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Weighted Sharing of Three Values

Indrajit Lahiri

Abstract. Using the notion of weighted sharing of values we prove some uniqueness theorems for meromorphic functions which improve some earlier results.

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1. Introduction

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. For $b \in \mathbb{C} \cup \{\infty\}$ we say that f and g *share the value* b CM (*counting multiplicities*) if f and g have the same b-points with the same multiplicities. If the multiplicities are ignored, we say that f and g *share the value* b IM (*ignoring multiplicities*). Though for the standard notations and definitions of Nevanlinna theory we refer [2], we now explain some notations and definitions which will be needed in the sequel.

Definition 1. [3, 13] Let s be a positive integer.

(i) We denote by $\overline{N}(r,a;f \mid\geq s)$ the counting function of those a-points of f whose multiplicities are greater than or equal to s, where each a-point is counted only once. The counting function $\overline{N}(r,a;f \mid\leq s)$ is defined likewise.

(ii) We denote by $N_s(r,a;f)$ the counting function of a-points of f, where an a-point with multiplicity m is counted m times if $m \leq s$ and s times if $m > s$. We put $N_\infty(r,a;f) \equiv N(r,a;f)$.

(iii) We denote by $N(r,a;f \mid\leq s)$ the counting function of those a-points of f whose multiplicities are less than or equal to s, where an a-point is counted according to its multiplicity.

Let f and g share a value a IM. Let z be an a-point of f and g with

Department of Mathematics, University of Kalyani, West Bengal 741235, India; ilahiri@hotmail.com, ilahiri@vsnl.com

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multiplicities $p_f(z)$ and $p_g(z)$ respectively. We put

$$\overline{\nu}_f(z) = \begin{cases} 1 & \text{if } \ p_f(z) > p_g(z) \\ 0 & \text{if } \ p_f(z) \le p_g(z) \end{cases} \quad \text{and} \quad \overline{\mu}_f(z) = \begin{cases} 1 & \text{if } \ p_f(z) < p_g(z) \\ 0 & \text{if } \ p_f(z) \ge p_g(z). \end{cases}$$

Let

$$\overline{n}(r,a;f>g) = \sum_{|z|\le r} \overline{\nu}_f(z) \quad \text{and} \quad \overline{n}(r,a;f<g) = \sum_{|z|\le r} \overline{\mu}_f(z).$$

Then, we denote by $\overline{N}(r,a;f>g)$ and $\overline{N}(r,a;f<g)$ the *integrated counting functions* obtained from $\overline{n}(r,a;f>g)$ and $\overline{n}(r,a;f<g)$ respectively. Finally we put

$$\overline{N}_*(r,a;f,g) = \overline{N}(r,a;f>g) + \overline{N}(r,a;f<g).$$

Again, for $a \in \mathbb{C} \cup \{\infty\}$ we use the following notations:

$$\begin{aligned} \delta_s(a;f) &= 1 - \limsup_{r\to\infty} \frac{N_s(r,a;f)}{T(r,f)} \\ \delta_{s)}(a;f) &= 1 - \limsup_{r\to\infty} \frac{N(r,a;f\mid\le s)}{T(r,f)} \\ \Theta_{s)}(a;f) &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a;f\mid\le s)}{T(r,f)}, \end{aligned}$$

where s is a positive integer.

Definition 2. For $a, b \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r,a;f \mid g \ne b)$ the counting function of those a-points of f which are not the b-points of g, where an a-point is counted according to its multiplicity. The reduced counting function $\overline{N}(r,a;f \mid g \ne b)$ is defined analogously.

H. Ueda [9] proved the following result.

Theorem A. *Let f and g be two distinct nonconstant entire functions sharing 0, 1 CM and let $a(\ne 0,1)$ be a finite complex number. If a is lacunary for f then $1-a$ is lacunary for g and $(f-a)(g+a-1) \equiv a(1-a)$.*

Improving Theorem A, H. X. Yi [11] proved the following result.

Theorem B. *Let f and g be two distinct nonconstant entire functions sharing 0, 1 CM and let $a(\ne 0,1)$ be a finite complex number. If $\delta(a;f) > \frac{1}{3}$ then a and $1-a$ are Picard exceptional values of f and g respectively and $(f-a)(g+a-1) \equiv a(1-a)$.*

S. Z. Ye [10] extended Theorem B to meromorphic functions and proved the following result.

Theorem C. *Let f and g be two distinct nonconstant meromorphic functions such that f and g share* 0, 1, ∞ *CM. Let $a(\neq 0,1)$ be a finite complex number. If $\delta(a;f)+\delta(\infty;f)>\frac{4}{3}$ then a and $1-a$ are Picard exceptional values of f and g respectively and ∞ is also a Picard exceptional value of both f and g and $(f-a)(g+a-1)\equiv a(1-a)$.*

The following two examples show that in the above theorems the sharing of 0 and 1 cannot be relaxed from CM to IM.

Example 1. [5, 7] Let $f=e^z-1$ and $g=(e^z-1)^2$ and $a=-1$. Then f, g share 0 IM and 1, ∞ CM. Also $N(r,\infty;f)\equiv 0$ and $N(r,a;f)\equiv 0$ but $(f-a)(g+a-1)\not\equiv a(1-a)$.

Example 2. [5, 7] Let $f=2-e^z$, $g=e^z(2-e^z)$ and $a=2$. Then f, g share 1 IM and 0, ∞ CM. Also $N(r,\infty;f)\equiv 0$ and $N(r,a;f)\equiv 0$ but $(f-a)(g+a-1)\not\equiv a(1-a)$.

Motivated by these examples, in [5, 7] the following question is asked:

Is it possible in any way to relax the nature of sharing of values in the theorems stated above?

