

Strict Equilibria of Multi-Valued Maps and Common Fixed Points

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Abstract. In the paper we deal with strict equilibria of continuous multi-valued maps. This allows us to study the invariance problem for differential inclusions in non-invariant closed subsets of Euclidean spaces. We also give a new method in studying common fixed points of families of maps.

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1. Introduction

As one knows, existence of equilibria of single- or multi-valued maps is one of the most important problems in many fields of mathematics. In particular, in nonlinear analysis, we look for critical points of functionals or stationary solutions for differential equations or differential inclusions. In the paper we fix our attention on the problem of existence of a *strict equilibrium* of a multi-valued map F , i.e. a point x with $F(x) = \{0\}$.

The first motivation to look for strict equilibria is taken from invariance problems. For a given closed set K we want to find a point $x_0 \in K$ such that *each* trajectory for $\dot{x}(t) \in F(x(t))$ starting from x_0 remains forever in K (is viable in K). If F is single-valued and implies uniqueness, there are several methods to solve such invariance problem (we use Nagumo's condition [19], Ważewski's retract method [23], the Conley homotopy index [9]) while for ordinary differential equations without uniqueness and for differential inclusions there are only a few results on this topic. It occurs that the invariance problem is essentially more difficult than the viability one, where existence of *at least*

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one viable trajectory is required of points from K . In Section 2 (Preliminaries) we briefly describe differences and difficulties. The Invariance Theorem [3: Theorem 5.3.4] assumes the strong tangency condition ¹⁾ $F(x) \subset T_K(x)$ on the boundary of K . As far as the author knows, there had been no invariance results for differential inclusions without this strong tangency condition until the recent paper [8] was published, where the authors made use of strict equilibria of multi-valued maps. As we note in Section 3, under suitable regularity of F , from a strict equilibrium point there starts only a stationary trajectory which obviously remains in K .

The second motivation for studying strict equilibria is that they correspond with strict fixed points ($A(x) = \{x\}$) of suitable multi-valued maps and, therefore, play an important role in multi-valued discrete dynamical systems. Namely, they are end-points (strictly stationary points) of the system (see [5]). In the framework of control theory this means that such end-point \bar{x} is a fixed point $\bar{x} = f(\bar{x}, u)$ for every control $u \in U$.

The notion of a strict equilibrium has not been well investigated yet. In particular, there have been no topological tools (homotopically invariant) to look for strict equilibria while there are several for studying equilibria or fixed points of multi-valued maps (topological degree, fixed point index, Lefschetz number; see [14] and references therein). Moreover, in [5] only dissipative systems are considered while the technique presented in [8] is essentially finite-dimensional. This became a motivation for further study of strict equilibria in general Banach spaces.

It is important that the idea of studying strict equilibria can be applicable in other mathematical problems. It occurs that the common fixed point problem, which has been intensively studied by the metric fixed point theory methods, may be described in terms of strict equilibria. We devote Section 4 to investigate this relation.

Let us explain how the paper is organized and which results are the most important.

At first, let us pay attention to Section 2, where some difficulties in the invariance problem are discussed. Examples 2.1 - 2.2 are a good base to understand types of assumptions arising in next sections. Section 3 is the main part of the paper. We investigate the notion of a strict equilibrium, give a general procedure to find strict equilibria and, after that, we present two different approaches described in two subsections. Namely, Subsection 3.1 deals with a topological degree approach and the main result is the following:

¹⁾ Recall that $T_K(x) = \{v \in \mathbb{R}^n : \liminf_{h \rightarrow 0^+} \frac{\text{dist}(x+hv, K)}{h} = 0\}$ is the Bouligand contingent cone to K in x .

Theorem 1.1. *Let U be an open subset of a Banach space E and let $\Phi : \bar{U} \multimap E$ be a multi-valued continuous compact vector field with compact convex values. Assume the following:*

- (A1) *For every $x \in \bar{U}$ with $\Phi(x) \neq \{0\}$ one has $0 \notin \overline{\text{conv}}\{y \in \Phi(x) : \|y\| = \|\Phi(x)\|\}$, where $\|\Phi(x)\| = \sup\{\|y\| : y \in \Phi(x)\}$.*
- (A2) $\text{Deg}(\Phi, U) \neq 0$.

Then there exists a strict equilibrium of Φ .

Note that Lemmas 3.3 - 3.4 are pivotal in the proof.

In Subsection 3.2 we use the generalized (Conley-type) homotopy index for multi-valued maps which has been constructed in [12], and obtain the following result which is a generalization of [8: Theorem 1.2].

Theorem 1.2. *Let $K = \overline{\text{Int } K} \subset \mathbb{R}^n$ be a compact sleek ²⁾ set and let $\Phi : \mathbb{R}^n \multimap \mathbb{R}^n$ be a continuous compact-convex-valued map with at most a linear growth and satisfying on $\text{Int } K$ condition (A1) or*

- (A3) $0 \notin \text{Int}_{\text{span } \Phi(x)} \Phi(x)$ for every $x \in X$ with $\Phi(x) \neq \{0\}$, $\text{Int}_{\text{span } \Phi(x)} \Phi(x)$ denoting a relative interior of $\Phi(x)$ in the subspace $\text{span } \Phi(x) \subset \mathbb{R}^n$ spanned by $\Phi(x)$.

Assume that the exit set $K^-(\Phi)$ (see Section 2) is a closed deformation retract of some open neighbourhood U in K , with $\text{Int } T_K(x) \neq \emptyset$ for every $x \in K \setminus K^-(\Phi)$, and that $0 \notin \Phi(\partial K)$ and $\chi(K, K^-(\Phi)) \neq 0$, where $\chi(K, K^-(\Phi)) := \chi(K) - \chi(K^-(\Phi))$ stands for the relative Euler characteristic of the pair $(K, K^-(\Phi))$. Then there is a strict equilibrium of Φ in $\text{Int } K$.

The results from Section 3 are applied in Section 4 to study common fixed points of families of continuous maps. Usual assumptions of commutativity and non-expansiveness type are replaced by geometrical and degree conditions.

Finally, we give some concluding remarks on possible topological methods for studying strict equilibria and strict fixed points.

