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Non-Oscillatory Solutions of Higher Order Nonlinear Neutral Delay Difference Equations

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Abstract. Consider the forced higher order nonlinear neutral delay difference equation

$$L_m \;\, x(n) + cx(n-\tau) \;\, + F \;\, n, x(\sigma(n)) \;\, = g(n) \quad (n \geq n_0).$$

We obtain a global result (with respect to c), which consists of some sufficient conditions for the existence of a non-oscillatory solution of the above equation. Our results improve and extend a number of existing results.

Keywords: *Neutral difference equations, non-oscillatory solutions*

AMS subject classification: 39A10

1. Introduction

Consider the forced higher order nonlinear neutral delay difference equation

$$L_m\big(x(n) + cx(n-\tau)\big) + F\big(n, x(\sigma(n))\big) = g(n) \quad (n \geq n_0) \tag{1}$$

where

$$\begin{aligned} L_0 x(n) &= x(n) \\ L_i x(n) &= \frac{1}{r_i(n)} \Delta(L_{i-1} x(n)) \quad (i = 1, 2, ..., m) \\ \Delta x(n) &= x(n+1) - x(n) \end{aligned}$$

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and

$$\begin{aligned} n_0, \tau &\in \mathbb{N}_0 \\ N(n_0) &= \{n_0, n_0+1, \dots\} \\ r_i: N(n_0) &\to (0,\infty) \quad (i=1,2,...,m-1),\ r_m \equiv 1 \\ \sigma: N(n_0) &\to N(n_0),\ \sigma(n)\to\infty \quad (n\to\infty) \\ g: N(n_0) &\to \mathbb{R} \\ F: N(n_0)\times\mathbb{R} &\to \mathbb{R} \text{ is continuous} \end{aligned}$$

i.e. F is continuous as a map from the topological space $N(n_0)\times\mathbb{R}$ into the topological space $\mathbb{R}$, the topology on $N(n_0)$ being the discrete one.

Oscillation theory of higher order neutral difference equations has developed very rapidly in recent years. It has concerned itself largely with the oscillatory and non-oscillatory properties of solutions (see, e.g., [1 - 5, 7 - 16] and the references cited therein).

In [14], Zhang and Sun study the existence of a non-oscillatory solution of the forced nonlinear difference equation

$$\Delta\big(x(n) + cx(n-\tau)\big) + F\big(n, x(\sigma(n))\big) = g(n) \quad (n \ge n_0). \tag{2}$$

The following is one of their main results:

Theorem A [14]. *Assume the following:*

(C$_1$) $|F(n,x)| \le |F(n,y)| \quad (|x| \le |y|)$

(C$_2$) *For each $0 < d_1 < d_2$ there exists $L(n)$ such that $|f(n,x) - f(n,y)| \le L(n)|x-y| \quad (x,y \in [d_1,d_2])$ and $\sum_{i=n_0}^{\infty} L(i) < \infty$.*

(C$_3$) $-1 < c < 1$.

Further, assume $\sum_{i=n_0}^{\infty} |F(i,d)| < \infty$ for some $d \ne 0$ and $\sum_{i=n_0}^{\infty} |g(i)| < \infty$. Then equation (2) has a bounded non-oscillatory solution.

In [3], Agarwal, Grace and O'Regan investigate the existence of non-oscillatory solutions of nonlinear second order neutral difference equations of the form

$$\Delta\Big(\frac{1}{a(n)}\Delta\big(x(n) + cx(n-\tau)\big)\Big) + F\big(n+1, x(n+1-\sigma)\big) = 0 \tag{3}$$

for $n \ge n_0$ where $a: N(n_0) \to (0,\infty)$, τ and σ are fixed non-negative integers. They proved the following:

Theorem B [3]. *Assume the following:*

(C$_4$) $|c| \neq 1$ *and* $F : \mathbb{N} \times (0, \infty) \to [0, \infty)$.

Further, assume there exist $K > 0$ *and* $n_0 \in \mathbb{N}$ *with*

$$\sum_{n=n_0}^{\infty} a(n) \sum_{i=n}^{\infty} \sup_{w \in [\frac{K}{2}, K]} F(i+1, w) < \infty.$$

Then equation (3) *has a bounded non-oscillatory solution.*

The authors of [3] remarked that Theorem B could be established using Krasnoselskii's fixed point theorem and it could be extended to higher order equations.

