

Tensor Algebras and Displacement Structure III: Asymptotic properties

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Abstract. We continue to investigate some classes of Szegő type polynomials in several variables. We focus on asymptotic properties of these polynomials and we extend several classical results of G. Szegő to this setting.

Keywords: *Spectral factorization, polynomials in several non-commuting variables, asymptotic properties*

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1. Introduction

An extension to several non-commuting variables of the Szegő orthogonal polynomials on the unit circle was considered in [6]. Some of the basic algebraic results on these polynomials were also obtained, including recurrence equations, Christoffel-Darboux formulae, and a Favard type result. Also, it was explained their connection with displacement structure theory. Our main goal is to continue to investigate this kind of polynomials, and in this paper we focus on some of their asymptotic properties. There are several fundamental results of G. Szegő involving asymptotic properties of orthogonal polynomials on the unit circle. Thus, let \mathbb{T} be the unit circle and let μ be a positive Borel measure on \mathbb{T} with $\log \mu' \in L^1$. Also let $\{\varphi_n\}_{n \geq 0}$ be the family of orthogonal polynomials associated to μ and $\varphi_n^\#(z) = z^n \varphi_n(1/\bar{z})$, $n \geq 0$. It is well-known (see [17]) that

$$\varphi_n \rightarrow 0 \tag{1}$$

and

$$\frac{1}{\varphi_n^\#} \rightarrow \Theta_\mu, \tag{2}$$

where Θ_μ is the spectral factor of μ and the convergence is uniform on the compact subsets of the unit disk \mathbb{D} . The second limit (2) is related to the so-

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called Szegő limit theorems concerning the asymptotic behaviour of Toeplitz determinants. Thus,

$$\frac{\det T_n}{\det T_{n-1}} = \frac{1}{|\varphi_n^\sharp(0)|^2},$$

where $T_n = [s_{i-j}]_{i,j=0}^n$ and $\{s_k\}_{k \in \mathbb{Z}}$ is the set of the Fourier coefficients of μ . As a consequence of the previous relation and (2) we deduce Szegő's first limit theorem,

$$\lim_{n \rightarrow \infty} \frac{\det T_n}{\det T_{n-1}} = |\Theta_\mu(0)|^2 = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log \mu'(t) dt\right). \tag{3}$$

The second (strong) Szegő limit theorem improves (3) by showing that

$$\lim_{n \rightarrow \infty} \frac{\det T_n}{g^{n+1}(\mu)} = \exp\left(\frac{1}{\pi} \int \int_{|z| \leq 1} |\Theta'_\mu(z)/\Theta_\mu(z)|^2 d\sigma(z)\right), \tag{4}$$

where $g(\mu)$ is the limit in (3) and σ is the planar Lebesgue measure. These two limits (3) and (4) have a useful interpretation in terms of asymptotics of angles in the geometry of a stochastic process associated to μ (see [12]).

Our goal in this paper is to extend these results to the class of orthogonal polynomials in several non-commuting variables introduced in [6]. The paper is organized as follows. In Section 2 we review notation and a framework for studying orthogonal polynomials associated to polynomial relations on several non-commuting variables. In Section 3 we analyse the case of no relation in dimension one. It turns out that this is, in fact, the most general situation, and for this reason we treat this case separately. The main result is Theorem 3.3, which extends the asymptotic properties (1) and (2). Theorem 3.4 contains extensions of the relations (3) and (4). In Section 4 we discuss a few examples. First, we show how to recapture the classical setting of orthogonal polynomials on the unit circle and on the real line. Then, we turn our attention to the orthogonal polynomials considered in [6].

2. Preliminaries

We introduce some necessary terminology and notation. Especially, we briefly review a rather familiar setting for orthogonal polynomials associated to relations on several variables (for some details, see [5] and [6]).

2.1. Tensor Algebras. Let \mathbb{F}_N^+ be the unital free semigroup on N generators $1, \dots, N$ with lexicographic order \prec . The empty word is the identity element, and the length of the word σ is denoted by $|\sigma|$. The length of the empty word is 0, and $l(\sigma)$ denotes the number of words $\tau \preceq \sigma$.

The tensor algebra over \mathbb{C}^N is defined by the algebraic direct sum

$$\mathcal{T}_N = \bigoplus_{k \geq 0} (\mathbb{C}^N)^{\otimes k},$$

where $(\mathbb{C}^N)^{\otimes k}$ denotes the k -fold tensor product of \mathbb{C}^N with itself. The addition is the componentwise addition and the multiplication is defined by juxtaposition:

$$(x \otimes y)_n = \sum_{k+l=n} x_k \otimes y_l.$$

If $\{e_1, \dots, e_N\}$ is the standard basis of \mathbb{C}^N , then $\{e_{i_1} \otimes \dots \otimes e_{i_k} \mid 1 \leq i_1, \dots, i_k \leq N\}$ is a basis of \mathcal{T}_N . If $\sigma = i_1 \dots i_k$ then we write e_σ instead of $e_{i_1} \otimes \dots \otimes e_{i_k}$, so that any element of \mathcal{T}_N can be uniquely written in the form $x = \sum_{\sigma \in \mathbb{F}_N^+} c_\sigma e_\sigma$, where only finitely many of the complex numbers c_σ are different from 0.

Another construction of \mathcal{T}_N is given by the algebra \mathcal{P}_N of polynomials in N noncommuting indeterminates X_1, \dots, X_N with complex coefficients. Each element $P \in \mathcal{P}_N$ can be uniquely written in the form $P = \sum_{\sigma \in \mathbb{F}_N^+} c_\sigma X_\sigma$ with $c_\sigma \neq 0$ for finitely many σ 's and $X_\sigma = X_{i_1} \dots X_{i_k}$ where $\sigma = i_1 \dots i_k \in \mathbb{F}_N^+$. The linear extension Φ_1 of the mapping $e_\sigma \rightarrow X_\sigma$, $\sigma \in \mathbb{F}_N^+$, gives an isomorphism of \mathcal{T}_N with \mathcal{P}_N .

Another known realization of the tensor algebra was used in [6] in order to establish a connection with the displacement structure theory. This was useful since many results for the tensor algebra could be seen just as particular instances of more general results in the triangular algebra. Thus, let \mathcal{E} be a Hilbert space and define: $\mathcal{E}_0 = \mathcal{E}$ and for $k \geq 1$,

$$\mathcal{E}_k = \underbrace{\mathcal{E}_{k-1} \oplus \dots \oplus \mathcal{E}_{k-1}}_{N \text{ terms}} = \mathcal{E}_{k-1}^{\oplus N}. \tag{5}$$

For $\mathcal{E} = \mathbb{C}$ we have that \mathbb{C}_k can be identified with $(\mathbb{C}^N)^{\otimes k}$, and \mathcal{T}_N is isomorphic to the algebra \mathcal{L}_N of lower triangular operators $T = [T_{k,j}] \in \mathcal{L}(\bigoplus_{k \geq 0} \mathbb{C}_k)$ with the property

$$T_{k,j} = \underbrace{T_{k-1,j-1} \oplus \dots \oplus T_{k-1,j-1}}_{N \text{ terms}} = T_{k-1,j-1}^{\oplus N} \tag{6}$$

for $k \geq j$, $k, j \geq 1$, and $T_{j,0} = 0$ for all sufficiently large j 's. The isomorphism is given by the map Φ_2 defined as follows: let $x = (x_0, x_1, \dots) \in \mathcal{T}_N$ ($x_p \in (\mathbb{C}^N)^{\otimes p}$ is the p -th homogeneous component of x); then $x_p = \sum_{|\sigma|=p} c_\sigma e_\sigma$ and for $j \geq 0$, $T_{j,0}$ is given by the column matrix $[c_\sigma]_{|\sigma|=j}^t$, where "t" denotes the matrix transpose. Then $T_{j,0} = 0$ for all sufficiently large j 's, and we can define $T \in \mathcal{L}(\bigoplus_{k \geq 0} \mathbb{C}_k)$ by using (6). Finally, set $\Phi_2(x) = T$.

