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Integral Estimates for the Laplace-Beltrami and Green's Operators Applied to Differential Forms on Manifolds

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Abstract. We obtain $A_r(M)$-weighted boundedness for compositions of Green's operator and the Laplace-Beltrami operator applied to differential forms on manifolds. As applications, we also prove $A_r(M)$-weighted Sobolev-Poincaré embedding theorems for Green's operator and norm comparison theorems for solutions of the A-harmonic equation on manifolds. These results can be used in developing the L^p theory of differential forms and the Hodge decomposition.

Keywords: *Differential forms, Sobolev embedding, Laplace-Beltrami and Green's operators*

AMS subject classification: Primary 58A10, secondary 46E35, 58A14, 35J60

1. Introduction

Our purpose is to study the L^p theory of the Laplace-Beltrami operator $\Delta = dd^\star + d^\star d$ and Green's operator G acting upon differential forms on manifolds. Both operators play an important role in many fields, including partial differential equations, harmonic analysis and quasiconformal mappings (see [9, 13, 15]). We establish some norm inequalities both for Δ and G and their compositions that are applied to differential forms on a compact, orientable, C^∞-smooth Riemannian manifold M without boundary. We also obtain $A_r(M)$-weighted estimates for these compositions and prove Sobolev-Poincaré embedding theorems for Green's operator. As applications of our local and global results, we derive some norm inequalities for solutions of the non-homogeneous as well as the homogeneous A-harmonic equations. These results will provide effective tools for studying of behavior of solutions of A-harmonic equations and related differential systems on manifolds.

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Let $e_1, \dots, e_n$ be the standard unit basis of $\mathbb{R}^n$ and assume that $\Lambda^l = \Lambda^l(\mathbb{R}^n)$ is the linear space of l-vectors, generated by the exterior products $e_I = e_{i_1} \wedge \ldots \wedge e_{i_l}$, corresponding to all ordered l-tuples $I = (i_1, \dots, i_l)$ $(1 \leq i_1 < \ldots < i_l \leq n;\, l = 0, 1, \dots, n)$. The Grassman algebra $\Lambda = \oplus \Lambda^l$ is a graded algebra with respect to the exterior products. For $\alpha = \sum \alpha^I e_I \in \Lambda$ and $\beta = \sum \beta^I e_I \in \Lambda$, the inner product in Λ is given by $\langle \alpha, \beta \rangle = \sum \alpha^I \beta^I$ with summation over all l-tuples $I = (i_1, \dots, i_l)$ and all integers $l = 0, 1, \dots, n$. We define the Hodge star operator $\star : \Lambda \to \Lambda$ by the rule

$$\begin{aligned} \star 1 &= e_1 \wedge \cdots \wedge e_n \\ \alpha \wedge \star\beta &= \beta \wedge \star\alpha = \langle \alpha, \beta \rangle (\star 1) \qquad (\alpha, \beta \in \Lambda). \end{aligned}$$

The norm of $\alpha \in \Lambda$ is given by the formula

$$|\alpha|^2 = \langle \alpha, \alpha \rangle = \star(\alpha \wedge \star\alpha) \in \Lambda^0 = \mathbb{R}.$$

The Hodge star is an isometric isomorphism on Λ with $\star : \Lambda^l \to \Lambda^{n-l}$ and $\star\star\, (-1)^{l(n-l)} : \Lambda^l \to \Lambda^l$.

Throughout this paper, we always assume that M is a Riemannian, compact, oriented and C^∞-smooth manifold without boundary on $\mathbb{R}^n$ and Ω is an open subset of $\mathbb{R}^n$. We write $\mathbb{R} = \mathbb{R}^1$. A differential l-form ω on M is a de Rham current (see [12]) on M with values in $\Lambda^l(\mathbb{R}^n)$. Let $\Lambda^l M$ be the l-th exterior power of the cotangent bundle and $C^\infty(\Lambda^l M)$ be the space of smooth l-forms on M. We use $D'(M, \Lambda^l)$ to denote the space of all differential l-forms and $L^p(\Lambda^l M)$ to denote the l-forms

$$\omega(x) = \sum_I \omega_I(x)\, dx_I = \sum \omega_{i_1 \cdots i_l}(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_l}$$

on M satisfying $\int_M |\omega_I|^p < \infty$ for all ordered l-tuples I. Thus $L^p(\Lambda^l M)$ is a Banach space with norm

$$\|\omega\|_{p,M} = \left(\int_M |\omega(x)|^p dx \right)^{\frac{1}{p}} = \left(\int_M \left(\sum_I |\omega_I(x)|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}.$$

We denote the exterior derivative by

$$d :\, D'(M, \Lambda^l) \to D'(M, \Lambda^{l+1}) \qquad (l = 0, 1, \dots, n).$$

The Hodge codifferential operator

$$d^\star :\, D'(M, \Lambda^{l+1}) \to D'(M, \Lambda^l)$$

is given by $d^\star = (-1)^{nl+1} \star d\star$ on $D'(M, \wedge^{l+1})$ $(l = 0, 1, \ldots, n)$, and the Laplace-Beltrami operator Δ is defined by

$$\Delta = dd^\star + d^\star d.$$

Also, we always use G to denote Green's operator throughout this paper. Further, we use B to denote a ball and σB to denote the ball with the same center as B and with diameter $\operatorname{diam}(\sigma B) = \sigma \operatorname{diam}(B)$. We do not distinguish balls from cubes in this paper.

The n-dimensional Lebesgue measure of a set $E \subseteq \mathbb{R}^n$ is denoted by $|E|$. We call w a weight if $w \in L^1_{loc}(\mathbb{R}^n)$ and $w > 0$ a.e. For $0 < p < \infty$ we denote the weighted L^p-norm of a measurable function f over E by

$$\|f\|_{p,E,w^\alpha} = \left(\int_E |f(x)|^p w(x)^\alpha dx \right)^{\frac{1}{p}}$$

where α is a real number.

T. Iwaniec and A. Lutoborski proved the following result in [9]:

Let $D \subset \mathbb{R}^n$ be a bounded, convex domain. Then to each $y \in D$ there corresponds a linear operator $K_y : C^\infty(D, \Lambda^l) \to C^\infty(D, \Lambda^{l-1})$ defined by

$$(K_y\omega)(x; \xi_1, \ldots, \xi_l) = \int_0^1 t^{l-1} \omega\big(tx + y - ty; x - y, \xi_1, \ldots, \xi_{l-1}\big) dt$$

and the decomposition $\omega = d(K_y\omega) + K_y(d\omega)$.

