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Energy Form on a Closed Fractal Curve

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Abstract. The energy form on a closed fractal curve F is constructed. As F is neither self-similar nor nested, it is regarded as a "fractal manifold". The energy is obtained by integrating the Lagrangian on F.

Keywords: *Dirichlet form, Lagrangian, non self-similar fractals*

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1. Introduction

In this paper we consider a closed fractal curve F, the so-called von Koch snowflake (see, e.g., Falconer [4]) and we construct the energy form associated to a free diffusion process on it. The main feature is that F is not a self-similar fractal, hence it is not possible to make use of the by-now well established theory of potential analysis (see, e.g., Kusuoka [13, 14] and Kigami [12]). In order to define an energy form on F, we regard F as a fractal manifold and define the energy form $\mathcal{E}_F$ on F by integrating a local energy or *Lagrangian* on F (see Fukushima, Oshima and Takeda [7], Mosco [18 - 21] and Strichartz [22]).

Two different decompositions of F into three Koch curves K_i are possible, namely $F = \bigcup_{i=1}^{3} K_i$ or $F = \bigcup_{i=4}^{6} K_i$ (see Figures 2.a and 2.b on Page **122**). It turns out that $\mathcal{E}_F$ can also be obtained as sum of energies associated with the three von Koch curves of which F is made by, independent of the decomposition (see Theorem 4.6). Indeed, in [8], for a certain class of fractal sets which are the finite union of nested fractals with possibly different

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Hausdorff dimensions, the energy is defined by using such sums of energies; in this sense, the set F considered here is a particular case of those fractals considered in [8]. Nevertheless, in the present paper the special geometry of F requires to analyze which is the role of the "junction points", because they match two "pieces" of same "shape" and same Hausdorff dimension. In this context, we are able to prove – from the probabilistic point of view – that the free diffusion process on F satisfies in the junctions points a strong reflection principle. This explains how the free diffusion on F corresponds to the three "reflecting" diffusion processes on K_1, K_2 and K_3 (or on K_4, K_5 and K_6, respectively) which are associated with the energy forms on the corresponding Koch curves.

Our Lagrangian approach can be obviously extended to more general geometries where F is the finite union of fractal sets with different Hausdorff dimensions, but also to the more interesting case of fractals which are images of a nested fractal under a C^1-diffeomorphic mapping (see [6]).

The plan of the paper is the following. In Section 2 we recall the definition and properties of the von Koch curve K and we define the energy form and the *Lagrangian* on K. The definition of the snowflake F as a "fractal manifold" is given in Section 3. In Section 4 the *Lagrangian* on the snowflake F is introduced as image measure of the *Lagrangian* on the Koch curve by using the description of F as a "fractal manifold". This leads to the definition of the energy form $\mathcal{E}_F$ on F, and it turns out that $\mathcal{E}_F$ is a closed, strongly local, regular Dirichlet form defined on $L^2(F, \mu_F)$, where μ_F is proportional to the D_f-dimensional Hausdorff measure on F (see Corollary 4.11). In Section 5 the Laplacian Δ_F is introduced as operator associated to the energy form $\mathcal{E}_F$. We show that the Laplacian on F is "locally" given by the Laplacian of the corresponding von Koch curve (see Theorem 5.3). In Section 6 we give a stochastic interpretation in terms of a strong reflection principle.

2. Preliminaries

Through this paper we are in the real plane $\mathbb{R}^2$, equipped with the Euclidean distance $|p-q|$ $(p, q \in \mathbb{R}^2)$. By $\overline{A}$ we denote the closure of a set A in $\mathbb{R}^2$. By $\mathcal{C}(A)$ we denote the space of real-valued, continuous functions on A, by $\mathcal{C}(A)'$ its dual, and by $\mathcal{C}_0(A)$ the space of continuous functions compactly supported on A.

2.1 The von Koch curve. Let us first recall the definition and some main properties of the von Koch curve K. This self-similar fractal belongs to the class of so-called nested fractals introduced by Lindstrøm [17] and is obtained as follows (see, e.g., [4]):

Pose $A=(0,0)$ and $B=(1,0)$, remove from the segment $\overline{AB}$ the middle (open) third and put up above this hole the two other sides of a regular triangle. Do the same with the four segments of length $\frac{1}{3}$ of the arising set, and do so on (see Figure 1).

Figure 1: Construction of the von Koch curve: first and second iteration

The limit set is the well known von Koch curve.

On the other hand, this fractal is given as unique non-empty set which is self-similar with respect to the family of affine contractions $\Psi=\{\psi_1,\dots,\psi_4\}$ where the mappings are given by

$$
\begin{aligned}
\psi_1(z) &= \frac{z}{3} \\
\psi_2(z) &= \frac{z}{3}e^{\mathrm{i}\frac{\pi}{3}}+\frac{1}{3} \\
\psi_3(z) &= \frac{z}{3}e^{-\mathrm{i}\frac{\pi}{3}}+\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{6} \\
\psi_4(z) &= \frac{z+2}{3}
\end{aligned}
$$

and z denotes an element of $\mathbb{C}$ (for the moment we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$). It is proved that there exists a unique non-empty compact set K such that $K=\bigcup_{i=1}^{4}\psi_i(K)$, (i.e. K consists of smaller similar copies of itself, hence K is self-similar; see Hutchinson [9]).

Furthermore, one can obtain K as the attractor of the dynamical system $\{\psi^n\}_{n\geq 1}$, where ψ^n denotes the n^{th} composition of ψ with itself, the map $\psi(A)=\bigcup_{i=1}^{4}\psi_i(A)\quad(A\subseteq\mathbb{R}^2)$ acts on the Banach space of all non-empty compact subsets of $\mathbb{R}^2$ equipped with the Hausdorff metric. Moreover, one has free choice of a (non-empty) closed starting set. The choice of $V_0=\{A,B\}$ – which is just the set of essential fixed points of the iterated function system Ψ (see [17] for details) – leads to the following approximation of K by an increasing sequence of finite sets of isolated points.

Setting $V_0=\{A,B\}$, for arbitrary n-tuples of indices $j_1,\dots,j_n\in\{1,\dots,4\}$ we define

$$
\begin{aligned}
\psi_{j_1\cdots j_n} &= \psi_{j_1}\circ\cdots\circ\psi_{j_n} \\
V_{j_1\cdots j_n} &= \psi_{j_1\cdots j_n}(V_0)
\end{aligned}
$$

and

$$
V_n=\bigcup_{j_1,\dots,j_n=1}^{4}V_{j_1\cdots j_n}.
$$

It is easy to see that $\sharp V_n = 4^n + 1$. Every point p in $V_n \setminus \{A, B\}$ has two neighbors $q \in V_n$, which are called n-neighbors of p, denoted in the following by $q \sim_n p$. We say that $p, q \in V_n$ are n-neighbors, if there exists a n-tuple of indices $j_1, ..., j_n \in \{1, ..., 4\}$ such that $p, q \in V_{j_1 \cdots j_n}$. They both have distance 3^{-n} from p. Further, we set

$$V_* = \bigcup_{n \geq 0} V_n = \lim_{n \to \infty} V_n.$$

There holds $K = \overline{V_*}$.

Moreover, there exists a unique Borel probability measure μ which is self-similar with respect to the family Ψ, i.e.

$$\mu(A) = \frac{1}{4} \sum_{i=1}^{4} \mu(\psi_i^{-1} A)$$

for any Borel set $A \subseteq \mathbb{R}^2$, and $\operatorname{supp} \mu = K$. Note that μ is given by the normalized D_f-dimensional Hausdorff measure $\mathcal{H}^{D_f}$, restricted to K, where $D_f = \frac{\ln 4}{\ln 3}$ (see [9]). Further, for any $n \geq 1$ we define a discrete measure μ^n on V_n by

$$\mu^n = \frac{1}{4^n} \sum_{p \in V_n} \delta_{\{p\}} \tag{2.1}$$

where $\delta_{\{p\}}$ denotes the Dirac measure at the point p. Note that $\mu^n(V_n) = 1 + \frac{1}{4^n}$.

In [15] the following result is proved:

Proposition 2.1. *The sequence* $(\mu^n)_{n \geq 1}$ *is weakly convergent (i.e. in* $\mathcal{C}(K)'$*) to the measure* μ.

2.2 Energy form on the von Koch curve. In this subsection we recall the construction of the energy form on the von Koch curve K. It is based on finite difference schemes and follows general lines described in [14] for nested fractals.