2.2. Spectral Factorization. We briefly review the spectral factorization of positive definite kernels on the set \mathbb{N}_0 of nonnegative integers. For more details, see [4]. Let \mathcal{E} be a Hilbert space and let $\mathcal{P}_+(\mathcal{E})$ be the set of positive definite kernels on \mathbb{N}_0 with values in $\mathcal{L}(\mathcal{E})$. The order on $\mathcal{P}_+(\mathcal{E})$ is: $K_1 \leq K_2$ if $K_2 - K_1$ belongs to $\mathcal{P}_+(\mathcal{E})$. Next consider a family $\mathcal{F} = \{\mathcal{F}_n\}_{n \geq 0}$ of Hilbert spaces and call a lower triangular array a family $\Theta = \{\Theta_{k,j}\}_{k,j \geq 0}$ of operators $\Theta_{k,j} \in \mathcal{L}(\mathcal{E}, \mathcal{F}_k)$ such that $\Theta_{k,j} = 0$ for $k < j$ and each column $c_j(\Theta) = [\Theta_{k,j}]_{k \geq 0}$, $j \geq 0$, belongs to $\mathcal{L}(\mathcal{E}, \oplus_{k \geq j} \mathcal{F}_k)$. Denote by $H^2(\mathcal{E}, \mathcal{F})$ the set of all lower triangular arrays as above. A lower triangular array is called *outer* if the set $\{c_j(\Theta)\mathcal{E} \mid j \geq k\}$ is total in $\oplus_{j \geq k} \mathcal{F}_j$ for all $k \geq 0$. If Θ is an outer triangular array, then the formula

$$K_\Theta(k, j) = c_k(\Theta)^* c_j(\Theta)$$

defines an element of $\mathcal{P}_+(\mathcal{E})$. For the proof of the following result see [4, Chapter 5].

Theorem 2.1. *Let K be an element of $\mathcal{P}_+(\mathcal{E})$. Then there exists a family $\mathcal{F} = \{\mathcal{F}_n\}_{n \geq 0}$ of Hilbert spaces and an outer triangular array $\Theta \in H^2(\mathcal{E}, \mathcal{F})$, referred to as the spectral factor of K , such that*

- (a) $K_\Theta \leq K$.
- (b) For any other family $\mathcal{F}' = \{\mathcal{F}'_n\}_{n \geq 0}$ of Hilbert spaces and any outer triangular array $\Theta' \in H^2(\mathcal{E}, \mathcal{F}')$ such that $K_{\Theta'} \leq K$, we have $K_{\Theta'} \leq K_\Theta$.
- (c) Θ is uniquely determined by (a) and (b) up to a left unitary diagonal factor.

2.3. Orthogonal Polynomials. Let \mathcal{P}_{2N} be the algebra of polynomials in $2N$ non-commuting indeterminates $X_1, \dots, X_N, X_{N+1}, \dots, X_{2N}$ with complex coefficients. An involution \mathcal{J} can be introduced on \mathcal{P}_{2N} as follows:

$$\begin{aligned} \mathcal{J}(X_k) &= X_{N+k}, & k = 1, \dots, N \\ \mathcal{J}(X_l) &= X_{l-N}, & l = N + 1, \dots, 2N, \end{aligned}$$

on monomials

$$\mathcal{J}(X_{i_1} \dots X_{i_k}) = \mathcal{J}(X_{i_k}) \dots \mathcal{J}(X_{i_1}),$$

and finally, if $Q = \sum_{\sigma \in \mathbb{F}_{2N}^+} c_\sigma X_\sigma$, then $\mathcal{J}(Q) = \sum_{\sigma \in \mathbb{F}_{2N}^+} \bar{c}_\sigma \mathcal{J}(X_\sigma)$. Thus, \mathcal{P}_{2N} is a unital, associative $*$ -algebra over \mathbb{C} , and we notice that \mathcal{P}_N is a subalgebra of \mathcal{P}_{2N} .

We say that $\mathcal{A} \subset \mathcal{P}_{2N}$ is \mathcal{J} -symmetric if $P \in \mathcal{A}$ implies $c\mathcal{J}(P) \in \mathcal{A}$ for some $c \in \mathbb{C} - \{0\}$. We construct an associative algebra $\mathcal{T}_N(\mathcal{A})$ as the quotient of \mathcal{P}_{2N} by the two-sided ideal $\mathcal{E}(\mathcal{A})$ generated by \mathcal{A} . We notice that $\mathcal{T}_N(\emptyset) = \mathcal{P}_{2N}$. We let $\pi = \pi_{\mathcal{A}} : \mathcal{P}_{2N} \rightarrow \mathcal{T}_N(\mathcal{A})$ be the quotient map, and since \mathcal{A} is \mathcal{J} -symmetric,

$$\mathcal{J}_{\mathcal{A}}(\pi(P)) = \pi(\mathcal{J}(P)) \tag{7}$$

gives an involution on $\mathcal{T}_N(\mathcal{A})$. We will be interested in linear functionals ϕ on $\mathcal{T}_N(\mathcal{A})$ with the property that $\phi(\mathcal{J}_A(\pi(P))\pi(P)) \geq 0$ for all $P \in \mathcal{P}_N$, and we will say that ϕ is a *positive functional* on $\mathcal{T}_N(\mathcal{A})$. We notice that $\phi(\mathcal{J}_A(\pi(P))) = \overline{\phi(\pi(P))}$ for $P \in \mathcal{P}_N$ and

$$|\phi(\mathcal{J}_A(\pi(P_1))\pi(P_2))|^2 \leq \phi(\mathcal{J}_A(\pi(P_1))\pi(P_1))\phi(\mathcal{J}_A(\pi(P_2))\pi(P_2))$$

for $P_1, P_2 \in \mathcal{P}_N$.

We now consider the Gelfand-Naimark-Segal construction associated to ϕ . Thus, we define on $\pi(\mathcal{P}_N)$

$$\langle \pi(P_1), \pi(P_2) \rangle_\phi = \phi(\mathcal{J}_A(\pi(P_2))\pi(P_1)) \tag{8}$$

and factor out the subspace $\mathcal{N}_\phi = \{\pi(P) \mid P \in \mathcal{P}_N, \langle \pi(P), \pi(P) \rangle_\phi = 0\}$. Completing this quotient with respect to the norm induced by (8) we obtain a Hilbert space \mathcal{H}_ϕ . From now on we will assume that ϕ is strictly positive, that is, $\phi(\mathcal{J}_A(\pi(P))\pi(P)) > 0$ for all $P \in \mathcal{P}_N - \mathcal{E}(\mathcal{A})$, so that $\mathcal{N}_\phi = \{0\}$ and $\pi(\mathcal{P}_N)$ can be viewed as a subspace of \mathcal{H}_ϕ . The *index set of \mathcal{A}* , $G \subset \mathbb{F}_N^+$, is chosen as follows: let $\emptyset \in G$; if $\alpha \in G$, choose the next element in G to be the least $\beta \in \mathbb{F}_N^+$ such that the set of elements $\pi(X_{\alpha'}), \alpha' \preceq \alpha$, and $\pi(X_\beta)$ is linearly independent. Define $F_\alpha = \pi(X_\alpha)$ for $\alpha \in G$ and set $\mathcal{B} = \{F_\alpha\}_{\alpha \in G}$. Let $G_n = \{\alpha \in G \mid |\alpha| = n\}$, then $G_0 = \{\emptyset\}$ and $\{G_n\}_{n \geq 0}$ is a partition of G .