A homotopy operator

$$T : C^\infty(D, \Lambda^l) \to C^\infty(D, \Lambda^{l-1})$$

is defined by averaging K_y over all points y in D:

$$T\omega = \int_D \varphi(y) K_y\omega \, dy \tag{1.1}$$

where $\varphi \in C_0^\infty(D)$ is normalized by $\int_D \varphi(y)\, dy = 1$, and the norm is estimated by

$$\|T\omega\|_{s,D} \le C \operatorname{diam}(D) \|\omega\|_{s,D}. \tag{1.2}$$

We define the l-form $\omega_D \in D'(D, \wedge^l)$ by

$$\omega_D = \begin{cases} |D|^{-1} \int_D \omega(y)\, dy & \text{for } l = 0 \\ d(T\omega) & \text{for } l = 1, \ldots, n \end{cases}$$

for all $\omega \in L^p(D, \Lambda^l)$, $1 \le p < \infty$. Then

$$\omega_D = \omega - T(d\omega) \tag{1.3}$$

and

$$\|d(Tu)\|_{s,D} \le \|u\|_{s,D} + C|D|\text{diam}(D)\|du\|_{s,D}. \tag{1.4}$$

By substituting $z = tx + y - ty$, (1.1) reduces to

$$T\omega(x,\xi) = \int_D \omega\big(z, \zeta(z, x-z), \xi\big) dz$$

where the vector function $\zeta : D \times \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$\zeta(z,h) = h \int_0^\infty s^{l-1}(1+s)^{n-1}\varphi(z - sh)\, ds.$$

Integral (1.1) defines a bounded operator

$$T : L^s(D, \wedge^l) \to W^{1,s}(D, \wedge^{l-1}) \qquad (l = 1, \dots, n)$$

with norm estimated by

$$\|Tu\|_{W^{1,s}(D)} \le C|D|\, \|u\|_{s,D}. \tag{1.5}$$

2. L^p-estimates and Sobolev-Poincaré embedding theorem

We say that $u \in L^1_{loc}(\Lambda^l M)$ has a generalized gradient if, for each coordinate system, the pullbacks of the coordinate function of u have generalized gradient in the familiar sense (see [14]). We write

$$\mathcal{W}(\Lambda^l M) = \Big\{u \in L^1_{loc}(\Lambda^l M) : u \text{ has generalized gradient}\Big\}.$$

As usual, harmonic l-fields are defined by

$$\mathcal{H}(\Lambda^l M) = \Big\{u \in \mathcal{W}(\Lambda^l M) : du = d^\star u = 0,\ u \in L^p \text{ for some } 1 < p < \infty\Big\},$$

the orthogonal complement $\mathcal{H}^\perp$ of $\mathcal{H}$ in L^1 by

$$\mathcal{H}^\perp = \big\{u \in L^1 : \langle u, h\rangle = 0 \text{ for all } h \in \mathcal{H}\big\}$$

and Greens' operator

$$G : C^\infty(\Lambda^l M) \to \mathcal{H}^\perp \cap C^\infty(\Lambda^l M)$$

by assigning $G(u)$ the unique element of $\mathcal{H}^\perp \cap C^\infty(\Lambda^l M)$ satisfying Poisson's equation

$$\Delta G(\omega) = \omega - H(\omega)$$

where H is either the harmonic projection or sometimes the harmonic part of ω.

From [13] we have the following lemma about L^s-estimates for Green's operator G.

Lemma 2.1. *Let* $u \in C^\infty(\Lambda^l M)$ $(l = 0, 1, \ldots, n)$ *and* $1 < s < \infty$. *Then there exists a constants* C, *independent of* u, *such that*

$$\begin{gathered}\|dd^\star G(u)\|_{s,M} + \|d^\star dG(u)\|_{s,M} \\ +\|dG(u)\|_{s,M} + \|d^\star G(u)\|_{s,M} + \|G(u)\|_{s,M} \le C\|u\|_{s,M}.\end{gathered} \tag{2.1}$$

Theorem 2.2. *Let* $u \in C^\infty(\Lambda^l M)$ $(l = 0, 1, \ldots, n)$ *and* $1 < s < \infty$. *Then there exists a constant* C, *independent of* u, *such that*

$$\|\Delta(G(u))\|_{s,M} \le C\|u\|_{s,M}.$$

Proof. From the definition of the Laplace-Beltrami operator Δ and Minkowski's inequality, using (2.1) we have

$$\begin{aligned}\|\Delta(G(u))\|_{s,M} &= \|(dd^\star + d^\star d)G(u)\|_{s,M} \\ &\le \|dd^\star G(u)\|_{s,M} + \|d^\star dG(u)\|_{s,M} \\ &\le C_1\|u\|_{s,M} + C_2\|u\|_{s,M} \\ &\le C\|u\|_{s,M}\end{aligned}$$

and the proof is complete ∎

Theorem 2.3. *Let* $u \in C^\infty(\Lambda^l M)$ $(l = 0, 1, \ldots, n)$ *and* $1 < s < \infty$. *Then there exists a constant* C, *independent of* u, *such that*

$$\|G(\Delta u)\|_{s,M} \le C\|u\|_{s,M}.$$

Proof. We know that Green's operator G commutes with d and $d^\star$ (see [16]), that is, for any differential form $u \in C^\infty(\wedge^l M)$ we have

$$\begin{aligned}dG(u) &= Gd(u) \\ d^\star G(u) &= Gd^\star(u).\end{aligned} \tag{2.2}$$

Using this, (2.1) and Minkowski's inequality we obtain

$$\begin{aligned}\|G(\Delta u)\|_{s,M} &= \|G(dd^\star + d^\star d)u\|_{s,M} \\ &= \|G(dd^\star(u)) + G(d^\star d(u))\|_{s,M} \\ &\le \|G(dd^\star(u))\|_{s,M} + \|G(d^\star d(u))\|_{s,M} \\ &= \|dd^\star(G(u))\|_{s,M} + \|d^\star d(G(u))\|_{s,M} \\ &\le C_1\|u\|_{s,M} + C_2\|u\|_{s,M} \\ &= C_3\|u\|_{s,M}.\end{aligned}$$

and the proof is complete ∎

Using Minkowski's inequality and combining Theorems 2.2 and 2.3, we obtain immediately the following

