A Simple Proof
of the
Mountain Circle Theorem

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Abstract. In this paper we give an alternative proof of Fournier-Willem's Mountain Circle Theorem. Our proof, which is simpler than the original one proposed by these authors, is based on a Lusternik-Schnirelman-type theory and some results of elementary Brouwer degree theory.

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1. Introduction

The Mountain Circle Theorem has interesting applications to the theory of ordinary differential equations in dynamics (see, for example, Fournier and Willem [10]). The purpose of this note is to give an elementary proof of that theorem due to Fournier and Willem [10]. As the name suggests, this theorem is similar to the well-known Mountain Pass Theorem by Ambrosetti and Rabinowitz, but, under slightly different hypotheses, the result by Fournier and Willem asserts the existence of at least two critical points (instead of one) of a C^1 functional. The main tool used by Fournier and Willem in their proof is a Lusternik-Schnirelman type theory, which they introduce in [7, 8, 10], together with some advanced techniques of cohomology theory (see [10]).

In the present paper, to prove the Mountain Circle Theorem we still use the Lusternik-Schnirelman-type theory introduced by Fournier and Willem but, instead of cohomology, we use some elementary results of Brouwer degree theory.

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2. Lusternik-Schnirelman category

In this section, X will be a topological space.

Definition 2.1. A subset A of X is said to be *contractible* in X if there exist a point $x_0 \in X$ and a homotopy $h : A \times [0,1] \to X$ such that $h(\cdot, 0) = \mathbf{1}_A$ and $h(A, 1) = \{x_0\}$.

Obviously, each subset of a contractible set is contractible.

We recall that a subset A of X is said to have *category n* in X if n is the least positive integer such that A can be covered by n closed sets which are contractible in X. If A cannot be covered by a finite number of such sets, then by definition A has category $+\infty$, while the category of the empty set is always set to be 0. The category of A in X is denoted by $\mathrm{cat}_X(A)$.

The following proposition shows some easy consequences of the definition of Lusternik-Schnirelman category.

Proposition 2.1 (see, for example, Mawhin and Willem [14: Lemma 4.6]). *Let A and B be subsets of a topological space X. Then:*

    1. *If $A \subset B \subset X$, then $\mathrm{cat}_X(A) \leq \mathrm{cat}_X(B)$.*

    2. $\mathrm{cat}_X(A \cup B) \leq \mathrm{cat}_X(A) + \mathrm{cat}_X(B)$.

    3. *If A is closed and $h : A \times [0,1] \to X$ is a homotopy such that $h(x, 0) = x$ for each $x \in A$, then $\mathrm{cat}_X(A) \leq \mathrm{cat}_X(h(A, 1))$.*

Obviously, the category of a set A depends on the space in which A is considered, as the following example shows.

Example 2.1. Let $A = S^1 \subset \mathbb{R}^2$. We have $\mathrm{cat}_{\mathbb{R}^2}(A) = 1$ because, as $\mathbb{R}^2$ is contractible in itself, A is contractible in $\mathbb{R}^2$. On the other hand, $\mathrm{cat}_A(A) = 2$ since A is not contractible in itself and can be covered by two closed sets which are contractible in A.

Definition 2.2. A normal space X is called an *absolute neighborhood retract* if, given any normal space Z in which X is imbedded as a closed subspace, X is a retract of an open set in Z.

The next proposition shows the continuity property of the category on compact subsets of metric absolute neighborhood retracts. In the following $\overline{U}$ denotes the closure of U.

Proposition 2.2 (see, for example, Mawhin and Willem [14: Lemma 4.7] or Palais [17: Theorem 6.3). *Suppose that X is a metric absolute neighborhood retract and $A \subset X$ is compact. Then $\mathrm{cat}_X(A) < +\infty$ and there exists an open neighborhood U of A such that $\mathrm{cat}_X(\overline{U}) = \mathrm{cat}_X(A)$.*

3. Fournier-Willem relative categories

In this section we will introduce two relative categories due to Fournier and Willem [7, 10], which generalize the Lusternik-Schnirelman category and are used to estimate the number of critical points of a C^1 functional. Relative categories play an important role in the proof of Fournier-Willem's Mountain Circle Theorem.