In [8], Graef and Thandapani obtained an existence criteria for non-oscillatory solutions of the forced third order delay difference equation

$$\Delta^3 x(n) + p(n) f(x(n - \tau)) = q(n) \tag{4}$$

where $\{p(n)\}$ and $\{q(n)\}$ are sequences of real numbers. They proved the following result by using the Schauder fixed point theorem:

Theorem C [8]. *Assume the following:*

(C$_5$) $f : \mathbb{R} \to \mathbb{R}$ *is a non-decreasing function.*

Further, assume

$$\sum_{i=n_0}^{\infty} i^2 |p(i)|, \sum_{i=n_0}^{\infty} i^2 |q(i)| < \infty.$$

Then equation (4) *has a bounded non-oscillatory solution.*

Recently, Yang and Liu [12] used the Banach contraction mapping principle to obtain an existence criteria for non-oscillatory solutions of higher order difference equations of the form

$$\Delta^m \big(x(n) + cx(n - \tau)\big) + p(n) f\big(x(\sigma(n))\big) = 0 \quad (n \geq n_0). \tag{5}$$

They proved the following

Theorem D [12]. *Assume the following:*

(C$_6$) $m \geq 2$ *is an even integer.*

(C$_7$) $c \neq \pm 1$, $(0 \not\equiv)\ p(n) \geq 0$.

(C$_8$) *For any* $x, y \in \mathbb{R}$, $|f(x) - f(y)| \leq L|x - y|$, *where* $L \in \mathbb{R}^+$.

Further, assume

$$\sum_{i=0}^{\infty}\sum_{j=i}^{\infty}\frac{(i-j+m-2)^{(m-2)}}{(m-2)!}\,p(i)<\infty. \tag{6}$$

Then equation (5) *has a bounded non-oscillatory solution.*

Note that (6) is equivalent to the simple condition

$$\sum_{i=n_0}^{\infty} i^{m-1}p(i)<\infty. \tag{7}$$

In a recent paper [16], the present author obtained the following result by using the Banach contraction mapping principle:

Theorem E [16]. *Assume the following:*

(C_9) *$m \geq 1$ is an odd integer.*

(C_{10}) *$c \neq -1$, and $f(x) = x$ for any $x \in \mathbb{R}$.*

Further, assume that $p(n) : N(n_0) \to \mathbb{R}$ and $\sum_{i=n_0}^{\infty} i^{m-1}|p(i)| < \infty$. Then equation (5) *has a bounded non-oscillatory solution.*

Our aim in this paper is to investigate the existence of non-oscillatory solutions of equation (1). First we extend Theorem B to equation (1) with $c \neq -1$ by using Krasnoselskii's fixed point theorem. Next, by using Schauder's fixed point theorem and some new techniques, we obtain a sufficient condition for the existence of a non-oscillatory solution of equation (1) in the critical case $c = -1$. To the best of our knowledge, there is no result in this critical case for a nonlinear neutral difference equation (1). In particular, our results improve and extend Theorems A - E by removing the restrictive conditions (C_1) - (C_{10}).

As is customary, a solution $\{x(n)\}$ of equation (1) is said to *oscillate about zero* or simply to *oscillate*, if its terms $x(n)$ are neither eventually all positive nor eventually all negative. Otherwise, the solution is called *non-oscillatory.*

2. Preliminaries

The space l^∞ is the set of all real sequences defined on the set of positive integers $\mathbb{N}$ where any individual sequence is bounded with respect to the usual supremum norm. It is well known that under this norm l^∞ is a Banach space. A subset Ω of a Banach space X is relatively compact, if every sequence in Ω has a subsequence converging to an element of X.

Definition 1 [5] A set Ω of sequences in l^∞ is *uniformly Cauchy* (or equi-Cauchy) if for every $\varepsilon > 0$ there exists an integer N such that $|x(i) - x(j)| < \varepsilon \ \ (i, j > N)$ for any $x = \{x(k)\}$ in Ω.

Lemma 1 [5] (Discrete Arzela-Ascoli Theorem). *A bounded, uniformly Cauchy subset Ω of l^∞ is relatively compact.*

The following fixed point theorems will be used to prove the main results in Section 3.