Since ϕ is strictly positive it follows that \mathcal{B} is a linearly independent family in \mathcal{H}_ϕ and the Gram-Schmidt procedure gives a family $\{\varphi_\alpha\}_{\alpha \in G}$ of elements in $\pi(\mathcal{P}_N) \subset \mathcal{T}_N(\mathcal{A})$ such that

$$\varphi_\alpha = \sum_{\beta \preceq \alpha} a_{\alpha,\beta} F_\beta, \quad a_{\alpha,\alpha} > 0; \tag{9}$$

$$\langle \varphi_\alpha, \varphi_\beta \rangle_\phi = \delta_{\alpha,\beta}, \quad \alpha, \beta \in G. \tag{10}$$

The elements $\varphi_\alpha, \alpha \in G$, will be called the *orthogonal polynomials* associated to ϕ . Typically, the theory of orthogonal polynomials deals with the study of algebraic and asymptotic properties of the orthogonal polynomials associated to strictly positive functionals on $\mathcal{T}_N(\mathcal{A})$. We also notice that the use of the Gram-Schmidt process depends on the order that we have chosen on \mathbb{F}_N^+ . A different order would yield a different family of orthogonal polynomials. Due to the natural grading on \mathbb{F}_N^+ it is possible to develop a base free approach to orthogonal polynomials. In the case of orthogonal polynomials in several commuting variables this is presented in [8]. However, in the present paper we stick to the lexicographic order on \mathbb{F}_N^+ (and on the index set G).

An explicit formula for the orthogonal polynomials can be obtained in the same manner as in the classical (one scalar variable) case. Define

$$s_{\alpha,\beta} = \phi(\mathcal{J}_A(F_\alpha)F_\beta) = \langle F_\beta, F_\alpha \rangle_\phi, \quad \alpha, \beta \in G, \tag{11}$$

and

$$D_\alpha = \det [s_{\alpha',\beta'}]_{\alpha',\beta' \preceq \alpha} > 0, \quad \alpha \in G. \tag{12}$$

We notice that ϕ is a positive functional on $\mathcal{T}_N(\mathcal{A})$ if and only if $K_\phi(\alpha, \beta) = s_{\alpha,\beta}$, $\alpha, \beta \in G$, is a positive definite kernel on G . While the kernel K_ϕ characterizes the positivity of the functional ϕ and contains the basic information for the construction of the orthogonal polynomials, in general it does not determine ϕ uniquely. We will occasionally say that the orthogonal polynomials are associated to the kernel K_ϕ rather than ϕ itself. One typical situation when K_ϕ determines ϕ is when $\{\mathcal{J}(X_k) - X_k \mid k + 1, \dots, N\} \subset \mathcal{A}$; another example is provided by the Wick polynomials

$$X_i \mathcal{J}(X_j) - \delta_{ij} + \sum_{k,l=1}^N T_{ij}^{kl} \mathcal{J}(X_l) X_k, \quad i, j = 1, \dots, N,$$

where T_{ij}^{kl} are complex numbers and δ_{ij} is the Kronecker symbol (see [14]).

From now on, $\tau - 1$ denotes the predecessor of τ with respect to the lexicographic order on \mathbb{F}_N^+ , while $\sigma + 1$ denotes the successor of σ . It is shown in [5] that $\varphi_\emptyset = s_{\emptyset,\emptyset}^{-\frac{1}{2}}$ and for $\emptyset \prec \alpha$,

$$\varphi_\alpha = \frac{1}{\sqrt{D_{\alpha-1} D_\alpha}} \det \begin{bmatrix} [s_{\alpha',\beta'}]_{\alpha' \prec \alpha; \beta' \preceq \alpha} \\ F_\emptyset \quad \dots \quad F_\alpha \end{bmatrix}, \tag{13}$$

with an appropriate interpretation of the determinant. In most of the cases, the formula (13) is not very useful for the actual computation of the orthogonal polynomials or for their study. Instead of (13) there are used recurrence formulae. We discuss several examples in the next sections.

3. The case $\mathcal{A} = \emptyset$, $N = 1$

It turns out that this is, in fact, the most general situation. For this reason we treat this case separately. For $\mathcal{A} = \emptyset$ and $N = 1$, the index set is $G = \mathbb{N}_0$ and a linear functional on \mathcal{P}_2 is positive if and only if $K_\phi(n, m) = \phi(\mathcal{J}(X_1^n) X_1^m)$, $n, m \in \mathbb{N}_0$, is a positive definite kernel on \mathbb{N}_0 . However, K_ϕ does not completely determine ϕ . For example, there is no way to deduce $\phi(X_1 \mathcal{J}(X_1))$ from K_ϕ in general. Still, we notice that there is no other restriction on K_ϕ in the sense that given a positive definite kernel K on \mathbb{N}_0 , there exist positive functionals ϕ on \mathcal{P}_2 such that $K_\phi = K$. This could be done as follows. Let ϕ_0 be a complex valued function defined on monomials X_σ , $\sigma \in \mathbb{F}_2^+$, such that $\phi_0(X_2^n X_1^m) = K(n, m)$, $n, m \in \mathbb{N}_0$. Then ϕ_0 can be extended by linearity to a positive linear functional ϕ on \mathcal{P}_2 which has the property that $K_\phi = K$.

We now review a certain structure of the kernel K_ϕ associated to a strictly positive functional ϕ on \mathcal{P}_2 . More details can be found in [4] and [6]. According to Theorem 1.5.3 in [4], the kernel K_ϕ is uniquely determined by a family $\{\gamma_{k,j}\}_{0 \leq k < j}$ of complex numbers with the property that $|\gamma_{k,j}| < 1$ for $0 \leq k < j$. The coefficients $s_{k,j} = K_\phi(k, j)$ of the kernel K_ϕ can be explicitly computed in terms of the numbers $\gamma_{k,j}$. Thus, we let $J(\gamma_{l,m})$ denote the *Julia operator* associated to $\gamma_{l,m}$ by the formula

$$J(\gamma_{l,m}) = \begin{bmatrix} \gamma_{l,m} & d_{l,m} \\ d_{l,m} & -\bar{\gamma}_{l,m} \end{bmatrix},$$

where $d_{k,j} = (1 - |\gamma_{k,j}|^2)^{\frac{1}{2}}$. Then we define $(j - k + 1) \times (j - k + 1)$ unitary matrices $U_{k,j}$ recursively: $U_{k,k} = 1$, and for $k < j$,

$$U_{k,j} = (J(\gamma_{k,k+1}) \oplus 1_{n-1})(1 \oplus J(\gamma_{k,k+2}) \oplus 1_{n-2})(1_{n-1} \oplus J(\gamma_{k,j}))(U_{k+1,j} \oplus 1),$$

where 1_m denotes the $m \times m$ identity matrix. With this notation it can be shown that for $k < j$,

$$K_\phi(k, j) = s_{k,j} = s_{k,k}^{\frac{1}{2}} s_{j,j}^{\frac{1}{2}} \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} U_{k,j} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{14}$$