Corollary 2.4. *Let $u \in C^\infty(\wedge^l M)$ $(l = 0, 1, \dots, n)$ and $1 < s < \infty$. Then there exists a constant C, independent of u, such that*

$$\|(G\Delta + \Delta G)u\|_{s,M} \le C\|u\|_{s,M}.$$

Theorem 2.5. *Let $u \in C^\infty(\Lambda^l M)$ $(l = 0, 1, \dots, n)$ and $1 < s < \infty$. Then there exists a constant C, independent of u, such that*

$$\|(G(u))_D\|_{s,D} \le C\|u\|_{s,D} \tag{2.3}$$

for any convex and bounded set D with $D \subset M$.

Proof. If $1 \le l \le n$, applying (1.4) and (2.1) we have

$$\begin{aligned}
\|(G(u))_D\|_{s,D} &= \|d(T(G(u)))\|_{s,D} \\
&\le \|G(u)\|_{s,D} + C_1|D|\mathrm{diam}(D)\|d(G(u))\|_{s,D} \\
&\le C_2\|G(u)\|_{s,D} \\
&\le C_3\|u\|_{s,D}.
\end{aligned}$$

If $l = 0$, using (2.1) and the Hölder inequality with $1 = \frac{1}{s} + \frac{1}{q}$ we find that

$$\begin{aligned}
\|(G(u))_D\|_{s,D} &= \left(\int_D |(G(u))_D|^s dx\right)^{\frac{1}{s}} \\
&= \left(\int_D \left|\frac{1}{|D|}\int_D G(u(y))\,dy\right|^s dx\right)^{\frac{1}{s}} \\
&\le \left(\left(\frac{1}{|D|}\int_D |G(u(y))|\,dy\right)^s \int_D 1\,dx\right)^{\frac{1}{s}} \\
&= \frac{1}{|D|}|D|^{\frac{1}{s}} \int_D |G(u(y))|\,dy \\
&\le |D|^{\frac{1}{s}-1}\left(\int_D |G(u(y))|^s dy\right)^{\frac{1}{s}} \left(\int_D 1^q dy\right)^{\frac{1}{q}} \\
&= \|G(u)\|_{s,D} \\
&\le C_2\|u\|_{s,D}
\end{aligned}$$

and the proof is complete ∎

Corollary 2.6. *Let $u \in C^\infty(\Lambda^l M)$ $(l = 0, 1, \dots, n)$ and $1 < s < \infty$. Then for any convex and bounded set D with $D \subset M$ there exists a constant C, independent of u, such that*

$$\|G(u) - (G(u))_D\|_{s,D} \le C\|G(u) - c\|_{s,D} \tag{2.4}$$

for any closed form c and

$$\|G(u) - (G(u))_D\|_{s,D} \le C\|u\|_{s,D}. \tag{2.5}$$

Proof. We know that $c_D = c$ if c is closed. Hence, by Theorem 2.5, we have

$$\begin{aligned}\|G(u) - (G(u))_D\|_{s,D} &\le \big\|\big(G(u) - c\big) - \big((G(u))_D - c_D\big)\big\|_{s,D} \\ &\le \|G(u) - c\|_{s,D} + \|(G(u) - c)_D\|_{s,D} \\ &\le \|G(u) - c\|_{s,D} + C_1\|G(u) - c\|_{s,D} \\ &\le C_2\|G(u) - c\|_{s,D}.\end{aligned}$$

Hence (2.4) holds. From (2.3) and (2.1) we find that

$$\begin{aligned}\|G(u) - (G(u))_D\|_{s,D} &\le \|G(u)\|_{s,D} + \|(G(u))_D\|_{s,D} \\ &\le C_1\|u\|_{s,D} + C_2\|u\|_{s,D} \\ &\le C_3\|u\|_{s,D}\end{aligned} \tag{2.6}$$

which ends the proof ∎

We use $W^{1,p}(M, \Lambda^l)$ to denote the Sobolev space of l-forms which equals $L^p(\Lambda^l M) \cap L_1^p(\Lambda^l M)$, with norm

$$\|\omega\|_{W^{1,p}(M)} = \operatorname{diam}(M)^{-1}\|\omega\|_{p,M} + \|\nabla\omega\|_{p,M}. \tag{2.7}$$

Here ω is the vector-valued differential form $\nabla\omega = \big(\frac{\partial\omega}{\partial x_1}, \dots, \frac{\partial\omega}{\partial x_n}\big)$ that consists of differential forms $\frac{\partial\omega}{\partial x_i} \in D'(M, \Lambda^l)$, where partial differentiation is applied to the coefficients of ω. The notations $W^{1,p}_{loc}(M, \mathbb{R})$ and $W^{1,p}_{loc}(M, \Lambda^l)$ are self-explanatory. For $0 < p < \infty$ and a weight w, the weighted norm of $\omega \in W^{1,p}(M, \Lambda^l)$ over M is denoted by

$$\|\omega\|_{W^{1,p}(M),w^\alpha} = \operatorname{diam}(M)^{-1}\|\omega\|_{p,M,w^\alpha} + \|\nabla\omega\|_{p,M,w^\alpha} \tag{2.8}$$

where α is a real number.

Next, we prove an analogue of the Poincaré inequality for Green's operator G.

Theorem 2.7. *Let $u \in C^\infty(\Lambda^l M)$ $(l = 0, 1, \ldots, n)$ and $1 < s < \infty$. Then there exists a constant C, independent of u, such that*

$$\|G(u) - (G(u))_B\|_{s,B} \le C\mathrm{diam}(B)\|du\|_{s,B} \tag{2.9}$$

for all balls B with $B \subset M$.

