,p(\cdot)}(\Omega)$ is endowed with the norm

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|u\|_{W^{1,p(\cdot)}(\mathbb{R})}.$$

If $1 < p^- \le p^+ < \infty$, then $W_0^{1,p(\cdot)}(\Omega)$ is a reflexive Banach space [9: Theorems 3.1 and 3.6]. Let $\Omega \subset \mathbb{R}$ be an open set and let $w \in W^{1,p(\cdot)}(\Omega)$. The energy operator corresponding to the boundary value function w acting on the space

$$\big\{u \in W^{1,p(\cdot)}(\Omega) \colon u - w \in W_0^{1,p(\cdot)}(\Omega)\big\}$$

is defined by

$$I_{\Omega,w}^{p(\cdot)}(u) = \int_\Omega |u'(x)|^{p(x)} dx.$$

The general problem is to find a function that minimizes values of the operator $I_{\Omega,w}^{p(\cdot)}$. It was shown in [10: Section 5] that this operator has a unique continuous minimizer provided $1 < p^- \le p^+ < \infty$.

3. The Dirichlet energy integral minimizer on an interval

Let us start by stating the Dirichlet energy integral problem on an interval. We will assume that the interval under consideration is $I = (0,1)$. It follows from the definitions in the previous section that $u \in W_0^{1,p(\cdot)}(0,1)$ if and only if $u(0) = u(1) = 0$. Hence the Dirichlet energy integral problem reduces to finding $u \in W^{1,p(\cdot)}(0,1)$ with $u(0) = 0$ and $u(1) = a > 0$ which minimizes

$$\int_0^1 |u'(y)|^{p(y)} dy.$$

Let us denote by $I_a^{p(\cdot)}$ the energy integral operator acting on such functions.

Energy integral minimizers

If p is fixed, then the minimizer is linear, $u(x) = ax$. The next example shows that the variable exponent adds some interest to this minimization question.

Example. Let $p\colon (0,1) \to (1,\infty)$ be defined by

$$p(x) = \begin{cases} 3 & \text{for } 0 < x \le \frac{1}{2} \\ 2 & \text{for } \frac{1}{2} < x < 1. \end{cases}$$

Suppose that $u \in W^{1,p(\cdot)}(0,1)$ is the minimizer for the boundary values 0 and $a > 0$. Denote $u(\frac{1}{2}) = b$. Then $u|_{(0,\frac{1}{2})}$ is the solution to the classical energy integral problem with boundary values 0 and b, and $u|_{(\frac{1}{2},1)}$ is the solution with boundary values b and a. Therefore these functions are linear, and so

$$u(x) = \begin{cases} 2bx & \text{for } 0 < x \le \frac{1}{2} \\ 2b + 2(a-b)(x-\frac{1}{2}) & \text{for } \frac{1}{2} < x < 1. \end{cases}$$

For this u we have $I_a^{p(\cdot)}(u) = 4b^3 + 2(a-b)^2$. It is easy to see that the function $b \mapsto 2b^3 + (a-b)^2$ has a minimum at $b = \frac{\sqrt{1+12a}-1}{6}$, which determines the minimizer of the variable exponent problem. The minimizing functions for some a's are shown in the figure above. As can be seen in that figure, and as can be confirmed by calculation, the minimizer is convex if $a > \frac{2}{3}$, concave if $a < \frac{2}{3}$ and linear for $a = \frac{2}{3}$.

It is in fact possible to give an explicit formula for the minimizer, as shown in the next theorem. The formula is not quite transparent, however, so we will also prove some properties of the minimizers later on.

Theorem 3.2. *Let p be bounded on $I = (0,1)$ and strictly greater than 1 almost everywhere on $(0,1)$, and let $a > 0$ be a constant. Then the operator $I_a^{p(\cdot)}$ has a unique minimizer if and only if either*

$$a \le \int_0^1 \left(\frac{1}{p(y)}\right)^{\frac{1}{p(y)-1}} dy \tag{3.1}$$

or there exists a constant $c > 1$ such that

$$\int_0^1 c^{\frac{1}{p(y)-1}}\, dy < \infty. \tag{3.2}$$

In this case the minimizer is given by

$$u(x) = \int_0^x \left(\frac{\tilde{c}}{p(y)}\right)^{\frac{1}{p(y)-1}} dy$$

for some constant $\tilde{c} > 0$.

