

Classes of Multiplication Operators and Their Limit Operators

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Abstract. Limit operators have proven to be a device for the study of several properties of an operator including Fredholmness and invertibility at infinity, but also the applicability of approximation methods. For band-dominated operators, the question of existence and structure of their limit operators essentially reduces to the study of multiplication operators and their limit operators, which is the topic of this paper.

Keywords: *Limit operator, multiplication operator, band-dominated operator*

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1. Introduction and main items

Limit operators have been introduced as a device for the study of several properties of an operator including Fredholmness [5 - 7, 9] and invertibility at infinity [3, 10], but also the applicability of approximation methods [4, 9, 10]. The first time limit operator techniques were applied to the general class of band-dominated operators was in 1985 by Lange and Rabinovich in [1, 2].

The motivation behind the concept of limit operators is to study the behaviour of an operator A at infinity. One therefore takes a sequence h of points h_m tending to infinity and watches the sequence of operators $V_{-h_m}AV_{h_m}$ as $m \rightarrow \infty$. If convergence of that sequence takes place in a certain sense (similar to strong convergence, see [3] or [9]), we will regard its limit as the *limit operator* of A with respect to the sequence h and denote it by A_h . Collecting all possible limit operators in this manner results in the so called *operator spectrum* $\sigma^{\text{op}}(A)$ of A . We will regard A as a *rich operator* if it possesses sufficiently many limit operators in the following sense: Every sequence h tending to infinity has an infinite subsequence g such that A_g exists.

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For a rich operator A , all necessary information about its behaviour at infinity is accurately stored in $\sigma^{\text{op}}(A)$, and for such operators, the typical criterion for the applications mentioned above says that an operator A is subject to the property under consideration if and only if all limit operators (or associated operators to these) of A are invertible and their inverses are uniformly bounded.

But in order to really work with these criteria, one still has to gain some knowledge on the objects it is dealing with:

Q1 How do we recognize rich operators?

Q2 How do their limit operators look like?

This paper is essentially concerned with answering these questions for some practically relevant classes of operators. To see why everything reduces to the study of multiplication operators, we have to take some closer look at the operators under consideration and their limit operators in Section 2. The study of multiplication operators and their limit operators is then done in Section 3.

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2. Preliminaries

2.1 Band-dominated operators. By ℓ^p and L^p we denote the usual spaces of complex-valued sequences on \mathbb{Z}^n and functions on \mathbb{R}^n , respectively. The Lebesgue parameter p is in $[1, \infty]$, as usual, and the dimension n is some fixed positive integer.

An operator $A \in \mathcal{L}(\ell^p)$ is a *band operator* if its matrix representation $[a_{\alpha\beta}]$ with respect to the standard basis in ℓ^p is a band matrix, i.e. $a_{\alpha\beta} = 0$ if $|\alpha - \beta|$ exceeds some fixed number – the so called *band width* of A . The set of band operators clearly turns out to be an algebra – but it is not closed. Hence, it is a natural desire to pass to its closure with respect to the norm in $\mathcal{L}(\ell^p)$, which is a Banach algebra then. The elements of the latter are called *band-dominated operators*.

For $\alpha \in \mathbb{Z}^n$, let V_α denote the so called *shift operator* on ℓ^p , acting by the rule $(V_\alpha u)_\beta = u_{\beta-\alpha}$ on every $u \in \ell^p$, i.e. shifting u by α components. Without introducing a new symbol, we will say that V_τ is the *shift operator* on L^p where $(V_\tau u)(x) = u(x - \tau)$ for every $u \in L^p$.

The important observation is that A is band operator if and only if it is a finite sum-product of shifts V_α and discrete multiplication operators (i.e.

diagonal matrices). Consequently, band-dominated operators are composed of these two ingredients as well.

By cutting \mathbb{R}^n into cubes of size 1 and identifying a function $f \in L^p$ with the ℓ^p -sequence of restrictions of f to these cubes, the notion of a band(-dominated) operator can be transferred to operators on L^p as well. It turns out that also here, huge classes of practically relevant operators are only composed (via addition, composition and taking norm limits) by operators of multiplication and shift-invariant operators like convolutions and shifts. We hereby call an operator A *shift-invariant* if it coincides with $V_{-c}AV_c$ for every vector c .

2.2 The set of rich operators. We do not know any algorithm that answers question Q1 from the introduction, i.e. tells if a given operator is rich or not. But it is not hard to see (e.g., in [9]) that the set of rich band-dominated operators actually forms a Banach algebra.

So what we can do is decomposing our operator A into its basic components, namely multiplication operators and shift-invariant operators, and examine their rich property: Clearly, shift-invariant operators S are always rich since the sequence $V_{-h_m}SV_{h_m}$ is constant for every $h = (h_m)$. For multiplication operators this question is extremely non-trivial, but we will find some answers in Section 3.

2.3 Computing limit operators. Concerning question Q2 from our introduction, we make extensive use of a fundamental property of limit operators (see, e.g., [3] or [9]):

For every fixed sequence h tending to infinity, the mapping $A \mapsto A_h$ is compatible with addition, composition, scalar multiplication, passing to adjoints and to norm-limits. That is, the equations

$$\begin{aligned}(A + B)_h &= A_h + B_h \\ (AB)_h &= A_h B_h \\ (\lambda A)_h &= \lambda A_h \\ (A^*)_h &= (A_h)^* \\ (\lim_{m \rightarrow \infty} A^{(m)})_h &= \lim_{m \rightarrow \infty} A_h^{(m)}\end{aligned}$$

hold, provided all limit operators on the right-hand sides exist.

So computing a limit operator of A can be done by decomposing A into its basic components, namely multiplication operators and shift-invariant operators, computing their limit operators, and puzzling these together again as A was composed by its components. Again, limit operators of a shift-invariant operator S are trivially equal to S itself, and limit operators of multiplication

operators are the essential problem, discussed in Section 3 for some classes of multipliers.

3. Limit operators of a multiplication operator

3.1 Notations and abbreviations. A discrete multiplication operator M_b on ℓ^p acts by the rule $(M_b u)_\alpha = b_\alpha u_\alpha$, where $b = (b_\alpha)$ is an element of ℓ^∞ . Without introducing a new symbol, let M_b denote the operator of multiplication in L^p by the bounded function $b \in L^\infty$.