2. Preliminaries

We start with some notations we use in the paper. By $\text{Int } A$, \bar{A} and ∂A we mean respectively the interior, the closure and the boundary of a subset A of a metric space X . The open ball centered in x_0 and with a radius r is denoted by $B(x_0, r)$ while the r -neighbourhood of a set A by $N_r(A) = \{x \in X : \text{dist}(x, M) < r\}$, where $\text{dist}(x, A) = \inf\{d(x, a) : a \in A\}$ is a distance

²⁾ A closed set $K \subset E$ is sleek if the Bouligand cone map $T_K(\cdot)$ is lower semicontinuous.

from x to A . So, $B(x_0, r) = N_r(\{x_0\})$. By $|\cdot|, \|\cdot\|, d_H$ we mean respectively an Euclidean norm, a norm in a Banach space and the Hausdorff metric.

Consider the problem

$$(P) \quad \begin{cases} \dot{x}(t) \in F(x(t)) & \text{for a.e. } t \geq 0 \\ x(t) \in K \subset \mathbb{R}^n & \text{for every } t \geq 0 \end{cases}$$

where K is closed in \mathbb{R}^n and $F : \mathbb{R}^n \multimap \mathbb{R}^n$ is upper semicontinuous with compact convex values and at most a linear growth, i.e.

$$|F(x)| = \sup\{|y| : y \in F(x)\} \leq c(1 + |x|)$$

for some constant $c \geq 0$ and every $x \in \mathbb{R}^n$. The set of all (absolutely continuous) solutions to the Cauchy problem $\dot{x}(t) \in F(x(t)), x(0) = x_0$ is denoted by $S_F(x_0)$. It is a non-empty compact R_δ -set (see, e.g., [2]).

We have the following two questions:

- (V) (**Viability problem**) Do there exist $x_0 \in K$ and a trajectory $x \in S_F(x_0)$ solving problem (P)?
- (I) (**Invariance problem**) Does there exist $x_0 \in K$ such that each trajectory $x \in S_F(x_0)$ solves problem (P)?

The invariance problem can be stated as follows: Is the *invariance kernel*

$$\text{Inv}_F(K) = \left\{ x_0 \in K : \forall x \in S_F(x_0), x \text{ solves problem } (P) \right\}$$

non-empty? The well-known Invariance Theorem [3: Theorem 5.3.4] says that, if F is locally Lipschitz and

$$F(x) \subset T_K(x) \quad \text{for every } x \in K, \tag{1}$$

then $\text{Inv}_F(K) = K$, which means that K is invariant.

In [20] it has been proved that $\text{Inv}_F(K) = K$ for upper semicontinuous maps satisfying condition (1) and for any proximal retract K but F has been defined only on K . In this case we can extend F on some neighbourhood in such a way that K remains invariant. It is easy to find examples that for F *a priori* given on larger sets than K , an upper semicontinuous (or continuous) regularity of F is too weak.

If the strong tangency condition (1) is not satisfied, Problem (I) is much more complicated, even for very regular right-hand side F . There are several reasons. At first, at a point on a boundary of K some trajectories can leave

the set and some of them can go inside simultaneously. In general, two exit sets appears:

$$K^-(F) = \left\{ x_0 \in \partial K : \forall x \in S_F(x_0), x \text{ leaves } K \text{ immediately} \right\}$$

$$K_e(F) = \left\{ x_0 \in \partial K : \exists x \in S_F(x_0) \text{ leaving } K \text{ immediately} \right\}$$

where “ x leaves K immediately” means that, for all $\varepsilon > 0$, there exists $0 < t < \varepsilon$ such that $x(t) \notin K$. We refer the reader to [13] for discussion about the role of both exit sets in the viability problem (V).

Two further difficulties are described in examples below.

Example 2.1. Consider the set $K = [-2, 2] \times [-2, 2]$ and the Lipschitz map $F : \mathbb{R}^2 \multimap \mathbb{R}^2$ defined by $F(x, y) = [x - 1, x + 1] \times \{-\frac{y}{2}\}$.

It is easy to check that

$$K^-(F) = K_e(F) = (\{-2\} \times [-2, 2]) \cup (\{2\} \times [-2, 2]),$$

$K^-(F)$ is compact and disconnected with $\chi(K, K^-(F)) = -1 \neq 0$. Notice that such behaviour of F on ∂K guarantees existence of equilibria. Indeed, we can take any Lipschitz selection of F and apply [18: Theorem 4.1]. In our example the Steiner selection $\sigma(F(\cdot))$ (see [4: Theorem 9.4.3]) has an equilibrium in the point $(0, 0)$ so, in one of points where a value of F contains *two opposite directions*. Although topological properties of the exit set are as good as possible, the invariance kernel is empty.

Example 2.2. Let $K = \overline{B((0, 0), 2)} \setminus B((0, 0), 1) \subset \mathbb{R}^2$ and let $F : \mathbb{R}^2 \multimap \mathbb{R}^2$ be given by $F(x, y) = \{(sx + y, -x + ty) : s, t \in [-\frac{1}{2}, \frac{1}{2}]\}$, that is, the

rotation is a selection of F . Obviously, $\text{Inv}_F(K) = \emptyset$. Here F in no point contains opposite directions but notice that $K^-(F) = \emptyset$ and $K_e(F) = \partial K$ which implies $\chi(K, K^-(F)) = 0 = \chi(K, K_e(F))$.

A deep analysis of the above examples shall lead us to the notion of a strict equilibrium (see Section 3) and allow us to solve Problem (I).

Let us recall some properties of a topological degree defined in [17]. For any open subset U of a Banach space E , each upper semicontinuous compact map $F : \bar{U} \multimap E$ induces a so-called *compact vector field* $\Phi : \bar{U} \multimap E$, $\Phi(x) = x - F(x)$, and for a class of such compact vector fields with compact convex values the topological degree $\text{Deg}(\Phi, U)$ has been constructed. It has the following main properties:

Proposition 2.3 [17: Theorems 7.7, 8.1, 10.1].