Proof. Applying (1.3) to $G(u)$ we find that

$$(G(u))_B = G(u) - T(d(G(u))).$$

Combining this with (1.2), (2.2) and (2.1) yields

$$\begin{aligned}\|G(u) - (G(u))_B\|_{s,B} &= \|Td(G(u))\|_{s,B} \\ &\le C_1\mathrm{diam}(B)\|d(G(u))\|_{s,B} \\ &= C_1\mathrm{diam}(B)\|G(du)\|_{s,B} \\ &\le C_2\mathrm{diam}(B)\|du\|_{s,B}.\end{aligned}$$

Thus, inequality (2.9) holds. This ends the proof ∎

As application of Theorem 2.7 we prove the following Sobolev-Poincaré embedding theorem about Green's operator G applied to a differential form u.

Theorem 2.8. *Let $u \in C^\infty(\Lambda^l M)$ $(l = 0, 1, \ldots, n)$ and $1 < s < \infty$. Then there exists a constant C, independent of u, such that*

$$\|G(u) - (G(u))_B\|_{W^{1,s}(B)} \le C\|du\|_{s,B}$$

for all balls B with $B \subset M$.

Proof. Since u_D is a closed form for any form u, then $(G(u))_B$ is a closed form and

$$\big\|d\big(G(u) - (G(u))_B\big)\big\|_{s,B} = \|d(G(u))\|_{s,B}. \tag{2.10}$$

Note that $\|\nabla u\|_{s,B} = \|du\|_{s,B}$. Using (2.7) and (2.10) we obtain

$$\begin{aligned}&\|G(u) - (G(u))_B\|_{W^{1,s}(B)} \\ &\quad = \mathrm{diam}(B)^{-1}\|G(u) - (G(u))_B\|_{s,B} + \big\|\nabla\big(G(u) - (G(u))_B\big)\big\|_{s,B} \\ &\quad = \mathrm{diam}(B)^{-1}\|G(u) - (G(u))_B\|_{s,B} + \big\|d\big(G(u) - (G(u))_B\big)\big\|_{s,D} \\ &\quad = \mathrm{diam}(B)^{-1}\|G(u) - (G(u))_B\|_{s,B} + \|d(G(u))\|_{s,B}.\end{aligned}$$

From here, (2.9) and (2.2) we get

$$\begin{aligned}
&\|G(u) - (G(u))_B\|_{W^{1,s}(B)} \\
&\qquad \le \mathrm{diam}(B)^{-1}\|G(u) - (G(u))_B\|_{s,B} + \|d(G(u))\|_{s,B} \\
&\qquad \le \mathrm{diam}(B)^{-1}C_1\mathrm{diam}(B)\|d(G(u))\|_{s,B} + \|d(G(u))\|_{s,B} \\
&\qquad \le C_2\|d(G(u))\|_{s,B} \\
&\qquad = C_2\|G(du)\|_{s,B} \\
&\qquad \le C_3\|du\|_{s,B}
\end{aligned}$$

and the proof is complete ∎

Remark. We can also prove Theorem 2.8 by applying [9: Corollary 4.1] to du and then using (2.2) and (2.1).

3. $A_r(M)$-weighted norm inequalities

The study of different versions of the A-harmonic equation for differential forms has developed rapidly in recent years. Many interesting results concerning A-harmonic tensors have been established recently (see [1 - 4, 9, 11, 13]). Early work about harmonic tensors can be found in [6]. In this section, we prove $A_r(M)$-weighted norm inequalities for solutions to the non-homogeneous A-harmonic equation

$$A(x, g + du) = h + d^\star v \tag{3.1}$$

for differential forms, where $g, h \in D'(M, \Lambda^l)$ and $A: M \times \Lambda^l(R^n) \to \Lambda^l(R^n)$ satisfies the conditions

$$|A(x,\xi)| \le a|\xi|^{p-1}$$
$$\langle A(x,\xi), \xi\rangle \ge |\xi|^p$$

for almost every $x \in M$ and all $\xi \in \Lambda^l(\mathbb{R}^n)$. Here $a > 0$ is a constant and $1 < p < \infty$ is a fixed exponent associated with (3.1).

Definition 3.1. We call u and v a *pair of conjugate A-harmonic tensors* in M if u and v satisfy the conjugate A-harmonic equation

$$A(x, du) = d^\star v \tag{3.2}$$

in M. Similarly, we call u an *A-harmonic tensor* in M if u satisfies the A-harmonic equation

$$d^\star A(x, du) = 0.$$

Note that $du = d^* v$ is an analogue of a Cauchy-Riemann system in $\mathbb{R}^n$. A differential l-form $u \in D'(M, \Lambda^l)$ is called a closed form if $du = 0$ in M.

Similarly, a differential $(l+1)$-form $v \in D'(M, \Lambda^{l+1})$ is called a coclosed form if $d^\star v = 0$. For example, $du = d^* v$ is an analogue of a Cauchy-Riemann system in $\mathbb{R}^n$. Clearly, the A-harmonic equation is not affected by adding a closed form to u and coclosed form to v. Therefore, any type of estimates between u and v must be modulo such forms.

Throughout this paper, we always assume that $\frac{1}{p} + \frac{1}{q} = 1$.

Definition 3.2. A weight w is called an A_r-*weight* for some $r > 1$ on a subset $E \subset \mathbb{R}^n$, write $w \in A_r(E)$, if $w(x) > 0$ a.e. and

$$\sup_B \left(\frac{1}{|B|} \int_B w \, dx \right) \left(\frac{1}{|B|} \int_B \left(\frac{1}{w} \right)^{\frac{1}{r-1}} dx \right)^{r-1} < \infty$$

for any ball $B \subset E$.

See [7] or [8] for properties of $A_r(E)$-weights. We will need the following generalized Hölder's inequality.

Lemma 3.3. *Let $0 < \alpha, \beta < \infty$ and $\frac{1}{s} = \frac{1}{\alpha} + \frac{1}{\beta}$. If f and g are measurable functions on $\mathbb{R}^n$, then $\|fg\|_{s,E} \le \|f\|_{\alpha,E} \|g\|_{\beta,E}$ for any $E \subset \mathbb{R}^n$.*

We also need the following lemma [7].

Lemma 3.4. *If $w \in A_r(E)$, then there exist constants $\beta > 1$ and C, independent of w, such that $\|w\|_{\beta,B} \le C|B|^{\frac{1-\beta}{\beta}} \|w\|_{1,B}$ for all balls $B \subset E$.*

The following weak reverse Hölder inequality appears in [11].