Proof. We start by considering a different, related, minimizing problem: for $f \in L^{p(\cdot)}(I)$, minimize $\int_0^1 |f(x)|^{p(x)} dx$ under the constraint $\int_0^1 f(x)\, dx = a$. This problem can be solved using the classical variational method (cf. [8]). It is clear that we can assume $f \ge 0$ when looking for the minimizer. Let (f_i) be a minimizing sequence. By Fatou's lemma, $\tilde{f}(x) = \liminf f_i(x)^{p(x)}$ exists almost everywhere and is integrable. Therefore $f(x) = \tilde{f}(x)^{1/p(x)}$ is a minimizer, if $\int_0^1 f(x)\, dx = a$. Let us call such a function f a tentative minimizer. Let then f be any tentative minimizer. Let $\delta\colon (0,1) \to \mathbb{R}$ be bounded and measurable such that $\int_0^1 \delta = 0$. Let $\varepsilon > 0$. If (f_i) is a minimizing sequence tending to f, then $(f_i + \delta)$ is a sequence tending to $f + \delta$. Since the first sequence was minimizing, we have $\int_0^1 (f(x) + \varepsilon\delta(x))^{p(x)} dx \geq \int_0^1 f(x)^{p(x)} dx$. Therefore

$$\int_0^1 \frac{|f(x) + \varepsilon\delta(x)|^{p(x)} - f(x)^{p(x)}}{\varepsilon} dx \geq 0.$$

Since p and δ are bounded, it follows by dominated convergence, as $\varepsilon \to 0$, that

$$\int_0^1 p(x)|f(x)|^{p(x)-1}\delta(x)\, dx \geq 0.$$

Suppose that $p(x)(f(x))^{p(x)-1}$ is not constant almost everywhere. Let then $d_1 < d_2$ be such that

$$A_1 = \{x \in (0,1) \colon p(x)(f(x))^{p(x)-1} < d_1\}$$
$$A_2 = \{x \in (0,1) \colon p(x)(f(x))^{p(x)-1} > d_2\}$$

have positive measure. Let $A_1' \subset A_1$ and $A_2' \subset A_2$ be such that $|A_1'| = |A_2'| > 0$. Define

$$\delta(x) = \begin{cases} 1 & \text{for } x \in A_1' \\ -1 & \text{for } x \in A_2' \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} &\int_0^1 p(x)|f(x)|^{p(x)-1}\delta(x)\, dx \\ &\quad = \int_{A_1'} p(x)|f(x)|^{p(x)-1} dx - \int_{A_2'} p(x)|f(x)|^{p(x)-1} dx \\ &\quad \leq (d_1 - d_2)|A_1'| \\ &\quad < 0 \end{aligned}$$

contrary to what was shown earlier. Therefore $p(x)(f(x))^{p(x)-1} = c$ for some constant c and almost every $x \in (0,1)$. We have shown that every tentative minimizer is of the form $f_c(x) = \left(\frac{c}{p(x)}\right)^{\frac{1}{p(x)-1}}$. Therefore we see that f is a minimizer if and only if c can be chosen so that $\int_0^1 f_c\, dx = a$.

Let us next prove that $c \mapsto \int_0^1 f_c\, dx$ is continuous when c is such that the integral converges. Let $c > 0$ be such that the integral is finite. Fix $\varepsilon > 0$ and define

$$A_\lambda = \{x \in (0,1) \colon p(x) > \lambda\}.$$

We choose a $\lambda > 1$ such that $\int_{(0,1)\setminus A_\lambda} f_c\,dx < \varepsilon$. In A_λ the exponent $\frac{1}{p(x)-1}$ is bounded from above and so we can choose $d > 0$ such that $\int_{A_\lambda} (f_c - f_{c-d})\,dx < \varepsilon$. Then

$$\begin{aligned}\int_{(0,1)} (f_c - f_{c-d})\,dx &\leq \int_{A_\lambda} (f_c - f_{c-d})\,dx + \int_{(0,1)\setminus A_\lambda} (f_c + f_{c-d})\,dx \\ &\leq 3\varepsilon.\end{aligned}$$

Clearly, $\int_0^1 f_0\,dx = 0$ and $c \mapsto \int_0^1 f_c\,dx$ is increasing. Since $\int_0^1 f_1\,dx < \infty$, we see that there exists a suitable $c \leq 1$, if $a \leq \int_0^1 f_1\,dx$, i.e. if (3.1) holds. If, on the other hand, (3.2) holds, then $\int_0^1 f_c\,dx$ increases continuously to ∞ so that there again exists a suitable c. If neither (3.1) nor (3.2) is satisfied, then there does not exist a c such that $\int_0^1 f_c\,dx = a$.