For any function $u : V_* \to \mathbb{R}$ we define

$$\mathcal{E}_n[u] = \frac{1}{2} 4^n \sum_{p \in V_n} \sum_{q \sim_n p} \big(u(p) - u(q)\big)^2 \tag{2.2}$$

where $q \sim_n p$ means that q is an n-neighbor of p. It can be shown (see [14]) that the sequence $(\mathcal{E}_n[u])_{n \geq 0}$ is non-decreasing, the limit of the right-hand side of (2.2) exists and the limit form

$$\mathcal{E}[u] = \lim_{n \to \infty} \mathcal{E}_n[u] \tag{2.3}$$

is non-trivial ($\mathcal{E} \neq \infty$) with domain

$$\mathcal{D}_*(\mathcal{E}) = \{u : V_* \to \mathbb{R} |\, \mathcal{E}[u] < \infty\}.$$

Every function $u \in \mathcal{D}_*(\mathcal{E})$ can be uniquely extended to an element of $\mathcal{C}(K)$. We denote this extension still by u and set

$$\mathcal{D} = \{u \in \mathcal{C}(K) : \mathcal{E}[u] < \infty\}$$

where $\mathcal{E}[u] = \mathcal{E}[u_{|V_*}]$. Hence $\mathcal{D} \subseteq \mathcal{C}(K) \subseteq L^2(K,\mu)$, where $L^2(K,\mu)$ is the Hilbert space of square summable functions on K with respect to the self-similar measure μ.

We now define the space $\mathcal{D}(\mathcal{E})$ as completion of $\mathcal{D}$ in the norm

$$\|u\|_{\mathcal{E}} = \left(\|u\|^2_{L^2(K,\mu)} + \mathcal{E}[u]\right)^{1/2}. \tag{2.4}$$

$\mathcal{D}(\mathcal{E})$ is injected in $L^2(K,\mu)$ and is a Hilbert space with scalar product associated to norm (2.4). Then we extend $\mathcal{E}$ as usual on the completed space $\mathcal{D}(\mathcal{E})$. By $\mathcal{E}(\cdot,\cdot)$ we denote the bilinear form defined on $\mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E})$ by polarization, i.e.

$$\mathcal{E}(u,v) = \frac{1}{2}\big(\mathcal{E}[u+v] - \mathcal{E}[u] - \mathcal{E}[v]\big) \qquad \big(u,v \in \mathcal{D}(\mathcal{E})\big).$$

It is easy to see that, for any pair $u,v \in \mathcal{D}(\mathcal{E})$, $\mathcal{E}(u,v)$ is the limit of the sequence $(\mathcal{E}_n(u,v))$ given by

$$\mathcal{E}_n(u,v) = \frac{1}{2}\, 4^n \sum_{p \in V_n} \sum_{q \sim_n p} [u(p) - u(q)]\,[v(p) - v(q)]. \tag{2.5}$$

The form $\mathcal{E}(\cdot,\cdot)$ with domain $\mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E})$ is a closed Dirichlet form in the Hilbert space $L^2(K,\mu)$. It is regular and strongly local. Regularity means that $\mathcal{D}(\mathcal{E}) \cap \mathcal{C}(K)$ is dense both in $\mathcal{C}(K)$ with respect to the uniform norm and in $\mathcal{D}(\mathcal{E})$ with respect to the intrinsic norm (2.4). This property implies that $\mathcal{D}(\mathcal{E})$ is not trivial (i.e. not made by only the constant functions). Moreover, the functions in $\mathcal{D}(\mathcal{E})$ posses a continuous representative, which is actually Hölder continuous on K (see [15: Corollary 3.3]). In the following we will use that $\mathcal{D}(\mathcal{E}) \cap \mathcal{C}_0(K)$ is dense in $\mathcal{C}_0(K)$.

Proposition 2.2. *The space $\mathcal{D}(\mathcal{E})$ is continuously embedded into $\mathcal{C}^{0,\beta}(K)$, the space of Hölder continuous functions with exponent $\beta = \frac{\ln 4}{2 \ln 3} = \frac{D_f}{2}$.*

In the following we identify $u \in \mathcal{D}(\mathcal{E})$ with its continuous representative, still denoted by u.

2.3 The Lagrangian on the von Koch curve. In this subsection, we construct the Lagrangian on the von Koch curve. For the concept of Lagrangians on fractals, i.e. the notion of a measure-valued local energy, we refer to [7, 18, 20] (see also [2, 21]).

We observe that the approximating energy forms $\mathcal{E}_n$ on V_n, defined in (2.5), can be written as

$$\mathcal{E}_n(u,v) = \int_{V_n} \nabla_n u \cdot \nabla_n v \, d\mu^n \tag{2.6}$$

where μ^n is the discrete measure given in (2.1). For every $n \geq 0$, μ^n is a measure on K supported on V_n, and for any $p \in V_n$ the "discrete gradient" is given by

$$\nabla_n u \cdot \nabla_n v(p) = \frac{1}{2} \sum_{q \sim_n p} \frac{u(p)-u(q)}{|p-q|^\delta} \, \frac{v(p)-v(q)}{|p-q|^\delta} \qquad (u,v \in \mathcal{D}(\mathcal{E}))$$

where $\delta = \frac{\ln 4}{\ln 3}$ (see [21]). Then there holds:

Proposition 2.3. *Let A be any subset of K. For every $u, v \in \mathcal{D}(\mathcal{E})$, the sequence of measures given by*

$$\mathcal{L}_K^{(n)}(u,v)(A) = \int_{A \cap V_n} \nabla_n u \cdot \nabla_n v \, d\mu^n \qquad (n \geq 0) \tag{2.7}$$

weakly converges in $\mathcal{C}(K)'$ to a signed finite Radon measure $\mathcal{L}_K(u,v)$ on K as $n \to \infty$, the so-called Lagrangian measure on K. Moreover,

$$\mathcal{E}(u,v) = \int_K d\mathcal{L}_K(u,v) \qquad (u,v \in \mathcal{D}(\mathcal{E})).$$

Proof. First, let us restrict ourselves to the quadratic case. Fix $u \in \mathcal{D}(\mathcal{E})$ and set $\mathcal{L}_K^{(n)}[u] = \mathcal{L}_K^{(n)}(u,u) \quad (n \geq 0)$. From (2.6) and (2.3) it follows that $(\mathcal{L}_K^{(n)}[u](K))_{n\geq 0}$ is a uniformly bounded sequence, because

$$\mathcal{L}_K^{(n)}[u](K) = \int_K d\mathcal{L}_K^{(n)}[u] = \mathcal{E}_n[u] \leq \mathcal{E}[u] < \infty \qquad (n \geq 0).$$

Let $n \in \mathbb{N}$ be fixed. It can be proved by straightforward calculations that, for every $u \in \mathcal{D}(\mathcal{E})$ and every $\varphi \in \mathcal{D}(\mathcal{E}) \cap C_0(K)$, the identity

$$\int_{V_n} \varphi d\mathcal{L}_K^{(n)}[u] = \mathcal{E}_n(\varphi u, u) - \frac{1}{2}\mathcal{E}_n(\varphi, u^2) \tag{2.8}$$

holds. As the energy form $\mathcal{E}[u]$ is a Dirichlet form of diffusion type, it admits an integral representation (see [16]): there exists a unique positive Radon measure, which we call $\mathcal{L}_K[u]$, such that $\mathcal{E}[u] = \int_K d\mathcal{L}_K[u]$ and which is uniquely defined by

$$\int_K \varphi d\mathcal{L}_K[u] = \mathcal{E}(\varphi u, u) - \frac{1}{2}\mathcal{E}(\varphi, u^2) \qquad \big(\varphi \in \mathcal{D}(\mathcal{E}) \cap C_0(K)\big) \tag{2.9}$$

(see [18]). Passing to the limit as $n \to \infty$ in (2.8), from (2.3), taking into account the regularity of the form, it follows that the right-hand side of (2.8) tends to the right-hand side of (2.9). Hence we have proved that

$$\mathcal{L}_K^{(n)}[u] \rightharpoonup \mathcal{L}_K[u] \qquad (n \to \infty). \tag{2.10}$$