In the following, X will be a topological space and $Y \subset X$ a closed subset.

Definition 3.1 (Fournier and Willem [7, 10]). A subset A of X is said to be *strongly deformable* into a subset B of X relative to Y in X (denoted by $A <_Y B$ in X) if there exists a homotopy $h : A \times [0,1] \to X$ such that $h(\cdot, 0) = \mathbf{1}_A$, $h(A, 1) \subset B$ and $h(y, t) = y$ for every $y \in A \cap Y$ and every $t \in [0, 1]$.

Observe that, if $X \supset A \supset Y$, A is strongly deformable into Y relative to Y in X if and only if Y is a strong deformation retract of A in X.

Definition 3.2 (Fournier and Willem [7, 10]). A subset A of X is said to have *weak category n* in X relative to Y (or to be of the *n-th weak category* in X relative to Y) if n is the least non-negative integer such that $A \subseteq \bigcup_{i=0}^n A_i$, where A_i is closed for any $i \geq 0$, A_i is contractible in X for each $i \geq 1$ and A_0 is strongly deformable into Y relative to Y in X.

We will denote the weak category of A in X relative to Y by $\mathrm{cat}_{X,Y}(A)$, set $\mathrm{cat}_{X,Y}(A) = +\infty$ if A has no finite covering as above and $\mathrm{cat}_{X,Y}(\emptyset) = 0$.

Definition 3.3 (Fournier and Willem [7, 10]). A subset A of X is said to be *touch and stop deformable* into a subset B of X relative to Y in X (denoted by $A \ll_Y B$ in X) if there exists a homotopy $h : A \times [0,1] \to X$ such that $h(\cdot, 0) = \mathbf{1}_A$, $h(A, 1) \subset B$ and, if $h(x, t) = y \in Y$ for some $x \in A$ and $t \in [0, 1]$, then $h(x, s) = y$ for every $s \geq t$.

Definition 3.4 (Fournier and Willem [7, 10]). A subset A of X is said to have *strong category n* in X relative to Y (or to be of the *n-th strong category* in X relative to Y) if n is the least non-negative integer such that $A \subseteq \bigcup_{i=0}^n A_i$, where A_i is closed for any $i \geq 0$, A_i is contractible in $X \setminus Y$ for each $i \geq 1$ and A_0 is touch and stop deformable into Y relative to Y in X.

We denote the strong relative category by $\mathrm{Cat}_{X,Y}(A)$ and set $\mathrm{Cat}_{X,Y}(A) = +\infty$ if A has no finite covering as above and $\mathrm{Cat}_{X,Y}(\emptyset) = 0$.

Clearly, in both Definitions 3.2 and 3.4, if $A \subset X$ is closed, the inclusion $A \subseteq \bigcup_{i=0}^n A_i$ can be replaced with the equality $A = \bigcup_{i=0}^n A_i$. Observe also that, from the definitions, it is easily seen that $\mathrm{cat}_{X,Y}(A) \leq \mathrm{Cat}_{X,Y}(A)$ for every $A \subset X$.

The next proposition will be used in the proof of the Mountain Circle Theorem. In the papers by Fournier and Willem it is proved by using cohomology theory. In the present paper, we will give a simpler proof based on Brouwer degree theory.

Before giving the result, we introduce some notation. Namely, given $\rho > 0$, we will denote by $B(0, \rho) \subset \mathbb{R}^n$ the open ball of radius ρ centered at the origin in $\mathbb{R}^n$ and by $\overline{B(0, \rho)}$ its closure.

The following easy consequence of finite-dimensional degree theory is crucial in the proof of Proposition 3.1 below.