Lemma 3.5. *Let u be an A-harmonic tensor in M, $\rho > 1$ and $0 < s, t < \infty$. Then there exists a constant C, independent of u, such that $\|u\|_{s,B} \le C|B|^{\frac{t-s}{st}} \|u\|_{t,\rho B}$ for all balls or cubes B with $\rho B \subset M$.*

Theorem 3.6. *Let $u \in C^\infty(\Lambda^l M)$ $(l = 1, 2, \ldots, n)$ be an A-harmonic tensor on a manifold M, let $\rho > 1$, $1 < s < \infty$ and $w \in A_r(M)$ for some $r > 1$. Then there exists a constant C, independent of u, such that*

$$\|G(\Delta u)\|_{s,B,w^\alpha} \le C \|u\|_{s,\rho B, w^\alpha}. \tag{3.3}$$

for any ball $B \subset M$ and any real number α with $0 < \alpha \le 1$.

Proof. We first show that (3.3) holds for $0 < \alpha < 1$. Let $t = \frac{s}{1-\alpha}$. Using Lemma 3.3, we get

$$\begin{aligned} \|G(\Delta u)\|_{s,B,w^\alpha} &= \left(\int_B \left(|G(\Delta u)| w^{\frac{\alpha}{s}} \right)^s dx \right)^{\frac{1}{s}} \\ &\le \|G(\Delta u)\|_{t,B} \left(\int_B w^{\frac{t\alpha}{t-s}} dx \right)^{\frac{t-s}{st}} \\ &= \|G(\Delta u)\|_{t,B} \left(\int_B w dx \right)^{\frac{\alpha}{s}}. \end{aligned} \tag{3.4}$$

Choosing M to be a ball B in Theorem 2.3, we have

$$\|G(\Delta u)\|_{t,B} \le C_1 \|u\|_{t,B}. \tag{3.5}$$

Choose $m = \frac{s}{1+\alpha(r-1)}$. Then $m < s$. Substituting (3.5) into (3.4) and using Lemma 3.5, we have

$$\begin{aligned} \|G(\Delta u)\|_{s,B,w^\alpha} &\le C_1 \|u\|_{t,B} \left(\int_B w dx\right)^{\alpha/s} \\ &\le C_2 |B|^{(m-t)/mt} \|u\|_{m,\rho B} \left(\int_B w dx\right)^{\alpha/s}. \end{aligned} \tag{3.6}$$

Using Lemma 3.3 again with $\frac{1}{m} = \frac{1}{s} + \frac{s-m}{sm}$, we have

$$\begin{aligned} \|u\|_{m,\rho B} &= \left(\int_{\rho B} |u|^m dx\right)^{\frac{1}{m}} \\ &= \left(\int_{\rho B} \left(|u| w^{\frac{\alpha}{s}} w^{-\frac{\alpha}{s}}\right)^m dx\right)^{\frac{1}{m}} \\ &\le \|u\|_{s,\rho B,w^\alpha} \left(\int_{\rho B} \left(\frac{1}{w}\right)^{\frac{1}{r-1}} dx\right)^{\frac{\alpha(r-1)}{s}} \end{aligned}$$

for all balls B with $\rho B \subset M$. Substituting this into (3.6) we obtain

$$\begin{aligned} \|G(\Delta u)\|_{s,B,w^\alpha} \le C_2 |B|^{\frac{m-t}{mt}} \|u\|_{s,\rho B,w^\alpha} \\ \times \left(\int_B w dx\right)^{\frac{\alpha}{s}} \left(\int_{\rho B} \left(\frac{1}{w}\right)^{\frac{1}{r-1}} dx\right)^{\frac{\alpha(r-1)}{s}}. \end{aligned} \tag{3.7}$$

Note that $w \in A_r(M)$. Then

$$\begin{aligned} &\|w\|_{1,B}^{\frac{\alpha}{s}} \left\|\frac{1}{w}\right\|_{\frac{1}{r-1},\rho B}^{\frac{\alpha}{s}} \\ &\le \left(\left(\int_{\rho B} w\, dx\right) \left(\int_{\rho B} \left(\frac{1}{w}\right)^{\frac{1}{r-1}} dx\right)^{r-1}\right)^{\frac{\alpha}{s}} \\ &= \left(|\rho B|^r \left(\frac{1}{|\rho B|} \int_{\rho B} w\, dx\right) \left(\frac{1}{|\rho B|} \int_{\rho B} \left(\frac{1}{w}\right)^{\frac{1}{r-1}} dx\right)^{r-1}\right)^{\frac{\alpha}{s}} \\ &\le C_3 |B|^{\frac{\alpha r}{s}}. \end{aligned}$$

Combining this with (3.7) we find that $\|G(\Delta u)\|_{s,B,w^{\alpha}} \leq C_4\|u\|_{s,\rho B,w^{\alpha}}$ for all balls B with $\rho B \subset M$ and we have proved that (3.3) is true if $0 < \alpha < 1$.

Next, we prove that (3.3) is true for $\alpha = 1$, that is, we need to show that

$$\|G(\Delta u)\|_{s,B,w} \leq C\|u\|_{s,\rho B,w}. \tag{3.8}$$

By Lemma 3.4, there exist constants $\beta > 1$ and $C_5 > 0$ such that

$$\|w\|_{\beta,B} \leq C_5|B|^{\frac{1-\beta}{\beta}}\|w\|_{1,B} \tag{3.9}$$

for any cube or any ball $B \subset \mathbb{R}^n$. Choose $t = \frac{s\beta}{\beta-1}$. Then $1 < s < t$ and $\beta = \frac{t}{t-s}$. Since $\frac{1}{s} = \frac{1}{t} + \frac{t-s}{st}$, by Lemma 3.3, Theorem 2.3 and (3.9) we have

$$\begin{aligned}
&\left(\int_B |G(\Delta u)|^s w\,dx\right)^{\frac{1}{s}} \\
&= \left(\int_B \left(|G(\Delta u)|w^{\frac{1}{s}}\right)^s dx\right)^{\frac{1}{s}} \\
&\leq \left(\int_B |G(\Delta u)|^t dx\right)^{\frac{1}{t}} \left(\int_B (w^{\frac{1}{s}})^{\frac{st}{t-s}}\,dx\right)^{\frac{t-s}{st}} \\
&\leq C_6\|G(\Delta u)\|_{t,B}\|w\|^{\frac{1}{s}}_{\beta,B} \\
&\leq C_6\|u\|_{t,B}\|w\|^{\frac{1}{s}}_{\beta,B} \\
&\leq C_7|B|^{\frac{1-\beta}{\beta s}}\|w\|^{\frac{1}{s}}_{1,B}\|u\|_{t,B} \\
&\leq C_7|B|^{-\frac{1}{t}}\|w\|^{\frac{1}{s}}_{1,B}\|u\|_{t,B}.
\end{aligned} \tag{3.10}$$