In [5, 7] this problem is studied using the notion of weighted sharing of values introduced in [3, 4] which measures how close a shared value is to being shared IM or to being shared CM.

Definition 3. Let k be a nonnegative integer or infinity . For $a\in C\cup\{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m\leq k$ and $k+1$ times if $m>k$. If $E_k(a;f)=E_k(a;g)$, we say that f, g share the value a with weight k.

The definition implies that if f , g share a value a with weight k then z_o is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_o is an a-point of f with multiplicity $m(>k)$ if and only if it is an a-point of g with multiplicity $n(>k)$ where m is not necessarily equal to n.

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k) then f, g share (a,p) for all integers p, $0\leq p<k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

In [5] the following theorem is proved.

Theorem D. *Let f and g be two distinct meromorphic functions sharing $(0,1)$, $(1,\infty)$, (∞,∞). If $a(\neq 0,1,\infty)$ is a complex number such that $3\delta_{2)}(a;f)+2\delta_{1)}(\infty;f)>3$ then a and $1-a$ are Picard exceptional values of f and g and ∞ is also a Picard exceptional value of both f and g and $(f-a)(g+a-1)\equiv a(1-a)$.*

Also in [7], the following two theorems are proved.

Theorem E. *Let f and g be two distinct meromorphic functions sharing $(0,1)$, $(1,\infty)$, $(\infty,11)$. If $a(\neq 0,1,\infty)$ is a complex number such that $3\delta_2(a;f)+3\delta(\infty;f)>4$ then a and $1-a$ are Picard exceptional values of f and g respectively and also ∞ is a Picard exceptional value of both f and g and $(f-a)(g+a-1)\equiv a(1-a)$.*

Theorem F. *Let f and g be two distinct meromorphic functions sharing $(0,1)$, $(1,\infty)$, $(\infty,0)$. If $a(\neq 0,1,\infty)$ is a complex number such that $3\delta_2(a;f)+14\ \delta(\infty;f)>15$ then a and $1-a$ are Picard exceptional values of f and g respectively and also ∞ is a Picard exceptional value of both f and g and $(f-a)(g+a-1)\equiv a(1-a)$.*

2. Results

The purpose of the paper is to improve Theorem D, Theorem E and Theorem F either by reducing the weight of sharing the values or by relaxing the condition on deficiencies. We now state the main results of the paper.

Theorem 1. *Let f and g be two distinct nonconstant meromorphic functions sharing $(0,1)$, $(1,m)$, (∞,k), where $(m-1)(mk-1)>(1+m)^2$. If $a(\neq 0,1,\infty)$ is a complex number such that*

$$3\delta_{2)}(a;f)+2\delta_{1)}(\infty;f)>3,$$

then a and $1-a$ are Picard exceptional values of f and g respectively and also ∞ is a Picard exceptional value of both f and g and $(f-a)(g+a-1)\equiv a(1-a)$.

This theorem improves Theorem D and Theorem E.

Corollary 1. Theorem 1 holds for the pairs of values $(m,k)=(3,4),(4,3),(2,6),(6,2)$.

Note 1. Considering $f=e^z/(1-e^z)$, $g=1/(2-2e^z)$, $a=-1$ we see that the condition $3\delta_{2)}(a;f)+2\delta_{1)}(\infty;f)>3$ is sharp.

Theorem 2. *Let f, g be two distinct nonconstant meromorphic functions sharing $(0,1)$, $(1,m)$, $(\infty,0)$ where $m\geq 2$. If $a(\neq 0,1,\infty)$ is a complex number such that*

$$3\delta_2(a;f)+\frac{11m+13}{m-1}\Theta(\infty;f)>\frac{12m+12}{m-1}$$

then a and $1-a$ are Picard exceptional values of f and g respectively and also ∞ is a Picard exceptional value of f and g and $(f-a)(g+a-1)\equiv a(1-a)$.

Corollary 2. Theorem F holds if $3\delta_2(a;f)+11\Theta(\infty;f)>12$.

Theorem 3. *Theorem 2 holds if*

$$3\delta_{2)}(a;f)+\frac{76m+52}{m-1}\Theta(\infty;f)>\frac{77m+51}{m-1}.$$

Corollary 3. Theorem F holds if $3\delta_{2)}(a;f)+76\Theta(\infty;f)>77$.

Note 2. Considering $f=e^z(1-e^z)$, $g=e^{-z}(1-e^{-z})$ and $a=\frac{1}{4}$ we see that the conditions

(i) $3\delta_{2)}(a;f)+2\delta_{1)}(\infty;f)>3$ of Theorem 1

(ii) $3\delta_{2)}(a;f)+76\Theta(\infty;f)>77$ of Corollary 3

cannot be replaced by the following weaker ones respectively:

(I) $3\Theta_{2)}(a;f)+2\delta_{1)}(\infty;f)>3$

(II) $3\Theta_{2)}(a;f)+76\Theta(\infty;f)>77$.

Throughout the paper we denote by f, g two nonconstant meromorphic functions in $\mathbb{C}$.

3. Lemmas

In this section we present some lemmas which are needed to prove the main results.

Lemma 1. [1, 3] *If f, g share $(0,0)$, $(1,0)$, $(\infty,0)$. Then*

(i) $T(r,g)\le 3T(r,f)+S(r,g)$

(ii) $T(r,f)\le 3T(r,g)+S(r,f)$.

This lemma shows that $S(r,f)=S(r,g)$, and we denote them by $S(r)$.