For every measurable set $U \subset \mathbb{R}^n$, P_U is the operator of multiplication by the characteristic function of U . Clearly, P_U is a projector. We will refer to its complementary projector $I - P_U$ by Q_U .

For a complex number z and some $\varepsilon > 0$, put $U_\varepsilon(z) = \{y \in \mathbb{C} : |y - z| < \varepsilon\}$. The hypercube $[-1, 1]^n$ will be abbreviated by C in what follows.

3.2 The discrete case. We first cite a result from the discrete case ℓ^p saying that, in this situation, every (discrete) multiplication operator is rich.

Proposition 3.1. *Every discrete multiplication operator M_b on ℓ^p is rich, and every limit operator of M_b is a discrete multiplication operator again.*

As a consequence, we get that every band-dominated operator on ℓ^p is rich, and question Q1 is ridiculous in this setting. The function case L^p is much more interesting here, and we will henceforth pay our attention to this one. Note that some results concerning the structure of limit operators (see, for instance, Subsection 3.6) have their discrete analogon in ℓ^p .

3.3 The function case. The proof of Proposition 3.1 (see [9]) uses some diagonal argument in connection with an enumeration of \mathbb{Z}^n and the Bolzano-Weierstrass theorem. Unfortunately, this proof is not portable to the case of (usual) multiplication operators M_b on L^p for some reasons:

- 1) \mathbb{R}^n cannot be enumerated.
- 2) By trying some workaround to reason 1) and writing $L^p \cong \ell^p(\mathbb{Z}^n, X)$ with $X = L^p([0, 1]^n)$, we get discrete multiplication operators with values in an infinite-dimensional space. Consequently, the relative-compactness (i.e. Bolzano-Weierstrass) argument is not applicable.

There is some very good reason for this proof being not portable to L^p : Proposition 3.1 is not true there. The only fact that can be rescued is that every limit operator of M_b is an operator of multiplication again, say M_c , with $c \in L^\infty$.

From $V_{-h_m} M_b V_{h_m} = M_{V_{-h_m} b}$ one can easily conclude that $(M_b)_h = M_c$ if and only if for every bounded and measurable set $U \subset \mathbb{R}^n$

$$\|P_U(V_{-h_m} b - c)\|_\infty = \operatorname{ess\,sup}_{u \in U} |b(h_m + u) - c(u)| \rightarrow 0 \quad (m \rightarrow \infty).$$

We will frequently abbreviate this fact by

$$b|_{h_m+U} \rightarrow c|_U \quad (m \rightarrow \infty) \tag{1}$$

(uniform convergence on U), where we agree in writing $b|_{h_m+U}$ instead of the much clumsier notation $V_{-h_m}(b|_{h_m+U})$ or $(V_{-h_m} b)|_U$.

Remark 3.2. We do not pay much attention to the general definition of limit operators in this paper because this would require several additional notations and technical journeys. For our purposes – the multiplication operators – relation (1) perfectly substitutes this definition. However, one aspect should be discussed:

Since the limit operator method grew up with the discrete case, the sequences $h = (h_m)$ were naturally restricted to \mathbb{Z}^n . In the function case, we could easily drop this restriction and pass to $h_m \in \mathbb{R}^n$. (Which would change nothing up to this point!) We will however resist this temptation and stay in the integers, which will result in some technical efforts in Subsections 3.4 - 3.7, but afterwards, in Remark 3.19 we will state the reason for doing so.

Definition 3.3. For a function $f \in L^\infty$ and a bounded and measurable set $U \subset \mathbb{R}^n$, we define

$$\operatorname{osc}_U(f) = \operatorname{ess\,sup}_{u,v \in U} |f(u) - f(v)|$$

$$\operatorname{osc}_x(f) = \operatorname{osc}_{x+C}(f),$$

the latter is referred to as *local oscillation* of f at x .

Lemma 3.4. *If $b \in L^\infty$ and $h = (h_m) \rightarrow \infty$ leads to a limit operator of M_b , say M_c , then, for every bounded and measurable set $U \subset \mathbb{R}^n$, $\operatorname{osc}_{h_m+U}(b) \rightarrow \operatorname{osc}_U(c)$ as $m \rightarrow \infty$.*

Proof. Take an arbitrary $\varepsilon > 0$ and a bounded and measurable $U \subset \mathbb{R}^n$. By (1), there is some m_0 such that, for every $m > m_0$, $\|b|_{h_m+U} - c|_U\|_\infty < \frac{\varepsilon}{2}$. For almost all $u, v \in U$ we then have

$$\begin{aligned} & |b(h_m + u) - b(h_m + v)| \\ & \leq |b(h_m + u) - c(u)| + |c(u) - c(v)| + |c(v) - b(h_m + v)| \\ & \leq \frac{\varepsilon}{2} + \operatorname{osc}_U(c) + \frac{\varepsilon}{2} \end{aligned}$$

and by passing to the essential supremum for $u, v \in U$, we get $\operatorname{osc}_{h_m+U}(b) \leq \operatorname{osc}_U(c) + \varepsilon$. Completely analogously, we derive $\operatorname{osc}_U(c) \leq \operatorname{osc}_{h_m+U}(b) + \varepsilon$, and taking this together we see that $|\operatorname{osc}_{h_m+U}(b) - \operatorname{osc}_U(c)| < \varepsilon$ for all $m > m_0$ which proves our claim ■

Definition 3.5. We call a function $b \in L^\infty$ *rich* if M_b is a rich operator. Otherwise we call b *ordinary*.

There are even functions $b \in L^\infty$ for which no sequence $h \rightarrow \infty$ in \mathbb{Z}^n at all leads to a limit operator of M_b . Such functions will be referred to as *poor* functions. We denote the set of rich functions by L_\S^∞ .

Since the set of rich operators is a Banach algebra, we have that also L_\S^∞ is closed under addition, multiplication and supremum norm, i.e. L_\S^∞ is a Banach subalgebra of L^∞ .

