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Explicit Upper Bound for Entropy Numbers

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Abstract. We give an explicit upper bound for the entropy numbers of the embedding $I : W^{r,p}(Q^l) \to C(\overline{Q^l})$ where $Q^l = (-l,l)^m \subset \mathbb{R}^m$, $r \in \mathbb{N}$, $p \in (1,\infty)$ and $rp > m$.

Keywords: *entropy numbers, embeddings of Sobolev spaces*

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1. Introduction

It is a well known fact that the embedding $I : W^{r,p}(\Omega) \to C(\overline{\Omega})$ is compact if Ω is a bounded domain in $\mathbb{R}^m$ with Lipschitz boundary, $r \in \mathbb{N}$, $p \in (1,\infty)$ and $rp > m$.

Birman and Solomyak [1] proved that the entropy numbers of this embedding satisfy $e_k(I) \sim k^{-\frac{r}{m}}$ if Ω is a cube (see Section 2 for the definition of entropy numbers). They have used this fact in a study of the spectrum of integral operators and nonsmooth elliptic boundary value problems. For the history and properties of entropy numbers we refer the reader to the books [2] and [4].

Sometimes it is not enough to know that there is $C > 0$ such that $e_k(I) \leq Ck^{-\frac{r}{m}}$; further knowledge of the value of C can be useful. For example, Cucker and Smale [3] ask for an explicit bound on C in the case of the embedding of the fractional Sobolev space $H^s(\Omega)$ in $C(\overline{\Omega})$ (recall that $H^s(\Omega) = W^{s,2}(\Omega)$ if $s \in \mathbb{N}$). Our aim is to find an explicit upper bound for the entropy numbers of the embedding $I : W^{r,p}((-l,l)^m) \to C([-l,l]^m)$ if $l > 0$, $r \in \mathbb{N}$, $p \in (1,\infty)$ and $rp > m$. In Theorem 6.2 we give explicit values of constants $\alpha = \alpha(r,p,m,l) > 0$ and $\beta = \beta(r,p,m,l) > 0$ such that

$$e_k\big(I : W^{r,p}((-l,l)^m) \to C([-l,l]^m)\big) \leq \frac{\alpha}{(k-\beta)^{\frac{r}{m}}}$$

if $k > \beta$. This gives an answer to Cucker and Smale's question when s is a natural number and Ω is a cube. Our proof follows the ideas of Birman and

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Solomyak [1]. We do not claim that the upper bound we obtain is sharp in its dependence on r, m and p.

2. Preliminaries

By Q^l we will always denote a cube $(-l,l)^m$. Throughout the paper we will use letters Q and Δ only for cubes and the term cube only for open cubes with sides parallel to the coordinate axes. The closure of a set $A \subset \mathbb{R}^m$ is denoted by $\overline{A}$. We say that a set of cubes $\Theta = \{Q_1, \ldots, Q_n\}$ forms a partition of a cube Q if Q_i are pairwise disjoint and $\bigcup_{i=1}^n \overline{Q_i} = \overline{Q}$.

Given any Banach space X we use the notation B_X for its unit ball (i.e. $B_X = \{x \in X : \ ||x||_X < 1\}$). Suppose that Y is a Banach space and $A \subset Y$ is a bounded domain. The *entropy numbers of the set* A are defined by

$$\varepsilon_i(A) = \inf\{\varepsilon > 0 : \ A \subset \bigcup_{j=1}^{i} (y_j + \varepsilon B_Y), \ y_j \in Y\}, \ i \in \mathbb{N} \ .$$

Let $T : X \to Y$ be a bounded linear mapping between two Banach spaces. The entropy numbers $\varepsilon_i(T)$of this mapping are defined by $\varepsilon_i(T) = \varepsilon_i(T(B_X))$. We also define the dyadic entropy numbers by $e_k(T) = \varepsilon_{2^{k-1}}(T)$ for $k \in \mathbb{N}$ as usual. We write $\lceil r \rceil$ for the upper integer part of a number $r > 0$ (i.e. for the smallest $i \in N$ such that $r \leq i$).