Lemma 2 [6] (Krasnoselskii's Fixed Point Theorem). *Let X be a Banach space, let Ω be a bounded closed convex subset of X and let $S_1, S_2 : \Omega \to X$ be maps such that $S_1x + S_2y \in \Omega$ for every pair $x, y \in \Omega$. If S_1 is a contraction and S_2 is completely continuous, then the equation $S_1x + S_2x = x$ has a solution in Ω.*

Lemma 3 [6] Schauder's Fixed Point Theorem). *Let Ω be a closed, convex and non-empty subset of a Banach space X and let $S : \Omega \to \Omega$ be a continuous mapping such that $S\Omega$ is a relatively compact subset of X. Then S has at least one fixed point in Ω, i.e. there exists an $x \in \Omega$ such that $Sx = x$.*

Lemma 4 [13] *If $r(n) \geq 0 \ \ (n \in N(n_0))$ and $\tau > 0$, then the statements*

i) $\displaystyle\sum_{s=n_0}^{\infty} sr(s) < \infty$

and

ii) $\displaystyle\sum_{j=0}^{\infty} \textstyle\sum_{s=n_0+j\tau}^{\infty} r(s) < \infty$

are equivalent.

3. Main results

Now we come to our main results.

Theorem 1. *Assume that $c \neq -1$ in equation* (1) *and that there exists some interval $[a, b] \subset \mathbb{R}^+$ such that*

$$\sum_{s_1=n_0}^{\infty} r_1(s_1) \sum_{s_2=s_1}^{\infty} r_2(s_2) \cdots \cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1}) \sum_{s=s_{m-1}}^{\infty} \sup_{w \in [a,b]} |F(s, w)| < \infty \tag{8}$$

and

$$\sum_{s_1=n_0}^{\infty} r_1(s_1) \sum_{s_2=s_1}^{\infty} r_2(s_2) \cdots \cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1}) \sum_{s=s_{m-1}}^{\infty} |g(s)| < \infty. \tag{9}$$

Then equation (1) *has a bounded non-oscillatory solution.*

Proof. The proof will be divided into Cases 1 - 5 in terms of the constant c. Let $l_{n_0}^{\infty}$ be the set of all real sequence $x = \{x(n)\}_{n=n_0}^{\infty}$ with norm $\|x\| = \sup_{n \geq n_0} |x(n)| < \infty$. Then $l_{n_0}^{\infty}$ is a Banach space. We define a closed, bounded and convex subset Ω of $l_{n_0}^{\infty}$ by

$$\Omega = \Big\{ x = \{x(n)\} \in l_{n_0}^{\infty} : \, a \leq x(n) \leq b \quad (n \geq n_0) \Big\}.$$

Case 1. For the case $-1 < c \leq 0$, by (8) and (9), we choose an $N > n_0$ sufficiently large such that

$$\frac{1}{(m-1)!} \sum_{s_1=N}^{\infty} r_1(s_1) \sum_{s_2=s_1}^{\infty} r_2(s_2) \cdots \cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1}) \times \sum_{s=s_{m-1}}^{\infty} \left(\sup_{w \in [a,b]} |F(s,w)| + |g(s)| \right) < \frac{(c+1)(b-a)}{2}.$$

Define two maps $S_1, S_2 : \Omega \to l_{n_0}^{\infty}$ by

$$S_1 x(n) = \begin{cases} \frac{(c+1)(a+b)}{2} - cx(n-\tau) & \text{for } n \geq N \\ S_1 x(N) & \text{for } n_0 \leq n < N \end{cases}$$

$$S_2 x(n) = \begin{cases} \left. \begin{aligned} & \frac{(-1)^{m+1}}{(m-1)!} \\ & \times \sum_{s_1=n}^{\infty} r_1(s_1) \sum_{s_2=s_1}^{\infty} r_2(s_2) \cdots \\ & \cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1}) \\ & \times \sum_{s=s_{m-1}}^{\infty} \big(F(s, x(\sigma(s))) - g(s) \big) \end{aligned} \right\} & \text{for } n \geq N \\ S_2 x(N) & \text{for } n_0 \leq n < N. \end{cases}$$

i) We shall show first that $S_1x + S_2y \in \Omega$ for any $x, y \in \Omega$. In fact, for every $x, y \in \Omega$ and $n \geq N$ we get