It is time to look at some examples now. Let $n = 1$. For instance, $b_1(x) = (-1)^{[x/\pi]}$, where $[y]$ denotes the integer part of y , is a poor function. The function b_1 is 2π -periodic and has a jump at every multiple of π . That is why $(b_1|_{h_m+U})_m$ cannot be a Cauchy sequence in $L^\infty(U)$ for any sequence of integers (h_m) . Here the cause for b_1 being poor is clearly the condition that all h_m have to be integers.

Another example of a poor function is $b_2(x) = \sin(x^2)$. One can easily show that no sequence (of integers or reals) tending to infinity leads to a limit operator of M_{b_2} .

3.4 Step functions. Loosely spoken, step functions are piecewise constant functions on a lattice of hypercubes. Let therefore $H := (0, 1)^n$.

Definition 3.6. Take some positive real number ℓ . A function $f \in L^\infty$ is called *step function* with *steps* of size ℓ if there is an $x_0 \in \mathbb{R}^n$ with the property that f is constant on all hypercubes $H_\alpha = x_0 + \ell(\alpha + H)$ with α running through \mathbb{Z}^n . The set of these functions will be denoted by T_ℓ . Finally, put

$$T_\mathbb{Q} = \bigcup_{p,q \in \mathbb{N}} T_{p/q}.$$

Our example, b_1 is obviously in T_π and has proven to be ordinary (even poor). We will see that this is a consequence of the irrationality of π .

Proposition 3.7. *Step functions with rational step size are rich, $T_\mathbb{Q} \subset L_\S^\infty$.*

Proof. Pick some arbitrary $p, q \in \mathbb{N}$. Since $T_{p/q} \subset T_{1/q}$, it remains to show that all functions $b \in T_{1/q}$ are rich. So put $\ell = \frac{1}{q}$, and take arbitrary $b \in T_\ell$, bounded and measurable $U \subset \mathbb{R}^n$ and $h = (h_m) \subset \mathbb{Z}^n$ with $h_m \rightarrow \infty$. The sets $h_m + U$ ($m \geq 1$) differ by an integer translation. Since $\ell q = 1$, we can unify q^n adjacent steps of b to one hypercube of size 1, which is determined by the q^n function values of b at the respective steps. In this sense, our step function b can be identified with an object in \mathbb{C}^{q^n} -valued ℓ^∞ . In this case (finite-dimensional-valued ℓ^∞), we have Proposition 3.1, and so we are done with $b \in T_{1/q}$ as well ■

3.5 Bounded and uniformly continuous functions. Let BC and BUC denote the Banach algebras of all continuous and all uniformly continuous L^∞ -functions, respectively.

Proposition 3.8. *Every $b \in \text{BUC}$ is rich. Moreover, if M_c is a limit operator of M_b , then $c \in \text{BUC}$ as well.*

Proof. Pick some arbitrary $b \in \text{BUC}$. To every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$b(x + \delta C) \subset U_\varepsilon(b(x)) \quad (x \in \mathbb{R}^n). \quad (2)$$

So this is true for some $\delta' \in \mathbb{Q}$ with $0 < \delta' \leq \delta$ as well. Consequently, there is a step function $s \in T_{\delta'}$ with $\|b - s\|_\infty < \varepsilon$ which tells that $T_{\mathbb{Q}}$ is dense in BUC. Proposition 3.7 and the norm-closedness of $L_{\mathbb{S}}^\infty$ show that $\text{BUC} \subset L_{\mathbb{S}}^\infty$.

Now suppose M_c is the limit operator of M_b with respect to some sequence $h = (h_m) \rightarrow \infty$. Take an arbitrary $\varepsilon > 0$, and choose $\delta > 0$ such that (2) holds. By (2) we have $\text{osc}_{h_m+U}(b) \leq 2\varepsilon$ ($m \geq 1$) for every bounded and measurable $U \subset \mathbb{R}^n$ with diameter not exceeding 2δ . But Lemma 3.4 then shows that $\text{osc}_U(c) \leq 2\varepsilon$ for every bounded and measurable U whose diameter does not exceed 2δ , i.e. $c \in \text{BUC}$ ■

So all uniformly continuous BC-functions are rich. But are there any more rich functions among the others in BC? To answer this question, we first have a look at an example of such a function in $\text{BC} \setminus \text{BUC}$:

Take a real-valued continuous function f on the axis which is only supported in the intervals $m^2 + (-\frac{1}{m}, \frac{1}{m})$ ($m \geq 1$) with $0 \leq f(x) \leq f(m^2) = 1$ for all $x \in \mathbb{R}$. Clearly, M_f has no limit operator with respect to any subsequence of $h = (m^2)$. (But there are sequences like $(m^2 + m)$ which lead to a limit operator of M_f . So f is not poor.) However, f is just ordinary. Indeed, one can show that every function in $\text{BC} \setminus \text{BUC}$ is a little bit like f and thus: ordinary.

Theorem 3.9. *A bounded and continuous function is rich if and only if it is uniformly continuous, $\text{BUC} = \text{BC} \cap L_{\mathbb{S}}^\infty$.*

Proof. Thinking of Proposition 3.8, it remains to show that all functions in $\text{BC} \setminus \text{BUC}$ are ordinary. So take a bounded and (not uniformly) continuous function b . Since b is uniformly continuous on every compact, the reason for its non-uniform continuity lies at infinity, i.e. there is an $\varepsilon_0 > 0$ and a sequence $(x_m) \subset \mathbb{R}^n$ with $x_m \rightarrow \infty$ and

$$b\left(x_m + \frac{1}{m}C\right) \not\subset U_{\varepsilon_0}(b(x_m)) \quad (m \geq 1). \quad (3)$$

Now choose the integer sequence $h = (h_m) = ([x_m])$, where $[y]$ denotes component-wise integer parts. We will see that M_b does not possess a limit

operator with respect to any subsequence of h . Suppose there is such a limit operator M_c , and let $U := C$. Then b is uniformly continuous on every hypercube $h_m + U$, and (1) shows that c is uniformly continuous on U , i.e. for every $\varepsilon > 0$ there is a $\delta > 0$ such that $c(x + \delta C) \subset U_\varepsilon(c(x))$ for all $x \in U = C$. So, especially,

$$c((x_m - h_m) + \delta C) \subset U_{\varepsilon_0/4}(c(x_m - h_m)) \quad (m \geq 1).$$

Because of (3), for all sufficiently large m (with $\frac{1}{m} \leq \delta$) the functions $b|_{h_m+U}$ and $c|_U$ differ on a translate of δC by at least $\frac{\varepsilon_0}{4}$, which contradicts (1) ■

3.6 Slowly oscillating functions. Here we will study another interesting class of functions which have the property that limit operators of their multiplication operators behave especially nice.