For illustration, we can write the formula in several particular cases. For instance,

$$\begin{aligned} s_{01} &= s_{00}^{\frac{1}{2}} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \gamma_{01} & d_{01} \\ d_{01} & -\bar{\gamma}_{01} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} s_{11}^{\frac{1}{2}} = s_{00}^{\frac{1}{2}} \gamma_{01} s_{11}^{\frac{1}{2}}; \\ s_{02} &= s_{00}^{\frac{1}{2}} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \\ &\quad \times \begin{bmatrix} \gamma_{01} & d_{01} & 0 \\ d_{01} & -\bar{\gamma}_{01} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \gamma_{02} & d_{02} \\ 0 & d_{02} & -\bar{\gamma}_{02} \end{bmatrix} \\ &\quad \times \begin{bmatrix} \gamma_{12} & d_{12} & 0 \\ d_{12} & -\bar{\gamma}_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} s_{22}^{\frac{1}{2}} \\ &= s_{00}^{\frac{1}{2}} (\gamma_{01} \gamma_{12} + d_{01} \gamma_{02} d_{12}) s_{22}^{\frac{1}{2}}; \end{aligned}$$

$$\begin{aligned}
 s_{03} &= s_{00}^{\frac{1}{2}} \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \\
 &\times \begin{bmatrix} \gamma_{01} & d_{01} & 0 & 0 \\ d_{01} & -\bar{\gamma}_{01} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_{02} & d_{02} & 0 \\ 0 & d_{02} & -\bar{\gamma}_{02} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_{03} & d_{03} \\ 0 & 0 & d_{03} & -\bar{\gamma}_{03} \end{bmatrix} \\
 &\times \begin{bmatrix} \gamma_{12} & d_{12} & 0 & 0 \\ d_{12} & -\bar{\gamma}_{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \gamma_{13} & d_{13} & 0 \\ 0 & d_{13} & -\bar{\gamma}_{13} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 &\times \begin{bmatrix} \gamma_{23} & d_{23} & 0 & 0 \\ d_{23} & -\bar{\gamma}_{23} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s_{33}^{\frac{1}{2}} \\
 &= s_{00}^{\frac{1}{2}} (\gamma_{01}\gamma_{12}\gamma_{23} + \gamma_{01}d_{12}\gamma_{13}d_{23} + d_{01}\gamma_{02}d_{12}\gamma_{23} \\
 &\quad - d_{01}\gamma_{02}\bar{\gamma}_{12}\gamma_{13}d_{23} + d_{01}d_{02}\gamma_{03}d_{13}d_{23}) s_{33}^{\frac{1}{2}}.
 \end{aligned}$$

A natural combinatorial question related to (14) would be the calculation of the number $N(s_{k,j})$ of additive terms in the expression of $s_{k,j}$. Thus, for $k \geq 0$,

$$\begin{aligned}
 N(s_{01}) &= N(s_{k,k+1}) = 1, \\
 N(s_{02}) &= N(s_{k,k+2}) = 2, \\
 N(s_{03}) &= N(s_{k,k+3}) = 5.
 \end{aligned}$$

The general formula is given by the following result.

Theorem 3.1. $N(s_{k,k+l})$ is given by the Catalan number $\frac{1}{l+1} \binom{2l}{l}$.

Proof. The first step of the proof considers the realization of $s_{k,j}$ through a time varying transmission line (or lattice) (see [4, Chapter 4] for more details). For illustration we consider the case of s_{03} in Figure 1. Each box in Figure 1 represents the action of the unitary matrix

$$\begin{bmatrix} \gamma_{k,j} & d_{k,j} \\ d_{k,j} & -\bar{\gamma}_{k,j} \end{bmatrix},$$

and we see in Figure 1 that the number of additive terms in the formula of s_{03} is given by the number of paths from A to B . It is clear that to each path from A

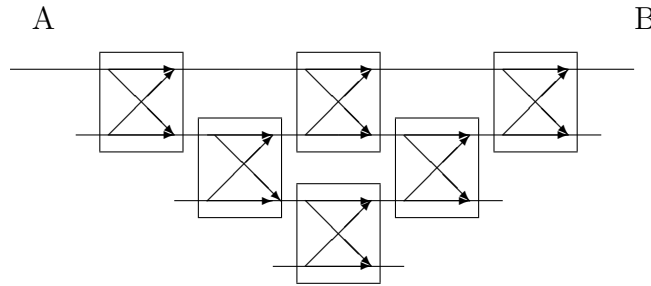


Figure 1: Lattice representation for s_{03}

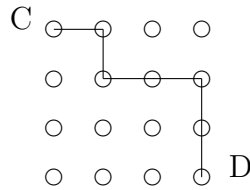


Figure 2: A Dyck path from C to D

to B in Figure 1 it corresponds a so-called Dyck path from C to D in Figure 2, that is a path that never steps below the diagonal and goes only to the right or downward.

More precisely, each box in Figure 1 corresponds to a point strictly above the diagonal in Figure 2. Once this one-to-one correspondence is established, we can use the well-known fact that the number of Dyck paths like the one in Figure 2 is given exactly by the Catalan numbers. ■

Returning to orthogonal polynomials, we notice that they obey the following recurrence relations (see [6, formulae (3.10) and (3.11)]):

$$\varphi_0(X, l) = \varphi_0^\sharp(X, l) = s_{l,l}^{-\frac{1}{2}}, \quad l \in \mathbb{N}_0, \tag{15}$$

and for $n \geq 1, l \in \mathbb{N}_0$,

$$\varphi_n(X, l) = \frac{1}{d_{l,n+l}} \left(X \varphi_{n-1}(X, l+1) - \gamma_{l,n+l} \varphi_{n-1}^\sharp(X, l) \right) \tag{16}$$

$$\varphi_n^\sharp(X, l) = \frac{1}{d_{l,n+l}} \left(-\bar{\gamma}_{l,n+l} X \varphi_{n-1}(X, l+1) + \varphi_{n-1}^\sharp(X, l) \right) \tag{17}$$

where $\varphi_n(X) = \varphi_n(X, 0)$ and $\varphi_n^\sharp(X) = \varphi_n^\sharp(X, 0)$. Formula (14) provides a possibility to recover the numbers $\gamma_{k,j}$ from the kernel K_ϕ . However, in the context of this paper it is more useful to recover those numbers directly from the orthogonal polynomials.

It follows from the proof of Theorem 3.2 in [6] that $\{\varphi_n(X, l)\}_{n \geq 0}$ is the family of orthogonal polynomials associated to the kernel $K_\phi^l(\alpha, \beta) = s_{\alpha+l, \beta+l}$, $\alpha, \beta \in \mathbb{N}_0$. Let k_n^l be the leading coefficient of $\varphi_n(X, l)$. We obtain the following formula for the parameters $\gamma_{k,j}$.

Theorem 3.2. *For $l \in \mathbb{N}_0$ and $n \geq 1$ holds*

$$\gamma_{l,n+l} = -\varphi_n(0, l) \frac{k_0^{l+1} \cdots k_{n-1}^{l+1}}{k_0^l \cdots k_n^l}.$$

Proof. We deduce from (16) that

$$\varphi_n(0, l) = -\frac{\gamma_{l,n+l}}{d_{l,n+l}} \varphi_{n-1}^\#(0, l),$$

and from formula (17) we deduce

$$\varphi_n^\#(0, l) = \frac{1}{d_{l,n+l}} \varphi_{n-1}^\#(0, l) = \dots = s_{l,l}^{-\frac{1}{2}} \prod_{p=1}^n \frac{1}{d_{l,p+l}},$$

hence

$$\varphi_n(0, l) = -s_{l,l}^{-\frac{1}{2}} \gamma_{l,n+l} \prod_{p=1}^n \frac{1}{d_{l,p+l}}.$$

Let $D_{m,l}$ denote the determinant of the matrix $[s_{k,j}]_{l \leq k, j \leq m}$. Using Theorem 1.5.10 in [4], we deduce that

$$\prod_{p=1}^n d_{l,p+l}^2 = s_{l,l}^{-1} \frac{D_{l,l+n}}{D_{l+1,l+n}}$$

so that,

$$\gamma_{l,n+l} = -\varphi_n(0, l) \sqrt{\frac{D_{l,l+n}}{D_{l+1,l+n}}}. \tag{18}$$

On the other hand, (16) gives that

$$k_n^l = s_{l+n,l+n}^{-\frac{1}{2}} \prod_{p=1}^{n-1} \frac{1}{d_{l+p,l+n}}, \quad n \geq 1.$$

Using once again Theorem 1.5.10 in [4], we deduce

$$k_n^l = \sqrt{\frac{D_{l,l+n-1}}{D_{l,l+n}}}, \quad n \geq 1.$$

This implies that

$$k_0^l \cdots k_n^l = \frac{1}{\sqrt{D_{l,l+n}}},$$

and this can be used in (18) in order to conclude the proof. ■

We now develop an analogue of the asymptotic properties (1) and (2). The formulae (16) and (17) suggest that it is more convenient to work in a larger algebra. Thus, we consider the set \mathcal{R}_1 of lower triangular arrays $a = [a_{k,j}]_{k,j \geq 0}$ with complex entries. No boundedness assumption is made on these arrays. The addition in \mathcal{R}_1 is defined by entry-wise addition and the multiplication is defined as follows: for two elements $a = [a_{k,j}]_{k \geq j}, b = [b_{k,j}]_{k,j \geq 0}$ of \mathcal{R}_1 ,

$$(ab)_{k,j} = \sum_{l \geq 0} a_{k,l} b_{l,j}$$

(the sum is finite since both a and b are lower triangular). Thus, \mathcal{R}_1 becomes an associative, unital algebra.