Lemma 2. [6] *Let f, g share $(0,1)$, $(1,m)$, (∞,k) and $f\not\equiv g$, where $(m-1)(mk-1)>(1+m)^2$. Then*

(i) $\overline{N}(r,0;f\mid\ge 2)+\overline{N}(r,1;f\mid\ge 2)+\overline{N}(r,\infty;f\mid\ge 2)=S(r)$

(ii) $\overline{N}(r,0;g\mid\ge 2)+\overline{N}(r,1;g\mid\ge 2)+\overline{N}(r,\infty;g\mid\ge 2)=S(r)$.

Lemma 3. *Let f, g share $(0,1)$, $(1,m)$, $(\infty,0)$ and $f\not\equiv g$, where $m\ge 2$. Then*

(i) $\overline{N}(r,0;f\mid\ge 2)\le\frac{m+1}{m-1}\overline{N}_*(r,\infty;f,g)+S(r)$

(ii) $\overline{N}(r,1;f\mid\ge m+1)\le\frac{2}{m-1}\overline{N}_*(r,\infty;f,g)+S(r)$.

Proof. Let $\phi_1=\frac{f'}{f-1}-\frac{g'}{g-1}$ and $\phi_2=\frac{f'}{f}-\frac{g'}{g}$. We suppose that $\overline{N}(r,a;f)\ne S(r)$ for $a=0,1$ because otherwise the lemma is trivial. Since $f\not\equiv g$, it follows that $\phi_i\not\equiv 0$ for $i=1,2$. Now

$$\begin{aligned}\overline{N}(r,0;f\mid\geq 2) &\leq N(r,0;\phi_1)\\ &\leq T(r,\phi_1)+O(1)\\ &= N(r,\infty;\phi_1)+S(r)\\ &\leq \overline{N}(r,1;f\mid\geq m+1)+\overline{N}_*(r,\infty;f,g)+S(r),\end{aligned}\tag{1}$$

and, analogously,

$$\begin{aligned}m\overline{N}(r,1;f\mid\geq m+1) &\leq N(r,0;\phi_2)\\ &\leq T(r,\phi_2)+O(1)\\ &= N(r,\infty;\phi_2)+S(r)\\ &\leq \overline{N}(r,0;f\mid\geq 2)+\overline{N}_*(r,\infty;f,g)+S(r).\end{aligned}\tag{2}$$

Hence, from (1) and (2) we get (i). Further, from (2) we get

$$\begin{aligned}\overline{N}(r,1;f\mid\geq m+1) &\leq \frac{1}{m}\overline{N}(r,0;f\mid\geq 2)+\frac{1}{m}\overline{N}_*(r,\infty;f,g)+S(r)\\ &\leq \frac{1}{m}\Big(\frac{m+1}{m-1}+1\Big)\,\overline{N}_*(r,\infty;f,g)+S(r)\\ &= \frac{2}{m-1}\overline{N}_*(r,\infty;f,g)+S(r),\end{aligned}$$

which is (ii). This proves the lemma. ∎

Lemma 4. *Let f, g share $(0,0)$, $(1,0)$, $(\infty,0)$ and $f\not\equiv g$. If $\alpha=(f-1)/(g-1)$ and $h=g/f$, then*

(i) $\overline{N}(r,0;\alpha)=\overline{N}(r,\infty;f<g)+\overline{N}(r,1;f>g)$

(ii) $\overline{N}(r,\infty;\alpha)=\overline{N}(r,\infty;f>g)+\overline{N}(r,1;f<g)$

(iii) $\overline{N}(r,0;h)=\overline{N}(r,0;f<g)+\overline{N}(r,\infty;f>g)$

(iv) $\overline{N}(r,\infty;h)=\overline{N}(r,0;f>g)+\overline{N}(r,\infty;f<g)$.

The proof is straightforward and omitted.

Lemma 5. *Let f, g share $(0,1)$, $(1,m)$, $(\infty;0)$ and α, h be defined as in Lemma 4, where $m\geq 2$. If $a\alpha h+b\alpha\equiv c$ for nonzero constants a, b, c, then*

$$T(r,f)\leq\frac{4(m+1)}{m-1}\overline{N}(r,\infty;f<g)+S(r).$$

Proof. If one of α and αh is constant then from the given condition we see that the other is constant and so $f=\frac{1-\alpha}{1-\alpha h}$ becomes a constant, which is a contradiction. So α and αh are nonconstant.

Let z_0 be a pole of f and g with multiplicities p and q respectively. If $p>q$ then z_0 is a zero of $\frac{1}{\alpha}$ and h. So from $ah+b\equiv\frac{c}{\alpha}$, it follows that $b=0$, which is a contradiction. So $N(r,\infty;f>g)\equiv 0$.

Let z_0 be a zero of f and g with multiplicities p and q respectively. If $p > q$ then z_0 is a pole of h and z_0 is a regular point of α with $\alpha(z_0) = 1$. Since $ah \equiv \frac{c}{\alpha} - b$, this implies a contradiction. So $N(r, 0; f > g) \equiv 0$.

Let z_0 be an 1-point of f and g with multiplicities p and q respectively. If $p > q$ then z_0 is a zero of α and z_0 is a regular point of h with $h(z_0) = 1$. Since $a\alpha h + b\alpha = c$, it follows that $c = 0$, which is a contradiction. So $N(r, 1; f > g) \equiv 0$.