We have now shown that the minimizing problem has a solution if and only if (3.1) or (3.2) holds. Let us return then to our original minimizing problem. Suppose that (3.1) or (3.2) holds and let $f = f_c$ so that $\int_0^1 f_c\,dx = a$. If $u \in W^{1,p(\cdot)}(I)$ is such that $f = u'$ and $u(0) = 0$, then clearly u is the minimizer we are looking for. Define therefore $u(x) = \int_0^x f(y)\,dy$ for $x \in (0,1]$. Since u is bounded by a, $u \in L^{p(\cdot)}(I)$. Further, we have

$$\varrho_{p(\cdot)}(u') = \int_0^1 \left(\frac{c}{p(x)}\right)^{\frac{p(x)}{p(x)-1}} dx \leq c \int_0^1 c^{\frac{1}{p(x)-1}}\,dx < \infty$$

if $c \leq 1$ or by assumption (3.2). Therefore $u \in W^{1,p(\cdot)}(I)$ and we are done. On the other hand, if u is a minimizer of the original problem, then $f = u'$ is a minimizer of the problem considered in this proof. So then (3.1) or (3.2) holds. Therefore these conditions are both necessary and sufficient ∎

Corollary 3.3. *If $p\colon (0,1) \to [m, M]$ for $1 < m \leq M < \infty$, then the operator $I_a^{p(\cdot)}$ has a unique minimizer given by*

$$u(x) = \int_0^x \left(\frac{c}{p(y)}\right)^{\frac{1}{p(y)-1}} dy$$

for some constant $c > 0$.

The following example shows that the Dirichlet energy integral does not always have a minimizer.

Example 3.4. For $p(x) = 1 + x$ the operator $I_a^{p(\cdot)}$ does not have a minimizer for large a. For, let $c > 1$. Then

$$\int_0^1 c^{\frac{1}{p(x)-1}}\,dx = \int_0^1 c^{\frac{1}{x}}\,dx \geq \log c \int_0^1 \frac{dx}{x} = \infty$$

so that the condition of Theorem 3.2 is not satisfied for $a > \int_0^1 p(x)^{\frac{1}{1-p(x)}}\,dx$.

4. Convexity and concavity of the minimizers

In the previous section we gave an explicit formula for the minimizer of the energy integral on an open unit interval. This formula is, however, somewhat complicated and not very transparent. So in this section and the next one we give simple conditions that guarantee some regularity of the minimizer.

Example 4.1. Using the previous theorem we plot some minimizers of the energy integral for $p(x) = 1.1 + x$. The number on the right is again the second boundary value, a. It looks as if there is a shift from convex to concave minimizers as the difference between the boundary values is large enough.

Energy integral minimizers when $p(x) = 1.1 + x$

The next theorem shows that if p is increasing, then the minimizer possesses quite a bit of regularity, namely it has zero or one point of inflection. The same holds also for decreasing p, only then concave and convex should be swapped.

Corollary 4.2. *Let $p\colon [0,1] \to [m, M]$ for $1 < m \leq M < \infty$ be increasing. Then the energy integral minimizer is either*

(1) *convex,*

(2) *concave, or*

(3) *concave on $(0, z)$ and convex on $(z, 1)$, for some $z \in (0,1)$.*

Proof. Let us define $F(z) = \frac{\log \frac{c}{z}}{z-1}$. Then the derivative of the minimizer equals $\exp(F(p(x)))$, by Theorem 3.2. The minimizer is convex if its derivative is increasing. Since $z \mapsto \exp(z)$ and $x \mapsto p(x)$ are increasing, this is equivalent to $z \mapsto F(z)$ being increasing. We find that

$$\frac{dF(z)}{dz} = \frac{-\frac{z-1}{z} - \log\frac{c}{z}}{(z-1)^2}.$$

Since $z > 1$, $z \mapsto F(z)$ is increasing if and only if $\frac{1}{z} + \log z \geq 1 + \log c$. Since the function $z \mapsto z^{-1} + \log z$ is increasing for $z > 1$, we see that the minimizer is convex if $\frac{1}{p(0)} + \log p(0) \geq 1 + \log c$ and concave if $\frac{1}{p(1)} + \log p(1) \leq 1 + \log c$. If neither of these conditions hold, then there exists a $z \in (0,1)$ such that $x \mapsto \frac{1}{p(x)} + \log p(x) - (1 + \log c)$ changes sign at z, and then the minimizer is concave on $(0, z)$ and convex on $(z, 1)$ ∎

Corollary 4.3. *Let $p\colon [0,1] \to [m, M]$ for $1 < m \leq M < \infty$ be increasing. If $a \leq \exp(-\frac{1}{p(0)})$, then the minimizer of $I_a^{p(\cdot)}$ is convex. If, on the other hand,*

$$a \geq \Big(\frac{p(1)}{p(0)}\Big)^{\frac{1}{p(0)-1}} \exp\Big(-\frac{1}{p(1)}\frac{p(1)-1}{p(0)-1}\Big),$$

then the minimizer of $I_a^{p(\cdot)}$ is concave. Both conditions are the best possible in terms of only $p(0)$ and $p(1)$.