Next we associate the element Φ_n of \mathcal{R}_1 to the polynomials $\varphi_n(X, l) = \sum_{k=0}^n a_{n,k}^l X^k, n, l \geq 0$, by the formula

$$(\Phi_n)_{k,j} = \begin{cases} a_{n,k-j}^j & k \geq j \\ 0 & k < j \end{cases} \tag{19}$$

similarly, the element Φ_n^\sharp of \mathcal{R}_1 is associated to the family of polynomials $\varphi_n^\sharp(X, l) = \sum_{k=0}^n b_{n,k}^l X^k, n, l \geq 0$, by the formula

$$(\Phi_n^\sharp)_{k,j} = \begin{cases} b_{n,k-j}^j & k \geq j \\ 0 & k < j. \end{cases} \tag{20}$$

We notice that since K_ϕ is a scalar-valued kernel, the Hilbert spaces $\mathcal{F}_n, n \geq 0$, given by Theorem 2.1 are at most one-dimensional (see [4, Section 5.1] for details). This implies that we can uniquely determine the spectral factor Θ_ϕ of K_ϕ by the requirement that $(\Theta_\phi)_{n,n} \geq 0$ for all $n \geq 0$. Also, $\Theta_\phi \in \mathcal{R}_1$. From now on we assume that $\inf_{n \geq 0} (\Theta_\phi)_{n,n} > 0$, and we say that in this case ϕ (or K_ϕ) belongs to the Szegö class. By formula (5.1.5) in [4] it follows that ϕ belongs to the Szegö class if and only if

$$\inf_{k \geq 0} s_{k,k}^{\frac{1}{2}} \prod_{n > k} d_{k,n} > 0. \tag{21}$$

This implies, in particular, that Φ_n^\sharp is invertible in \mathcal{R}_1 for all $n \geq 0$. Finally, we say that a sequence $\{a_n\} \subset \mathcal{R}_1$ converges to $a \in \mathcal{R}_1$ if $\{(a_n)_{k,j}\}$ converges to $a_{k,j}$ for all $k, j \geq 0$ (and we write $a_n \rightarrow a$). We now obtain the following generalization of (1) and (2).

Theorem 3.3. *Let ϕ belong to the Szegö class. Then*

$$\Phi_n \rightarrow 0 \tag{22}$$

and

$$(\Phi_n^\sharp)^{-1} \rightarrow \Theta_\phi. \tag{23}$$

Proof. First we show (22). It is convenient to consider the natural derivation on \mathcal{P}_1 ; for $P = \sum_{k=0}^n a_k X^k \in \mathcal{P}_1$,

$$P^{(1)} = \sum_{k=1}^n k a_k X^{k-1},$$

and then, for $k \geq 1$,

$$P^{(k)} = (P^{(k-1)})^{(1)}.$$

We see that (22) is equivalent to

$$\varphi_n^{(k)}(0, l) \rightarrow 0$$

for each fixed $k, l \geq 0$. We claim that

$$\sum_{n \geq 0} |\varphi_n^{(k)}(0, l)|^2 < \infty \tag{24}$$

and for each $l, k \geq 0$,

$$\lim_{n \rightarrow \infty} (\varphi_n^\#)^{(k)}(0, l) \text{ exists and is finite.} \tag{25}$$

We prove these statements by induction on $k \geq 0$. For $k = 0$ we use the formula

$$\varphi_n(0, l) = -s_{l,l}^{-\frac{1}{2}} \gamma_{l,n+l} \prod_{p=1}^n \frac{1}{d_{l,p+l}}.$$

obtained in the proof of Theorem 3.2 in order to deduce that

$$\sum_{n \geq 0} |\varphi_n(0, l)|^2 = s_{l,l}^{-1} \sum_{n \geq 0} |\gamma_{l,n+l}|^2 \prod_{p=1}^n \frac{1}{d_{l,p+l}^2}.$$

Since ϕ belongs to the Szegő class, we have that

$$g_l = s_{l,l}^{\frac{1}{2}} \prod_{n>l} d_{l,n} > 0.$$

We deduce that $\prod_{p=1}^n \frac{1}{d_{l,p+l}^2} \leq c_l$ for some $c_l > 0$ and all $n \geq 0$. Also, for all $n \geq 0$,

$$\sum_{n \geq 0} |\gamma_{l,n+l}|^2 < \infty.$$

In particular, $\gamma_{l,n+l} \rightarrow 0$ as $n \rightarrow \infty$; all of these give relations (24) and (25) for $k = 0$ and all $l \geq 0$.

We now proceed to prove the general case. From (16) and (17) we also deduce

$$\varphi_n^{(k)}(0, l) = \frac{1}{d_{l,n+l}} \left(k\varphi_{n-1}^{(k-1)}(0, l+1) - \gamma_{l,n+l}(\varphi_{n-1}^\#)^{(k)}(0, l) \right) \quad (26)$$

$$(\varphi_n^\#)^{(k)}(0, l) = \frac{1}{d_{l,n+l}} \left(-k\bar{\gamma}_{l,n+l}\varphi_{n-1}^{(k-1)}(0, l+1) + (\varphi_{n-1}^\#)^{(k)}(0, l) \right) \quad (27)$$

for $k \geq 1$.

Since $k \geq 1$, $(\varphi_0^\#)^{(k)}(0, l) = 0$, and we deduce from (27) that

$$(\varphi_n^\#)^{(k)}(0, l) = -k \left(\prod_{p=1}^n \frac{1}{d_{l,p+l}} \right) \sum_{j=1}^n \bar{\gamma}_{l,j+l} \left(\prod_{q=1}^{j-1} d_{l,q+l} \right) \varphi_{j-1}^{(k-1)}(0, l+1),$$

with the convention that $\prod_{q=1}^0 d_{l,q+l} = 1$. By Schwarz inequality,

$$\begin{aligned} & \sum_{j \geq 1} |\bar{\gamma}_{l,j+l} \left(\prod_{q=1}^{j-1} d_{l,q+l} \right) \varphi_{j-1}^{(k-1)}(0, l+1)| \\ & \leq \left(\sum_{j \geq 1} |\gamma_{l,j+l}|^2 \prod_{q=1}^{j-1} d_{l,q+l}^2 \right)^{\frac{1}{2}} \left(\sum_{j \geq 1} |\varphi_{j-1}^{(k-1)}(0, l+1)|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Again, since ϕ belongs to the Szegö class and $\prod_{q=1}^{j-1} d_{l,q+l}^2 \leq 1$, we deduce that

$$\begin{aligned} & \sum_{j \geq 1} |\bar{\gamma}_{l,j+l} \varphi_{j-1}^{(k-1)}(0, l+1) \prod_{q=1}^{j-1} d_{l,q+l}| \\ & \leq C \left(\sum_{j \geq 1} |\varphi_{j-1}^{(k-1)}(0, l+1)|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

This and the induction hypothesis give that the series

$$\sum_{j \geq 1} \bar{\gamma}_{l,j+l} \left(\prod_{q=1}^{j-1} d_{l,q+l} \right) \varphi_{j-1}^{(k-1)}(0, l+1)$$

converges absolutely and since

$$\lim_{n \rightarrow \infty} \prod_{p=1}^n \frac{1}{d_{l,p+l}} = \frac{s_{l,l}^{\frac{1}{2}}}{g_l} < \infty,$$

we deduce that $\lim_{n \rightarrow \infty} (\varphi_n^\#)^{(k)}(0, l)$ exists and is finite.