$$\begin{aligned}
&S_1x(n) + S_2y(n) \\
&\quad \leq \frac{(c+1)(a+b)}{2} - cb + \frac{1}{(m-1)!} \sum_{s_1=n}^{\infty} r_1(s_1) \sum_{s_2=s_1}^{\infty} r_2(s_2) \cdots \\
&\qquad \cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1}) \sum_{s=s_{m-1}}^{\infty} \Big(\sup_{w \in [a,b]} |F(s,w)| + |g(s)| \Big) \\
&\quad \leq \frac{(c+1)(a+b)}{2} - cb + \frac{(c+1)(b-a)}{2} \\
&\quad = b.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&S_1x(n) + S_2y(n) \\
&\quad \geq \frac{(c+1)(a+b)}{2} - ca - \frac{1}{(m-1)!} \sum_{s_1=n}^{\infty} r_1(s_1) \sum_{s_2=s_1}^{\infty} r_2(s_2) \cdots \\
&\qquad \cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1}) \sum_{s=s_{m-1}}^{\infty} \Big(\sup_{w \in [a,b]} |F(s,w)| + |g(s)| \Big) \\
&\quad \geq \frac{(c+1)(a+b)}{2} - ca - \frac{(c+1)(b-a)}{2} \\
&\quad = a.
\end{aligned}$$

Hence

$$a \leq S_1x(n) + S_2y(n) \leq b \quad (n \geq n_0).$$

Thus we have proved that $S_1x + S_2y \in \Omega$ for any $x, y \in \Omega$.

ii) We shall show next that S_1 is a contraction mapping on Ω. In fact, for $x, y \in \Omega$ and $n \geq N$ we have

$$\big|S_1x(n) - S_1y(n)\big| \leq -c\big|x(n-\tau) - y(n-\tau)\big| \leq -c\|x - y\|.$$

This implies $\|S_1x - S_1y\| \leq -c\|x - y\|$. Since $0 < -c < 1$, we conclude that S_1 is a contraction mapping on Ω.

iii) We finally show that S_2 is completely continuous. First, we will show that S_2 is continuous. For this, let $x_k = \{x_k(n)\} \in \Omega$ be a sequence such that $x_k(n) \to x(n)$ as $k \to \infty$. Because Ω is closed, $x = \{x(n)\} \in \Omega$. For $n \geq N$

we have

$$\begin{aligned}
&\big|S_2x_k(n) - S_2x(n)\big| \\
&\quad \le \frac{1}{(m-1)!}\sum_{s_1=n}^{\infty} r_1(s_1)\sum_{s_2=s_1}^{\infty} r_2(s_2)\cdots \\
&\qquad \cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1})\sum_{s=s_{m-1}}^{\infty}\Big(\big|F\big(s,x_k(\sigma(s))\big) - F\big(s,x(\sigma(s))\big)\big|\Big) \\
&\quad \le \frac{1}{(m-1)!}\sum_{s_1=N}^{\infty} r_1(s_1)\sum_{s_2=s_1}^{\infty} r_2(s_2)\cdots \\
&\qquad \cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1})\sum_{s=s_{m-1}}^{\infty}\Big(\big|F\big(s,x_k(\sigma(s))\big) - F\big(s,x(\sigma(s))\big)\big|\Big).
\end{aligned}$$

Since

$$\big|F(n,x_k(\sigma(n))) - F(n,x(\sigma(n)))\big| \to 0 \quad (k\to\infty),$$

by applying the Lebesgue dominated convergence theorem we conclude that $\lim_{k\to\infty}\big\|S_2x_k(n) - S_2x(n)\big\| = 0$. This means that S_2 is continuous.

Next we show that $S_2\Omega$ is relatively compact. Indeed, for any $\varepsilon > 0$, by (8) and (9), there exists $N^* \ge N$ such that

$$\begin{aligned}
&\sum_{s_1=N^*}^{\infty} r_1(s_1)\sum_{s_2=s_1}^{\infty} r_2(s_2)\cdots \\
&\cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1})\sum_{s=s_{m-1}}^{\infty}\Big(\sup_{w\in[a,b]}|F(s,w)| + |g(s)|\Big) < \frac{\varepsilon}{2}.
\end{aligned}$$

Then, for any sequence $x = \{x(n)\} \in \Omega$ and $n_2 > n_1 \ge N^*$,

$$\begin{aligned}
&\big|S_2x(n_2) - S_2x(n_1)\big| \\
&\quad \le \sum_{s_1=n_2}^{\infty} r_1(s_1)\sum_{s_2=s_1}^{\infty} r_2(s_2)\cdots \\
&\qquad \cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1})\sum_{s=s_{m-1}}^{\infty}\Big(\sup_{w\in[a,b]}|F(s,w)| + |g(s)|\Big) \\
&\qquad + \sum_{s_1=n_1}^{\infty} r_1(s_1)\sum_{s_2=s_1}^{\infty} r_2(s_2)\cdots
\end{aligned}$$

$$
\begin{aligned}
&\cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1}) \sum_{s=s_{m-1}}^{\infty} \left(\sup_{w\in[a,b]} |F(s,w)| + |g(s)| \right) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}
$$