Let $m = \frac{s}{r}$. From Lemma 3.5 we find that

$$\|u\|_{t,B} \leq C_8|B|^{\frac{m-t}{mt}}\|u\|_{m,\rho B}. \tag{3.11}$$

Using Lemma 3.3 yields

$$\begin{aligned}
\|u\|_{m,\rho B} &= \left(\int_{\rho B} \left(|u|w^{\frac{1}{s}}w^{-\frac{1}{s}}\right)^m dx\right)^{\frac{1}{m}} \\
&\leq \left(\int_{\rho B} |u|^s w\,dx\right)^{\frac{1}{s}} \left(\int_{\rho B} \left(\frac{1}{w}\right)^{\frac{1}{r-1}} dx\right)^{\frac{r-1}{s}}
\end{aligned} \tag{3.12}$$

for all balls B with $\rho B \subset M$. Note that $w \in A_r(M)$. Then

$$\begin{aligned}
&\|w\|_{1,B}^{\frac{1}{s}} \left\| \frac{1}{w} \right\|_{\frac{1}{r-1},\rho B}^{\frac{1}{s}} \\
&\quad \le \left(\left(\int_{\rho B} w\,dx \right) \left(\int_{\rho B} \left(\frac{1}{w} \right)^{\frac{1}{r-1}} dx \right)^{r-1} \right)^{\frac{1}{s}} \\
&\quad = \left(|\rho B|^r \left(\frac{1}{|\rho B|} \int_{\rho B} w\,dx \right) \left(\frac{1}{|\rho B|} \int_{\rho B} \left(\frac{1}{w} \right)^{\frac{1}{r-1}} dx \right)^{r-1} \right)^{\frac{1}{s}} \\
&\quad \le C_9 |B|^{\frac{r}{s}}.
\end{aligned} \tag{3.13}$$

Combining (3.10) - (3.13) we get

$$\begin{aligned}
\|G(\Delta u)\|_{s,B,w} &\le C_{10} |B|^{-\frac{1}{t}} \|w\|_{1,B}^{\frac{1}{s}} |B|^{\frac{m-t}{mt}} \|u\|_{m,\rho B} \\
&\le C_{10} |B|^{-\frac{1}{m}} \|w\|_{1,B}^{\frac{1}{s}} \left\| \frac{1}{w} \right\|_{\frac{1}{r-1},\rho B}^{\frac{1}{s}} \|u\|_{s,\rho B,w} \\
&\le C_{11} \|u\|_{s,\rho B,w}
\end{aligned}$$

for all balls B with $\rho B \subset M$. Hence, (3.8) follows and the proof is complete ∎

Using the same method developed in the proof of Theorem 3.6 we can extend Theorem 2.2 into the following $A_r(M)$-weighted version.

Theorem 3.7. *Let $u \in C^\infty(\Lambda^l M)$ $(l = 1, 2, \ldots, n)$ be an A-harmonic tensor on a manifold M, let $\rho > 1$, $1 < s < \infty$ and $w \in A_r(M)$ for some $r > 1$. Then there exists a constant C, independent of u, such that*

$$\|\Delta(G(u))\|_{s,B,w^\alpha} \le C \|u\|_{s,\rho B,w^\alpha}$$

for any ball $B \subset M$ and any real number α with $0 < \alpha \le 1$.

Combining Theorems 3.6 and 3.7 we get the following $A_r(M)$-weighted inequality.

Corollary 3.8. *Let $u \in C^\infty(\Lambda^l M)$ $(l = 1, 2, \ldots, n)$ be an A-harmonic tensor on a manifold M, let $\rho > 1$, $1 < s < \infty$ and $w \in A_r(M)$ for some $r > 1$. Then there exists a constant C, independent of u, such that*

$$\|\Delta(G(u)) + G(\Delta u)\|_{s,B,w^\alpha} \le C \|u\|_{s,\rho B,w^\alpha}$$

for any ball $B \subset M$ and any real number α with $0 < \alpha \le 1$.

Now, we prove the following $A_r(M)$-weighted Sobolev-Poincaré embedding theorem for Green's operator G.

Theorem 3.9. *Let $G(u) \in C^\infty(\Lambda^l M)$ $(l = 1, 2, \ldots, n)$ be an A-harmonic tensor on a manifold M, let $\rho > 1$, $1 < s < \infty$ and $w \in A_r(M)$ for some $r > 1$. Then there exists a constant C, independent of u, such that*

$$\|G(u) - (G(u))_B\|_{W^{1,s}(B),w} \le C\|du\|_{s,\rho B,w} \tag{3.14}$$

for all balls B with $\rho B \subset M$.

Proof. Applying the Poincaré inequality established in [4] to $G(u)$ we get

$$\|G(u) - (G(u))_B\|_{s,B,w} \le C_1 \mathrm{diam}(B)\|d(G(u))\|_{s,\rho B,w}. \tag{3.15}$$

Note that $(G(u))_B$ is a closed form and

$$\begin{aligned} \big\|\nabla\big(G(u) - (G(u))_B\big)\big\|_{s,B,w} &= \big\|d\big(G(u) - (G(u))_B\big)\big\|_{s,B,w} \\ &= \|d(G(u))\|_{s,B,w}. \end{aligned} \tag{3.16}$$

Hence, using (2.8), (3.15) - (3.16), (2.2) and (2.1) we find that

$$\begin{aligned} &\|G(u) - (G(u))_B\|_{W^{1,s}(B),w} \\ &\quad = \mathrm{diam}(B)^{-1}\|G(u) - (G(u))_B\|_{s,B,w} + \|\nabla(G(u) - (G(u))_B)\|_{s,B,w} \\ &\quad \le \mathrm{diam}(B)^{-1} C_1 \mathrm{diam}(B)\|d(G(u))\|_{s,B,w} + \|d(G(u))\|_{s,B,w} \\ &\quad \le C_1\|d(G(u))\|_{s,B,w} + \|d(G(u))\|_{s,B,w} \\ &\quad \le C_2\|d(G(u))\|_{s,B,w} \\ &\quad \le C_2\|G(du)\|_{s,B,w} \\ &\quad \le C_3\|du\|_{s,B,w}. \end{aligned}$$

Therefore, inequality (3.14) holds ∎

Using a method similar to that in the proof of Theorem 3.6 we can extend inequalities (2.3) and (2.5) to the following $A_r(M)$-weighted version.