The (signed) Radon measure $\mathcal{L}_K^{(n)}(u,v)$ $(u, v \in \mathcal{D}(\mathcal{E}))$ is given by polarization:

$$\mathcal{L}_K^{(n)}(u,v) = \frac{1}{2}\Big\{\mathcal{L}_K^{(n)}(u+v, u+v) - \mathcal{L}_K^{(n)}(u,u) - \mathcal{L}_K^{(n)}(v,v)\Big\}.$$

These are Radon measures on K uniquely associated with every $u, v \in \mathcal{D}(\mathcal{E})$. The weak convergence of the sequence $(\mathcal{L}_K^{(n)}(u,v))_{n\geq 0}$ to the (signed) Radon measure $\mathcal{L}_K(u,v)$ for any $u, v \in \mathcal{D}(\mathcal{E})$ follows from the polarization formula and (2.10) (see [18]) ∎

Remark 2.4. The measure-valued map $\mathcal{L}_K$ on $\mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E})$ is bilinear, symmetric and positive (i.e. $\mathcal{L}_K[u]$ is a positive measure). This measure-valued Lagrangian takes on the fractal K the role of the Euclidean Lagrangian $d\mathcal{L}(u,v) = \nabla u \cdot \nabla v dx$. Note that in the case of the Koch curve the Lagrangian $\mathcal{L}_K$ is absolutely continuous with respect to the volume measure μ (see [3]). On the contrary, this is not true on most fractals (see [20]).

3. The von Koch snowflake

3.1 Intuitive description. By a von Koch snowflake F we will denote the union of three standard von Koch curves K_1, K_2 and K_3 as shown in Figure 2a. We assume that the junction points x_1, x_3 and x_5 are the vertices of a regular triangle with unit side length, i.e. $|x_1 - x_3| = |x_1 - x_5| = |x_3 - x_5| = 1$. Obviously, F can also be seen as the union of the three other standard von Koch curves K_4, K_5 and K_6 (with junction points x_2, x_4 and x_6), as shown in Figure 2b.

From now on we assume that a clockwise orientation is given on F.

Figure 2: a) first decomposition b) second decomposition

Remark 3.1. We observe that the points x_1, x_3 and x_5, which are the endpoints of K_1, K_2 and K_3, are interior points of K_4, K_5 and K_6, respectively; viceversa, the points x_2, x_4 and x_6 are interior points of K_1, K_2 and K_3, respectively.

By $\mathring{K}_i$ $(1 \leq i \leq 6)$ we denote the curve K_i without its endpoints. Of course, the Hausdorff dimension of the von Koch snowflake is also given by $D_f = \frac{\ln 4}{\ln 3}$. But unfortunately, this fractal is no longer self-similar (and, hence, not nested). So we cannot apply the theory of defining an energy form on a nested – or, more general, on a post critically finite – fractal which was developed by several authors (see, e.g., [12, 13, 17]).

One can define, in a natural way, a finite Borel measure μ_F supported on F by

$$\mu_F = \mu_1 + \mu_2 + \mu_3 \tag{3.1}$$

where μ_i denotes the normalized D_f-dimensional Hausdorff measure, restricted to K_i $(i = 1, 2, 3)$. There holds also that $\mu_F = \mu_4 + \mu_5 + \mu_6$, where μ_i is the normalized D_f-dimensional Hausdorff measure restricted to K_i $(i = 4, 5, 6)$.

Obviously, K_1 is the uniquely determined self-similar set with respect to four suitable contractions $\psi_1^{(1)}, \ldots, \psi_4^{(1)}$, with the same ratio $\frac{1}{3}$, which can be obtained from $\psi_1, \ldots, \psi_4$ by composing them with rotations and translations, as it will be pointed out in the next section.

As before, we approximate K_1 from below by a sequence of finite sets of points. For this, set

$$V_0^{(1)} = \{x_1, x_3\}$$
$$V_{j_1 \cdots j_n}^{(1)} = \psi_{j_1}^{(1)} \circ \cdots \circ \psi_{j_n}^{(1)}(V_0^{(1)})$$

and

$$V_n^{(1)} = \bigcup_{j_1, \ldots, j_n = 1}^{4} V_{j_1 \cdots j_n}^{(1)}.$$

Further, we set

$$V_*^{(1)} = \bigcup_{m\geq 0} V_m^{(1)}.$$

There follows $K_1 = \overline{V_*^{(1)}}$. In a similar way, we approximate the von Koch curves $K_2, \ldots, K_6$ by sequences $(V_m^{(2)})_{m\geq 0}, \ldots, (V_m^{(6)})_{m\geq 0}$ and denote their limits by $V_*^{(2)}, \ldots, V_*^{(6)}$, respectively.

In order to approximate F, we define the increasing sequence of finite sets of points $\mathcal{V}_m = \bigcup_{i=1}^3 V_m^{(i)} = \bigcup_{i=4}^6 V_m^{(i)} \quad (m \geq 1)$ and $\mathcal{V}_* = \bigcup_{m\geq 1} \mathcal{V}_m$. There follows $\mathcal{V}_* = \bigcup_{i=1}^3 V_*^{(i)} = \bigcup_{i=4}^6 V_*^{(i)}$ and $F = \overline{\mathcal{V}_*}$.

3.2 The von Koch snowflake as a manifold. We assume that we are given a von Koch snowflake F as described above. For our purposes it is convenient to regard F as a "fractal manifold". We cover the snowflake by sets $U_i \quad (i \geq 1)$, which are open subsets of the snowflake and which can be mapped by a corresponding set of homeomorphisms $\{\varphi_i\}_{i\geq 1}$ to certain "fractal reference sets". Here "open in the snowflake" means open with respect to the trace topology on F of the Euclidean one on $\mathbb{R}^2$.

Because of the simple geometry of the snowflake it seems reasonable to choose

$$U_i = \mathring{K}_i \qquad (i = 1, \ldots, 6)$$

(according to Figures 2a and 2b) and to define the mappings $\varphi_i : \mathbb{R}^2 \to \mathbb{R}^2$ as uniquely determined orientation preserving Euclidean motions such that every φ_i maps the set K_i to the reference von Koch curve K defined in Subsection 2.1. Obviously, such a map φ_i is given as composition of a rotation and a translation of the plane $\mathbb{R}^2$, i.e. $\varphi_i(p) = e^{\mathrm{i}\,\theta_i} p + b_i \quad (i = 1, \ldots, 6)$, where θ_i is the rotational angle and $b_i \in \mathbb{R}^2$ is a vector; obviously, $\varphi_i(V_0^{(i)}) = V_0$. By means of these functions we choose the maps $\psi_j^{(i)} \quad (j = 1, ..., 4;\ i = 1, ..., 6)$ as

$$\psi_j^{(i)}(\cdot) = \varphi_i^{-1}\big(\psi_j(\varphi_i(\cdot))\big).$$

Further, each map φ_i preserves the property of n-neighborhood:

Lemma 3.2. *For any $n \geq 1$ and any $i = 1, \ldots, 6$ the following holds: p and q are n-neighbors in $V_n^{(i)}$ if and only if $\varphi_i(p)$ and $\varphi_i(q)$ are n-neighbors in V_n.*

Proof. Let $p, q \in V_n^{(i)}$ be n-neighbors. Then there exists an n-tuple $j_1, ..., j_n$ such that

$$p = \psi_{j_1}^{(i)} \circ \cdots \circ \psi_{j_n}^{(i)}(\xi^{(i)})$$
$$q = \psi_{j_1}^{(i)} \circ \cdots \circ \psi_{j_n}^{(i)}(\eta^{(i)})$$

where $\{\xi^{(i)}, \eta^{(i)}\} = V_0^{(i)}$. Then

$$\varphi_i(p) = \psi_{j_1} \cdots \psi_{j_n}(\varphi_i(\xi^{(i)}))$$
$$\varphi_i(q) = \psi_{j_1} \cdots \psi_{j_n}(\varphi_i(\eta^{(i)})),$$

thus the thesis follows from the properties of the functions φ_i. As the matrix $e^{\mathrm{i}\,\theta_i}$ is orthogonal, $|\varphi_i(p) - \varphi_i(q)| = |p - q| \quad (p, q \in V_n^{(i)};\ i = 1, \dots, 6)$ ∎

Corollary 3.3. *For any $n \geq 1$ and any $i = 1, \dots, 6$, the map $\varphi_i^{-1} : K \to K_i$ preserves the property of n-neighborhood on V_n.*

From the above consideration a natural definition of n-neighbors can be given for the points in $\mathcal{V}_m$. Every $p \in \mathcal{V}_m$ has two neighbors q in the following denoted by $p \sim_m q$.