Since $ah - \frac{c}{\alpha} \equiv -b$, it follows from the first and second fundamental theorems and Lemma 4

$$\begin{aligned} T(r,\alpha) &\leq \overline{N}(r,\infty;\alpha) + \overline{N}(r,0;\alpha) + \overline{N}(r,0;h) + S(r,\alpha) \\ &= \overline{N}(r,1;f<g) + \overline{N}(r,\infty;f<g) + \overline{N}(r,0;f<g) + S(r,\alpha). \end{aligned}$$

Again since $ah + b \equiv \frac{c}{\alpha}$, it follows from Lemma 4 that

$$\begin{aligned} \overline{N}(r,-\frac{b}{a};h) &= \overline{N}(r,\infty;\alpha) = \overline{N}(r,1;f<g) \\ \overline{N}(r,0;h) &= \overline{N}(r,0;f<g) \\ \overline{N}(r,\infty;h) &= \overline{N}(r,0;\alpha) = \overline{N}(r,\infty;f<g). \end{aligned}$$

By the second fundamental theorem we get

$$\begin{aligned} T(r,h) &\leq \overline{N}(r,\infty;h) + \overline{N}(r,0;h) + \overline{N}(r,-b/a;h) + S(r,h) \\ &= \overline{N}(r,\infty;f<g) + \overline{N}(r,0;f<g) + \overline{N}(r,1;f<g) + S(r,h). \end{aligned}$$

It follows from Lemma 1 and the first fundamental theorem $S(r,\alpha) = S(r)$ and $S(r,h) = S(r)$. Since

$$\frac{1}{f} = 1 - \frac{h-1}{\frac{1}{\alpha} - 1},$$

it follows from the first fundamental theorem that

$$\begin{aligned} T(r,f) &\leq T(r,\alpha) + T(r,h) + O(1) \\ &\leq 2\overline{N}(r,\infty;f<g) + 2\overline{N}(r,0;f<g) + 2\overline{N}(r,1;f<g) + S(r). \end{aligned} \tag{3}$$

Since f, g share $(0,1)$, $(1,m)$, $(\infty,0)$, it follows from Lemma 3 that

$$\begin{aligned} \overline{N}(r,0;f<g) &\leq \overline{N}(r,0;f\mid\geq 2) \leq \frac{m+1}{m-1}\,\overline{N}(r,\infty;f<g) + S(r) \\ \overline{N}(r,1;f<g) &\leq \overline{N}(r,1;f\mid\geq m+1) \leq \frac{2}{m-1}\,\overline{N}(r,\infty;f<g) + S(r). \end{aligned}$$

So from (3) we get

$$T(r,f) \leq \frac{4(m+1)}{m-1}\overline{N}(r,\infty;f<g) + S(r).$$

This proves the lemma. ∎

Lemma 6. [6] *For a meromorphic function f it holds*

$$\overline{N}(r,0;f') \le 2\overline{N}(r,0;f) + \overline{N}(r,\infty;f) + S(r,f).$$

Lemma 7. [5] *If f, g share $(0,1)$, $(1,\infty)$, (∞,∞) and $f \not\equiv g$, then for any $a(\neq 0,1,\infty)$*

(i) $\overline{N}(r,a;f\mid\ge 3) = S(r)$

(ii) $\overline{N}(r,a;g\mid\ge 3) = S(r)$.

Lemma 8. *Let f, g share $(0,1)$, $(1,m)$, $(\infty,0)$ and $f \not\equiv g$, where $m \ge 2$. Then for any $a(\neq 0,1,\infty)$*

(i) $\overline{N}(r,a;f\mid\ge 3) \le \frac{13m+7}{m-1}\,\overline{N}_*(r,\infty;f,g) + S(r)$

(ii) $\overline{N}(r,a;g\mid\ge 3) \le \frac{13m+7}{m-1}\,\overline{N}_*(r,\infty;f,g) + S(r)$.

Proof. Let α and h be defined as in Lemma 4. If α or h is constant then clearly f, g share $(0,\infty)$, $(1,\infty)$, (∞,∞) and so the result follows from Lemma 7. We now suppose that α and h are nonconstant.

Since $f = \frac{1-\alpha}{1-\alpha h}$, it follows that

$$f - a = \frac{(1-a)+\alpha(ah-1)}{1-\alpha h}.$$

Let z_0 be a zero of $f-a$ with multiplicity ≥ 3. Then z_0 is a zero of

$$\frac{d}{dz}\Big[(1-a)+\alpha(ah-1)\Big] = \alpha'\Big[ah - 1 + \frac{a\alpha h'}{\alpha'}\Big]$$

with multiplicity ≥ 2. So z_0 is a zero of α' or z_0 is a zero of

$$\frac{d}{dz}\Big[ah - 1 + \frac{a\alpha h'}{\alpha'}\Big] = ah'\Big[2 + \frac{\alpha h''}{\alpha' h'} - \frac{\alpha\alpha''}{(\alpha')^2}\Big].$$

Therefore

$$\begin{aligned}
\overline{N}(r,a;f\mid\ge 3) &\le \overline{N}(r,0;\alpha') + \overline{N}(r,0;h') + T\Big(r, 2 + \frac{\alpha h''}{\alpha' h'} - \frac{\alpha\alpha''}{(\alpha')^2}\Big)\\
&\le \overline{N}(r,0;\alpha') + \overline{N}(r,0;h') + T\Big(r,\frac{h''}{h'}\Big) + 2T\Big(r,\frac{\alpha'}{\alpha}\Big)\\
&\quad + T\Big(r,\frac{\alpha''}{\alpha'}\Big) + O(1)\\
&\le 2\overline{N}(r,0;\alpha') + 2\overline{N}(r,0;h') + 2\overline{N}(r,0;\alpha) + 3\overline{N}(r,\infty;\alpha)\\
&\quad + \overline{N}(r,\infty;h) + S(r).
\end{aligned}$$