(i) (Existence) *If $0 \notin \Phi(\partial U)$ and $\text{Deg}(\Phi, U) \neq 0$, then there is $x \in U$ with $0 \in \Phi(x)$.*

(ii) (Additivity) *If $U_1 \cap U_2 = \emptyset$, $U_1 \cup U_2 \subset U$ and $0 \notin \Phi(\bar{U} \setminus (U_1 \cup U_2))$, then $\text{Deg}(\Phi, U) = \text{Deg}(\Phi, U_1) + \text{Deg}(\Phi, U_2)$.*

(iii) (Homotopy) *If $H : \bar{U} \times [0, 1] \multimap E$ is a compact map such that $0 \notin x - H(x, t)$ for every $x \in \partial U$ and $t \in [0, 1]$, then $\text{Deg}(H(\cdot, t), U)$ is constant for every $t \in [0, 1]$.*

We finish this preliminary section with some basic information on the generalized (Conley-type) homotopy index for multi-valued maps obtained recently by the author in [12]. This index slightly differs from the one defined by Kunze in [16], where the *a priori* condition is assumed that the maximal weakly invariant subset of K does not meet a boundary of K . Under some regularity conditions we can describe the index in terms of the exit set $K^-(F)$. Let us add that under the *a priori* assumption mentioned above, these two indices (defined in [16] and [12]) coincide.

The Conley index approach is useful when we look for equilibria of F , the map F is not tangent to a prescribe set K and some suitable topological properties of the exit set are known.

We assume the following:

(A5) $F : \mathbb{R}^n \multimap \mathbb{R}^n$ is a compact-convex-valued upper semicontinuous map with at most a linear growth, and $K = \overline{\text{Int } K} \subset \mathbb{R}^n$ is a compact sleek set such that the exit set $K^-(F)$ is closed and $\text{Int } T_K(x) \neq \emptyset$ for every $x \in K \setminus K^-(F)$.

In assumption $\text{Int } T_K(x) \neq \emptyset$ we forbid too sharp corners outside $K^-(F)$. It is satisfied for e.g. all C^1 n -manifolds in \mathbb{R}^n , but also for a larger class of sets.

Denote by $\mathcal{I}_F(K)$ the largest closed subset of K such that, for each $x_0 \in \mathcal{I}_F(K)$, there is a trajectory x with $\dot{x}(t) \in F(x(t))$ for a.e. $t \in \mathbb{R}$, $x(0) = x_0$

and $x(t) \in K$ for every $t \in \mathbb{R}$. For a single-valued Lipschitz map F the set $\mathcal{I}_F(K)$ is called a *maximal invariant subset of K* , so we adopt this notion also for multi-valued maps and call the set $\mathcal{I}_F(K)$ a *maximal weakly invariant subset of K* .

Then the *generalized homotopy index* $I(\mathcal{I}_F(K), F)$ for F is defined ³⁾ and has the following properties:

Proposition 2.4 [12: Propositions 3.10 - 3.12].

1. (Existence) *If $I(\mathcal{I}_F(K), F) \neq \bar{0}$, then $\mathcal{I}_F(K) \neq \emptyset$.*
2. (Localization) *Assume $S \subset \text{Int } K$ is such that $\mathcal{I}_F(S) = S$. If*
 - (i) *S is isolated, i.e. $S = \mathcal{I}_F(P) \subset \text{Int } P$ for some $P = \overline{\text{Int } P} \subset \text{Int } K$,*

or

- (ii) *there is a sleek set $P = \overline{\text{Int } P} \subset \text{Int } K$ such that $S = \mathcal{I}_F(P)$, $P^-(F)$ is closed, and $\text{Int } T_P(x) \neq \emptyset$ for every $x \in P \setminus P^-(F)$,*

and $I(\mathcal{I}_F(K), F) \neq I(S, F)$, then there exists an invariant trajectory for F in $K \setminus \text{Int } P$.

3. (Homotopy) *Let $H : \mathbb{R}^n \times [0, 1] \multimap \mathbb{R}^n$ be an upper semicontinuous map with compact convex values and such that each $H_\lambda = H(\cdot, \lambda)$ has at most a linear growth. Assume the following:*

- (i) *$K^-(H_\lambda)$ is closed for every $\lambda \in [0, 1]$, and $\text{Int } T_K(x) \neq \emptyset$ for every $x \in K \setminus \bigcap_{\lambda \in [0, 1]} K^-(H_\lambda)$.*
- (ii) *For every $\lambda_0 \in [0, 1]$ and any open neighbourhood Ω of $K^-(H_{\lambda_0})$ in K , there is $\delta_0 > 0$ such that, if $|\lambda - \lambda_0| \leq \delta_0$ and $\lambda \in [0, 1]$, then $K^-(H_\lambda) \subset \Omega$.*
- (iii) *For every $\lambda_0 \in [0, 1]$ and $\varepsilon > 0$, there is $\delta > 0$ such that, if $|\lambda - \lambda_0| \leq \delta$ and $\lambda \in [0, 1]$, then $H(x, \lambda_0) \subset H(x, \lambda) + \varepsilon B_1$ for every $x \in \partial K$ (i.e. H is lower semicontinuous with respect to λ , uniformly on ∂K).*

Then $I(\mathcal{I}_{H_\lambda}(K), H_\lambda)$ is independent of the choice of $\lambda \in [0, 1]$.

Remark 2.5. In the construction of $I(\mathcal{I}_F(K), F)$ one finds a suitable sufficiently near approximation of F with an exit set which is close to $K^-(F)$, and with the maximal invariant subset of K contained in $\text{Int } K$. Then one shows that for arbitrary two such approximations, their ordinary Conley indices coincide.

Two main results which use the above generalized index are the following:

³⁾ $I(\mathcal{I}_F(K), F)$ is a pointed homotopy type $[P_1/P_2, [P_2]]$ of a pointed space $(P_1/P_2, [P_2])$ for a suitable index pair (P_1, P_2) for $\mathcal{I}_F(K)$. Recall that by the *trivial pointed homotopy type* we mean $\bar{0} = [*, \{*\}]$.

Proposition 2.6 [12: Theorem 4.1]. *Let $K = \overline{\text{Int } K} \subset \mathbb{R}^n$ be a compact sleek set and let $F : \mathbb{R}^n \multimap \mathbb{R}^n$ be a compact-convex-valued upper semicontinuous map with at most a linear growth and such that*

$$\left. \begin{array}{l} K^-(F) \text{ is a closed deformation retract} \\ \text{of some open neighbourhood } U \text{ in } K \\ \text{and } \text{Int } T_K(x) \neq \emptyset \text{ for every } x \in K \setminus K^-(F). \end{array} \right\} \quad (2)$$

Then

$$I(\mathcal{I}_F(K), F) = [K/K^-(F), [K^-(F)]].$$

In particular, if $[K/K^-(F), [K^-(F)]] \neq \bar{0}$, then there is an invariant trajectory for F in K .