Therefore, $\{S_2 x : x \in \Omega\}$ is a bounded and uniformly Cauchy subset. Hence, by Lemma 1, $S_2\Omega$ is relatively compact. By Lemma 2, there is an $x_0 \in \Omega$ such that $S_1 x_0 + S_2 x_0 = x_0$. It is easy to see that $\{x_0(n)\}$ is a bounded non-oscillatory solution of equation (1). This completes the proof in this case.

Case 2. For the case $c < -1$, by (8) and (9), we choose an $N > n_0$ sufficiently large such that

$$
\begin{aligned}
-\frac{1}{c(m-1)!} &\sum_{s_1=N+\tau}^{\infty} r_1(s_1) \sum_{s_2=s_1}^{\infty} r_2(s_2) \cdots \\
&\cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1}) \\
\times &\sum_{s=s_{m-1}}^{\infty} \left(\sup_{w\in[a,b]} |F(s,w)| + |g(s)| \right) < \frac{(c+1)(b-a)}{2c}.
\end{aligned}
$$

Define two maps $S_1, S_2 : \Omega \to l_{n_0}^{\infty}$ by

$$
S_1 x(n) = \begin{cases} \frac{(c+1)(a+b)}{2c} - \frac{x(n+\tau)}{c} & \text{for } n \geq N \\ S_1 x(N) & \text{for } n_0 \leq n < N \end{cases}
$$

$$
S_2 x(n) = \begin{cases} \left. \begin{aligned} &\frac{(-1)^{m+1}}{c(m-1)!} \sum_{s_1=n+\tau}^{\infty} r_1(s_1) \sum_{s_2=s_1}^{\infty} r_2(s_2) \cdots \\ &\cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1}) \\ &\times \sum_{s=s_{m-1}}^{\infty} \big(F(s, x(\sigma(s))) - g(s)\big) \end{aligned} \right\} & \text{for } n \geq N \\ S_2 x(N) & \text{for } n_0 \leq n < N. \end{cases}
$$

The rest of the proof is similar to that of the case 1 and it is thus omitted.

Case 3. For the case $0 \leq c < 1$, by (8) and (9), we choose an $N > n_0$

sufficiently large such that

$$\frac{1}{(m-1)!}\sum_{s_1=N}^{\infty} r_1(s_1)\sum_{s_2=s_1}^{\infty} r_2(s_2)\cdots \cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1}) \times \sum_{s=s_{m-1}}^{\infty}\left(\sup_{w\in[a,b]}|F(s,w)|+|g(s)|\right) < \frac{(1-c)(b-a)}{2}.$$

Defining two maps $S_1, S_2 : \Omega \to l_{n_0}^{\infty}$ as in case 1, the rest of the proof is similar to that of case 1 and it is thus omitted.

Case 4. For the case $c > 1$, by (8) and (9), we choose an $N > n_0$ sufficiently large such that

$$\frac{1}{c(m-1)!}\sum_{s_1=N+\tau}^{\infty} r_1(s_1)\sum_{s_2=s_1}^{\infty} r_2(s_2)\cdots \cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1}) \times \sum_{s=s_{m-1}}^{\infty}\left(\sup_{w\in[a,b]}|F(s,w)|+|g(s)|\right) < \frac{(c-1)(b-a)}{2c}.$$

Defining two maps $S_1, S_2 : \Omega \to l_{n_0}^{\infty}$ as in the case 2, the rest of the proof is also similar to that of the case 2 and it is thus omitted.