So by the Lemmas 3, 4 and 6 we get

$$\begin{aligned}\overline{N}(r,a;f\mid\geq 3) &\leq 6\overline{N}(r,0;\alpha)+5\overline{N}(r,\infty;\alpha)+4\overline{N}(r,0;h)\\ &\quad +3\overline{N}(r,\infty;h)+S(r)\\ &\leq \frac{13m+7}{m-1}\,\overline{N}_*(r,\infty;f,g)+S(r),\end{aligned}$$

which is (i). Similarly we can prove statement (ii). This proves the lemma. ∎

Lemma 9. [12] *Let f_1, f_2, f_3 be nonconstant meromorphic functions such that $f_1+f_2+f_3\equiv 1$ and let $g_1=-f_1/f_3$, $g_2=1/f_3$, $g_3=-f_2/f_3$. If f_1, f_2, f_3 are linearly independent, then g_1, g_2, g_3 are linearly independent.*

Lemma 10. [8] *Let f_1, f_2, f_3 be nonconstant meromorphic functions such that $f_1+f_2+f_3\equiv 1$. If f_1, f_2, f_3 are linearly independent, then*

$$T(r,f_1)\leq \sum_{i=1}^{3} N_2(r,0;f_i)+\sum_{i=1}^{3}\overline{N}(r,\infty;f_i)+\sum_{i=1}^{3}S(r,f_i).$$

Lemma 11. *Let f, g share $(0,1)$, $(1,m)$, $(\infty,0)$ and $f\not\equiv g$, where $m\geq 2$. Let*

$$f_1=\frac{(f-a)(1-\alpha h)}{1-a},\quad f_2=\frac{-a\alpha h}{1-a},\quad \text{and } f_3=\frac{\alpha}{1-a},$$

where $a(\neq 0,1,\infty)$ be a complex number and α, h are defined as in Lemma 4. If f_1, f_2, f_3 are linearly independent, then

(i) $\overline{N}(r,0;f)\leq N_2(r,a;f)+\frac{5m+9}{m-1}\overline{N}_*(r,\infty;f,g)+S(r)$

(ii) $\overline{N}(r,1;f)\leq N_2(r,a;f)+\frac{5m+5}{m-1}\overline{N}_*(r,\infty;f,g)+S(r)$.

Proof. Since $(1-a)f_1=(1-\alpha)-a(1-\alpha h)$ and at a pole of f, $\alpha h=\frac{g(f-1)}{f(g-1)}$ has no pole, it follows that

$$\begin{aligned}\overline{N}(r,\infty;f_1) &\leq \overline{N}(r,\infty;f>g)+\overline{N}(r,0;f>g)+\overline{N}(r,1;f<g)\\ \overline{N}(r,\infty;f_2) &\leq \overline{N}(r,0;f>g)+\overline{N}(r,1;f<g)\\ \overline{N}(r,\infty;f_3) &\leq \overline{N}(r,\infty;f>g)+\overline{N}(r,1;f<g).\end{aligned}$$

If α is a constant, $\overline{N}(r,0;f)\equiv 0$ because $f-1\equiv\alpha(g-1)$, $f\not\equiv g$ and f, g share $(0,1)$. So we suppose that α is nonconstant. Since $\sum_{i=1}^{3}S(r,f_i)=S(r)$, we get by Lemma 10

$$\begin{aligned}T(r,\alpha) &\leq \sum_{i=1}^{3}N_2(r,0;f_i)+\sum_{i=1}^{3}\overline{N}(r,\infty;f_i)+S(r)\\ &\leq N_2(r,0;f_1)+2\overline{N}(r,0;f_2)+2\overline{N}(r,0;f_3)+\sum_{i=1}^{3}\overline{N}(r,\infty;f_i)+S(r).\end{aligned}$$

Further, since

$$\begin{aligned}\overline{N}(r,0;f_2) &\leq \overline{N}(r,0;g>f)+\overline{N}(r,1;f>g)\\ \overline{N}(r,0;f_3) &\leq \overline{N}(r,\infty;f<g)+\overline{N}(r,1;f>g),\end{aligned}$$

it follows that

$$\begin{aligned}T(r,\alpha) \leq\ & N_2(r,0;f_1)+2\overline{N}_*(r,0;f,g)+3\overline{N}_*(r,1;f,g)\\ &+\overline{N}(r,1;f>g)+2\overline{N}_*(r,\infty;f,g)+S(r).\end{aligned} \tag{4}$$

We see that $(1-a)f_1 \equiv (f-a)(1-\alpha h) \equiv (1-\alpha)-a(1-\alpha h)$ and $f \equiv \frac{1-\alpha}{1-\alpha h}$. So z_0 is a possible zero of f_1 if either z_0 is a zero of $f-a$ or z_0 is a common zero of $1-\alpha$ and $1-\alpha h$. Therefore

$$N_2(r,0;f_1) \leq N_2(r,a;f)+N(r,0;1-\alpha h \mid \alpha \neq \infty)-N(r,\infty;f \mid \alpha \neq \infty). \tag{5}$$

Since $f \equiv \frac{1-\alpha}{1-\alpha h}$ and the possible poles of α occur only at the poles and 1-points of f, it follows in view of (4) and (5)

$$\begin{aligned}\overline{N}(r,0;f) &\leq N(r,0;1-\alpha)-N(r,0;1-\alpha h \mid \alpha \neq \infty)\\ &\quad +N(r,\infty;f \mid \alpha \neq \infty)+\overline{N}(r,\infty;\alpha h \mid \alpha \neq \infty)\\ &\leq T(r,\alpha)-N(r,0;1-\alpha h)+N(r,\infty;f \mid 1-\alpha h=0)\\ &\quad +\overline{N}(r,\infty;\alpha h \mid \alpha \neq \infty)+O(1)\\ &\leq N_2(r,a;f)+3\overline{N}_*(r,0;f,g)+4\overline{N}_*(r,1;f,g)\\ &\quad +2\overline{N}_*(r,\infty;f,g)+S(r).\end{aligned}$$

Since f, g share $(0,1)$, $(1,m)$ we get

$$\begin{aligned}\overline{N}_*(r,0;f,g) &\leq \overline{N}(r,0;f \mid \geq 2)\\ \overline{N}_*(r,1;f,g) &\leq \overline{N}(r,1;f \mid \geq m+1).\end{aligned}$$

So by Lemma 3 we obtain

$$\overline{N}(r,0;f) \leq N_2(r,a;f)+\frac{5m+9}{m-1}\,\overline{N}_*(r,\infty;f,g)+S(r),$$

which is assertion (i).