Proposition 2.7 [12: Theorem 4.4]. *Let $K = \overline{\text{Int } K} \subset \mathbb{R}^n$ be a compact sleek neighbourhood retract and let $F : \mathbb{R}^n \multimap \mathbb{R}^n$ be a compact-convex-valued upper semicontinuous map with at most a linear growth and satisfying assumption (2). If $\chi(K, K^-(F)) := \chi(K) - \chi(K^-(F)) \neq 0$, then there exists an equilibrium of F in K .*

3. Strict equilibria

In this section we give a definition and some properties of strict equilibria of multi-valued maps. Then, we present main theorems on existence of strict equilibria, where two different techniques are used – the topological degree and the generalized homotopy index approaches. The key-point to obtain a strict equilibrium lies in finding suitable selections of multi-valued maps (see Lemmas 3.3 - 3.4).

Let X be a topological space and E a Banach space (in general, X can be an arbitrary set and E a linear space).

Definition 3.1. We say that $x \in X$ is a *strict equilibrium* of a multi-valued map $\Phi : X \multimap E$, if $\Phi(x) = \{0\}$.

This notion coincides with the one of an equilibrium in a single-valued case while it brings an important and interesting information for multi-valued maps ⁴⁾. As we mentioned in Introduction, the problem of existence of strict equilibria is related to the invariance problem (I). Indeed, for sufficiently regular maps one has

⁴⁾ Note that by an *equilibrium* of a multi-valued map Φ one means a point x with $0 \in \Phi(x)$.

Proposition 3.2. *Let Ω be an open subset of a Banach space E , $K \subset \Omega$ be closed in E and let $F : \Omega \multimap E$ be a compact-convex-valued map, Lipschitz⁵⁾ in a neighborhood of a strict equilibrium $x_0 \in K$. Then from x_0 there starts only a stationary trajectory. In particular, $\text{Inv}_K(F) \neq \emptyset$.*

Proof. Let y be a solution to the Cauchy problem $\dot{x} \in F(x), x(0) = x_0$. Then $y(t) = x_0 + \int_0^t u(s) ds$, for some $u \in L^1$ with $u(s) \in F(y(s))$ ($s \in [0, \delta)$) and for every $t \in [0, \delta)$. Notice that

$$\|F(y(s))\| = d_H(F(y(s)), F(x_0)) \leq L\|y(s) - x_0\|.$$

Therefore,

$$\|y(t) - x_0\| \leq \int_0^t \|u(s)\| ds \leq L \int_0^t \|y(s) - x_0\| ds.$$

By the Gronwall inequality, $\|y(t) - x_0\| = 0$ which means that y is stationary ■

A further study of strict equilibria is the main aim of the section. We consider the class of continuous multi-valued maps as an appropriate class for this strict equilibrium problem. A semicontinuous assumption seems to be too weak and not compatible with our differential motivations and application to the common fixed point problem (see Section 4). As we shall see, as a consequence of the considerations in the section we will clarify difficulties described in Examples 2.1 - 2.2.

Let us determine the way we can find strict equilibria. In [8] the authors give the following procedure:

- (a) One finds a sufficiently near approximation f of Φ with a property that each equilibrium of f is a strict equilibrium of Φ , then
- (b) one ensures that this approximation f does have an equilibrium.

Some sufficient conditions for (a) and (b) has been given [8: Corollary 4.4, Example 4.5]. In particular, if for a continuous map $\Phi : X \multimap \mathbb{R}^n$

$$\Phi(x) \cap -\Phi(x) \subset \{0\} \quad \text{for every } x \in X, \tag{3}$$

then there exists a continuous selection f of Φ satisfying (a). Notice that condition (3) implies (A1) and (A3) and it can be expressed as a property that no value of Φ contains opposite directions.

Under assumption (3), since a codomain is assumed to be finite-dimensional, the Steiner selection technique can be used. Unfortunately, this technique

⁵⁾ We mean Lipschitzeanity with respect to the Hausdorff distance, i.e. the inequality $d_H(F(x), F(y)) \leq L\|x - y\|$ ($x, y \in \Omega$) holds for some constant $L \geq 0$.

can not be applied in an infinite-dimensional case. Therefore, in the sequel we give a different and new procedure to find strict equilibria which is appropriate for both finite- and infinite-dimensional spaces. We will proceed as follows:

- (P1) We find arbitrarily near *suitable* continuous approximations (or selections) of Φ with equilibria.
- (P2) We obtain a strict equilibrium of Φ as a *limit of equilibria* of such approximations.

3.1 Topological degree approach. In the present part of the paper we prove Theorem 1.1 stated in Introduction. A suitable selection-approximation technique will be the main tool in the proof. We describe this technique in lemmas below.

Lemma 3.3. *Let X be a metric and E a Banach spaces. Assume that $\Phi : X \multimap E$ is a continuous, compact-convex-valued map. Then the maps*

$$\begin{aligned}\Psi : X \multimap E & \quad \Psi(x) = \{y \in \Phi(x) : \|y\| = \|\Phi(x)\|\} \\ \Theta_\Phi : X \multimap E & \quad \Theta_\Phi(x) = \overline{\text{conv}}\Psi(x)\end{aligned}$$

are upper semicontinuous with compact values.