Case 5. For the case $c = 1$, by (8) and (9), we choose an $N > n_0$ sufficiently large such that

$$\frac{1}{(m-1)!}\sum_{s_1=N+\tau}^{\infty} r_1(s_1)\sum_{s_2=s_1}^{\infty} r_2(s_2)\cdots \cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1}) \times \sum_{s=s_{m-1}}^{\infty}\left(\sup_{w\in[a,b]}|F(s,w)|+|g(s)|\right) < \frac{(b-a)}{2}.$$

Define a map $S : \Omega \to l_{n_0}^{\infty}$ by

$$Sx(n) = \begin{cases} \left.\begin{array}{l} \dfrac{a+b}{2} + \dfrac{(-1)^{m+1}}{(m-1)!} \displaystyle\sum_{j=1}^{\infty} \\ \times \displaystyle\sum_{s_1=n+(2j-1)\tau}^{n+2j\tau} r_1(s_1) \sum_{s_2=s_1}^{\infty} r_2(s_2) \cdots \\ \cdots \displaystyle\sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1}) \\ \times \displaystyle\sum_{s=s_{m-1}}^{\infty} \big(F(s, x(\sigma(s)) - g(s)\big) \end{array}\right\} & \text{for } n \geq N \\ Sx(N) & \text{for } n_0 \leq n < N. \end{cases}$$

Proceeding similarly as in the proof of case 1 we obtain $S\Omega \subset \Omega$ and the mapping S is completely continuous. By Lemma 3, there is an $x_0 \in \Omega$ such that $Sx_0 = x_0$, therefore

$$\begin{aligned} &x_0(n) + x_0(n-\tau) \\ &\quad = a + b + \frac{(-1)^{m+1}}{(m-1)!} \sum_{s_1=n}^{\infty} r_1(s_1) \sum_{s_2=s_1}^{\infty} r_2(s_2) \cdots \\ &\qquad \cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1}) \sum_{s=s_{m-1}}^{\infty} \big(F(s, x_0(\sigma(s))) - g(s)\big). \end{aligned}$$

Clearly, $x_0 = x_0(n)$ is a bounded non-oscillatory solution of equation (1). This completes the proof of Theorem 1 ∎

Remark 1. For the critical case $c = -1$, it is also possible that equation (1) has no non-oscillatory solution in spite of the fact that (8) and (9) hold. For example, we consider the neutral difference equation

$$\Delta^m\big(x(n) - x(n-\tau)\big) + \frac{1}{n^\alpha} x(n-r) = 0 \quad (n \geq n_0) \tag{10}$$

where $\tau, r \in N(n_0)$ and $m < \alpha < m+1$. Clearly, (8) and (9) hold. But, by [15: Theorem 1], equation (10) has no non-oscillatory solution.

Theorem 2. *Assume that* $c = -1$ *and that there exists some interval* $[a,b] \subset \mathbb{R}^+$ *such that*

$$\begin{aligned} &\sum_{s_1=n_0}^{\infty} s_1 r_1(s_1) \sum_{s_2=s_1}^{\infty} r_2(s_2) \cdots \\ &\cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1}) \sum_{s=s_{m-1}}^{\infty} \sup_{w \in [a,b]} |F(s,w)| < \infty \end{aligned} \tag{11}$$

and

$$\sum_{s_1=n_0}^{\infty} s_1 r_1(s_1) \sum_{s_2=s_1}^{\infty} r_2(s_2) \cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1}) \sum_{s=s_{m-1}}^{\infty} |g(s)| < \infty. \tag{12}$$

Then equation (1) *has a bounded non-oscillatory solution.*

Proof. By Lemma 4, (11) and (12) are equivalent to

$$\sum_{j=0}^{\infty} \sum_{s_1=n_0+j\tau}^{\infty} r_1(s_1) \sum_{s_2=s_1}^{\infty} r_2(s_2) \cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1}) \sum_{s=s_{m-1}}^{\infty} \sup_{w\in[a,b]} |F(s,w)| < \infty \tag{13}$$

and

$$\sum_{j=0}^{\infty} \sum_{s_1=n_0+j\tau}^{\infty} r_1(s_1) \sum_{s_2=s_1}^{\infty} r_2(s_2) \cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1}) \sum_{s=s_{m-1}}^{\infty} |g(s)| < \infty, \tag{14}$$

respectively. We choose a sufficiently large $N > n_0$ such that

$$\frac{1}{(m-1)!} \sum_{j=1}^{\infty} \sum_{s_1=N+j\tau}^{\infty} r_1(s_1) \sum_{s_2=s_1}^{\infty} r_2(s_2) \cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1}) \times \sum_{s=s_{m-1}}^{\infty} \left(\sup_{w\in[a,b]} |F(s,w)| + |g(s)| \right) < \frac{b-a}{2}.$$