4. Lagrangian and energy form on the snowflake

4.1 Lagrangian on the snowflake. In this subsection, we define the Lagrangian $\mathcal{L}_F$ on the fractal snowflake F by using its representation as a "fractal manifold" (see Subsection 3.2).

Let $\mathcal{L}_K$ be the Lagrangian on the von Koch curve K introduced in Proposition 2.3. To this aim, we introduce the space

$$\mathcal{D}_F = \left\{ w : F \to \mathbb{R} \middle| w \circ \varphi_i^{-1} \in \mathcal{D}(\mathcal{E}) \ \ \forall\ i = 1, \dots, 6 \right\}. \tag{4.1}$$

Let w, z be two given functions in $\mathcal{D}_F$ defined on F. We want to define a measure $\mathcal{L}_F(w, z)$ on F.

Definition 4.1. Let A be a Borel set of K_i. We introduce the measure-valued Lagrangian $\mathcal{L}_F(u, v)$ of the set A as image measure (see, e.g., [5]) of the measure $\mathcal{L}_K\big(w \circ \varphi_i^{-1}, z \circ \varphi_i^{-1}\big)$ under the map φ_i^{-1}, i.e.

$$\mathcal{L}_F(w, z)(A) = \mathcal{L}_K\big(w \circ \varphi_i^{-1}, z \circ \varphi_i^{-1}\big)(\varphi_i(A)) \qquad (A \subseteq K_i).$$

Remark 4.2. Due to Proposition 2.2, for any $w \in \mathcal{D}_F$ the functions $w \circ \varphi_i^{-1} \ \ (i = 1, \dots, 6)$ are continuous on K. Hence, $w_{|K_i} = w \circ \varphi_i^{-1} \circ \varphi_i$ is continuous on K_i for every $i = 1, \dots, 6$. From Remark 3.1 the continuity of w on all of F follows. Thus, from now on we identify the elements of $\mathcal{D}_F$ with their continuous representatives.

We now show that the definition of $\mathcal{L}_F(w, z) \ \ (w, z \in \mathcal{D}_F)$ is well posed.

Proposition 4.3. *The above definition of the Lagrangian $\mathcal{L}_F$ is independent of the choice of the sets K_i, i.e. if $A \subset K_i \cap K_j \quad (i, j = 1, \ldots, 6; i \neq j)$, then*

$$\mathcal{L}_K\big(w \circ \varphi_i^{-1}, z \circ \varphi_i^{-1}\big)(\varphi_i(A)) = \mathcal{L}_K\big(w \circ \varphi_j^{-1}, z \circ \varphi_j^{-1}\big)(\varphi_j(A)) \tag{4.2}$$

for all $w, z \in \mathcal{D}_F$.

Proof. Choose two functions $w, z \in \mathcal{D}_F$ and two indices $i \neq j$. From Proposition 2.3 it follows that $\mathcal{L}_K$ is the weak limit of $\mathcal{L}_K^{(n)}$. In order to prove (4.2) it is sufficient to show that, for any $n \geq 1$ and for any $p \in K_i \cap K_j \cap \mathcal{V}_n$, the discrete gradients satisfy

$$\nabla_n(w \circ \varphi_i^{-1}) \cdot \nabla_n(z \circ \varphi_i^{-1})(\varphi_i(p)) = \nabla_n(w \circ \varphi_j^{-1}) \cdot \nabla_n(z \circ \varphi_j^{-1})(\varphi_j(p)).$$

From (4.1) we have that the functions $u = w \circ \varphi_i^{-1}$ and $v = z \circ \varphi_i^{-1}$, acting from K to $\mathbb{R}$, are in $\mathcal{D}(\mathcal{E})$. Set $r = \varphi_i(p)$. Then $r \in K \cap V_n$, and we have to show that, for any $n \geq 1$,

$$\begin{aligned} &\nabla_n(u) \cdot \nabla_n(v)(r) \\ &\quad = \nabla_n\big(u \circ (\varphi_i \circ \varphi_j^{-1})\big) \cdot \nabla_n\big(v \circ (\varphi_i \circ \varphi_j^{-1})\big)\big((\varphi_j \circ \varphi_i^{-1})(r)\big) \end{aligned} \tag{4.3}$$

holds. Setting $h = \varphi_j \circ \varphi_i^{-1}$, the right-hand side of (4.3) is given by

$$\begin{aligned} &\sum_{q \sim_n h(r)} \frac{(u \circ h^{-1})(h(r)) - (u \circ h^{-1})(q)}{|h(r) - q|^\delta} \, \frac{(v \circ h^{-1})(h(r)) - (v \circ h^{-1})(q)}{|h(r) - q|^\delta} \\ &\quad = \sum_{q' : h(q') \sim_n h(r)} \frac{u(r) - u(q')}{|h(r) - h(q')|^\delta} \, \frac{v(r) - v(q')}{|h(r) - h(q')|^\delta} \\ &\quad = \sum_{q' \sim_n r} \frac{u(r) - u(q')}{|r - q'|^\delta} \, \frac{v(r) - v(q')}{|r - q'|^\delta} \end{aligned}$$

where the last two equalities follow from Corollary 3.3. The last sum equals to the left-hand side of (4.3) ∎

Definition 4.4. If B is an arbitrary Borel subset of F, it can be regarded as disjoint union of sets $B_1, \ldots, B_6$ defined by $B_i = B \cap C_{i,i+1} \quad (i = 1, \ldots, 5)$ and $B_6 = B \cap C_{6,1}$, where $C_{i,i+1}$ denotes the set of all points of F located between x_i and x_{i+1}, including x_i and excluding x_{i+1}, and $C_{6,1}$ denotes the set of all points between x_6 and x_1, including x_6 and excluding x_1. Then any of the sets B_i is contained in one of the sets $K_1, \ldots, K_6$, and we define

$$\mathcal{L}_F(w, z)(B) = \sum_{i=1}^{6} \mathcal{L}_F(w, z)(B_i).$$

$\mathcal{L}_F$ is defined on $\mathcal{D}_F \times \mathcal{D}_F$.

We define the energy form on the fractal snowflake F in terms of its local energy measure $\mathcal{L}_F$.

Definition 4.5. We introduce on $\mathcal{D}_F \times \mathcal{D}_F$ the symmetric bilinear form

$$\mathcal{E}_F(u,v) = \int_F d\mathcal{L}_F(u,v) \qquad (u,v \in \mathcal{D}_F). \tag{4.4}$$

We note that

$$\mathcal{E}_F(u,v) = \sum_{i=1}^{3} \int_{K_i} d\mathcal{L}_F(u,v) = \sum_{i=4}^{6} \int_{K_i} d\mathcal{L}_F(u,v)$$

as follows from Remark 2.4 in this simpler situation and from [1: Theorem 5.2] in the more general case of post critically finite fractals.

4.2 An alternative definition of the energy form on the snowflake. In this subsection, we give a definition of an energy form on the fractal snowflake which does not make use of the notion of the Lagrangian. Later we will see that both approaches are equivalent. Here we refer to the set F no longer as a manifold but as union of three von Koch curves (see Figures 2a and 2b; see also [8]).

In order to introduce some notations, we recall the definition of the energy form on one of these curves, say K_1 (according to Subsection 2.2). For any function $u : V_*^{(1)} \to \mathbb{R}$ we define the non-decreasing sequence $(\mathcal{E}_m^{(1)}[u])_{m\geq 1}$ by

$$\mathcal{E}_m^{(1)}[u] = \frac{1}{2}\, 4^m \sum_{p\in V_m^{(1)}} \sum_{q\sim_m p} \big(u(p) - u(q)\big)^2.$$

On

$$\mathcal{D}_*(\mathcal{E}^{(1)}) = \Big\{ u : V_*^{(1)} \to \mathbb{R} \Big| \lim_{m\to\infty} \mathcal{E}_m^{(1)}[u] < \infty \Big\}$$

we set

$$\mathcal{E}^{(1)}[u] = \lim_{m\to\infty} \mathcal{E}_m^{(1)}[u].$$

As explained in Subsection 2.2, we identify each $\mathcal{D}_*(\mathcal{E}^{(1)})$-function by its continuous extension on K_1. Proceeding as in Subsection 2.2 we have that $(\mathcal{E}^{(1)}, \mathcal{D}(\mathcal{E}^{(1)}))$ is a strongly local Dirichlet form on $L^2(K_1,\mu_1)$ and $\mathcal{D}(\mathcal{E}^{(1)})$ is a Hilbert space equipped with the norm $\big(\|\cdot\|^2_{L^2(K_1,\mu_1)} + \mathcal{E}^{(1)}[\cdot]\big)^{1/2}$.