Lemma 3.1 (Degiovanni [4: Lemma 2.7]). *If U is a bounded open set in $\mathbb{R}^n$ and $0 \in U$, then ∂U is not contractible in $\mathbb{R}^n \setminus \{0\}$.*

Proof. Suppose by contradiction that ∂U is contractible in $\mathbb{R}^n \setminus \{0\}$, with a contraction $h : \partial U \times [0, 1] \to \mathbb{R}^n \setminus \{0\}$ such that $h(\partial U, 1) = \{y\}$. Since the topological degree is homotopy invariant and depends only on the values of functions on the boundary (see, for example, Deimling [5: Theorem 3.1]), we should have $0 = \deg(y, U, 0) = \deg(\mathbf{1}_U, U, 0) = 1$, which is a contradiction ∎

Propoposition 3.1. *Let $X = A = \overline{B(0, R)} \setminus B(0, r) \subset \mathbb{R}^n$ with $0 < r < R$ and $Y = \partial A = \partial B(0, R) \cup \partial B(0, r)$. Then $\mathrm{Cat}_{X,Y}(A) = 2$.*

Proof. Clearly, A can be covered by two closed sets which are contractible in $X \setminus Y$ and a closed set which is touch and stop deformable into Y relative to Y in X. Thus $\mathrm{Cat}_{X,Y}(A) \leq 2$.

We need to show that $\mathrm{Cat}_{X,Y}(A) > 1$. Suppose by contradiction $A = A_0 \cup A_1$, where A_0 and A_1 are both closed in $X = A$, $A_0 \ll_Y Y$ and A_1 is contractible in $X \setminus Y$. Without loss of generality we can suppose that $A_0 \supset Y$. So A_0 must be disconnected, precisely of the form $A_0 = A_{0,r} \cup A_{0,R}$ with $A_{0,r}$ and $A_{0,R}$ disjoint closed sets, $A_{0,r} \supset \partial B(0, r)$ and $A_{0,R} \supset \partial B(0, R)$. As $A_{0,r}$ and $A_{0,R}$ are disjoint compact sets, we have $\mathrm{dist}(A_{0,r}, A_{0,R}) = d > 0$. Fix $\varepsilon \in (0, \frac{d}{3}]$ and consider the bounded open set

$$U_\varepsilon = \big\{ x \in \mathbb{R}^n \ : \ \mathrm{dist}(x, B(0, r) \cup A_{0,r}) < \varepsilon \big\}.$$

Obviously, U_ε contains 0 and $\partial U_\varepsilon \subset A_1$. By Lemma 3.1, ∂U_ε is not contractible in $\mathbb{R}^n \setminus \{0\}$, so *a fortiori* it is not contractible in $X \setminus Y$. As A_1 is a superset of ∂U_ε, it is not contractible in $X \setminus Y$, and this is a contradiction ∎

The following example shows that we may have $\mathrm{cat}_{X,Y}(A) < \mathrm{Cat}_{X,Y}(A)$.

Example 3.1. Let $X = A = \overline{B(0, 2)} \subset \mathbb{R}^2$ and $Y = \partial B(0, 2) \cup \overline{B(0, 1)} \subset \mathbb{R}^2$. Then we have $\mathrm{cat}_{X,Y}(A) = 1$ because A can be covered by two closed subsets of X: one of these is strongly deformable into Y relative to Y in X,

while the other is contractible in X. On the other hand, by arguing as in Proposition 3.1 we can prove that $\mathrm{Cat}_{X,Y}(A) = 2$.

Remark 3.1. By an argument similar to that of Proposition 3.1 we can prove also that, if $X = A = \overline{B(0,R)}$ (with $R > 0$) and $Y = \partial B(0,R) \cup \{0\}$, then $\mathrm{Cat}_{X,Y}(A) = 2$.

The next proposition shows some of the main properties of relative categories, which are generalizations of those of the Lusternik-Schnirelman category. In the following, we will assume that A and B are closed in X.