We define a closed, bounded, and convex subset Ω of $l_{n_0}^{\infty}$ by

$$\Omega = \Big\{ x = \{x(n)\} \in l_{n_0}^{\infty} : \, a \le x(n) \le b \ \ (n \ge n_0) \Big\}.$$

Define a mapping $S : \Omega \to l^\infty_{n_0}$ by

$$Sx(n) = \begin{cases} \left. \begin{aligned} &\frac{a+b}{2} + \frac{(-1)^m}{(m-1)!} \sum_{j=1}^{\infty} \\ &\times \sum_{s_1=n+j\tau}^{\infty} r_1(s_1) \sum_{s_2=s_1}^{\infty} r_2(s_2) \cdots \\ &\cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1}) \\ &\times \sum_{s=s_{m-1}}^{\infty} \big(F(s, x(\sigma(s))) - g(s)\big) \end{aligned} \right\} & \text{for } n \geq N \\ Sx(N) & \text{for } n_0 \leq n < N. \end{cases}$$

We shall show that $S\Omega \subset \Omega$. In fact, for every $x \in \Omega$ and $n \geq N$, we get

$$\begin{aligned} Sx(n) &\leq \frac{a+b}{2} + \frac{1}{(m-1)!} \sum_{j=1}^{\infty} \sum_{s_1=n+j\tau}^{\infty} r_1(s_1) \sum_{s_2=s_1}^{\infty} r_2(s_2) \cdots \\ &\qquad \cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1}) \sum_{s=s_{m-1}}^{\infty} \big(|F(s, x(\sigma(s)))| + |g(s)|\big) \\ &\leq \frac{a+b}{2} + \frac{1}{(m-1)!} \sum_{j=1}^{\infty} \sum_{s_1=N+j\tau}^{\infty} r_1(s_1) \sum_{s_2=s_1}^{\infty} r_2(s_2) \cdots \\ &\qquad \cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1}) \sum_{s=s_{m-1}}^{\infty} \left(\sup_{w \in [a,b]} |F(s, w)| + |g(s)| \right) \\ &\leq \frac{a+b}{2} + \frac{b-a}{2} \\ &= b. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} Sx(n) \geq \frac{a+b}{2} - \frac{1}{(m-1)!} \sum_{j=1}^{\infty} \sum_{s_1=n+j\tau}^{\infty} r_1(s_1) \sum_{s_2=s_1}^{\infty} r_2(s_2) \cdots \\ \cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1}) \sum_{s=s_{m-1}}^{\infty} \big(|F(s, x(\sigma(s)))| + |g(s)|\big) \end{aligned}$$

$$\geq \frac{a+b}{2} - \frac{1}{(m-1)!} \sum_{j=1}^{\infty} \sum_{s-1=N+j\tau}^{\infty} r_1(s_1) \sum_{s_2=s_1}^{\infty} r_2(s_2) \cdots$$
$$\cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1}) \sum_{s=s_{m-1}}^{\infty} \left(\sup_{w\in[a,b]} |F(s,w)| + |g(s)| \right)$$
$$\geq \frac{a+b}{2} - \frac{b-a}{2}$$
$$= a.$$

Hence, $S\Omega \subset \Omega$.

We now show that S is continuous. Let $x_k = \{x_k(n)\} \in \Omega$ be such that $x_k(n) \to x(n)$ as $k \to \infty$. Because Ω is closed, $x = x(n) \in \Omega$. For $n \geq N$, we have

$$\left|Sx_k(n) - Sx(n)\right|$$
$$\leq \frac{1}{(m-1)!} \sum_{j=1}^{\infty} \sum_{s_1=N+j\tau}^{\infty} r_1(s_1) \sum_{s_2=s_1}^{\infty} r_2(s_2) \cdots$$
$$\cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1}) \sum_{s=s_{m-1}}^{\infty} \left|F(s, x_k(\sigma(s))) - F(s, x(\sigma(s)))\right|.$$

Since

$$\left|F(n, x_k(\sigma(n))) - F(x(\sigma(n)))\right| \to 0 \quad (k \to \infty),$$

by applying the Lebesgue dominated convergence theorem we conclude that $\lim_{k\to\infty} \|Sx_k(n) - Sx(n)\| = 0$. This means that S is continuous.