Proof. Suppose that Ψ is not upper semicontinuous, i.e. there are a point $x \in X$, an open neighbourhood U of $\Psi(x)$ in E , and a sequence $x_n \rightarrow x$ with $\Psi(x_n) \not\subset U$. Take $y_n \in \Psi(x_n) \setminus U$. Since Φ is upper semicontinuous with compact values, the set $\Phi(\{x_1, x_2, \dots\} \cup \{x\})$ is compact and, without any loss of generality, we can assume that $y_n \rightarrow y \notin U$. Moreover, $y \in \Phi(x)$. By the continuity of Φ ,

$$\left| \|y_n\| - \|\Phi(x)\| \right| = \left| \|\Phi(x_n)\| - \|\Phi(x)\| \right| \leq d_H(\Phi(x_n), \Phi(x)) \rightarrow 0$$

which implies that $\|y\| = \|\Phi(x)\|$ and hence $y \in \Psi(x)$; a contradiction. The second part of the proof follows from [14: Proposition 14.12] ■

Lemma 3.4. *Under the assumptions of Lemma 3.3, for every $\varepsilon > 0$ there exists an ε -approximation ⁶⁾ of Θ_Φ which is a selection of Φ .*

Proof. The statement is a consequence of [7: Lemma 5.1 and Remark 5.2] ■

Now we are in a position to prove the main result of this subsection.

⁶⁾ We say that a continuous map $f : X \rightarrow Y$ is an ε -approximation of a multi-valued map $\Phi : X \multimap Y$, if $f(x) \in N_\varepsilon \Phi(B(x, \varepsilon))$ for every $x \in X$.

Proof of Theorem 1.1. From Lemma 3.4 it follows that, for every $m \geq 1$, there exists a $\frac{1}{m}$ -approximation f_m of Θ_Φ which is a selection of Φ . Since $\deg(f_m, U) = \text{Deg}(\Phi, U) \neq 0$ by Assumption (A2), there is an equilibrium x_m of f_m . The set $\Phi_+^{-1}(0)$ is compact, so without any loss of generality we can assume that $x_m \rightarrow x$ with $0 \in \Phi(x)$. We show that x is a strict equilibrium of Φ .

Suppose, on the contrary, that there exists $0 \neq y \in \Phi(x)$. We use Assumption (A1) and properties of the map Θ_Φ , and obtain $d > 0$ such that $\Theta_\Phi(x) \cap B(0, 2d) = \emptyset$. Then, by the upper semicontinuity of Θ_Φ , there is an open neighbourhood $V = B(x, \delta)$ ($\delta > 0$) of x in U such that $\Theta_\Phi(z) \cap B(0, d) = \emptyset$ for every $z \in V$. Take $m \geq 1$ such that $\frac{1}{m} < \min\{d, \frac{\delta}{2}\}$ and $\|x_m - x\| < \frac{\delta}{2}$. Then

$$0 = f_m(x_m) \in N_{\frac{1}{m}}(\Theta_\Phi(B(x_m, \frac{1}{m}))) \subset N_{\frac{1}{m}}(\Theta_\Phi(B(x, \delta)))$$

which means that there is $z \in B(x, \delta)$ and $u \in \Theta_\Phi(z)$ such that $\|u\| < \frac{1}{m} < d$. This implies that $\Theta_\Phi(z) \cap B(0, d) \neq \emptyset$. This contradiction finishes the proof ■

Corollary 3.5. *Let U be an open subset of a Hilbert space E and let $\Phi : \bar{U} \multimap E$ be a multi-valued continuous compact vector field with compact convex values. Assume that Φ satisfies (A1) and there is a continuous compact vector field (a guiding map) $\gamma : \bar{U} \rightarrow E$ such that $\langle \gamma(x), y \rangle > -\|y\| \|\gamma(x)\|$ for every $x \in \partial U$ and $\deg(\gamma, U) \neq 0$. Then there exists a strict equilibrium of Φ .*

Proof. It is sufficient to define

$$H : \bar{U} \times [0, 1] \multimap E, \quad H(x, t) = t\gamma(x) + (1-t)\Phi(x)$$

and notice that $H(x, t) \neq 0$ for every $x \in \partial U$ and $t \in [0, 1]$. Indeed, if it were not true, then there would be $x \in \partial U$, $y \in \Phi(x)$ and $t \in [0, 1]$ such that $t\gamma(x) + (1-t)y = 0$. Obviously, $t \neq 1$ since $\gamma(x) \neq 0$. It would follow that $y = -\frac{t}{1-t}\gamma(x)$ and hence

$$-\frac{t}{1-t} \|\gamma(x)\|^2 = \langle y, \gamma(x) \rangle > -\|\gamma(x)\| \frac{t}{1-t} \|\gamma(x)\|$$

which is impossible. By the homotopy property of the degree, $\text{Deg}(\Phi, U) \neq 0$ and, by Theorem 1.1, there is a strict equilibrium of Φ in U ■

3.2 Homotopy index approach. Now, our goal is to use the generalized homotopy index (see Section 2) proving the existence of strict equilibria of a multi-valued map in a prescribed set $K \subset \mathbb{R}^n$.

Proof of Theorem 1.2. We proceed in several steps.

Step 1. By Proposition 2.6 the index $I(\mathcal{I}_\Phi(K), \Phi)$ is defined and equal to $[K/K^-(\Phi), [K^-(\Phi)]] \neq \bar{0}$. It is easy to check that, by assumption $0 \notin \Phi(\partial K)$ and continuity of Φ , there are an open neighbourhood $N_\eta(\partial K)$ of ∂K in K and $\gamma > 0$ such that $f_\gamma(x) \neq 0$ for every γ -approximation f_γ of Φ and every $x \in N_\eta(\partial K)$.

Step 2. We show that, for every $\varepsilon > 0$ and every continuous map $u : \mathbb{R}^n \rightarrow [0, 1]$ with $u(\mathbb{R}^n \setminus K) = \{0\}$, there exists $\varepsilon_1 > 0$ such that, if f, g are ε_1 -approximations of Φ , then the map $k(\cdot) = u(\cdot)f(\cdot) + (1 - u(\cdot))g(\cdot)$ is an ε -approximation of Φ .

Indeed, since Φ is upper semicontinuous, we can find a finite covering $\{B(u_i, \delta_i)\}_{i=1}^l$ of K with $\delta_i < \varepsilon$ such that $\Phi(z) \subset N_{\frac{\varepsilon}{2}}(\Phi(u_i))$ for every $z \in B(u_i, \delta_i)$. There is $0 < \varepsilon_1 < \frac{\varepsilon}{2}$ such that any ball with a center in a point of K and a diameter less than $2\varepsilon_1$ is contained in some $B(u_i, \delta_i)$.