Definition 3.10. Let S^{n-1} denote the unit sphere (with respect to the Euclidian norm $|\cdot|_E$) of \mathbb{R}^n . Let $s \in S^{n-1}$. Then we say that a sequence $(x_m) \subset \mathbb{R}^n$ tends to infinity in the direction s , and write $x_m \rightarrow \infty_s$ as $m \rightarrow \infty$, if for every $R > 0$ and every neighborhood $U \subset S^{n-1}$ of s there is a m_0 such that $|x_m|_E > R$ and $\frac{x_m}{|x_m|_E} \in U$ for all $m > m_0$.

Definition 3.11. We will say that a function $f \in L^\infty$ is *slowly oscillating* towards ∞_s and write $f \in \text{SO}_s$ if $\text{osc}_x(f) \rightarrow 0$ as $x \rightarrow \infty_s$. Finally, we put $\text{SO} = \bigcap_{s \in S^{n-1}} \text{SO}_s$.

Lemma 3.12. Let f be some arbitrary function in L^∞ and $s \in S^{n-1}$. Then:

a) The following three conditions are equivalent:

- (i) $f \in \text{SO}_s$.
- (ii) $\lim_{x \rightarrow \infty_s} \text{osc}_x(f) = 0$.
- (iii) $\lim_{x \rightarrow \infty_s} \text{osc}_{x+U}(f) = 0$ for all bounded and measurable $U \subset \mathbb{R}^n$.

b) If f is differentiable in some neighborhood of ∞_s , then $\text{grad}f(x) \rightarrow 0$ as $x \rightarrow \infty_s$ is sufficient for $f \in \text{SO}_s$, but not necessary.

Proof. Part a). The equivalence (i) \iff (ii) holds by Definition 3.11. The implication (iii) \Rightarrow (ii) is trivial since assertion (ii) is just assertion (iii) with $U = C$. The implication (ii) \Rightarrow (iii): If assertion (ii) holds, then we have

$$\lim_{x \rightarrow \infty_s} \text{osc}_{x+U}(f) = 0 \quad (4)$$

for $U = C$ and all subsets of C . If (4) holds for a set U , then it also holds for all sets of the form $t+U$ ($t \in \mathbb{R}^n$) in place of U . Finally, if it holds for $U = U_1$ and $U = U_2$ with $U_1 \cap U_2 \neq \emptyset$, then it clearly holds for $U = U_1 \cup U_2$ since $\text{osc}_{x+(U_1 \cup U_2)}(f) \leq \text{osc}_{x+U_1}(f) + \text{osc}_{x+U_2}(f)$. Taking all this together, it is

clear that (4) holds for all bounded and measurable U . This implies assertion (iii).

Part b) Pick some $\varepsilon > 0$. If $\text{grad}f(x) \rightarrow 0$ as $x \rightarrow \infty_s$, there is some neighborhood V_ε of ∞_s such that $\|\text{grad}f(x)\|_\infty < \frac{\varepsilon}{2n}$ for all $x \in V_\varepsilon$. But from

$$|f(x+u) - f(x+v)| = |\text{grad}f(\xi_{x,u,v}) \cdot (u-v)| < \varepsilon \quad (u, v \in C, \xi_{x,u,v} \in x+C)$$

we conclude that $\text{osc}_x(f) \leq \varepsilon$ if $x \in V_\varepsilon$. This is assertion (ii).

Check the function $f(x) = \frac{\sin x^2}{x}$ to see that $f'(x)$ need not tend to zero if $f \in \text{SO}_\pm$. Moreover, slowly oscillating functions need not even be continuous.

For instance, look at $f(x) = \frac{(-1)^{[\sqrt{x}]}}{x}$ ■

In what follows, we will often use property (iii) to characterize the sets SO_s . It is easy to observe that SO_s is a closed subalgebra of L^∞ . We will now study the set of limit operators of M_b when b is slowly oscillating.

The *local operator spectrum* $\sigma_s^{\text{op}}(A)$ of an operator A is the set of all limit operators A_h with h tending to infinity into direction $s \in S^{n-1}$. It is not surprising that for every operator A , the identity $\sigma^{\text{op}}(A) = \cup_{s \in S^{n-1}} \sigma_s^{\text{op}}(A)$ holds (see [3] or [9]).

Proposition 3.13. *If $b \in \text{SO}_s$, then the set of limit operators towards ∞_s of M_b is $\sigma_s^{\text{op}}(M_b) = \{cI : c \in b(\infty_s)\}$ where $b(\infty_s)$ refers to the essential cluster points (set of partial limits) of b at ∞_s .*

Proof. Pick some $c \in b(\infty_s)$, some bounded and measurable set $U \subset \mathbb{R}^n$ and an $\varepsilon > 0$. There is a sequence of points $(x_m) \subset \mathbb{R}^n$ tending to ∞_s and a sequence (c_m) of complex numbers $c_m \in b(x_m + U)$ such that $c_m \rightarrow c$ as $m \rightarrow \infty$. Since $U' = U + C$ is bounded and measurable and $b \in \text{SO}_s$, there is a m_0 such that for every $m > m_0$ the oscillation $\text{osc}_{x_m+U'}(b)$ is less than $\frac{\varepsilon}{2}$. If m_0 is taken large enough that in addition $|c_m - c| < \frac{\varepsilon}{2}$, then

$$\begin{aligned} |b(x_m + u) - c| &\leq |b(x_m + u) - c_m| + |c_m - c| \\ &\leq \text{osc}_{x_m+U'}(b) + |c_m - c| \\ &< \varepsilon \end{aligned}$$

for all $u \in U'$, i.e. $b(x_m + U') \subset U_\varepsilon(c)$ if $m > m_0$. Now define the sequence of integers $h = (h_m)$ by $h_m = [x_m]$. Then $h_m + U$ is contained in $x_m + U'$ and, consequently, $b(h_m + U)$ is contained in $U_\varepsilon(c)$ for all $m > m_0$. From (1) we get that cI is the limit operator of M_b with respect to the sequence h .