Proof. Let us estimate c in the derivative of the minimizer. We have

$$\sup_{x\in(0,1)} \Big(\frac{c}{p(x)}\Big)^{\frac{1}{p(x)-1}} \geq \int_0^1 \Big(\frac{c}{p(x)}\Big)^{\frac{1}{p(x)-1}} dx = a \geq \inf_{x\in(0,1)} \Big(\frac{c}{p(x)}\Big)^{\frac{1}{p(x)-1}}.$$

It follows from this that

$$\max_{x\in[0,1]} p(x)a^{p(x)-1} \geq c \geq \min_{x\in[0,1]} p(x)a^{p(x)-1}.$$

Let us start with the first claim of the theorem, convexity. Since $a \leq e^{-\frac{1}{p(0)}}$, the function $z \mapsto za^{z-1}$ is decreasing so that $c \leq p(0)a^{p(0)-1}$. It was shown in the previous theorem that the minimizer is convex if $\frac{1}{p(0)} + \log p(0) \geq 1 + \log c$. Therefore it suffices to show that $\frac{1}{p(0)} + \log p(0) \geq 1 + \log\big(p(0)a^{p(0)-1}\big)$, which is the condition of the first part of the corollary.

The argument for the second claim is almost the same. We find that for $z \in [p(0), p(1)]$ the function $z \mapsto za^{z-1}$ is increasing, provided $a \geq e^{-\frac{1}{p(1)}}$. We prove that

$$\Big(\frac{p(1)}{p(0)}\Big)^{\frac{1}{p(0)-1}} \exp\Big(-\frac{1}{p(1)}\frac{p(1)-1}{p(0)-1}\Big) \geq e^{-\frac{1}{p(1)}}.$$

When we raise both sides to the power $p(0)-1$, we see that this is equivalent to

$$\frac{p(1)}{p(0)} \geq \exp\Big(-\frac{p(0)-1}{p(1)} + \frac{p(1)-1}{p(1)}\Big).$$

Defining the new variable $z = \frac{p(1)}{p(0)}$, we find that this is the same as $ze^{\frac{1}{z}} \geq e$. But $z \geq 1$, so this is clear. Having established that $z \mapsto za^{z-1}$ is increasing, it follows that $c \geq p(0)a^{p(0)-1}$. It was shown in the previous theorem that the minimizer is concave if $\frac{1}{p(1)} + \log p(1) \leq 1 + \log c$. Therefore it suffices to show that $\frac{1}{p(1)} + \log p(1) \leq 1 + \log\big(p(0)a^{p(0)-1}\big)$, which is just the condition of the second part of the corollary.

We prove the sharpness in the first case only, since the proof in the second case is similar. Fix $p_1 > p_0 > 1$. Suppose that $a > \exp(-1/p_0)$. Then we see that $\frac{1}{p_0} + \log p_0 < 1 + \log(p_0 a^{p_0-1})$. We can therefore choose $\varepsilon > 0$ such that $\frac{1}{p_0} + \log p_0 < 1 + \log(p_0 a^{p_0-1} - \varepsilon)$. Let us set $p(x) = p_0$ if $x \in [0,t)$ and $p(x) = p_1$ otherwise for $t \in (0,1)$. Then

$$t\Big(\frac{c}{p_0}\Big)^{\frac{1}{p_0-1}} + (1-t)\Big(\frac{c}{p_1}\Big)^{\frac{1}{p_1-1}} = a.$$

Then by choosing t close to one, we get $c > p_0 a^{p_0-1} - \varepsilon$, But then $\frac{1}{p(0)} + \log p(0) < 1 + \log c$ and so the minimizer is not convex, by Corollary 4.2 ∎

Remark 4.4. It follows from Corollary 4.3 that if $a \leq \frac{1}{e}$, then the minimizer is always convex, irrespective of how p increases.

5. Regularity of the minimizers

Coscia and Mingione [3] and Acerbi and Mingione [1] have investigated the regularity of energy integral minimizers. Their approach is based on local minimizers and avoiding using variable exponent spaces explicitly. In the one-dimensional case we can now, for the first time, investigate the necessity of their assumption. In the following we ignore some technical differences between the way Mingione and his collaborators define minimizers and our definition.