Now, let f and g be ε_1 -approximations of Φ and let $k(x) = u(x)f(x) + (1 - u(x))g(x)$ for every $x \in \mathbb{R}^n$. Then k is an ε_1 -approximation of Φ outside K since $k|_{\mathbb{R}^n \setminus K} = g|_{\mathbb{R}^n \setminus K}$. For every $x \in K$, there are $z_1, z_2 \in B(x, \varepsilon_1)$ and $y_1 \in \Phi(z_1), y_2 \in \Phi(z_2)$ such that $|f(x) - y_1| < \varepsilon_1$ and $|g(x) - y_2| < \varepsilon_1$. Then $B(x, \varepsilon_1) \subset B(u_i, \varepsilon)$ and $y_1, y_2 \in N_{\frac{\varepsilon}{2}}(\Phi(u_i))$ for some $i \in \{1, \dots, l\}$. Since $w = ty_1 + (1 - t)y_2 \in N_{\frac{\varepsilon}{2}}(\Phi(u_i))$, it follows that there is $v \in \Phi(u_i)$ with $|w - v| < \frac{\varepsilon}{2}$. Hence,

$$\begin{aligned} |k(x) - v| &\leq |u(x)f(x) + (1 - u(x))g(x) - ty_1 - (1 - t)y_2| + |w - v| \\ &< \varepsilon_1 + \frac{\varepsilon}{2} \\ &< \varepsilon \end{aligned}$$

which implies that $k(x) \in N_\varepsilon(\Phi(B(x, \varepsilon)))$, that is, k is an ε -approximation of Φ .

Step 3. Following construction of the index (see Remark 2.5) we can find $0 < \varepsilon_0 < \varepsilon$ such that, for every $0 < \delta \leq \varepsilon_0$, there is a locally Lipschitz δ -approximation g of Φ such that $I(\mathcal{I}_g(K), g) = I(\mathcal{I}_\Phi(K), \Phi)$ and that each locally Lipschitz δ -approximation which coincides with g on a neighbourhood of ∂K has the same index. Take, for every $m \geq 1$ with $\frac{1}{m} \leq \varepsilon_0$, a number $0 < \varepsilon_m < \frac{1}{m}$ and a locally Lipschitz ε_m -approximation g_m of Φ with properties above and such that $k(\cdot) = u(\cdot)g_m(\cdot) + (1 - u(\cdot))f(\cdot)$ is a $\frac{1}{m}$ -approximation of Φ for every ε_m -approximation f of Φ and any continuous map $u : \mathbb{R}^n \rightarrow [0, 1]$ vanishing outside K .

Denote $W = N_{\frac{\eta}{2}}(\partial K) \cap K$ and assume (A1). Then we find a continuous selection h_m of Φ on $K \setminus W$ which is an $\frac{\varepsilon_m}{2}$ -approximation of Θ_Φ (see Lemma 3.4). We approximate h_m and find a Lipschitz ε_m -approximation f_m of both

Φ and Θ_Φ on $K \setminus W$. If (A3) is assumed, we take $h = \sigma(\Phi(\cdot))$, the Steiner selection of Φ on $K \setminus W$, and choose any Lipschitz ε_m -approximation f_m of h .

Let $u : \mathbb{R}^n \rightarrow [0, 1]$ be a Lipschitz function such that $u(x) = 0$ for every $x \in \mathbb{R}^n \setminus (K \setminus \overline{W})$ and $u(x) = 1$ for every $x \in K \setminus N_\eta(\partial K)$. Define

$$k_m : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad k_m(x) = u(x)g_m(x) + (1 - u(x))f_m(x).$$

Now k_m is a locally Lipschitz $\frac{1}{m}$ -approximation of Φ which coincides with g_m outside $K \setminus W$ and hence

$$I(\mathcal{I}_{k_m}(K), k_m) = I(\mathcal{I}_\Phi(K), \Phi) = [K/K^-(F), [K^-(F)]] \neq \bar{0}$$

which implies that there exists an equilibrium $x_m \in K \setminus N_\eta(\partial K)$ of k_m .

Step 4. Taking $m \rightarrow \infty$ we find an equilibrium $x \in K \setminus N_\eta(\partial K)$ of Φ . But in a neighbourhood of x one has $k_m = f_m$ for every sufficiently large m . Therefore, under Assumption (A1), we can repeat arguments from the proof of Theorem 1.1 and, using the map Θ_Φ , we show that x is a strict equilibrium of Φ . In the case with Assumption (A3) we notice that $0 = \sigma(\Phi(x))$ which implies that $\Phi(x) = \{0\}$ ■

As a consequence of Theorem 1.2 we get

Corollary 3.6 [8: Theorem 1.2]. *Let $K \subset \mathbb{R}^n$ and $\Phi : \mathbb{R}^n \multimap \mathbb{R}^n$ satisfy the following conditions:*

- (i) K is a compact $C^{1,1}$ n -manifold with a boundary.
- (ii) Φ is a Lipschitz map with compact convex values and $0 \notin \Phi(\partial K)$.
- (iii) $K^-(\Phi)$ is closed and, if it is non-empty, it is a $C^{1,1}$ $(n-1)$ -submanifold of ∂K with a boundary.
- (iv) $\chi(K, K^-(\Phi)) \neq 0$.
- (v) Φ satisfies condition (A3).

Then there exists a strict equilibrium of Φ in $\text{Int } K$.

Finally, let us come back to Examples 2.1 - 2.2 and compare them with the situation considered in Theorem 1.2. One can see that in Example 2.1 the map F does not satisfy assumption (A3) while in Example 2.2 another assumption is not fulfilled, namely, $\chi(K, K^-(F)) = 0$.

4. Common fixed points

Consider a family $\mathcal{F} = \{f_\alpha\}_{\alpha \in \Lambda}$ of maps $f_\alpha : K \rightarrow E$ for a subset K of an arbitrary Banach space E .

Definition 4.1. By a *common fixed point* of a family \mathcal{F} we mean a point $x \in K$ such that $x = f_\alpha(x)$ for every $\alpha \in \Lambda$.

The common fixed point theorems are met in the metric fixed point theory where contractive or non-expansive maps are intensively studied. On the other hand, in [10: Chapter II] the authors deal with common fixed points of so-called distal families of affine maps. In both situations some commutativity properties are assumed (e.g. a semigroup structure [10], (weak) commutativity [22], (weak) compatibility [1, 15]). In the present section we study the problem of existence of common fixed points for a family of maps without any non-expansiveness-like assumptions and commutativity. Instead, we use topological degree arguments together with a geometrical assumption (A1). Our new technique gives possibility to study common fixed points for a large class of maps.