Using (26), we get

$$\begin{aligned} & \sum_{n \geq 1} |\varphi_n^{(k)}(0, l)|^2 \\ & \leq k^2 \sum_{n \geq 1} \frac{1}{d_{l, n+l}^2} |\varphi_{n-1}^{(k-1)}(0, l+1)|^2 \\ & \quad + 2k \sum_{n \geq 1} \frac{1}{d_{l, n+l}^2} |\varphi_{n-1}^{(k-1)}(0, l+1) \gamma_{l, n+l} (\varphi_{n-1}^\#)^{(k)}(0, l)| \\ & \quad + \sum_{n \geq 1} \frac{1}{d_{l, n+l}^2} |\gamma_{l, n+l}|^2 |(\varphi_{n-1}^\#)^{(k)}(0, l)|^2. \end{aligned}$$

Since for sufficiently large n , $d_{l, n+l} \geq C_l > 0$ and $|(\varphi_{n-1}^\#)^{(k)}(0, l)| \leq C'_l$, another application of the Schwarz inequality, the fact that ϕ belongs to the Szegő class and the induction hypothesis give that $\sum_{n \geq 1} |\varphi_n^{(k)}(0, l)|^2 < \infty$. In particular, $\varphi_n^{(k)}(0, l) \rightarrow 0$ as $n \rightarrow \infty$, concluding the proof of (22).

A convenient proof of (23) can be based on the so-called Toeplitz embedding, systematically used in [9]. This approach would also explain the meaning of the elements $\Phi_n, \Phi_n^\#$ of \mathcal{R}_1 . We assume, without loss of generality, that $s_{l,l} = 1$ for all $l \geq 0$ and define, for $n \geq 1$,

$$(\Gamma_n)_{k,j} = \begin{cases} \gamma_{k,j} & j = k + n \\ 0 & \text{otherwise.} \end{cases}$$

Then, let $A = [A_{j-k}]_{k,j \geq 0}$ be the positive definite Toeplitz kernel associated by Proposition 1.5.6 in [4] to $\{\Gamma_n\}_{n \geq 1}$ and $A_0 = I$. Since ϕ belongs to the Szegő class, each Γ_n is a strict contraction. Therefore, right operator-valued orthogonal polynomials can be associated to the kernel A (see [1] for some details). By Proposition 1.6.10 (a) in [4], K_ϕ is just a compression of the kernel A . By Proposition 1.6.10 (b) and Theorem 5.1.2 in [4], the spectral factor Θ_ϕ of K_ϕ is a corresponding compression of the spectral factor of A . The key point of the proof is the connection between Φ_n and the right orthogonal polynomials of A . Thus, let $\{R_n\}_{n \geq 0}$ be the set of the right orthogonal polynomials of A ,

$$R_n(z) = \sum_{k=0}^n R_{n,k} z^k, \quad R_{n,n} \geq 0,$$

and define $R_n^\# = z^n R_n(1/\bar{z})^* = \sum_{k=0}^n n R_{n, n-k} z^k$. Also, define

$$\rho_n = [R_{n,n}^*, \dots, R_{n,0}^*]^t$$

and let $\tilde{\rho}_n$ be obtained by the canonical reshuffle of ρ_n ([15], Chapter 7). Then,

$$\Phi_n^\# = \begin{bmatrix} I_n & 0 \\ 0 & \tilde{\rho}_n \end{bmatrix},$$

where I_n denotes the $n \times n$ identity matrix. This relation can be easily checked by using the characterization of

$$[R_{n,0}, \dots, R_{n,n}]^t$$

as the unique solution of

$$[A_{j-k}]_{0 \leq k, j \leq n} [R_{n,0}, \dots, R_{n,n}]^t = [0, \dots, 0, D_n]^t,$$

where D_n is a positive operator, and the orthogonality properties of $\{\Phi_n\}_{n \geq 0}$. By Theorem 4.37 in [1], the coefficients of $(R_n^\sharp)^{-1}$ converge in the strong operator topology to the corresponding coefficients of the spectral factor of A . The above relations between the spectral factor of A and the spectral factor of K_ϕ , and between the coefficients of $(\Phi_n^\sharp)^{-1}$ and the coefficients of $(R_n^\sharp)^{-1}$ imply that $(\Phi_n^\sharp)^{-1} \rightarrow \Theta_\phi$. ■

In order to provide generalizations of Szegő's limit theorems (3) and (4) in this setting we consider first their geometrical interpretation. Thus, by a result of Kolmogorov, K_ϕ is the covariance kernel of a stochastic process $\{f_n\}_{n \geq 0} \subset L^2(\mu)$ for some probability space (X, \mathcal{M}, μ) . That is,

$$K_\phi(m, n) = \int_X f_n \bar{f}_m d\mu.$$

We can suppose, without loss of generality, that $\{f_n\}_{n \geq 0}$ is total in $L^2(\mu)$ and for $p \leq q$ we introduce the subspaces $\mathcal{H}_{p,q}$ given by the closure in $L^2(\mu)$ of the linear span of $\{f_k\}_{k=p}^q$.

The operator angle between two spaces \mathcal{E}_1 and \mathcal{E}_2 of $L^2(\mu)$ is defined by

$$B(\mathcal{E}_1, \mathcal{E}_2) = P_{\mathcal{E}_1} P_{\mathcal{E}_2} P_{\mathcal{E}_1},$$

where $P_{\mathcal{E}_1}$ is the orthogonal projection of $L^2(\mu)$ onto \mathcal{E}_1 . Also define

$$\Delta(\mathcal{E}_1, \mathcal{E}_2) = I - B(\mathcal{E}_1, \mathcal{E}_2).$$

The geometric interpretation of the limits (3) and (4) is discussed in [12] and nonstationary extensions are presented in [4], Chapter 6. The interpretation of the second Szegő limit theorem in [4] required a stochastic process indexed by the set of integers, which is not the case in our situation. So, we need a modification of that interpretation that fits into our setting. Thus, we consider first the scale of limits:

$$s - \lim_{r \rightarrow \infty} \Delta(\mathcal{H}_{0,n}, \mathcal{H}_{n+1,r}) = \Delta(\mathcal{H}_{0,n}, \mathcal{H}_{n+1,\infty}) \tag{28}$$

for $n \geq 0$, and then we let $n \rightarrow \infty$ and deduce

$$s - \lim_{n \rightarrow \infty} \Delta(\mathcal{H}_{0,n}, \mathcal{H}_{n+1,\infty}) = \Delta(\mathcal{H}_{0,\infty}, \bigcap_{n \geq 0} \mathcal{H}_{n,\infty}), \tag{29}$$

where $s - \lim$ denotes the strong operatorial limit.

We then deduce analogues of the Szegő limit theorems by expressing these limits in terms of the determinants $D_{r,q} = \det [K_\phi(r', q')]_{r \leq r', q' \leq q}$, $r \leq q$.