Corollary 3.10. *Let $u \in C^\infty(\Lambda^l M)$ $(l = 1, 2, \ldots, n)$ be an A-harmonic tensor on a manifold M, let $\rho > 1$, $1 < s < \infty$ and $w \in A_r(M)$ for some $r > 1$. Then there exists a constant C, independent of u, such that*

$$\begin{aligned} \|(G(u))_B\|_{s,B,w^\alpha} &\le C_1\|u\|_{s,\rho B,w^\alpha} \\ \|G(u) - (G(u))_B\|_{s,B,w^\alpha} &\le C_2\|u\|_{s,\rho B,w^\alpha} \end{aligned}$$

for all balls B with $\rho B \subset M$ and any real number α with $0 < \alpha \le 1$.

4. Applications to the A-harmonic equation

In this section, we discuss applications to the non-homogeneous as well as homogeneous A-harmonic equations.

In order to prove global $A_r(M)$-weighted results, we need the following lemma about Whitney covers appearing in [11] (see [14] for more properties of Whitney cubes).

Lemma 4.1. *Each Ω has a modified Whitney cover of cubes $\mathcal{V} = \{Q_i\}$ such that*

$$\bigcup_i Q_i = \Omega$$

$$\sum_{Q\in\mathcal{V}} \chi_{\sqrt{\frac{5}{4}}Q}(x) \le N\chi_\Omega(x)$$

for all $x \in \mathbb{R}^n$ and some $N > 1$, where χ_E is the characteristic function for a set E. Moreover, if $Q_i \cap Q_j \ne \phi$ for $i \ne j$, then there exists a cube R (this cube does not need to be a member of $\mathcal{V}$) in $Q_i \cap Q_j$ such that $Q_i \cup Q_j \subset NR$. Also, if Ω is δ-John, then there is a distinguished cube $Q_0 \in \mathcal{V}$ which can be connected with every cube $Q \in \mathcal{V}$ by a chain of cubes $Q_0, Q_1, \ldots, Q_k = Q$ from $\mathcal{V}$ and such that $Q \subset \rho Q_i \quad (i = 0, 1, \ldots, k)$ for some $\rho = \rho(n, \delta)$.

Now, we prove the following global $A_r(M)$-weighted norm inequalities for compositions of the Laplace-Beltrami operator Δ and Green's operator G on the manifold M.

Theorem 4.2. *Let M be a compact, orientable, C^∞-smooth Riemannian manifold without boundary, let $u \in C^\infty(\Lambda^l M) \quad (l = 1, 2, \cdots, n)$ be an A-harmonic tensor on M, let $\rho > 1$, $1 < s < \infty$ and $w \in A_r(M)$ for some $r > 1$. Then there exists a constant C, independent of u, such that*

$$\begin{aligned} \|G(\Delta(u))\|_{s,M,w^\alpha} &\le C\|u\|_{s,M,w^\alpha} \\ \|\Delta(G(u))\|_{s,M,w^\alpha} &\le C\|u\|_{s,M,w^\alpha} \\ \|G(\Delta(u)) + \Delta(G(u))\|_{s,M,w^\alpha} &\le C\|u\|_{s,M,w^\alpha} \end{aligned} \tag{4.1}$$

for any real number α with $0 < \alpha \le 1$.

Proof. Since M is compact, there is a finite coordinate chart cover

$$\{U_1, \ldots, U_m\}$$

of M such that $\cup_{k=1}^m U_k = M$. Note that we can give M a topology in unique way such that each U_k is open (see [10: Chapter II]). Hence we may assume that all U_k are open. Applying Theorem 3.6 and Lemma 4.1 to U_k (note that $\cup_{B\in\mathcal{V}} B = U_k$ now), we obtain

$$\begin{aligned}
\|G(\Delta(u))\|_{s,U_k,w^\alpha} &\le \sum_{B\in\mathcal{V}} \|G(\Delta(u))\|_{s,B,w^\alpha} \\
&\le \sum_{B\in\mathcal{V}} C_1\|u\|_{s,\rho B,w^\alpha} \\
&\le \sum_{B\in\mathcal{V}} C_1\|u\|_{s,U_k,w^\alpha} \\
&\le C_k\|u\|_{s,M,w^\alpha}.
\end{aligned}$$

Hence

$$\begin{aligned}
\|G(\Delta(u))\|_{s,M,w^\alpha} &= \sum_{k=1}^{m} \|G(\Delta(u))\|_{s,U_k,w^\alpha} \\
&\le \sum_{k=1}^{m} C_k\|u\|_{s,M,w^\alpha} \\
&\le \|u\|_{s,M,w^\alpha} \sum_{k=1}^{m} C_k \\
&\le C\|u\|_{s,M,w^\alpha}.
\end{aligned}$$

Thus, $(4.1)_1$ is true. Similarly, we can prove $(4.1)_2$ and $(4.1)_3$ using Lemma 4.1 and Theorem 3.7 and Corollary 3.8, respectively. The proof has been completed ∎

From [5], we have the following norm comparison statement on a manifold M.