At first we replace the common fixed point problem by the equilibrium one. To do this notice that for maps $g_\alpha : K \rightarrow E$, $g_\alpha(x) = x - f_\alpha(x)$, the point $x \in K$ is a common fixed point of \mathcal{F} if and only if x is a *common equilibrium* of the family $\mathcal{G} = \{g_\alpha\}$.

Definition 4.2. We say that a family $\mathcal{G} = \{g_\alpha\}$ is *directed* if

$$\overline{\text{conv}}\{g_\alpha(x)\} \cap -\overline{\text{conv}}\{g_\alpha(x)\} \subset \{0\} \quad \text{for every } x \in K. \quad (4)$$

It is easy to see that for a finite set $\Lambda = \{1, \dots, k\}$ condition (4) can be simplified and it takes the form

$$\left\{ \sum_{i=1}^k (p_i + q_i) g_i(x) = 0 \right\} \implies \left\{ \sum_{i=1}^k p_i g_i(x) = \sum_{i=1}^k q_i g_i(x) = 0 \right\}$$

for $\sum_{i=1}^k p_i = \sum_{i=1}^k q_i = 1$, $p_i, q_i \geq 0$.

Define the map $\Phi : K \rightarrow E$, $\Phi(x) = \overline{\text{conv}}\{g_\alpha(x)\}$. It is seen that $x \in K$ is a common equilibrium of the family $\mathcal{G} = \{g_\alpha\}$ if and only if x is a strict equilibrium of Φ . Notice that $\Phi(x) = x - F(x)$ for every $x \in K$, where $F(x) = \overline{\text{conv}}\{f_\alpha(x)\}$. Following suitable remarks in Section 3 we can see that, if \mathcal{G} is directed, then Φ satisfies condition (A1).

We can apply for the map Φ results from previous sections and obtain the following common fixed point theorems.

Theorem 4.3 (Corollary of Theorem 1.1). *Let E be a Banach space, $U \subset E$ be an open subset and let $\mathcal{F} = \{f_\alpha : \bar{U} \rightarrow E\}_{\alpha \in \Lambda}$ be a family of continuous maps satisfying the following conditions:*

- (i) *The map $F : \bar{U} \multimap E$, $F(x) = \overline{\text{conv}}\{f_\alpha(x)\}$ is compact.*
- (ii) *The map $\Phi : \bar{U} \multimap E$, $\Phi(x) = x - F(x)$ satisfies condition (A1) (or, in particular, the family $\{g_\alpha\}$, where $g_\alpha(x) = x - f_\alpha(x)$, is directed).*
- (iii) *$0 \notin \Phi(\partial U)$ and $\text{Deg}(\Phi, U) \neq 0$.*

Then there exists a common fixed point of \mathcal{F} .

Remark 4.4. If $\Lambda = \{1, \dots, k\}$, and the family $\{g_i\}$ is directed, then assumption (iii) holds whenever $g_i(\partial U) \not\ni 0$ for every $i \in \Lambda$ and there is $j \in \Lambda$ with $\text{deg}(g_j, U) \neq 0$.

Indeed, at first we can show that $0 \notin \Phi(\partial U)$. Suppose $0 = \sum_{i=1}^k p_i g_i(x)$ for some $x \in \partial U$ with $\sum_{i=1}^k p_i = 1$. Since $g_i(x) \neq 0$ for every $i \in \Lambda$, there are at least two non-zero numbers p_i . Denote $I = \{i \in \Lambda : p_i \neq 0\}$, $l = \#I$ and let $d = \min_{i \in I} \min\{p_i, 1 - p_i\}$. Take $\varepsilon \in (0, d)$. There is $m \in I$ such that $g_m(x) \neq \frac{1}{l-1} \sum_{i \in I \setminus \{m\}} g_i(x)$ since, otherwise, one obtains $g_i(x) = g_j(x)$ for each $i, j \in I$, and hence $g_i(x) = 0$ for $i \in I$, which is forbidden. Consider a point

$$y = (p_m + \varepsilon)g_m(x) + \sum_{i \in I \setminus \{m\}} \left(p_i - \frac{\varepsilon}{l-1}\right)g_i(x).$$

One can easily check that $y \neq 0$ and $y \in \Phi(x)$. Moreover,

$$\begin{aligned} -y &= -(p_m + \varepsilon)g_m(x) - \sum_{i \in I \setminus \{m\}} \left(p_i - \frac{\varepsilon}{l-1}\right)g_i(x) \\ &= (p_m - \varepsilon)g_m(x) + \sum_{i \in I \setminus \{m\}} \left(p_i + \frac{\varepsilon}{l-1}\right)g_i(x) \\ &\in \Phi(x) \end{aligned}$$

which implies that $\Phi(x) \cap -\Phi(x) \not\subset \{0\}$. But the family $\{g_i\}$ is directed and we have a contradiction.

Now, the homotopy

$$H : \bar{U} \times [0, 1] \multimap E, \quad H(x, t) = tg_j(x) + (1 - t)\Phi(x) \subset \Phi(x)$$

joins g_j and Φ in such a way that $\text{Deg}(\Phi, U) = \text{deg}(g_j, U) \neq 0$ (see the homotopy property, Proposition 2.3 (iii)).

Example 4.5. Consider two functions $f_1, f_2 : [-2, 2] \rightarrow \mathbb{R}$, $f_1(x) = x - \frac{1}{4}x^3$ and $f_2(x) = x - x^5$. Evidently, they are not non-expansive. We

define

$$\begin{aligned} g_1(x) &= x - f_1(x) = \frac{1}{4}x^3 & \text{and} & & \Phi(x) &= \text{conv}\{g_1(x), g_2(x)\}. \\ g_2(x) &= x - f_2(x) = x^5 \end{aligned}$$

Then $\text{Deg}(\Phi, (-2, 2)) = 1$ and $g_1(x)g_2(x) \geq 0$ for every $x \in [-2, 2]$. Of course, there is a common fixed point of f_i , namely $x = 0$, whose existence is implied by Theorem 4.3. On the other hand, f_1 and f_2 do not commute. Moreover, they are not weakly compatible ⁷⁾.