Conversely, by Lemma 3.4, it is clear that for all limit operators M_c towards ∞_s of M_b the local oscillation $\text{osc}_x(c)$ has to be zero at every $x \in \mathbb{R}^n$, i.e. c has to be a constant which is contained in the essential range of b at ∞_s ■

Corollary 3.14. *If $b \in \text{SO}$, then $\sigma^{\text{op}}(M_b) = \{cI : c \in b(\infty)\}$, where $b(\infty)$ is the set of all partial limits of b at infinity.*

Proposition 3.15. *Slowly oscillating functions are rich, i.e. $\text{SO} \subset L_{\mathfrak{s}}^{\infty}$.*

Proof. Take a $b \in \text{SO}$ and an arbitrary sequence $h = (h_m)$ of integers with $h_m \rightarrow \infty$. The partial limiting set of the local essential ranges of b in h_1, h_2, \dots is non-empty. So pick a complex number c in that partial limiting set and an appropriate subsequence g of h , and proceed as in the proof of Proposition 3.13 to show that cI is the limit operator of M_b with respect to g ■

In some sense, even the reverse of Corollary 3.14 is true!

Proposition 3.16. *If $b \in L_{\mathfrak{s}}^{\infty}$ and every limit operator of M_b is a multiple of the identity operator I , then $b \in \text{SO}$.*

Proof. Let the conditions of the proposition be fulfilled, and suppose that $b \notin \text{SO}$. Then there exist a bounded and measurable set $U \subset \mathbb{R}^n$, an $\varepsilon_0 > 0$ and a sequence of points $(x_m) \subset \mathbb{R}^n$ tending to infinity such that for every m the oscillation $\text{osc}_{x_m+U}(b)$ is larger than ε_0 .

Now let $U' := U + C$ and $h = (h_m) \subset \mathbb{Z}^n$ with $h_m = [x_m]$. Since $x_m + U \subset h_m + U'$, we have $\text{osc}_{h_m+U'}(b) \geq \text{osc}_{x_m+U}(b) > \varepsilon_0$ for all $m \geq 1$. Since $b \in L_{\mathfrak{s}}^{\infty}$, there is a subsequence g of h such that the limit operator of M_b exists with respect to $g = (g_m)$. Denote this limit operator by M_c . By Lemma 3.4 we then conclude that $\text{osc}_{U'}(c) \geq \varepsilon_0$ and hence c is certainly not constant on U' . So at least one limit operator of M_b is not a multiple of I which contradicts our assumption ■

Denoting the set of functions $b \in L^{\infty}$, for which every limit operator of M_b is a multiple of the identity, by

$$\text{CL} = \left\{ b \in L^{\infty} : \sigma^{\text{op}}(M_b) \subset \{cI : c \in \mathbb{C}\} \right\},$$

we can summarize Corollary 3.14 and Propositions 3.15 - 3.16 by the following theorem (which is also true in its local versions at ∞_s).

Theorem 3.17. *A function b is slowly oscillating if and only if it is rich and all limit operators of M_b are multiples of the identity, $\text{SO} = \text{CL} \cap L_{\mathfrak{s}}^{\infty}$.*

As a consequence, we get that every limit operator of a rich band-dominated operator A is shift-invariant if and only if all multiplication operators, which are components of A , are slowly oscillating!

3.7 Oscillating functions. In this subsection we restrict ourselves to functions on the axis, i.e. $n = 1$. Let \mathbb{T} denote the complex unit circle, and suppose

$f : \mathbb{T} \rightarrow \mathbb{C}$ is a bounded and non-constant function. Then for every $p > 0$, by $b(x) = f(\exp \frac{2\pi ix}{p})$ we get a periodic function in L^∞ with $b(x + p) = b(x)$ for all $x \in \mathbb{R}$. In this case, b is an oscillating function with a constant frequency.

Furthermore, we will study cases of oscillating functions whose frequency tends to zero and to infinity, respectively, and decide whether they are rich or not. For simplicity, we will restrict our studies to the local operator spectrum $\sigma_+^{\text{op}}(M_b)$ at plus infinity and will therefore demand the oscillation of a prescribed frequency only towards $+\infty$.

So let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a strictly monotonously increasing, differentiable function with $\lim_{x \rightarrow +\infty} g(x) = +\infty$, and put

$$b(x) = f\left(e^{2\pi i g(x)}\right) \quad (x \in \mathbb{R}). \tag{5}$$

The three cases under consideration are:

- (i) $g'(x) \rightarrow +\infty$ as $x \rightarrow +\infty$ (frequency tends to infinity)
- (ii) $g'(x) = \frac{1}{p}$ for all $x \in \mathbb{R}$ (constant frequency – the periodic case)
- (iii) $g'(x) \rightarrow 0$ as $x \rightarrow +\infty$ (frequency tends to zero).

Proposition 3.18. *Let f be continuous on \mathbb{T} , g subject to one of the cases (i) – (iii) and b be as in (5). Then:*

In case (i), b is always poor.

In case (ii), b is always rich with $\sigma_+^{\text{op}}(M_b) = \{M_{V_c b} : c \in E\}$, where

$$E = \begin{cases} \{0, \frac{1}{\ell}, \dots, \frac{k-1}{\ell}\} & \text{if } p = \frac{k}{\ell} \in \mathbb{Q}, \text{ where } \gcd(k, \ell) = 1 \\ [0, p) & \text{if } p \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \tag{6}$$

(so E , and consequently $\sigma_+^{\text{op}}(M_b)$, has k elements if $p = \frac{k}{\ell} \in \mathbb{Q}$).