Lemma 4.3. *Let M be a compact, orientable, C^∞-smooth Riemannian manifold without boundary, and let $u \in \Lambda^{l-1}M$ $(l = 1, \ldots, n)$ and $v \in \Lambda^{l+1}M$ $(l = 0, 1, \cdots, n-1)$ be a pair of solutions to the conjugate A-harmonic equation (3.2). Then $du \in L^p(M, \Lambda^l)$ if and only if $d^\star v \in L^q(M, \Lambda^l)$. Moreover, there exist constants C_1 and C_2, independent of u and v, such that*

$$C_1\|du\|^p_{p,M} \le \|d^\star v\|^q_{q,M} \le C_2\|du\|^p_{p,M}. \tag{4.2}$$

Theorem 4.4. *Let $u \in C^\infty(\Lambda^{l-1}M)$ $(l = 1, \ldots, n)$ and $v \in C^\infty(\Lambda^{l+1}M)$ $(l = 0, 1, \ldots, n-1)$ be a pair of solutions to the conjugate A-harmonic equation (3.2) on a manifold M. Then there exists a constant C, independent of u and v, such that*

$$\|G(du) - (G(du))_D\|^p_{p,D} \le C\|d^\star v\|^q_{q,D}$$

for any convex and bounded domain D with $D \subset M$.

Proof. Applying (2.6) to du and using Lemma 3.3, we have

$$\|G(du)) - (G(du))_D\|_{p,D}^p \le C_1\|du\|_{p,D}^p \le C_2\|d^\star v\|_{q,D}^q$$

and the statement is proved ∎

We prove the following global Sobolev-Poincaré type embedding theorem for Green's operator G.

Theorem 4.5. *Let $u \in C^\infty(\Lambda^{l-1}M)$ $(l = 1, \cdots, n)$ and $v \in C^\infty(\Lambda^{l+1}M)$ $(l = 0, 1, \ldots, n-1)$ be a pair of solutions to the conjugate A-harmonic equation (3.2) on a manifold M. Then there exists a constant C, independent of u and v, such that*

$$\|G(u) - (G(u))_D\|_{W^{1,p}(D)}^p \le C\|d^\star v\|_{q,D}^q$$

for any convex and bounded domain D with $D \subset M$.

Proof. From (1.3), (1.5), (2.2) and (2.1) we obtain

$$\begin{aligned}\|G(u) - (G(u))_D\|_{W^{1,p}(D)} &= \|Td(G(u))\|_{W^{1,p}(D)}\\ &\le C_1|D|\,\|d(G(u))\|_{p,D}\\ &\le C_1|D|\,\|G(du)\|_{p,D}\\ &\le C_2|D|\,\|du\|_{p,D}.\end{aligned}$$

Using this and (4.2) yields

$$\begin{aligned}\|G(u) - (G(u))_D\|_{W^{1,p}(D)}^p &\le C_3|D|^p\|du\|_{p,D}^p\\ &\le C_4|D|^p\|d^\star v\|_{q,D}^q\\ &\le C_5\|d^\star v\|_{q,D}^q\end{aligned}$$

since D is bounded and the proof has been completed ∎

Theorem 4.6. *Let $u \in C^\infty(\Lambda^{l-1}M)$ $(l = 1, \ldots, n)$ and $v \in C^\infty(\Lambda^{l+1}M)$ $(l = 0, 1, \ldots, n-1)$ be a pair of solutions to the conjugate A-harmonic equation (3.2) on a manifold M. Then there exists a constant C, independent of u and v, such that*

$$\|G(d^\star v) - (G(d^\star v))_D\|_{q,D}^q \le C\|du\|_{p,D}^p$$

for any convex and bounded domain D with $D \subset M$.

Proof. Using (1.3), (1.5), Lemma 2.1 and (4.2) we have

$$\begin{aligned}\|G(d^\star v) - (G(d^\star v))_D\|_{W^{1,q}(D)}^q &= \|Td(G(d^\star v))\|_{W^{1,q}(D)}^q\\ &\le C_1|D|^q\|d(G(d^\star v))\|_{q,D}^q\\ &\le C_2|D|^q\|d^\star v\|_{q,D}^q\\ &\le C_3\|du\|_{p,D}\end{aligned}$$

since D is bounded and the proof has been completed ∎

Theorem 4.7. *Let M be a compact, orientable, C^∞-smooth Riemannian manifold without boundary, let $u \in C^\infty(\Lambda^l M)$ $(l = 1, \dots, n)$ be an A-harmonic tensor on M and let $\rho > 1$, $1 < s < \infty$ and $w \in A_r(M)$ for some $r > 1$. Then there exists a constant C, independent of u, such that*

$$\begin{aligned} &\|dd^\star G(u)\|_{s,M,w^\alpha} + \|d^\star dG(u)\|_{s,M,w^\alpha} \\ &+\|dG(u)\|_{s,M,w^\alpha} + \|d^\star G(u)\|_{s,M,w^\alpha} + \|G(u)\|_{s,M,w^\alpha} \le C\|u\|_{s,M,w^\alpha}. \end{aligned} \tag{4.3}$$

for any real number α with $0 < \alpha \le 1$.

Proof. From inequality (2.1) we have the inequalities

$$\begin{aligned} \|dd^\star G(u)\|_{s,B} &\le C_1\|u\|_{s,B} \\ \|d^\star dG(u)\|_{s,B} &\le C_2\|u\|_{s,B} \\ \|dG(u)\|_{s,B} &\le C_3\|u\|_{s,B} \\ \|d^\star G(u)\|_{s,B} &\le C_4\|u\|_{s,B} \\ \|G(u)\|_{s,B} &\le C_5\|u\|_{s,B} \end{aligned}$$

for any ball B with $B \subset M$. By the same method as we used in the proof of Theorem 3.6 we can extend the above inequalities to weighted versions. Then, similar to the proof of Theorem 4.2, we obtain the global versions

$$\begin{aligned} \|dd^\star G(u)\|_{s,M,w^\alpha} &\le C_6\|u\|_{s,M,w^\alpha} \\ \|d^\star dG(u)\|_{s,M,w^\alpha} &\le C_7\|u\|_{s,M,w^\alpha} \\ \|dG(u)\|_{s,M,w^\alpha} &\le C_8\|u\|_{s,M,w^\alpha} \\ \|d^\star G(u)\|_{s,M,w^\alpha} &\le C_9\|u\|_{s,M,w^\alpha} \\ \|G(u)\|_{s,M,w^\alpha} &\le C_{10}\|u\|_{s,M,w^\alpha}. \end{aligned}$$

Adding these inequalities we obtain (4.3) which ends the proof ∎

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