If h is a constant then $\overline{N}(r,1;f) \equiv 0$ because $g=hf$, $f \not\equiv g$ and f, g share $(1,m)$. So we suppose that h is nonconstant. Let

$$g_1=\frac{-f_1}{f_3}=\frac{-(f-a)(1-\alpha h)}{\alpha},\ g_2=\frac{1}{f_3}=\frac{1-a}{\alpha},\ g_3=\frac{-f_2}{f_3}=ah.$$

Then $g_1 + g_2 + g_3 \equiv 1$, and by Lemma 9 **the functions** g_1, g_2, g_3 are linearly independent. Since $\sum_{i=1}^{3} S(r, g_i) = S(r, f)$, applying Lemma 10 to g_1, g_2, g_3 we get

$$\begin{aligned} T(r,h) &\le \sum_{i=1}^{3} N_2(r,0;g_i) + \sum_{i=1}^{3} \overline{N}(r,\infty;g_i) + S(r) \\ &\le N_2(r,0;g_1) + 2\overline{N}(r,0;g_2) + 2\overline{N}(r,0;g_3) + \sum_{i=1}^{3} \overline{N}(r,\infty;g_i) + S(r) \\ &\le N_2(r,0;g_1) + 2\overline{N}(r,\infty;\alpha) + 2\overline{N}(r,0;h) + \overline{N}(r,\infty;g_1) \\ &\quad +\overline{N}(r,0;\alpha) + \overline{N}(r,\infty;h) + S(r). \end{aligned}$$

We get by Lemma 4

$$\begin{aligned} T(r,h) \le\ & N_2(r,0;g_1) + \overline{N}(r,\infty;g_1) + \overline{N}_*(r,1;f,g) + \overline{N}(r,1;f<g) \\ & +\overline{N}_*(r,0;f,g) + \overline{N}(r,0;f<g) + 2\overline{N}_*(r,\infty;f,g) \\ & +\overline{N}(r,\infty;f>g). \end{aligned} \tag{6}$$

Since $g_1 = \left(1 - \frac{a}{f}\right)\left(1 - \frac{g-1}{f-1}\right)$ and f, g share $(0,1)$, $(1,m)$, $(\infty,0)$, it follows that possible poles of g_1 occur at the zeros, 1-points and poles of f and g.
Let z_o be a zero of f and g with multiplicities l and n respectively. Then in some neighbourhood of z_o we get

$$g_1(z) = \frac{\{(z-z_o)^l\phi - a\}\{(z-z_o)^l\phi - (z-z_o)^n\psi\}}{(z-z_o)^l\phi\{(z-z_o)^l\phi - 1\}},$$

where ϕ, ψ are analytic at z_o and $\phi(z_o) \ne 0$, $\psi(z_o) \ne 0$. This shows that z_o is a pole of g_1 only if $l > n$. Again since $g_1 = \left(1 - \frac{a}{f}\right)\left(1 - \frac{1}{\alpha}\right)$, it follows in view of Lemma 4

$$\overline{N}(r,\infty;g_1) \le \overline{N}(r,0;f>g) + \overline{N}(r,\infty;f<g) + \overline{N}(r,1;f>g).$$

So from (6) we get

$$\begin{aligned} T(r,h) \le\ & N_2(r,0;g_1) + 2\overline{N}_*(r,0;f,g) + 2\overline{N}_*(r,1;f,g) \\ & +3\overline{N}_*(r,\infty;f,g) + S(r). \end{aligned} \tag{7}$$

We see that

$$g_1 = \frac{-(f-a)(1-\alpha h)}{\alpha} = \frac{a(1-\alpha h) - (1-\alpha)}{\alpha}\ ,\ f = \frac{1-\alpha}{1-\alpha h}\ .$$

So z_o is a possible zero of g_1 if

(1) z_o is a zero of $f-a$

(2) z_o is a common zero of $1-\alpha$ and $1-\alpha h$

(3) z_o is a pole of α.

If z_0 is a pole of α then z_0 is either a pole of f or an 1-point of f. Since $g_1 = \left(1-\frac{a}{f}\right)\left(1-\frac{1}{\alpha}\right)$, it follows that if z_0 is a pole of f then $g_1(z_0)=1$ and if z_0 is an 1-point of f then $g_1(z_0)=1-a(\neq 0)$. Therefore

$$N_2(r,0;g_1) \le N_2(r,a;f) + N(r,0;1-\alpha h \mid \alpha \neq \infty) - N(r,\infty;f \mid \alpha \neq \infty). \quad (8)$$

Since $f-1=\frac{(1-h)\alpha}{1-\alpha h}$ and a zero of α occurs at a pole of f or at an 1-point of f, we get in view of Lemma 4 and (7), (8)

$$\begin{aligned}\overline{N}(r,1;f) &\le N(r,1;h) - N(r,0;1-\alpha h \mid \alpha\neq\infty) + N(r,\infty;f \mid \alpha \neq \infty)\\ &\quad +\overline{N}_*(r,1;f,g)\\ &\le T(r,h) - N(r,0;1-\alpha h \mid \alpha\neq\infty) + N(r,\infty;f \mid \alpha \neq \infty)\\ &\quad +\overline{N}_*(r,1;f,g)+O(1)\\ &\le N_2(r,a;f)+2\overline{N}_*(r,0;f,g)+3\overline{N}_*(r,1;f,g)\\ &\quad +3\overline{N}_*(r,\infty;f,g)+S(r).\end{aligned}$$