In case (iii), b is always rich with $\sigma_+^{\text{op}}(M_b) = \{cI : c \in f(\mathbb{T})\}$.

Proof. First of all, note that $f \in \text{BUC}(\mathbb{T})$ since \mathbb{T} is compact. Secondly, let $e : \mathbb{R} \rightarrow \mathbb{T}$ refer to the mapping $t \mapsto \exp(2\pi it)$, which is uniformly continuous as well. Moreover, note that g is reversible, and put g^{-1} such that $g^{-1} \circ g = \text{id} = g \circ g^{-1}$.

In case (i), we clearly have $b \in \text{BC} \setminus \text{BUC}$ and hence, by Proposition 3.9, b is ordinary. But moreover, by the same arguments as in the proof of Proposition 3.9, it is readily seen that for every sequence (of reals or integers) $h = (h_m) \rightarrow +\infty$ and every bounded interval U the sequence $(b|_{h_m+U})$ cannot be a Cauchy sequence. Hence, b is even poor.

In case (ii), we obviously have $b = f \circ e \circ g \in \text{BUC}$ since f, e, g are all uniformly continuous. So, by Proposition 3.8, b is rich. To compute the set of limit operators of M_b , note that $V_{-h_m} M_b V_{h_m}$ is just a multiplication by $V_{-h_m} b$. But $V_{-h_m} b = V_{-h_m \bmod p} b$ since b is p -periodic. For convergence of $V_{-h_m} b$, the sequence $(-h_m \bmod p)_{m=1}^\infty$ needs to converge ¹⁾ to some value c . Conversely,

¹⁾ Without loss of generality, we suppose that b has no period less than p .

if a sequence (c_m) converges to some c , then also $\|V_{c_m} b - V_c b\|_\infty \rightarrow 0$ since $b \in \text{BUC}$.

It remains to check which values can be attained by $z \bmod p$ if $z \in \mathbb{Z}$, and then to compute the closure of this set:

If $p = \frac{k}{\ell} \in \mathbb{Q}$, where $k, \ell \in \mathbb{N}$ have no common divisor, then the answer is $0, \frac{1}{\ell}, \dots, \frac{k-1}{\ell}$.

If p is irrational, then the answer is a dense subset of $[0, p)$ since 1 and p are incommensurable. So the set of limits c equals $[0, p)$. (The limit $c = p$ corresponds to $c = 0$ in terms of $V_c b$.)

Finally, case (iii) strongly reminds us of the slowly oscillating functions from Subsection 3.6. Indeed, from Lemma 3.12 b) we get that g has the slowly oscillating property towards $+\infty$ (ignoring the unboundedness of g which is unimportant for the proof of Lemma 3.12 b)). Since e and f are uniformly continuous, an easy computation shows that the composition $b = f \circ e \circ g$ is in SO_+ , and from Propositions 3.15 and 3.13 we get that b is rich (towards $+\infty$) and that every limit operator of M_b is of the form cI with $c \in b(+\infty) = f(\mathbb{T})$. Conversely, for every $c \in f(\mathbb{T})$ choose $t \in [0, 1)$ such that $f(e(t)) = c$, and put $h_m := [g^{-1}(t + m)] \in \mathbb{Z}$. Then $b(h_m) \rightarrow c$ as $m \rightarrow \infty$, and $h = (h_m)$ leads to the limit operator $(M_b)_h = cI$ ■

So if f is continuous, all answers in cases (i) - (iii) are given – including an explicit description of the operator spectra. The situation changes completely as soon as f has a single discontinuity, say a jump at $1 \in \mathbb{T}$:

Case (i) remains poor which is shown similarly as in the continuous case.

Case (ii) is rich if and only if the period p is rational. (Note the incidence with step functions of rational/irrational step length!)

Most interesting is case (iii). Here one cannot give such a precise statement – especially not one that is independent from the exact knowledge of the function g . To demonstrate this, we will consider $g(x) = \log_a x$, where $a > 1$ is fixed. The function b then jumps at every $x = a^k$ ($k \geq 0$) and is continuous at every other point (where the local oscillation outside of the jumps becomes smaller, the closer we come to $+\infty$).

Suppose the basis a is an integer. Then all jumps of b are at integer points and hence, their differences are all integer. Now it is an easy observation that there is a step function b_s with step length 1 and a function $b_0 \in L_0^\infty$ (see Subsection 3.8) such that $b = b_s + b_0$. As a sum of two rich functions, b is rich.

Suppose $a = \sqrt{2}$. Then the jumps a^k of b are integers for even k , and those, where k is odd, are multiples of $\sqrt{2}$. So it is easily seen that the sequence $h = (h_m) = ([a^{2m+1}])_m$ has no subsequence leading to a limit operator of M_b ,

while the sequence $h = (a^{2^m}) = (2^m)$ leads to a limit operator. So here, b is ordinary – but not poor.

If a is a transcendent number, then the jumps of b absolutely do not fit together modulo 1 (otherwise we had integers m_1 and m_2 such that $a^{m_1} - a^{m_2}$ is an integer k , i.e. a solves the equation $x^{m_1} - x^{m_2} - k = 0$). So no subsequence of $h = ([a^m])_m$ leads to a limit operator, and hence, b is just ordinary.

For completeness, we remark that b is never poor in case (iii) (and this is independent from the explicit structure of g) since there are many sequences leading to limit operators, for instance, $h = ([g^{-1}(t + m)])_m$, where $t \in (0, 1)$ is fixed.

It is not hard to see that the results from f having one jump can be extended to f having finitely many jumps.

Remark 3.19. In Subsections 3.4 - 3.7 we have experienced the consequences of the restriction of h to integer sequences, as already discussed in Remark 3.2. We have seen that step functions with rational step length and (non-continuous) periodic functions with rational period are always rich while their irrational counterparts are ordinary or even poor in general. This seems a bit unnatural indeed since re-scaling axes a little bit will change the rich-or-ordinary-situation completely. But note that, if we would have considered real sequences h , no non-convergent step function and no non-continuous periodic function would be rich at all – regardless if rational or irrational parameters. (Take multiples of $\sqrt{2} \ell$ or $\sqrt{2} p$ as elements of h , and observe that there is no subsequence of h leading to a limit operator.)