Proposition 3.2 (Fournier and Willem [10: Proposition 2.4]). *Let X be a topological space and $A, B \subset X$ closed subsets. Then:*

  1. *If $A \subset B$, then $\mathrm{cat}_{X,Y}(A) \leq \mathrm{cat}_{X,Y}(B)$.*

  2. *If $A <_Y B$, then $\mathrm{cat}_{X,Y}(A) \leq \mathrm{cat}_{X,Y}(B)$.*

  3. *$\mathrm{cat}_{X,Y}(A \cup B) \leq \mathrm{cat}_{X,Y}(A) + \mathrm{cat}_X(B)$.*

  4. *$\mathrm{cat}_{X,Y}(A) = 0$ if and only if $A <_Y Y$.*

Furthermore, in each of these properties we can replace cat *by* Cat, *provided we replace $<_Y Y$ by $\ll_Y Y$ and, in Assertion 3, $\mathrm{cat}_X(B)$ by $\mathrm{cat}_{X \setminus Y}(B)$ with the further condition $X \setminus Y \supset B$.*

The next result shows the behavior of relative categories in relation to supersets and retractions.

Proposition 3.3 (Fournier and Willem [10: Proposition 2.5]). *Let $X' \supset X \supset A$, $X' \supset Y' \supset Y$, $X' \supset A' \supset A$ and $X \supset Y$. Then:*

  1. *$\mathrm{cat}_{X',Y}(A) \leq \mathrm{cat}_{X,Y}(A)$.*

  2. *$\mathrm{cat}_{X,Y'}(A) \geq \mathrm{cat}_{X,Y}(A)$, provided $Y' \subset X$ and $\mathrm{cat}_{X,Y}(Y') = 0$.*

  3. *$\mathrm{cat}_{X',Y'}(A') \geq \mathrm{cat}_{X,Y}(A)$, if there exists a retraction $r : X' \to X$ such that $r(A') \supset A$ and $r^{-1}(Y) = Y'$.*

  4. *$\mathrm{cat}_{X',Y'}(A') \leq \mathrm{cat}_{X,Y}(A)$, if $A' \setminus A = X' \setminus X = Y' \setminus Y$ and X is closed in X'.*

Moreover, in each of these properties it is possible to replace cat *by* Cat.

The next proposition holds for cat, but not for Cat.

Proposition 3.4 (Fournier and Willem [10: Proposition 2.6]) *Let $X' \supset X \supset A$, $X \supset Y$ and $X' \supset Y' \supset Y$. If there exists a retraction $r : X' \to X$ such that $r(Y') <_Y Y$ in X, then $\mathrm{cat}_{X',Y'}(A') \geq \mathrm{cat}_{X,Y}(A)$, provided $A \subset A'$.*

In the following property it is not possible to replace Cat by cat.

Proposition 3.5 (Excision property; Fournier and Willem [10: Proposition 2.7]). *Given $A \subset X$ and $Y \subset X$, we have*

$$\mathrm{Cat}_{X,Y}(A) = \mathrm{Cat}_{X \setminus V, Y \setminus V}(A \setminus V) = \mathrm{Cat}_{X \cap F, Y \cap F}(A \cap F),$$

for any subset V of the interior of Y and any closed set F such that $X = F \cup Y$.

Now we give a proposition which is valid for Cat but not for cat.

Proposition 3.6 (Fournier and Willem [10: Proposition 2.8]). *Suppose $\{X_j\}_{j \in J}$ is a finite family of disjoint non-empty closed sets whose union is X. Then*

$$\mathrm{Cat}_{X,Y}(A) = \sum_{j \in J} \mathrm{Cat}_{X_j, Y \cap X_j}(A \cap X_j).$$

Now, let us give a critical point result by Fournier and Willem, based on the strong relative category, which will be used to prove the Mountain Circle Theorem. First, we recall the Palais-Smale condition.

Definition 3.5 (see, for example, Mawhin and Willem [14] or Struwe [20]). Let X be a Banach space. A map $f \in C^1(X, \mathbb{R})$ satisfies the *Palais-Smale condition* if

(P-S) Any sequence (x_n) in X such that $(f(x_n))$ is bounded and $df(x_n) \to 0$ admits a convergent subsequence.

For any $c \in \mathbb{R}$ we will denote by f^c the subset of M where $f \leq c$.