We proceed analogously for the von Koch curves $K_2, \ldots, K_6$. We denote the corresponding energy forms by $\mathcal{E}^{(2)}, \ldots, \mathcal{E}^{(6)}$, obtained as limits of sequences $(\mathcal{E}_m^{(2)})_{m\geq 1}, \ldots, (\mathcal{E}_m^{(6)})_{m\geq 1}$, respectively. Finally, we denote the domains of these strongly local Dirichlet forms by $\mathcal{D}(\mathcal{E}^{(2)}), \ldots, \mathcal{D}(\mathcal{E}^{(6)})$ and the corresponding Lagrangian on K_i by $\mathcal{L}_{K_i}[\cdot]$ (see Proposition 2.3).

In order to define the energy form on F, we proceed as follows. For any function $u : \mathcal{V}_* \to \mathbb{R}$ we define

$$\widetilde{\mathcal{E}}_m[u] = \frac{1}{2}\, 4^m \sum_{p\in\mathcal{V}_m} \sum_{q\sim_m p} \big(u(p) - u(q)\big)^2 \qquad (m \geq 1).$$

$(\widetilde{\mathcal{E}}_m[u])_{m\geq 1}$ is a sequence non-decreasing in m. Further, we introduce the domain

$$\widetilde{\mathcal{D}} = \Big\{u \in \mathcal{C}(F) \Big| \, \widetilde{\mathcal{E}}_F[u] := \lim_{m\to\infty} \widetilde{\mathcal{E}}_m[u] < \infty\Big\}.$$

Hence, $\widetilde{\mathcal{D}} \subseteq \mathcal{C}(F) \subseteq L^2(F, \mu_F)$, where μ_F is defined in Subsection 3.1. We now define the space $\mathcal{D}(\widetilde{\mathcal{E}}_F)$ as completion of $\tilde{\mathcal{D}}$ in the norm

$$\|u\|_{\mathcal{D}(\widetilde{\mathcal{E}}_F)} = \big(\|u\|^2_{L^2(F,\mu_F)} + \widetilde{\mathcal{E}}_F[u]\big)^{1/2}. \tag{4.6}$$

$\mathcal{D}(\widetilde{\mathcal{E}}_F)$ is injected into $L^2(F, \mu_F)$ and is a Hilbert space with scalar product associated to norm (4.6). Then we extend $\widetilde{\mathcal{E}}_F$ as usual on $\mathcal{D}(\widetilde{\mathcal{E}}_F)$.

We recall that F can be thought as union of three von Koch curves according to Figure 2a as well as to Figure 2b.

We now show that $\widetilde{\mathcal{E}}_F$ is independent of the chosen tiling of F. Namely, we have:

Theorem 4.6. *A function u is in $\mathcal{D}(\widetilde{\mathcal{E}}_F)$ if and only if $u \in \mathcal{C}(F)$ and $u_{|K_i} \in \mathcal{D}(\mathcal{E}^{(i)})$ $(i = 1, ..., 6)$. Moreover, in this case we have*

$$\begin{aligned} \widetilde{\mathcal{E}}_F[u] &= \mathcal{E}^{(1)}[u_{|K_1}] + \mathcal{E}^{(2)}[u_{|K_2}] + \mathcal{E}^{(3)}[u_{|K_3}] \\ &= \mathcal{E}^{(4)}[u_{|K_4}] + \mathcal{E}^{(5)}[u_{|K_5}] + \mathcal{E}^{(6)}[u_{|K_6}] \end{aligned} \qquad (u \in \mathcal{D}(\widetilde{\mathcal{E}}_F)). \tag{4.7}$$

Proof. We only prove the first equality, the second being analogous. As on F the clockwise orientation is given, every point p in $\mathcal{V}_m$ has a "preceding" and a "following" m-neighbor in $\mathcal{V}_m$, denoted in the following by p^{ml} and p^{mr}, respectively. First suppose $u \in \widetilde{D}$. According to Figure 2a, for any $u : \mathcal{V}_* \to \mathbb{R}$ and for any $m \geq 1$ we have

$$\begin{aligned} \widetilde{\mathcal{E}}_m[u] &= \frac{1}{2}\, 4^m \sum_{p\in\mathcal{V}_m} \sum_{q\sim_m p} [u(p) - u(q)]^2 \\ &= \frac{1}{2}\, 4^m \Bigg(\sum_{p\in V_m^{(1)}\setminus\{x_1,x_3\}} \sum_{q\sim_m p} [u(p) - u(q)]^2 \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{p\in V_m^{(2)}\setminus\{x_3,x_5\}} \sum_{q\sim_m p} [u(p)-u(q)]^2 \\
&+ \sum_{p\in V_m^{(3)}\setminus\{x_1,x_5\}} \sum_{q\sim_m p} [u(p)-u(q)]^2 + \sum_{q\sim_m x_1} [u(x_1)-u(q)]^2 \\
&+ \sum_{q\sim_m x_3} [u(x_3)-u(q)]^2 + \sum_{q\sim_m x_5} [u(x_5)-u(q)]^2 \Bigg) \\
= \frac{1}{2}\, 4^m \Bigg(&\sum_{p\in V_m^{(1)}\setminus\{x_1,x_3\}} \sum_{q\sim_m p} [u(p)-u(q)]^2 \\
&+ [u(x_1)-u(x_1^{mr})]^2 + [u(x_3)-u(x_3^{ml})]^2 \\
&+ \sum_{p\in V_m^{(2)}\setminus\{x_3,x_5\}} \sum_{q\sim_m p} [u(p)-u(q)]^2 \\
&+ [u(x_3)-u(x_3^{mr})]^2 + [u(x_5)-u(x_5^{ml})]^2 \\
&+ \sum_{p\in V_m^{(3)}\setminus\{x_1,x_5\}} \sum_{q\sim_m p} [u(p)-u(q)]^2 \\
&+ [u(x_5)-u(x_5^{mr})]^2 + [u(x_1)-u(x_1^{ml})]^2 \Bigg) \\
= \frac{1}{2}\, 4^m \Bigg(&\sum_{p\in V_m^{(1)}} \sum_{q\in V_m^{(1)}:q\sim_m p} [u(p)-u(q)]^2 \\
&+ \sum_{p\in V_m^{(2)}} \sum_{q\in V_m^{(2)}:q\sim_m p} [u(p)-u(q)]^2 \\
&+ \sum_{p\in V_m^{(3)}} \sum_{q\in V_m^{(3)}:q\sim_m p} [u(p)-u(q)]^2 \Bigg) \\
= \mathcal{E}_m^{(1)}&[u_{|V_*^{(1)}}] + \mathcal{E}_m^{(2)}[u_{|V_*^{(2)}}] + \mathcal{E}_m^{(3)}[u_{|V_*^{(3)}}].
\end{aligned}
$$

By passing to the limit as $m\to\infty$, it follows $u_{|K_i} \in D(\mathcal{E}^{(i)})\ \ (i=1,2,3)$. By a similar argument it also follows that $u_{|K_i} \in D(\mathcal{E}^{(i)})\ \ (i=4,5,6)$. From the definition of $D(\tilde{\mathcal{E}}_F)$ it follows that (4.7) holds for any $u\in D(\widetilde{\mathcal{E}}_F)$. Finally, $u\in\mathcal{C}(F)$ follows from Remark 3.1 ∎

Remark. From the proof of Theorem 4.6 it follows that for a function u to be in $D(\widetilde{\mathcal{E}}_F)$ it is sufficient that $u\in C(F)$ and $u_{|K_i}\in D(\mathcal{E}^{(i)})\ \ (i=1,2,3)$ (or, equivalently, $i=4,5,6$). In particular, from this follows that no matching conditions at the junction points are needed.

Corollary 4.7. *$\big(\widetilde{\mathcal{E}}_F, \mathcal{D}(\widetilde{\mathcal{E}}_F)\big)$ is a strongly local, closed, regular Dirichlet form on $L^2(F, \mu_F)$, where μ_F is defined by* (3.1).