So by restricting ourselves to integer sequences h , we are left with at least some subclasses of rich functions in these two (practically relevant!) classes.

3.8 Admissible additive perturbations. Looking for the set of all rich functions b which do not change anything (in terms of limit operators) when used as additive perturbations, one easily arrives at the set of all bounded functions vanishing at infinity,

$$L_0^\infty := \left\{ b \in L^\infty : \operatorname{ess\,sup}_{|x|>\tau} |b(x)| \rightarrow 0 \text{ as } \tau \rightarrow \infty \right\},$$

in short, $b \in L_0^\infty$ if and only if $b \in L^\infty$ and $\|Q_{\tau C} b\|_\infty \rightarrow 0$ as $\tau \rightarrow \infty$.

Proposition 3.20. *The function $b \in L^\infty$ is rich with $\sigma^{\text{op}}(M_b) = \{0\}$ if and only if $b \in L_0^\infty$.*

Proof. If $b \in L_0^\infty$, then trivially, b is rich with all limit operators of M_b being 0. The reverse implication follows from Proposition 3.16 and Corollary 3.14 ■

It is readily seen that L_0^∞ is a closed ideal in L^∞ (and, of course, in L_\S^∞). Its elements serve as additive perturbations. For instance, instead of studying operators of multiplication by BUC-functions, one can study such by functions in $\text{BUC} + L_0^\infty$ which essentially enlarges the class of functions under consideration without changing the property of being rich or the structure of any limit operator.

3.9 Slowly oscillating and continuous functions. Sometimes, the class $\text{SOC} = \text{SO} \cap \text{BC}$ of slowly oscillating and continuous functions is of interest.

Proposition 3.21. *A slowly oscillating and continuous function is uniformly continuous, $\text{SOC} \subset \text{BUC}$.*

Proof. Although there is a direct proof (using lots of ε 's and δ 's), we will do something different: By Proposition 3.15, we have $\text{SO} \subset L_\S^\infty$. Consequently, $\text{SOC} = \text{SO} \cap \text{BC} \subset L_\S^\infty \cap \text{BC} = \text{BUC}$, by Theorem 3.9 ■

SOC comes from SO by taking intersection with BC , and SO can be derived from SOC by adding L_0^∞ :

Proposition 3.22. *The relation $\text{SOC} + L_0^\infty = \text{SO}$ holds.*

Proof. If $f \in \text{SOC}$ and $g \in L_0^\infty$, then both are in SO , and hence, $f + g \in \text{SO}$.

For the reverse inclusion, take an arbitrary $f \in \text{SO}$, and put $H = [0, 1]^n$. Define a function g as follows: In the integer points $x \in \mathbb{Z}^n$, let $g(x)$ be some value from the local essential range of f at x , and then use some interpolation idea by setting $g(x + h)$ a convex combination (with coefficients depending on $h \in H$) of the function values in the 2^n corners of the hypercube $x + H$:

$$g(x + h) := \sum_{v \in \{0,1\}^n} \left(\prod_{i=1}^n (1 - |h_i - v_i|) \right) g(x + v)$$

where $h = (h_i) \in H$ and $v = (v_i) \in \{0, 1\}^n$. Then g is continuous, and $g(x + H) \subset \text{conv } f(x + H)$ for all $x \in \mathbb{Z}^n$, by our construction. Consequently, $\text{osc}_{x+H}(g) \leq \text{osc}_{x+H}(f) \rightarrow 0$ as $x \rightarrow \infty$, whence $g \in \text{SOC}$. Finally, from $\|(f - g)|_{x+H}\|_\infty \leq \text{osc}_{x+H}(f) \rightarrow 0$ as $x \rightarrow \infty$ we get $f - g \in L_0^\infty$, whence $f \in \text{SOC} + L_0^\infty$ ■

3.10 Interplay with convolution operators. The *commutator* of two operators A, B is the operator $AB - BA$, where the operators AB and BA are referred to as *semi-commutators* of A and B . Recall that convolution with a L^1 -function is a bounded linear operator on every space L^p with $1 \leq p \leq \infty$.

In [11] Shteinberg (using somewhat different notations) studied the classes Q_{SC} and Q_{C} of functions b for which the semi-commutators and the commutators of M_b with L^1 -convolutions are compact, respectively, in all spaces L^p ($1 < p < \infty$). This question arises since firstly, multiplication and convolution operators are the essential building stones for many interesting band-dominated operators on L^p , and secondly, compact operators are rich with their operator spectrum being equal to $\{0\}$ if $1 < p < \infty$.

It turns out (see [11]) that a function $b \in L^\infty$ is in Q_{SC} if and only if for each compact $U \subset \mathbb{R}^n$,

$$\|b|_{x+U}\|_1 \rightarrow 0 \quad (x \rightarrow \infty) \quad (7)$$

where $\|b|_{x+U}\|_1$ denotes the L^1 -norm of the restriction of b to the compact $x + U$. Moreover (still citing [11]), Q_{C} is a Banach subalgebra of L^∞ , Q_{SC} is a closed ideal in L^∞ (and hence, in Q_{C}), and both algebras are related by the equality

$$Q_{\text{C}} = Q_{\text{SC}} + \text{SOC}. \quad (8)$$

Actually, (8) can be refined a little bit:

Proposition 3.23. *The relation $Q_{\text{C}} = Q_{\text{SC}} + \text{SO}$ holds.*

Proof. From $L_0^\infty \subset Q_{\text{SC}}$ by (7) and both being linear spaces, we get $Q_{\text{SC}} = Q_{\text{SC}} + L_0^\infty$. Taking this together with (8) and Proposition 3.22 we conclude $Q_{\text{C}} = Q_{\text{SC}} + \text{SOC} = Q_{\text{SC}} + L_0^\infty + \text{SOC} = Q_{\text{SC}} + \text{SO}$ ■

We will start with a description of limit operators for $b \in Q_{\text{SC}}$:

Proposition 3.24. *If $b \in Q_{\text{SC}}$, then $\sigma^{\text{op}}(M_b)$ is either $\{0\}$ or \emptyset .*

Proof. Suppose M_b has a limit operator $(M_b)_h = M_c$. We have to show that $c = 0$. For every compact $U \subset \mathbb{R}^n$, we have $\|d_m\|_\infty \rightarrow 0$ as $m \rightarrow \infty$ by (1), where $d_m = b|_{h_m+U} - c|_U$. Consequently,

$$\begin{aligned} \|c|_U\|_1 &\leq \|b|_{h_m+U}\|_1 + \|d_m\|_1 \\ &\leq \|b|_{h_m+U}\|_1 + (\text{mes } U)\|d_m\|_\infty \\ &\rightarrow 0 \end{aligned}$$

as $m \rightarrow \infty$ by (7). So $c|_U = 0$ for every compact U , i.e. $c = 0$ ■

Now we are in the position to describe the set of rich functions among Q_{SC} and Q_{C} .