Theorem 3.4. *Let ϕ belong to the Szegő class. Then*

$$\frac{D_{r,q}}{D_{r+1,q}} = s_{r,r} \det \Delta(\mathcal{H}_{r,r}, \mathcal{H}_{r+1,q}) = \frac{1}{|\varphi_{q-r}^\#(0, r)|^2}, \tag{30}$$

$$\lim_{q \rightarrow \infty} \frac{D_{r,q}}{D_{r+1,q}} = s_{r,r} \det \Delta(\mathcal{H}_{r,r}, \mathcal{H}_{r+1,\infty}) = |\Theta_\phi(r, r)|^2 = s_{r,r} \prod_{j \geq 1} d_{r,r+j}^2. \tag{31}$$

If we define

$$L = \lim_{n \rightarrow \infty} \prod_{0 \leq k < n < j} d_{k,j}^2 > 0,$$

then

$$\lim_{n \rightarrow \infty} \frac{D_{0,n}}{\prod_{l=0}^n g_l^2} = \frac{1}{\det \Delta(\mathcal{H}_{0,\infty}, \cap_{n \geq 0} \mathcal{H}_{n,\infty})} = \frac{1}{L}. \tag{32}$$

Proof. The connection between the operator angles and determinants of type $D_{r,q}$ is given by the following formula which is a consequence of Lemma 6.4.1 in [4]: for $r \leq l \leq q$,

$$\det \Delta(\mathcal{H}_{r,l}, \mathcal{H}_{l+1,q}) = \frac{D_{r,q}}{D_{r,l} D_{l+1,q}}. \tag{33}$$

Then Theorem 1.5.10 in [4] allows the computation of $D_{r,q}$ in terms of the parameters $\gamma_{i,j}$. Noticing that $D_{r,r} = s_{r,r}$ and using the formula $\varphi_q^\#(0, r) = s_{r,r}^{-\frac{1}{2}} \prod_{l=1}^q \frac{1}{d_{r,r+l}}$ obtained in the proof of Theorem 3.3, we deduce that

$$\begin{aligned} \frac{D_{r,q}}{D_{r+1,q}} &= s_{r,r} \det \Delta(\mathcal{H}_{r,r}, \mathcal{H}_{r+1,q}) \\ &= s_{r,r} \frac{\prod_{r \leq k < j \leq q} d_{k,j}^2}{\prod_{r+1 \leq k \leq j \leq q} d_{k,j}^2} \\ &= s_{r,r} \prod_{j=1}^{q-r} d_{r,r+j}^2 \\ &= \frac{1}{|\varphi_{q-r}^\#(0, r)|^2}, \end{aligned}$$

which is (30). This relation and Theorem 6.2.2 in [4] imply (31).

Using again (33), we deduce for $n < r$, that

$$\det \Delta(\mathcal{H}_{0,n}, \mathcal{H}_{n+1,r}) = \frac{D_{0,r}}{D_{0,n}D_{n+1,r}} = \prod_{0 \leq k \leq n < j \leq r} d_{k,j}^2,$$

hence

$$\det \Delta(\mathcal{H}_{0,n}, \mathcal{H}_{n+1,\infty}) = \lim_{r \rightarrow \infty} \det \Delta(\mathcal{H}_{0,n}, \mathcal{H}_{n+1,r}) = \prod_{0 \leq k \leq n < j} d_{k,j}^2.$$

On the other hand,

$$\begin{aligned} \frac{\prod_{l=0}^n g_l^2}{D_{0,n}} &= \frac{\prod_{l=0}^n \prod_{j \geq 1} d_{l,l+j}^2}{\prod_{0 \leq k < j \leq n} d_{k,j}^2} \\ &= \prod_{0 \leq k \leq n < j} d_{k,j}^2, \end{aligned}$$

which shows that

$$\det \Delta(\mathcal{H}_{0,n}, \mathcal{H}_{n+1,\infty}) = \frac{\prod_{l=0}^n g_l^2}{D_{0,n}},$$

hence relation (32) holds. ■

Formula (31) would represent an analogue of (3), while (32) is an analogue of (4). It would be of interest to express the limit in (32) in terms of the spectral factor Θ_ϕ .

4. Some examples

4.1. Polynomials on the unit circle. Consider $\mathcal{A} = \{1 - \mathcal{J}(X_1)X_1\}$. In this case the index set is \mathbb{N}_0 , and if ϕ is a linear functional on $\mathcal{T}_1(\mathcal{A})$, then

$$K_\phi(n + k, m + k) = K_\phi(n, m), \quad m, n, k \in \mathbb{N}_0,$$

which means that K_ϕ is a Toeplitz kernel. It turns out that the parameters $\{\gamma_{k,j}\}$ also satisfy the Toeplitz condition $\gamma_{n+k,m+k} = \gamma_{n,m}$, $n < m$, $k \geq 1$. The orthogonal polynomials associated to ϕ are then the orthogonal polynomials on the unit circle and (16), (17) reduce to the classical recurrence equations in [17]. Also, Theorem 3.3 and Theorem 3.4 reduce to the classical results of Szegő [17].

4.2. Polynomials on the real line. Consider $\mathcal{A} = \{X_1 - \mathcal{J}(X_1)\}$. In this case the index set is still \mathbb{N}_0 , and if ϕ is a linear functional on $\mathcal{T}_1(\mathcal{A})$, this time the kernel K_ϕ has the Hankel property, that is

$$K_\phi(n, m + k) = K_\phi(n + k, m), \quad m, n, k \in \mathbb{N}_0.$$

The parameters $\{\gamma_{k,j}\}$ do not necessarily satisfy a similar Hankel property. In fact, it might be interesting to find a characterization of those families of parameters $\{\gamma_{k,j}\}$ producing Hankel forms. Traditionally, there are other parameters, usually called canonical moments, that are used. The canonical moments of ϕ can be calculated by using a Q - D (quotient-difference) algorithm (see [13]). Also, the recurrence formulas of type (16), (17) are replaced by a three-term recurrence equation,

$$x\varphi_n(x) = b_n\varphi_{n+1}(x) + a_n\varphi_n(x) + b_{n-1}\varphi_{n-1}(x), \tag{34}$$

with initial conditions $\varphi_{-1} = 0, \varphi_0 = 1$ ([17]).

Still, parameters $\{\gamma_{k,j}\}$ can be associated such that (16), (17) hold. Also, Theorem 3.3 and Theorem 3.4 provide asymptotic properties of the orthogonal polynomials and, respectively, Hankel determinants in the corresponding Szegő class.

We consider an example computing the parameters $\gamma_{k,j}$ of the Hilbert matrix,

$$H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and notice that the associated orthogonal polynomials satisfy the three-term recurrence equation (34) with

$$b_{n-1} = \frac{n}{2\sqrt{4n^2 - 1}}, \quad n \geq 1,$$

and $a_n = \frac{1}{2}, n \geq 0$. For example, the first 5 polynomials are:

$$\varphi_{-1} = 0, \quad \varphi_0 = 1, \quad \varphi_1(x) = \sqrt{3}(2x - 1),$$

$$\varphi_2(x) = \sqrt{5}(6x^2 - 6x + 1), \quad \varphi_3(x) = \sqrt{7}(20x^3 - 30x^2 + 12x - 1).$$

The canonical moments $\{p_n\}_{n \geq 0}$ can be calculated from the continued fraction expansion of the Stieltjes transform of the uniform measure on $[0, 1]$,

$$\int_0^1 \frac{dx}{z - x} = \frac{1}{z - \frac{\frac{1}{2}}{1 - \frac{\frac{2}{3}\frac{1}{3}}{z - \frac{\frac{2}{3}\frac{1}{2}}{1 - \dots}}}}},$$

which gives

$$p_{2k-1} = \frac{1}{2}, \quad p_{2k} = \frac{k}{2k+1}, \quad k \geq 1.$$

We deduce that, for $n \geq 1$,

$$\det \begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n} \end{bmatrix} = \left(\prod_{k=1}^{n+1} \frac{1}{2k-1} \right) \prod_{l=0}^{n-1} \prod_{k=1}^{n-l} \left(\frac{k}{k+2l+1} \right)^2.$$

This formula, (16) and (34) give that

$$\gamma_{0,l} = (-1)^{l-1} \frac{\sqrt{2l+1}}{l+1}, \quad l \geq 1.$$

Extending this argument (based on results from [17]), we deduce that

$$\gamma_{k,k+l} = (-1)^{l-1} \frac{\sqrt{(2k+1)(2k+2l+1)}}{2k+l+1}, \quad k \in \mathbb{N}_0, l \geq 1,$$

hence

$$d_{k,k+l} = \frac{l}{2k+l+1}.$$

These formulae show that the uniform measure on $[0, 1]$ does not belong to the Szegő class.