Proof. The proof follows from Theorem 4.6 and the corresponding properties of $\mathcal{E}^{(i)}$ on K_i ∎

Corollary 4.8. *$\mathcal{D}(\widetilde{\mathcal{E}}_F)$ is continuously embedded into $\mathcal{C}^{0,\beta}(F)$, $\beta = \frac{D_f}{2}$.*

Proof. The thesis follows from Theorem 4.6 and Proposition 2.2 ∎

4.3 Equivalence of both approaches. In this subsection, we show that the form $\widetilde{\mathcal{E}}_F$ coincides with the form $\mathcal{E}_F$. In the following we denote the restriction of $\mathcal{L}_F[u]$ to K_i $(i = 1, \ldots, 6)$ by $\mathcal{L}_F[u]_{|K_i}(\cdot) = \mathcal{L}_F[u](\cdot \cap K_i)$.

We preliminary prove the following

Lemma 4.9. *For any $u \in \mathcal{D}_F$ we have $u_{|K_i} \in \mathcal{D}(\mathcal{E}^{(i)})$,*

$$\int_{K_i} d\mathcal{L}_F[u] = \mathcal{E}^{(i)}[u_{|K_i}] \qquad (i = 1, \ldots, 6) \tag{4.8}$$

and $\mathcal{L}_{K_i}[u] = \mathcal{L}_F[u]_{|K_i}$ $(i = 1, \ldots, 6)$.

Proof. Without loss of generality we show the assertion for $i = 1$ only. For this, consider $\mathcal{L}_F[u]_{|K_1}$ which by Definition 4.1 is given by $\mathcal{L}_K[u \circ \varphi_1^{-1}]$. We recall that, for $u \circ \varphi_1^{-1} \in \mathcal{D}(\mathcal{E})$, $\mathcal{L}_K[u \circ \varphi_1^{-1}]$ is the weak limit of the sequence $(\mathcal{L}_K^{(n)}[u \circ \varphi_1^{-1}])$ defined in (2.7). Hence, the left-hand side of (4.8) can be written as

$$\begin{aligned}
\int_{K_1} d\mathcal{L}_F[u] &= \int_K d\mathcal{L}_K[u \circ \varphi_1^{-1}] \\
&= \lim_{n\to\infty} \int_{V_n} d\mathcal{L}_K^{(n)}[u \circ \varphi_1^{-1}] \\
&= \frac{1}{2} \lim_{n\to\infty} \sum_{p \in V_n} \sum_{q \in V_n : q \sim_n p} \frac{\big[u(\varphi_1^{-1}(p)) - u(\varphi_1^{-1}(q))\big]^2}{|p-q|^{2\delta}} \\
&= \frac{1}{2} \lim_{n\to\infty} \sum_{p' \in V_n^{(1)}} \sum_{q' \in V_n : q' \sim_n p'} \frac{[u(p') - u(q')]^2}{|p'-q'|^{2\delta}}
\end{aligned}$$

where the last equality follows from the fact that $\varphi_1^{-1} : K \to K_1$ preserves n-neighborhood (see Lemma 3.2). From the finiteness of the last limit it follows that $u_{|K_1} \in \mathcal{D}(\mathcal{E}^{(1)})$ and that

$$\frac{1}{2} \lim_{n\to\infty} \sum_{p' \in V_n^{(1)}} \sum_{q' \in V_n : q' \sim_n p'} \frac{[u(p') - u(q')]^2}{|p'-q'|^{2\delta}} = \mathcal{E}^{(1)}[u_{|K_1}].$$

Thus the statement is proved ∎

Theorem 4.10. *A function $u : F \to \mathbb{R}$ belongs to $\mathcal{D}_F$ if and only if it belongs to $\mathcal{D}(\widetilde{\mathcal{E}}_F)$. In this case,*

$$\mathcal{E}_F[u] = \widetilde{\mathcal{E}}_F[u]. \tag{4.9}$$

Proof. Let u be in $\mathcal{D}_F$. We prove that u belongs to $\mathcal{D}(\widetilde{\mathcal{E}}_F)$ and that (4.9) holds. Indeed, every $u \in \mathcal{D}_F$ is continuous on F (see Remark 4.2). From Lemma 4.9 and Theorem 4.6 it follows that $u \in \mathcal{D}(\widetilde{\mathcal{E}}_F)$. We note that $\mathcal{E}_F$ can be written as

$$\mathcal{E}_F[u] = \int_{K_1} d\mathcal{L}_F[u] + \int_{K_2} d\mathcal{L}_F[u] + \int_{K_3} d\mathcal{L}_F[u] \qquad (u \in \mathcal{D}_F)$$

(see (4.5)). From Theorem 4.6 we have

$$\widetilde{\mathcal{E}}_F[u] = \mathcal{E}^{(1)}[u_{|K_1}] + \mathcal{E}^{(2)}[u_{|K_2}] + \mathcal{E}^{(3)}[u_{|K_3}] \qquad (u \in \mathcal{D}(\widetilde{\mathcal{E}}_F)).$$

This together with Lemma 4.9 yields (4.9) for any $u \in \mathcal{D}_F$.

On the other hand, if a function u is in $\mathcal{D}(\widetilde{\mathcal{E}}_F)$, we obtain from Theorem 4.6 that $u_{|K_i} \in \mathcal{D}(\mathcal{E}^{(i)})$ $(i = 1, \ldots, 6)$. Proceeding as in the proof of Lemma 4.9, it follows that $u \circ \varphi_i^{-1} \in \mathcal{D}(\mathcal{E})$, hence $u \in \mathcal{D}_F$ and (4.9) holds ∎

From Theorem 4.10 and Corollary 4.7 we obtain

Corollary 4.11. *$(\mathcal{E}_F, \mathcal{D}_F)$ is a strongly local, closed, regular Dirichlet form on $L^2(F, \mu_F)$.*

Corollary 4.12.

i) *$\mathcal{D}_F$ is a Hilbert space with scalar product associated to the norm $\|u\|_{\mathcal{D}_F} = \left(\|u\|^2_{L^2(F,\mu_F)} + \mathcal{E}_F[u]\right)^{1/2}$.*

ii) *$\mathcal{D}_F$ is continuously embedded into $\mathcal{C}^{0,\beta}(F)$ with $\beta = \frac{D_f}{2}$.*

Because of the equivalence of both energy forms, from now on we use only the notation $(\mathcal{E}_F, \mathcal{D}_F)$.

Remark 4.13. We point out that the definition of energy forms in terms of a local energy measure, i.e. a Lagrangian, can be adopted to define energy forms on a wider range of more general fractals with a smaller (or, even empty) symmetry group; of course, in the case of the von Koch snowflake, the above equivalence is due to the simple geometry – i.e. the high symmetry – of the von Koch snowflake.

5. The Laplacian on the snowflake

We now define the Laplace operator on F. As $(\mathcal{E}_F, \mathcal{D}_F)$ is a strongly local, closed, regular Dirichlet form on $L^2(F, \mu_F)$, there exists (see [11: Chapter 6, Theorem 2.1]) a unique self-adjoint, non-positive operator Δ_F on $L^2(F, \mu_F)$ – with domain $\mathcal{D}(\Delta_F) \subseteq \mathcal{D}_F$, dense in $L^2(F, \mu_F)$ – such that

$$\mathcal{E}_F(u,v) = -\int_F (\Delta_F u) v \, d\mu_F \qquad \big(u \in \mathcal{D}(\Delta_F), v \in \mathcal{D}_F\big).$$

In a similar way, on each of the Koch curves $K_1, \ldots, K_6$ we define Laplacians Δ_{K_i} as unique self-adjoint, non-positive operators on $L^2(K_i, \mu_i)$ – with domains $\mathcal{D}(\Delta_{K_i}) \subseteq \mathcal{D}(\mathcal{E}^{(i)})$, dense in $L^2(K_i, \mu_i)$ – such that, for any $i = 1, \ldots, 6$,

$$\mathcal{E}^{(i)}(u,v) = -\int_{K_i} (\Delta_{K_i} u) v d\mu_i \qquad \big(u \in \mathcal{D}(\Delta_{K_i}), v \in \mathcal{D}(\mathcal{E}^{(i)})\big).$$

These are the "field operators" with homogeneous "Neumann" boundary conditions [19].