Indeed, $f_1(x) = f_2(x)$ for $x \in \{-\frac{1}{2}, 0, \frac{1}{2}\}$ but $f_1(f_2(\frac{1}{2})) \neq f_2(f_1(\frac{1}{2}))$. Thus, as far as the author knows, existence of the evident common fixed point $x = 0$ is not a consequence of known results from the metric fixed point theory.

Example 4.6. Consider a family of Cauchy problems

$$(\mathbf{P})_i \begin{cases} \dot{x}(t) = f_i(x(t)) & \text{for } t \geq 0 \\ x(0) = x_i \end{cases}$$

where $f_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are continuous for $i \in \{1, \dots, k\}$. Assume that the family $\{f_i\}$ is directed, there is a ball $B \subset \mathbb{R}^n$ such that $0 \notin f_i(\partial B)$ for every i and, finally, that there is $j \in \{1, \dots, k\}$ such that $x \neq \lambda f_j(x)$ for every $\lambda < 0$ and $x \in \partial B$. Under the above assumptions there exists a common stationary point \bar{x} for all problems $(\mathbf{P})_i$.

Indeed, one defines $F(x) = \text{conv}\{f_i(x) : i = 1, \dots, k\}$ and show that, by Remark 4.4, $\text{Deg}(F, B) = \text{deg}(f_j, B) = 1 \neq 0$. We apply Theorem 4.3 and obtain the desired \bar{x} .

Finally, we formulate a common fixed point result on compact sleek sets.

Theorem 4.7 (Corollary of Theorem 1.2). *Let $K = \overline{\text{Int } K} \subset \mathbb{R}^n$ be a compact sleek set and let $\mathcal{F} = \{f_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n\}_{\alpha \in \Lambda}$ be a family of continuous maps satisfying the following conditions:*

- (i) *The map $F : \mathbb{R}^n \multimap \mathbb{R}^n$, $F(x) = \overline{\text{conv}}\{f_\alpha(x)\}$ has at most a linear growth.*
- (ii) *The map $\Phi : \overline{U} \multimap E$, $\Phi(x) = x - F(x)$ satisfies condition (A1) (on $\text{Int } K$) and (2).*
- (iii) *$0 \notin \Phi(\partial K)$ and $\chi(K, K^-(\Phi)) \neq 0$.*

Then there is a common fixed point of \mathcal{F} in $\text{Int } K$.

⁷⁾ Two maps T, S are *weakly compatible*, if $S(u) = T(u)$ implies $ST(u) = TS(u)$ (see [15]).

5. Concluding remarks

We finish the paper with a short discussion on possibility of defining topological invariants appropriate for finding strict equilibria. As usual, the first question is what class of maps is suitable for such invariant. We could see in preceding sections that some geometrical assumptions were necessary. So, one can consider as an admissible class of maps

$$\mathcal{CG}(U, E) = \left\{ \Phi = i - F : \bar{U} \multimap E \left| \begin{array}{l} F \text{ is continuous compact} \\ \text{with compact convex values} \\ 0 \notin \Phi(\partial U) \text{ and } \Phi \text{ satisfies (A1)} \end{array} \right. \right\}$$

where $i : \bar{U} \rightarrow E$ is the inclusion map, with a homotopy *in* $\mathcal{CG}(U, E)$ as

$$H : \bar{U} \times [0, 1] \multimap E, \quad H(\cdot, t) \in \mathcal{CG}(U, E) \text{ for every } t \in [0, 1].$$

Then the degree (see Section 3) is defined for $\mathcal{CG}(U, E)$ and it has the standard additivity and homotopy (with respect to a homotopy in $\mathcal{CG}(U, E)$) properties. Moreover, the existence property takes a form

$$\{\text{Deg}(\Phi, U) \neq 0\} \implies \{\exists x \in U : \Phi(x) = \{0\}\}.$$

Since every single-valued map satisfies (A1), one also has the normalization property, and the degree in $\mathcal{CG}(U, E)$ is an obvious generalization of the Leray-Schauder degree for single-valued continuous compact vector fields.

Note that for obtaining a strict equilibrium of $H(\cdot, 1)$ (the end of the homotopy) it is not necessary to assume condition (A1) for each homotopy level $H(\cdot, t)$ (compare Corollary 3.5). This condition brings the information on existence of strict equilibria for each $t \in [0, 1]$. Note also that the evident *essentiality* property ⁸⁾ in $\mathcal{CG}(U, E)$:

For any isolated (in the set of all equilibria) strict equilibrium x of $\Phi : \bar{U} \multimap E$ and any isolating neighbourhood V of x in U with $\text{Deg}(\Phi, V) \neq 0$, there is $\varepsilon > 0$ such that each $\Psi \in \mathcal{CG}(U, E)$, $\sup_{x \in \bar{U}} d_H(\Phi(x), \Psi(x)) < \varepsilon$, has a strict equilibrium in V ,

is not true without assuming that Ψ satisfies (A1). The reason is that (A1) is not preserved under small perturbations, so it is really independent of the degree conditions. As an example we can consider $\Phi, \Psi : [-1, 1] \multimap \mathbb{R}$, $\Phi(x) = x$ and $\Psi(x) = [x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}]$, and notice that Ψ has no strict equilibrium.

In a similar way as above we can study *strict fixed points* of maps, i.e. points with $F(x) = \{x\}$, using a standard fixed point index or the Lefschetz

⁸⁾ The notion of essentiality was introduced by M. K. Fort in [11].

number for multi-valued maps. It is sufficient to add a suitable geometrical condition, for instance,

$$\{x\} \cap F(x) \subset \text{ext } F(x) \quad \text{for every } x \in U \quad (5)$$

where $\text{ext } F(x)$ denotes the set of all extreme points of the convex set $F(x)$. Notice that condition (5) is equivalent to (3) for $\Phi(x) = x - F(x)$. In particular, we obtain

Proposition 5.1. *Let X be a compact neighbourhood retract in a Banach space E and let $F : X \multimap X$ be a compact-convex-valued continuous map satisfying condition (5) and such that the Lefschetz number $\Lambda(F) \neq 0$. Then there is a strict fixed point of F .*

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