Since

$$\begin{aligned}\overline{N}_*(r,0;f,g) &\le \overline{N}(r,0;f \mid\ge 2)\\ \overline{N}_*(r,1;f,g) &\le \overline{N}(r,1;f \mid\ge m+1)\end{aligned}$$

we get by Lemma 3

$$\overline{N}(r,1;f) \le N_2(r,a;f) + \frac{5m+5}{m-1}\overline{N}_*(r,\infty;f,g)+S(r),$$

which is assertion (ii). This proves the lemma. ∎

Lemma 12. *Under the hypotheses of Lemma* 11 *we get*

(i) $\overline{N}(r,0;f) \le N(r,a;f \mid\le 2) + \frac{31m+23}{m-1}\overline{N}_*(r,\infty;f,g)+S(r)$

(ii) $\overline{N}(r,1;f) \le N(r,a;f \mid\le 2) + \frac{31m+19}{m-1}\overline{N}_*(r,\infty;f,g)+S(r)$.

Proof. Since $N_2(r,a;f) = N(r,a;f\mid\le 2)+2\overline{N}(r,a;f\mid\ge 3)$, the lemma follows from the Lemmas 8 and 11. ∎

Lemma 13. *If in Lemma* 11 *we suppose that* f, g *share* $(0,1)$, $(1,m)$ *and* (∞,k), *where* $(m-1)(mk-1)>(1+m)^2$, *then*

(i) $\overline{N}(r,0;f) \le N(r,a;f \mid\le 2)+S(r)$

(ii) $\overline{N}(r,1;f) \le N(r,a;f \mid\le 2)+S(r)$.

Proof. Since $\overline{N}_*(r,\infty;f,g) \le \overline{N}(r,\infty;f \mid\ge 2) = S(r)$ in view of Lemma 2, the lemma follows from Lemma 12. ∎

4. Proofs of the main results

Theorem 1 can be proved in the line of Theorem 2 using Lemmas 2, 4, 5, 8 and 13. Also Theorem 3 can be proved in the line of Theorem 2 using Lemmas 3, 4, 5, 8 and 12. So we will prove Theorem 2, only.

Proof of Theorem 2. Let f_1, f_2, f_3 be defined as in Lemma 11. It is possible to suppose that f_1, f_2, f_3 are linearly independent. Then by the second fundamental theorem and Lemma 11 we get

$$\begin{aligned} 2T(r,f) &\leq \overline{N}(r,0;f) + \overline{N}(r,1;f) + \overline{N}(r,\infty;f) + \overline{N}(r,a;f) + S(r,f) \\ &\leq 3N_2(r,a;f) + \frac{11m+13}{m-1}\,\overline{N}(r,\infty;f) + S(r,f) \end{aligned}$$

and so

$$3\delta_2(a;f) + \frac{11m+13}{m-1}\Theta(\infty;f) \leq \frac{12m+12}{m-1}\ ,$$

which is a contradiction. So there exist constants c_1, c_2, c_3, not all zero, such that

$$c_1 f_1 + c_2 f_2 + c_3 f_3 \equiv 0. \tag{9}$$

If $c_1 = 0$, then from (9) and the definition of f_2, f_3 it follows that h is a constant. Since $f \not\equiv g$, we see that $h \neq 1$, and so 1 becomes a Picard exceptional value of f because f, g share $(1, m)$ and $g \equiv hf$.

Since $g \equiv hf$, it follows that f, g share (∞, ∞), and since 1 is a Picard exceptional value of f and so of g, we see that $\alpha = \frac{f-1}{g-1}$ has no pole. Since

$$f \equiv \frac{1}{h} + \frac{h-1}{h(1-\alpha h)}$$

and α has no pole, it follows that $\frac{1}{h}$ is also a Picard exceptional value of f (in this case h is a nonzero constant). Since 1 and $\frac{1}{h}$ $(\neq 1, \infty)$ are Picard exceptional values of f, by the second fundamental theorem of Nevanlinna it follows that $\Theta(\infty; f) = 0$, which is a contradiction. Hence $c_1 \neq 0$.

Since $f_1 + f_2 + f_3 \equiv 1$, we get from (9)

$$cf_2 + df_3 \equiv 1\ , \tag{10}$$

where $|c| + |d| \neq 0$. We now consider the following cases.

`Case I:` Let $c \neq 0$ and $d \neq 0$. Then from (10) we get

$$\frac{-ac\alpha h}{1-a} + \frac{d\alpha}{1-a} \equiv 1.$$

Since f, g share $(0,1)$, $(1,m)$, $(\infty,0)$ we get by Lemma 5

$$\begin{aligned} T(r,f) &\le \frac{4(m+1)}{m-1}\,\overline{N}(r,\infty;f<g)+S(r,f) \\ &\le \frac{4(m+1)}{m-1}\,\overline{N}(r,\infty;f)+S(r,f) \end{aligned}$$

and so $\Theta(\infty;f)\le\frac{3m+5}{4m+4}$. Therefore

$$3\delta_2(a;f)+\frac{11m+13}{m-1}\,\Theta(\infty;f)\le\frac{12m+12}{m-1}-\frac{3m+5}{4m+4}\,,$$

which is a contradiction.