We now define the local operator $\Delta_{K_i, loc}$ on $\overset{\circ}{K}_i$ $(i = 1, ..., 6)$. Following [19], we define $\mathcal{D}(\mathcal{E}^{(i)})_{loc}$ as the space of all μ_i-measurable functions $u : K_i \to \mathbb{R}$ such that, for any open relatively compact subset $U \subset\subset \overset{\circ}{K}_i$, there exists a function $w \in \mathcal{D}(\mathcal{E}^{(i)})$ such that $u = w$ μ_i-a.e. on U. Given $u \in \mathcal{D}(\mathcal{E}^{(i)})_{loc}$, the measure $\mathcal{L}_{K_i}[u]$ is well defined on $\overset{\circ}{K}_i$ by putting $\mathcal{L}_{K_i}[u]_{|U} = \mathcal{L}_{K_i}[w]_{|U}$ for arbitrary U and w as above. The measure $\mathcal{L}_{K_i}(u,v)$ on $\overset{\circ}{K}_i$ for $u, v \in \mathcal{D}(\mathcal{E}^{(i)})_{loc}$ is given by polarization. We recall that, from Lemma 4.9, for every $u \in \mathcal{D}(\mathcal{E}^{(i)})_{loc}$ we have $\mathcal{L}_{K_i}[u] = (\mathcal{L}_F)_{|K_i}[u]$ on $\overset{\circ}{K}_i$ $(i = 1, \ldots, 6)$.

We now define the "local" Laplacian $\Delta_{K_i, loc}$ $(i = 1, \ldots, 6)$.

Definition 5.1. Fix $i \in \{1, \ldots, 6\}$. For $f \in L^2(K_i, \mu_i)$, we say that $u \in L^2(K_i, \mu_i) \cap \mathcal{D}(\mathcal{E}^{(i)})_{loc}$ is a *local weak solution* of the (formal) equation

$$-\Delta_{K_i} u = f \qquad \text{in } \overset{\circ}{K}_i \tag{5.1}$$

if

$$\int_{\overset{\circ}{K}_i} d\mathcal{L}_{K_i}(u,v) = \int_{\overset{\circ}{K}_i} f v \, d\mu_i \qquad \forall\, v \in \mathcal{D}(\mathcal{E}^{(i)}) \cap \mathcal{C}_0(\overset{\circ}{K}_i). \tag{5.2}$$

Given $f \in L^2(K_i, \mu_i)$, we denote the set of all weak solutions u of equation (5.1) by

$$R_0[\Delta_{K_i}]_{loc}(f) = \Big\{ u \in L^2(K_i, \mu_i) \cap \mathcal{D}(\mathcal{E}^{(i)})_{loc} : \ u \text{ satisfies } (5.2) \Big\}.$$

This defines $R_0[\Delta_{K_i}]_{loc}$ as a multi-valued operator

$$R_0[\Delta_{K_i}]_{loc}:\ L^2(K_i,\mu_i) \to L^2(K_i,\mu_i).$$

We denote by $-\Delta_{K_i,loc}$ the inverse of $R_0[\Delta_{K_i}]_{loc}$ in multi-valued sense. That is, $-\Delta_{K_i,loc}$ is the operator defined on the domain

$$\mathcal{D}(\Delta_{K_i}loc) = R_0[\Delta_{K_i}]_{loc}(L^2(K_i,\mu_i))$$

setting $f \in -\Delta_{K_i,loc}(u)$ for $u \in \mathcal{D}(-\Delta_{K_i,loc})$ if and only if $u \in R_0[\Delta_{K_i}]_{loc}(f)$ for $f \in L^2(K_i,\mu_i)$.

It can be proved (see [19]) that $-\Delta_{K_i,loc}$ is single-valued on its domain, and we write $f = -\Delta_{K_i,loc}(u)$ if $u \in \mathcal{D}(\Delta_{K_i}loc)$, with domain

$$\mathcal{D}(\Delta_{K_i,loc}) = R_0[\Delta_{K_i}]_{loc}(L^2(K_i,\mu_i))$$

being the space of all functions $u \in L^2(K_i,\mu_i) \cap \mathcal{D}(\mathcal{E}^{(i)})_{loc}$ such that u is the weak solution of equation (5.2) for a function $f \in L^2(K_i,\mu_i)$, and then we write $-\Delta_{K_i,loc}u = f$.

In order to describe properties of the free diffusion process associated to $\mathcal{E}_F$, from now on, just to fix ideas, we assume $F = \bigcup_{i=1}^3 K_i$.

Remark 5.2. We note that, as $F \backslash \{x_1, x_3, x_5\} = \bigcup_{i=1}^3 \overset{\circ}{K}_i$, according to Theorems 4.6 and 4.10

$$\begin{aligned}&\mathcal{D}_F \cap \mathcal{C}_0\Big(\bigcup_{i=1}^{3} \overset{\circ}{K}_i\Big)\\ &\qquad = \Big\{ v:\ F \to \mathbb{R} \Big| \, v_{|K_i} \in \mathcal{D}(\mathcal{E}^{(i)}) \cap \mathcal{C}_0(\overset{\circ}{K}_i) \quad (i=1,2,3)\Big\}.\end{aligned}$$

For every function v, which is in $\mathcal{D}(\mathcal{E}^{(i)}) \cap \mathcal{C}_0(\overset{\circ}{K}_i)$ for a fixed i, denote by $\tilde{v}$ the trivial extension

$$\tilde{v} = \begin{cases} v & \text{on } \overset{\circ}{K}_i \\ 0 & \text{elsewhere on } F. \end{cases}$$

Then $\tilde{v} \in \mathcal{D}_F \cap \mathcal{C}_0(\bigcup_{i=1}^3 \overset{\circ}{K}_i)$.

Theorem 5.3. *For any $i \in \{1,\ldots,3\}$ and every $u \in \mathcal{D}(\Delta_F)$, $u|_{K_i} \in \mathcal{D}(\Delta_{K_i,loc})$ and*

$$\Delta_{K_i,loc} u|_{K_i} = \Delta_F u \qquad \textit{on } \overset{\circ}{K}_i.$$

Proof. Without loss of generality we choose $i = 1$. Let $u \in \mathcal{D}(\Delta_F)$. From the definition of $\mathcal{D}(\Delta_F)$,

$$\int_F d\mathcal{L}_F(u,v) = \int_F fv\, d\mu_F \qquad \forall\ v \in \mathcal{D}_F \tag{5.3}$$

for some $f \in L^2(F, \mu_F)$. Now choose $v \in \mathcal{D}(\mathcal{E}^{(1)}) \cap \mathcal{C}_0(\mathring{K}_1)$ and let $\tilde{v}$ be its trivial extension (see Remark 5.2). From (5.3) and taking into account Theorem 4.5, Lemma 4.9 and the strong locality of the form $\mathcal{E}^{(i)}$ we obtain

$$\int_{\mathring{K}_1} d\mathcal{L}_{K_1}(u, v) = \int_{\mathring{K}_1} fv \, d\mu_1 \qquad \forall\, v \in \mathcal{D}(\mathcal{E}^{(1)}) \cap \mathcal{C}_0(\mathring{K}_1).$$

Hence, $u \in \mathcal{D}(\Delta_{K_1,loc})$ and $\Delta_{K_1,loc} u = \Delta_F u$ on $\mathring{K}_1$ ∎

From Theorem 5.3, we conclude immediately:

Corollary 5.4. *For any $u \in \mathcal{D}(\Delta_F)$,*

$$\Delta_F u = \sum_{i=1}^{3} \Delta_{K_i,loc}(u) \mathbf{1}_{\mathring{K}_i} \qquad \textit{on } F \backslash \{x_1, x_3, x_5\}.$$

When $F = \cup_{i=4}^{6} K_i$, a similar procedure can be carried on. Namely, for $u \in \mathcal{D}(\Delta_F)$, one can obtain

$$\Delta_F u = \sum_{i=4}^{6} \Delta_{K_i,loc}(u) \mathbf{1}_{\mathring{K}_i} \qquad \text{on } F \backslash \{x_2, x_4, x_6\}. \tag{5.4}$$

6. Reflected sets and some stochastic aspects

By the general theory of stochastic analysis (see, e.g., [7]) there is a one-to-one correspondence between local regular Dirichlet forms and stochastic processes. This correspondence can be expressed in terms of the corresponding Laplacian (and the associated semigroup) which is the so-called infinitesimal generator of the process. Roughly speaking, regularity of the form implies existence of an associated strong Markovian process, while locality of the form ensures continuity of the paths for quasi every starting point. From Corollary 4.11 we obtain that there exist a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a stochastic process $(X_t)_{t \geq 0}$ on it with state space F, equipped with the σ-algebra of Borel subsets of F, which is associated with $(\mathcal{E}_F, \mathcal{D}_F)$. Moreover, except for some exceptional set of starting points $x \in F$, $(X_t)_{t \geq 0}$ has continuous paths. Because F is a closed curve, we have no boundary conditions for $(X_t)_{t \geq 0}$, i.e. $(X_t)_{t \geq 0}$ moves free on F.