We can obtain explicit computation of $\{\gamma_{k,j}\}$ for other classes of classical orthogonal polynomials. The main point is to notice that if $\{\varphi_n\}_{n \geq 0}$ is the family of orthogonal polynomials associated to a certain weight $w(x)$, then $\{\varphi_n(x, l)\}_{n \geq 0}$ is the family of orthogonal polynomials associated to the weight $x^{2l}w(x)$. The polynomials associated to $x^{2l}w(x)$ are called the modified orthogonal polynomials, and their calculation for Hermite and Gegenbauer polynomials can be found, for instance, in [8]; for more references and further treatment, see [16]. Then Theorem 3.2 can be used to determine the parameters $\{\gamma_{k,j}\}$. Details will appear in [2].

4.3. Szegő polynomials in several non-commuting variables. The next examples are motivated in part by multiscale processes. These are stochastic processes indexed by the nodes of a tree. Isotropic processes on homogeneous trees were systematically studied, see [3] and the references therein. An extension to chordal graphs was recently given in [10]. Some classes of stochastic processes associated to the full binary (Cayley) tree were also considered (see, for instance, [11]).

We discuss here this last example. The vertices of the Cayley tree are indexed by \mathbb{F}_N^+ . Let (X, \mathcal{M}, μ) be a probability space and let $\{v_\sigma\}_{\sigma \in \mathbb{F}_N^+} \subset L^2(\mu)$ be a family of random variables. Its covariance kernel is

$$K(\sigma, \tau) = \int_X \bar{v}_\sigma v_\tau dP.$$

The processes is called stationary (see [11]), if

$$K(\tau\sigma, \tau\sigma') = K(\sigma, \sigma') \quad \text{for } \tau, \sigma, \sigma' \in \mathbb{F}_N^+ \tag{35}$$

$$K(\sigma, \tau) = 0 \quad \text{if there is no } \alpha \in \mathbb{F}_N^+ \text{ such that } \sigma = \tau\alpha \text{ or } \tau = \sigma\alpha. \tag{36}$$

Let $\mathcal{A}_S = \{1 - \mathcal{J}(X_k)X_k \mid k = 1, \dots, N\} \cup \{\mathcal{J}(X_k)X_l, k, l = 1, \dots, N, k \neq l\}$ and note that the index set of \mathcal{A}_S is \mathbb{F}_N^+ . We see that ϕ is a positive functional on $\mathcal{T}_N(\mathcal{A}_S)$ if and only if K_ϕ is the covariance of a stationary process as above. It was noticed in [6] that this happens if and only if

$$\gamma_{\tau\sigma, \tau\sigma'} = \gamma_{\sigma, \sigma'}, \quad \text{for } \tau, \sigma, \sigma' \in \mathbb{F}_N^+ \tag{37}$$

$$\gamma_{\sigma, \tau} = 0 \quad \text{if there is no } \alpha \in \mathbb{F}_N^+ \text{ such that } \sigma = \tau\alpha \text{ or } \tau = \sigma\alpha, \tag{38}$$

where $\{\gamma_{\sigma, \tau}\}_{\sigma \prec \tau}$ is the family of parameters associated to K_ϕ in [6]. The main consequence of these relations is that we can define $\gamma_\sigma = \gamma_{\emptyset, \sigma}$, $\sigma \in \mathbb{F}_N^+ - \emptyset$, and $\{\gamma_{\sigma, \tau}\}_{\sigma \prec \tau}$ is uniquely determined by $\{\gamma_\sigma\}_{\sigma \in \mathbb{F}_N^+ - \emptyset}$ due to the relation

$$[\gamma_{\sigma, \tau}]_{|\sigma|=j, |\tau|=k} = ([\gamma_{\sigma', \tau'}]_{|\sigma'|=j-1, |\tau'|=k-1})^{\oplus N}, \quad j, k \geq 1. \tag{39}$$

From now on we assume that ϕ is unital, $\phi(1) = 1$. Then we can show that the recurrence equations (16) and (17) simplify to $\varphi_\emptyset = 1$ and for $k \in \{1, \dots, N\}$, $\sigma \in \mathbb{F}_N^+$,

$$\varphi_{k\sigma} = \frac{1}{d_{k\sigma}}(X_k \varphi_\sigma - \gamma_{k\sigma} \varphi_{k\sigma-1}^\sharp), \tag{40}$$

where $\varphi_\emptyset^\sharp = 1$ and for $k \in \{1, \dots, N\}$, $\sigma \in \mathbb{F}_N^+$,

$$\varphi_{k\sigma}^\sharp = \frac{1}{d_{k\sigma}}(-\bar{\gamma}_{k\sigma} X_k \varphi_\sigma + \varphi_{k\sigma-1}^\sharp). \tag{41}$$

We also notice that the algebra \mathcal{T}_N is naturally embedded into \mathcal{R}_1 and $\Phi_n, \Phi_n^\sharp \in \mathcal{T}_N$. Then, Theorem 3.3 implies that Θ_ϕ belongs to \mathcal{T}_N (but this is also seen directly), and through the isomorphisms mentioned in Section 2.1, Θ_ϕ can be identified with an element of the full Fock space over \mathbb{C}^N , and therefore with a formal power series on variables X_1, \dots, X_N . Similarly, $(\Phi_n^\sharp)^{-1}$ can be identified with a formal power series, denoted $(\varphi_n^\sharp)^{-1}$, on variables X_1, \dots, X_N . Finally, we notice that ϕ belongs to the Szegö class if and only if

$$\prod_{\sigma \in \mathbb{F}_N^+ - \emptyset} d_\sigma > 0.$$

For two formal series on variables X_1, \dots, X_N , the sign \rightarrow means coefficient-wise convergence. The next result is a consequence of Theorem 3.3.

Theorem 4.1. *Let ϕ be a functional on $\mathcal{T}_N(\mathcal{A}_S)$ and belonging to the Szegő class. Then*

$$\varphi_n \rightarrow 0 \tag{42}$$

and

$$(\varphi_n^\#)^{-1} \rightarrow \Theta_\phi. \tag{43}$$

As a consequence of Theorem 3.4 we obtain the following result (which, aside the new geometrical interpretation, would be also a direct consequence of Theorem 6.4.5 in [4]).

Theorem 4.2. *Let ϕ be a functional on $\mathcal{T}_N(\mathcal{A}_S)$ and belonging to the Szegő class. Then*

$$\lim_{|\tau| \rightarrow \infty} \frac{D_{\emptyset, \tau}}{D_{1, \tau}} = |\Theta_\phi(0)|^2 = \prod_{\sigma \in \mathbb{F}_N^+ - \emptyset} d_\sigma^2. \tag{44}$$

If we denote the above limit by g and

$$L = \prod_{\sigma \in \mathbb{F}_N^+ - \emptyset} d_\sigma^{2|\sigma|} > 0,$$

then

$$\lim_{|\tau| \rightarrow \infty} \frac{D_{\emptyset, \tau}}{g^{l(\tau)}} = \frac{1}{L}. \tag{45}$$

Finally, we mention that similar results can be obtained for the commutative case, $\mathcal{A}_C = \mathcal{A}_S \cup \{X_k X_l - X_l X_k \mid k, l = 1, \dots, N\}$. Positive functionals on \mathcal{A}_C correspond to positive definite functions on \mathbb{Z}^N . Details as well as other examples will be given in [2].

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