`Case II:` Let $c=0$ and $d\ne 0$. Then, from (10) we see that α is a constant. Since $\alpha=\frac{f-1}{g-1}$ and $f\not\equiv g$, it follows that $\alpha\ne 1$. So $N(r,0;f)\equiv 0$ because f, g share $(0,1)$. Since $f=\frac{1-\alpha}{1-\alpha h}$, we get by the second fundamental theorem and Lemma 4

$$\begin{aligned} T(r,f) &\le \overline{N}(r,0;f)+\overline{N}(r,1-\alpha;f)+\overline{N}(r,\infty;f)+S(r,f) \\ &= \overline{N}(r,0;h)+\overline{N}(r,\infty;f)+S(r,f) \\ &= \overline{N}(r,\infty;f>g)+\overline{N}(r,\infty;f)+S(r,f) \\ &\le 2\overline{N}(r,\infty;f)+S(r,f) \end{aligned}$$

and so $\Theta(\infty;f)\le\frac{1}{2}$, which contradicts the given condition.

`Case III:` Let $c\ne 0$ and $d=0$. Then from (10) we see that $\alpha h=p$, say, a constant. Since $f\not\equiv g$ and $\alpha h=\frac{g(f-1)}{f(g-1)}$, it follows that $p\ne 1$. So we get

$$f-a\equiv\frac{(1+ap-a)-\alpha}{1-p}. \tag{11}$$

If $1+ap-\alpha\ne 0$, by the second fundamental theorem and Lemma 4 we get

$$\begin{aligned} T(r,\alpha) &\le \overline{N}(r,\infty;\alpha)+\overline{N}(r,0;\alpha)+\overline{N}(r,1+ap-a;\alpha)+S(r,\alpha) \\ &\le \overline{N}_*(r,\infty;f,g)+\overline{N}_*(r,1;f,g)+\overline{N}(r,a;f)+S(r,f) \\ &\le \overline{N}(r,\infty;f)+\overline{N}_*(r,1;f,g)+\overline{N}(r,a;f)+S(r,f). \end{aligned}$$

Since f, g share $(0,1)$, $(1,m)$, $(\infty,0)$, by Lemma 3 we get

$$\overline{N}_*(r,1;f,g)\le\overline{N}(r,1;f\mid\ge m+1)\le\frac{2}{m-1}\overline{N}_*(r,\infty;f,g)\le\frac{2}{m-1}\overline{N}(r,\infty;f).$$

Again, since $f=\frac{1-\alpha}{1-p}$, it follows that $T(r,f)=T(r,\alpha)+O(1)$. Hence, from above we get

$$T(r,f)\le N_2(r,a;f)+\frac{m+1}{m-1}\,\overline{N}(r,\infty;f)+S(r,f)$$

and so $(m-1)\delta_2(a;f)+(m+1)\Theta(\infty;f)\le m+1$, which contradicts the given condition. Therefore $1+ap-a=0$. So from (11) we get

$$f-a\equiv -a\alpha. \tag{12}$$

Since $g=hf$, we get from (12)

$$g+a-1\equiv \frac{a-1}{\alpha}. \tag{13}$$

From (12) and (13) we obtain $(f-a)(g+a-1)\equiv a(1-a)$. This proves the theorem. ∎

Proof of Corollary 2. We choose an $\varepsilon>0$ such that $3\delta_2(a;f)+11\Theta(\infty;f)>12+2\varepsilon$. Now it is possible to choose a sufficiently large positive integer m such that

$$\frac{11m+13}{m-1}>11-\varepsilon\ ,\quad \frac{12m+12}{m-1}<12+\varepsilon.$$

Since f, g share $(0,1)$, $(1,m)$, $(\infty,0)$ and

$$\begin{aligned} 3\delta_2(a;f)+\frac{11+13}{m-1}\,\Theta(\infty;f) &> 3\delta_2(a;f)+(11-\varepsilon)\Theta(\infty;f)\\ &> 12+2\varepsilon-\varepsilon\Theta(\infty;f)\\ &\ge 12+\varepsilon\\ &> \frac{12m+12}{m-1}\ , \end{aligned}$$

the corollary follows from Theorem 2. ∎

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References

[1] Gundersen, G. G: *Meromorphic functions that share three or four values.* J. London Math. Soc. Vol.20 (1979)(2), 457 – 466.

[2] Hayman, W. K.: *Meromorphic Functions.* Oxford: The Clarendon Press 1964.

[3] Lahiri, I.: *Weighted sharing and uniqueness of meromorphic functions.* Nagoya Math. J. 161 (2001), 193 – 206.

[4] Lahiri, I.: *Weighted value sharing and uniqueness of meromorphic functions.* **journal ?** Complex Variables 46 (2001)(3), 241 – 253.

[5] Lahiri, I.: *Weighted sharing of three values and uniqueness of meromorphic functions.* Kodai Math. J. 24 (2001), 421 – 435.

[6] Lahiri, I.: *On a result of Ozawa concerning uniqueness of meromorphic functions II.* J. Math. Anal. Appl. 283 (2003)(1), 66 – 76.

[7] Lahiri, I.: *Uniqueness of meromorphic functions and sharing of three values with some weight.* New Zealand J. Math. 32 (2003)(2), 161 – 171.

[8] Ping Li and Yang, C. C.: *Some further results on the unique range sets of meromorphic functions.* Kodai Math. J. 18 (1995), 437 – 450.

[9] Ueda, H.: *Unicity theorem for meromorphic or entire functions.* Kodai Math. J. 3 (1980), 457 – 471.

[10] Ye, S. Z.: *Uniqueness of meromorphic functions that share three values.* Kodai Math. J. 15 (1992), 236 – 243.

[11] Yi, H. X.: *Meromorphic functions that share three values.* Chinese Ann. Math. 9A (1988), 434 – 439.

[12] Yi, H. X.: *Meromorphic functions that share two or three values.* Kodai Math. J. 13 (1990), 363 – 372.

[13] Yi, H. X.: *On characteristic function of a meromorphic function and its derivative.* Indian J. Math. 13 (1991)(2), 119 – 133.

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