In the following, we show that $S\Omega$ is relatively compact. By (13) and (14), for any $\varepsilon > 0$, take $N^* \geq N$ large enough so that

$$\frac{1}{(m-1)!} \sum_{j=1}^{\infty} \sum_{s_1=N^*+j\tau}^{\infty} r_1(s_1) \sum_{s_2=s_1}^{\infty} r_2(s_2) \cdots$$
$$\cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1}) \sum_{s=s_{m-1}}^{\infty} \left(\sup_{w\in[a,b]} |F(s,w)| + |g(s)| \right) < \frac{\varepsilon}{2}.$$

Then, for $x \in \Omega$ and $n_2 > n_1 \geq N^*$,

$$\left|Sx(n_2) - Sx(n_1)\right|$$
$$\leq \frac{1}{(m-1)!} \sum_{j=1}^{\infty} \sum_{s_1=n_2+j\tau}^{\infty} r_1(s_1) \sum_{s_2=s_1}^{\infty} r_2(s_2) \cdots$$

$$\cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1}) \sum_{s=s_{m-1}}^{\infty} \Big(\sup_{w\in[a,b]} |F(s,w)| + |g(s)| \Big)$$
$$+ \frac{1}{(m-1)!} \sum_{j=1}^{\infty} \sum_{s_1=n_1+j\tau}^{\infty} r_1(s_1) \sum_{s_2=s_1}^{\infty} r_2(s_2) \cdots$$
$$\cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1}) \sum_{s=s_{m-1}}^{\infty} \Big(\sup_{w\in[a,b]} |F(s,w)| + |g(s)| \Big)$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$
$$= \varepsilon.$$

Therefore, $\{Sx : x \in \Omega\}$ is a bounded and uniformly Cauchy subset. Hence, by Lemma 1, $S\Omega$ is relatively compact. By Lemma 3, there is an $x_0 \in \Omega$ such that $Sx_0 = x_0$, therefore

$$\begin{aligned} x_0(n) - x_0(n-\tau) &= \frac{(-1)^{m+1}}{(m-1)!} \sum_{s_1=n}^{\infty} r_1(s_1) \sum_{s_2=s_1}^{\infty} r_2(s_2) \cdots \\ &\quad \cdots \sum_{s_{m-1}=s_{m-2}}^{\infty} r_{m-1}(s_{m-1}) \qquad\qquad (n \ge N). \\ &\quad \times \sum_{s=s_{m-1}}^{\infty} \big(F(s, x_0(\sigma(s)) - g(s) \big) \end{aligned}$$

Clearly, $x_0 = \{x_0(n)\}$ is a bounded non-oscillatory solution of equation (1). This completes the proof of Theorem 2 ∎

When $r_i(n) \equiv 1 \quad (i = 1, 2, ..., m)$ and $F(n, x) = p(n)f(x)$, where $p : N(n_0) \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$, equation (1) reduces to

$$\Delta^m \big(x(n) + cx(n-\tau) \big) + p(n)f(x(\sigma(n)) = g(n) \quad (n \ge n_0). \tag{15}$$

Clearly, equations (4) and (5) are special cases of equation (15). By using Theorems 1 and 2, we obtain the following results.

Corollary 1. *Assume that* $c \neq -1$ *and* $\sum_{i=n_0}^{\infty} i^{m-1}|p(i)| < \infty$ *as well as* $\sum_{i=n_0}^{\infty} i^{m-1}|g(i)| < \infty$. *Then equation* (15) *has a bounded non-oscillatory solution.*

Proof. We note that the finiteness of the series in the corollary is equivalent to

$$\left.\begin{array}{l}\displaystyle\sum_{s_1=n_0}^{\infty}\sum_{s_2=s_1}^{\infty}\cdots\sum_{s_{m-1}=s_{m-2}}^{\infty}\sum_{s=s_{m-1}}^{\infty}|p(s)| \\ \displaystyle\sum_{s_1=n_0}^{\infty}\sum_{s_2=s_1}^{\infty}\cdots\sum_{s_{m-1}=s_{m-2}}^{\infty}\sum_{s=s_{m-1}}^{\infty}|g(s)|\end{array}\right\} < \infty,$$

respectively. This implies that (8) and (9) hold, so the proof is complete ∎

Corollary 2. *Assume that* $c = -1$ *and that* $\sum_{i=n_0}^{\infty} i^m|p(i)| < \infty$ *as well as* $\sum_{i=n_0}^{\infty} i^m|g(i)| < \infty$*. Then equation* (15) *has a bounded non-oscillatory solution.*

The proof is similar to that of Corollary 1, it is therefore omitted.

Remark 2. Theorems 1 and 2 extend and improve Theorems A - E.

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