Theorem 3.1 (Fournier and Willem [7: Theorem 4.2]). *Let X be a Banach space and suppose $f \in C^1(X, \mathbb{R})$ satisfies the Palais-Smale condition (P-S). Fix two numbers $a, b \in \mathbb{R}$ with $a < b$ such that the critical sets K_a and K_b at level a and b respectively are empty. Then the number of critical points of f in $f^{-1}([a, b])$ is at least $\mathrm{Cat}_{f^b, f^a}(f^b)$.*

4. The Mountain Circle Theorem

We are now ready to prove the Mountain Circle Theorem. In the following, X will be a Banach space and $f \in C^1(M, \mathbb{R})$ a functional.

Theorem 4.1 (The Mountain Circle Theorem). *Let $a, b \in \mathbb{R}$ with $a < b$. Suppose f satisfies the Palais-Smale condition (P-S) in $f^{-1}([a, +\infty))$ and there exist $0 \leq r_1 < r_2 < r_3$ such that*

1. $f(x, v) \leq a$ *for $\|v\| = r_1$ or $\|v\| = r_3$ and for any $x \in X$*
2. $f(x, v) > b$ *for $\|v\| = r_2$ and for any $x \in X$.*

Then f has at least two critical points in $(X \times \mathbb{R}^n) \setminus f^a$.

Proof. We will use Theorem 3.1 to prove the assertion. Suppose first $r_1 > 0$. Let us prove that, for $\varepsilon > 0$ small enough, we have $\mathrm{Cat}_{X \times \mathbb{R}^n, f^{a+\varepsilon}}(X \times \mathbb{R}^n) \geq 2$. For $\varepsilon \in (0, b-a)$, $f^{a+\varepsilon}$ is disconnected. Set $A = \{v \in \mathbb{R}^n : r_1 \leq \|v\| \leq r_3\}$. The set $f^{a+\varepsilon} \cap (X \times A)$ is of the form $F_1 \cup F_2$, where F_1 and F_2 are disjoint closed sets. In fact, if we set respectively

$$F_1 = f^{a+\varepsilon} \cap (X \times A) \cap \left(X \times \overline{B(0, r_2)}\right)$$
$$F_2 = f^{a+\varepsilon} \cap (X \times A) \cap \left[X \times (\mathbb{R}^n \setminus B(0, r_2))\right]$$

we have easily $f^{a+\varepsilon} \cap (X \times A) = F_1 \cup F_2$ and $F_1 \cap F_2 = \emptyset$, because $f^{a+\varepsilon}$ does not meet $X \times \partial B(0, r_2)$. Moreover, we have $X \times \partial B(0, r_1) \subset F_1$ and $X \times \partial B(0, r_3) \subset F_2$. The homotopy

$$h : \left(f^{a+\varepsilon} \cap (X \times A)\right) \times [0, 1] \to X \times A$$

defined by

$$h(x, v, t) = \begin{cases} \left(x, (1-t)v + tr_1 \frac{v}{\|v\|}\right) & \text{if } (x, v) \in F_1, t \in [0, 1] \\ \left(x, (1-t)v + tr_3 \frac{v}{\|v\|}\right) & \text{if } (x, v) \in F_2, t \in [0, 1] \end{cases}$$

shows that $f^{a+\varepsilon} \cap (X \times A)$ is touch and stop deformable into $X \times \partial A$ relative to $X \times \partial A$. By Proposition 3.3 we have

$$\mathrm{Cat}_{X \times \mathbb{R}^n, f^{a+\varepsilon}}(X \times \mathbb{R}^n) \geq \mathrm{Cat}_{X \times \mathbb{R}^n, f^{a+\varepsilon}}(X \times A)$$
$$= \mathrm{Cat}_{X \times A, f^{a+\varepsilon} \cap (X \times A)}(X \times A)$$
$$\geq \mathrm{Cat}_{X \times A, X \times \partial A}(X \times A)$$
$$\geq \mathrm{Cat}_{A, \partial A}(A).$$

By Proposition 3.1 we have $\mathrm{Cat}_{A, \partial A}(A) = 2$, so the result is proved. If $r_1 = 0$, by applying Remark 3.1 we get also the result ∎

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