Now we want to give a stochastic interpretation of Theorem 4.6. For $i = 1, \ldots, 6$ and for quasi every starting point $x \in K_i$, a continuous stochastic process $(X_t^{(i)})_{t \geq 0}$ – associated with the Dirichlet form $(\mathcal{E}^{(i)}, \mathcal{D}(\mathcal{E}^{(i)}))$ – and state space K_i is given. From Theorem 5.3 it follows that these processes can be defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Note that the process $(X_t^{(i)})_{t\geq 0}$ is reflected at the endpoints of K_i $(i = 1, \ldots, 6)$ (see, e.g., [7]). As the K_i $(i = 1, \ldots, 6)$ are nested fractals, these diffusion processes $(X_t^{(i)})_{t\geq 0}$ can be obtained as limits of sequences of suitable renormalized discrete symmetric random walks $(X_m^{(i)})_{m\in\mathbb{N}}$ on $V_m^{(i)}$ (see [17]). Of course, such a limit process $(X_t^{(i)})_{t\geq 0}$ is still symmetric and strong Markovian, i.e. in particular, it satisfies a strong reflection principle on $\overset{\circ}{K}_i$ $(i = 1, \ldots, 6)$. Note that "symmetric" means symmetric with respect to reflections on F introduced below.

We now prove that $(X_t)_{t\geq 0}$ satisfies a strong reflection principle on all of F. In order to do this, we have to introduce the notion of reflection of any Borel subset A of the fractal F at a point $p \in F$.

For any $m \in \mathbb{N}$, we define a distance d_m on $\mathcal{V}_m$ in the following way: we connect m-neighbors each other by a line segment, i.e. we obtain a closed polygonal curve Q_m. For two points $p, q \in \mathcal{V}_m$, we define their distance $d_m(p, q)$ as the curve length of the shortest path from p to q along Q_m.

Fix $m \in \mathbb{N}$ and a point $p \in \mathcal{V}_m$. We define the reflection of a point $q \in \mathcal{V}_m \setminus \{p\}$ in p, denoted by $R_p^m(q)$, as the point $r \in \mathcal{V}_m$ which is different from q such that $d_m(r, p) = d_m(q, p)$ if it exists or $R_p^m(q) := q$ otherwise. We set $R_p^m(p) = p$.

For a set $A \subseteq \mathcal{V}_m$, the reflected set is given by

$$R_p^m(A) = \bigcup_{q\in A} R_p^m(q).$$

For a set $A \subseteq F$ and a point $p \in \mathcal{V}_*$, we define the reflection of A in p by

$$R_p(A) = \bigcup_{m\geq m_0} R_p^m(A \cap \mathcal{V}_m)$$

where m_0 is the smallest natural number such that $p \in \mathcal{V}_{m_0}$. By the density of $\mathcal{V}_*$ in F, this defines for any point $p \in F$ and any set $A \subseteq F$ a reflected set $R_p(A) \subseteq F$. Note that in general a set $A \subseteq F$ and its reflection $R_p(A)$ have not the same shape as in the Euclidean case.

Now we are able to formulate the strong reflection principle of a process $(X_t^{(1)})_{t\geq 0}$ with starting point $x \in K_1$. We refer for the definitions of a Markov process, a random (or, stopping) time, a filtration, and the conditioned expectation with respect to a σ-algebra etc. to [10].

For any $c \in K_1$, define the random time

$$\tau_c = \inf\left\{t > 0 :\, X_t^{(1)} = c\right\}$$

and denote the sub-σ-algebra of $\mathcal{F}$ of the τ_c-past by $\mathcal{G}_{\tau_c}$. Then, for any Borel set $A \subseteq K_1$ with $R_c(A) \subseteq K_1$,

$$\mathbf{P}\big(X^{(1)}_{\tau_c+s} \in A|\mathcal{G}_{\tau_c}\big) = \mathbf{P}(X^{(1)}_{\tau_c+s} \in R_c(A)|\mathcal{G}_{\tau_c}) \qquad (s>0), \tag{6.1}$$

i.e. from time τ_c on, the process "behaves symmetric" with respect to c. This follows from the theory of symmetric strong Markovian processes on nested fractals (see [14, 17]).

Now we come back to the interpretation of Theorem 4.6, i.e. we show how the free diffusion on F "splits" into three reflected processes on K_1, K_2 and K_3. Note that the infinitesimal generator Δ_F of the stochastic process $(X_t)_{t\geq 0}$ uniquely determines its so-called one-step-transition probabilities because, for any Borel set $B \subseteq F$, any point $x \in F$ and any time $t > 0$,

$$\mathbf{P}\big(X_t \in B|X_0 = x\big) = \mathbf{E}\big(\mathbf{1}_B(X_t)|X_0 = x\big) = (e^{\Delta_F t}\mathbf{1}_B)(x).$$

By the Chapman-Kolmogorov equality we obtain that the infinitesimal generator determines uniquely also finite-dimensional distributions of the associated Markovian process. Therefore, Corollary 5.4 yields

$$\begin{aligned} &X_t \mathbf{1}_{\{X_t \in F\setminus\{x_1,x_3,x_5\}\}} \\ &\quad \stackrel{d}{=} X^{(1)}_t \mathbf{1}_{\{X_t \in \mathring{K}_1\}} + X^{(2)}_t \mathbf{1}_{\{X_t \in \mathring{K}_2\}} + X^{(3)}_t \mathbf{1}_{\{X_t \in \mathring{K}_3\}} \end{aligned} \tag{6.2}$$

where the processes $(X^{(1)}_t)_{t\geq 0}, (X^{(2)}_t)_{t\geq 0}$ and $(X^{(3)}_t)_{t\geq 0}$ are chosen to be independent. By the same arguments (cf. (5.4)) we obtain

$$\begin{aligned} &X_t \mathbf{1}_{\{X_t \in F\setminus\{x_2,x_4,x_6\}\}} \\ &\quad \stackrel{d}{=} X^{(4)}_t \mathbf{1}_{\{X_t \in \mathring{K}_4\}} + X^{(5)}_t \mathbf{1}_{\{X_t \in \mathring{K}_5\}} + X^{(6)}_t \mathbf{1}_{\{X_t \in \mathring{K}_6\}}. \end{aligned} \tag{6.3}$$

From (6.2) and (6.1) we know that the process $(X_t)_{t\geq 0}$ satisfies a strong reflection principle in every point $c \in F\setminus\{x_1, x_3, x_5\}$; from (6.3) we conclude that the same holds also in the points x_1, x_3, x_5, because they can be regarded as inner points of K_4, K_5 and K_6, respectively. So we observe a strong reflection principle on all of F, i.e. for any $c \in F$ and any set $A \subseteq F$ we have

$$\mathbf{P}\big(X_{\tau_c+s} \in A|\mathcal{G}_{\tau_c}\big) = \mathbf{P}\big(X_{\tau_c+s} \in R_c(A)|\mathcal{G}_{\tau_c}\big) \qquad (s>0)$$

where τ_c is now given by

$$\tau_c = \inf\big\{t > 0 : X_t = c\big\}.$$

Therefore, $(X_t)_{t\geq 0}$ is a symmetric Markovian process on F, which corresponds stochastically to three reflected diffusions on the von Koch curves K_1, K_2 and

K_3. In other words, when the process arrives in say x_1 from the left, with probability $\frac{1}{2}$, it goes back and still behaves like $X_t^{(3)}$ $(t \geq 0)$ and, with probability $\frac{1}{2}$, it passes through x_1 and behaves in "the next small future" like $X_t^{(1)}$ $(t \geq 0)$. This means that, for the free diffusion on F, the (former junction) points x_1, x_3, x_5 are no longer exceptional points.

Obviously, $(X_t)_{t \geq 0}$ can also be obtained as limit of renormalized symmetric random walks on the approximating sets $\mathcal{V}_m$.

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