Acerbi and Mingione [1] proved that the minimizer is α-Hölder continuous for some $\alpha > 0$ if $p^- > 1$ and there exists a constant $C > 0$ such that

$$|p(x) - p(y)| \leq \frac{C}{-\log|x-y|}$$

for all $|x-y| \leq \frac{1}{2}$. Our next result indicates that the second assumption might be excessive.

Corollary 5.1. *If* $p\colon(0,1)\to[m,M]$ *for* $1<m\le M<\infty$, *then the minimizer of the operator* $I_a^{p(\cdot)}$ *is bi-Lipschitz continuous for every* a.

Proof. By Theorem 3.2, the minimizer has derivative $\left(\frac{c}{p(x)}\right)^{\frac{1}{p(x)-1}}$ for some constant $c>0$. Since

$$\begin{aligned}0&<\max\left\{\left(\frac{c}{p^+}\right)^{\frac{1}{p^+-1}},\left(\frac{c}{p^+}\right)^{\frac{1}{p^--1}}\right\}\\&\le\left(\frac{c}{p(x)}\right)^{\frac{1}{p(x)-1}}\\&\le\max\left\{1,c^{\frac{1}{p^--1}}\right\}\\&<\infty\end{aligned}$$

for all $x\in(0,1)$, it follows from the mean-value theorem that the minimizer is bi-Lipschitz continuous ∎

Coscia and Mingione [3] proved that the derivative of the minimizer is α-Hölder continuous provided $p^->1$ and p is β-Hölder continuous, where α depends on β. The next result shows that we cannot expect to get much stronger results than this.

Corollary 5.2. *If* $p\colon(0,1)\to[m,M]$ *for* $1<m\le M<\infty$, *then the derivative of the minimizer of the operator* $I_a^{p(\cdot)}$ *is* α*-Hölder continuous if and only if the exponent* p *is* α*-Hölder continuous.*

Proof. Let us denote $F(z)=(\frac{c}{z})^{\frac{1}{z-1}}$. Then the derivative of the minimizer equals $F(p(x))$. But F is differentiable on $(1,\infty)$, so

$$|F(p(x))-F(p(y))|=F'(\xi)\,|p(x)-p(y)|$$

where $\xi\in(p(x),p(y))$, by the mean-value theorem. It is easy to see that F' is bounded and bounded away from 0 on $[p^-,\infty]$, so that $F(p(x))$ possesses the same degree of regularity as $p(x)$ ∎

The assumption $p^->1$ has been made in all previous investigations of energy integral minimizers [1, 3, 9]. The next result shows that if we relax this assumption, then we are also liable to lose a lot of regularity of the minimizer.

Corollary 5.3. *Let* $p(x)=1+\left(\log\frac{1}{x}\right)^{-1}$ *for all* $x\in(0,1)$. *The minimizer of* $I_a^{p(\cdot)}$ *is* α*-Hölder continuous for some* α *depending on* $a>0$. *The derivative of the minimizer is not even uniformly continuous for large* a.

Proof. We have

$$\int_0^1 c^{\frac{1}{p(x)-1}}\,dx=\int_0^1 x^{-\log c}dx=\frac{1}{1-\log c}$$

provided $c < e$, so condition (3.2) is satisfied. Therefore the derivative of the minimizer is $\left(\frac{c}{p(x)}\right)^{\frac{1}{p(x)-1}}$ for some $c > 0$, by Theorem 3.2. Thus we have, for $0 < y < x < 1$,

$$\begin{aligned} |u(x) - u(y)| &= \left| \int_y^x x^{-\log c} p(x)^{-\frac{1}{p(x)-1}} \, dx \right| \\ &\leq \frac{x^{1-\log c} - y^{1-\log c}}{1 - \log c} \\ &\leq \frac{(x-y)^{1-\log c}}{1-\log c}. \end{aligned}$$

We see that u is $(1 - \log c)$-Hölder continuous. Moreover, if a is such that $c > 1$, then the derivative is unbounded, hence not uniformly continuous ∎

Example 5.4. Using the main theorem we plot some minimizers of the energy integral for $p(x) = 1 + (-\log(x))^{-1}$. The number on the right is again the second boundary value, a. The lower three curves are Lipschitz continuous, the following two are 0.738- and 0.530-Hölder continuous.

Energy integral minimizers when $p(x) = 1 + (-\log(x))^{-1}$

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