Theorem 3.25. *The relations*

- a) $Q_{\text{SC}} \cap L_{\mathfrak{S}}^\infty = L_0^\infty$
- b) $Q_{\text{C}} \cap L_{\mathfrak{S}}^\infty = \text{SO}$

follow.

Proof. If $b \in Q_{\text{SC}}$ and $b \in L_{\mathfrak{S}}^{\infty}$, then $\sigma^{\text{op}}(M_b) = \{0\}$ by Proposition 3.24, and Proposition 3.20 tells that $b \in L_0^{\infty}$. The reverse inclusion is trivial.

If $f \in Q_{\text{C}} \cap L_{\mathfrak{S}}^{\infty}$, then, by Proposition 3.23, we get $f = g + h$ with $g \in Q_{\text{SC}}$ and $h \in \text{SO} \subset L_{\mathfrak{S}}^{\infty}$ by Proposition 3.15. Consequently, $f \in L_{\mathfrak{S}}^{\infty}$ implies $g = f - h \in L_{\mathfrak{S}}^{\infty}$, and from assertion a) we get $g \in L_0^{\infty} \subset \text{SO}$. So $g, h \in \text{SO}$, whence $f = g + h \in \text{SO}$. For the reverse inclusion, recall Propositions 3.23 and 3.15 ■

Having this theorem, we will try to find alternative descriptions of Q_{SC} and Q_{C} . Clearly, Q_{SC} contains L_0^{∞} . But it is strictly larger, as it also contains functions b with property (7) and

$$\|b|_{x+U}\|_{\infty} \not\rightarrow 0 \quad (x \rightarrow \infty). \quad (9)$$

We will refer to functions which are subject to (7) and (9) as *noise*. A typical example of a noise function is the characteristic function of the set $\cup_{m=1}^{\infty} [m, m + \frac{1}{m}]$ for $n = 1$. The set of all noise functions will be denoted by \mathcal{N} .

Lemma 3.26. *We have $Q_{\text{SC}} = L_0^{\infty} \uplus \mathcal{N}$ where \uplus denotes the union of two disjoint sets.*

Proof. This is trivial: The set of functions subject to (7) decomposes into those fulfilling (7) and (9), which is \mathcal{N} , and those fulfilling (7) and not (9), which is L_0^{∞} ■

Corollary 3.27. *Noise is never rich, $\mathcal{N} \cap L_{\mathfrak{S}}^{\infty} = \emptyset$.*

Proof 1. Use Lemma 3.26 and Theorem 3.25 a) ■

Proof 2. Conversely, suppose $b \in \mathcal{N}$ is rich. Take an integer sequence $h = (h_m)$ tending to infinity with $\|b|_{h_m+C}\|_{\infty}$ being bounded away from zero. But from Proposition 3.24 we know that $(M_b)_h = 0$, which clearly contradicts (1) ■

For a concise description of Q_{SC} and Q_{C} , put $\mathcal{N}_0 := \mathcal{N} \cup \{0\}$.

Theorem 3.28. *Q_{SC} and Q_{C} decompose into a rich and a noisy part by \uplus and by $+$:*

- a) $Q_{\text{SC}} = L_0^{\infty} \uplus \mathcal{N}$ b) $Q_{\text{C}} = \text{SO} \uplus (\text{SO} + \mathcal{N})$.
- c) $Q_{\text{SC}} = L_0^{\infty} + \mathcal{N}_0$ d) $Q_{\text{C}} = \text{SO} + \mathcal{N}_0$.

Proof. The proof of Assertion a) is a repetition of that of Lemma 3.26. Considering Assertion b), we recall Proposition 3.23 a) and $L_0^\infty \subset \text{SO}$ to get

$$\begin{aligned} Q_C &= Q_{\text{SC}} + \text{SO} \\ &= (L_0^\infty \cup \mathcal{N}) + \text{SO} \\ &= (L_0^\infty + \text{SO}) \cup (\mathcal{N} + \text{SO}) \\ &= \text{SO} \cup (\mathcal{N} + \text{SO}). \end{aligned}$$

Clearly, SO and $\text{SO} + \mathcal{N}$ are disjoint since the earlier functions are always rich by Proposition 3.15, and the latter are always ordinary by Proposition 3.15 and Corollary 3.27.

Now Assertion c) follows from Assertion a) and $\mathcal{N} = \mathcal{N} + L_0^\infty$ by

$$Q_{\text{SC}} = L_0^\infty \cup \mathcal{N} = L_0^\infty \cup (L_0^\infty + \mathcal{N}) = L_0^\infty + (\mathcal{N} \cup \{0\}),$$

and Assertion d) follows from Assertion b) by

$$Q_C = (\text{SO} + \mathcal{N}) \cup \text{SO} = \text{SO} + (\mathcal{N} \cup \{0\}).$$

Thus the theorem is proved ■

We conclude this journey by a simple corollary:

Corollary 3.29.

- a) If $b \in Q_{\text{SC}}$ and $(M_b)_h = M_c$ exists, then $c = 0$.
- b) If $b \in Q_C$ and $(M_b)_h = M_c$ exists, then $c = \text{const.}$

Proof. The proof of Assertion a) is a repetition of that of Proposition 3.24, and Assertion b) follows from Theorem 3.28 b) and Corollary 3.14 ■

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