Given any continuous function f from the cube $U = (-1,1)^m \subset \mathbb{R}^m$ to $\mathbb{R}$, the following substitution formula is true :

$$\int_U f(x) \ dx = \int_{\partial U} \int_0^1 r^{m-1} f(rs) \ dr \ d\mathcal{H}^{m-1}(s) \ . \tag{2.1}$$

By $\#S$ we denote the number of elements of the set S and by $|A|$ we denote the Lebesgue measure of a set $A \subset \mathbb{R}^m$. Let ${\mathbb{N}_0}^m$ denote the subset of $\mathbb{R}^m$ which is formed by the elements with non-negative integer components. An element $\alpha = [\alpha_1, \ldots, \alpha_m] \in \mathbb{N}_0^m$ is called a multi-index and the length of α is $|\alpha| = \alpha_1 + \ldots + \alpha_m$. If $x \in \mathbb{R}^m$ and $\alpha \in \mathbb{N}_0^m$ then

$$\alpha! = \alpha_1! \cdots \alpha_m! \text{ and } x^\alpha = x_1^{\alpha_1} \cdots x_m^{\alpha_m} \ .$$

We shall need that

$$\#\{\alpha \in {\mathbb{N}_0}^m : \ |\alpha| = r\} = \binom{r+m-1}{m-1} \ . \tag{2.2}$$

This follows from the fact that there is a one to one correspondence between such multi-indices and the number of variants to choose $m-1$ balls from the row of $r+m-1$ balls. Number α_1 corresponds to the number of balls from the beginning to the first chosen ball, number α_2 corresponds to the the number of balls between the first and the second chosen ball, and so on. It follows that for every $r \geq m$ we have

$$\begin{aligned} &\#\{\beta \in \mathbb{N}_0{}^m : \ \beta_i \geq 1, \ |\beta| = r\} \\ &\qquad = \ \#\{\alpha \in \mathbb{N}_0{}^m : \ |\alpha| = r-m\} = \binom{r-1}{m-1} . \end{aligned} \tag{2.3}$$

Given $r \in \mathbb{N}$ and $p \in (1,\infty)$ we will denote by $W^{r,p}(\Omega)$ the corresponding Sobolev space. This space is equipped with the norm

$$||u||^p_{W^{r,p}(\Omega)} = \sum_{|\alpha|\leq r} \int_\Omega |D^\alpha u|^p .$$

Suppose that $r \in \mathbb{N}$, $p \in (1,\infty)$, $rp > m$ and $\Omega \subset \mathbb{R}^m$ is a bounded domain with Lipschitz boundary. It is a well-known fact that for any $u \in W^{r,p}(\Omega)$ there exists $\tilde{u} \in C(\overline{\Omega})$ such that $u = \tilde{u}$ almost everywhere on Ω. In the whole paper we will use only this representative which is defined and continuous on $\overline{\Omega}$. For $p \in (1,\infty)$ we denote its conjugate index by $p^* = p/(p-1)$ (i.e. $\frac{1}{p} + \frac{1}{p^*} = 1$). Given a function $u \in W^{r,p}(\Omega)$ we define $N_p(u,r,\Omega)$ by

$$(N_p(u,r,\Omega))^p = \sum_{|\alpha|=r} \int_\Omega |D^\alpha u|^p .$$

We shall need the following bound for the entropy numbers of the unit ball in a finite-dimensional space from [2, page 9].

Lemma 2.1. *Suppose that E is a real k-dimensional Banach space. Then for all $i \in \mathbb{N}$*

$$i^{-1/k} \leq \varepsilon_i(B_E) \leq 4i^{-1/k} .$$

3. Semi-additive functions

Let the non-negative function $J(Q)$ of cubes $Q \subset Q^l$ be lower semi-additive, i.e. for every cube $Q \subset Q^l$ and for any partition of Q into cubes Q_j we have

$$\sum_j J(Q_j) \leq J(Q) .$$

By J we will always denote only those non-negative lower semi-additive functions. Let us be given J, a partition Θ of Q^l into cubes and $a > 0$. We set

$$G_a(J,\Theta) = \max_{\Delta\in\Theta} |\Delta|^a J(\Delta) .$$

Lemma 3.1. *Let $s \in \mathbb{N}$, $a > 0$, $x_j \geq 0$ and $y_j \geq 0$ for $j \in \{1, \dots, s\}$. Suppose that $b > 0$ is such that*

$$\sum_{j=1}^{s} x_j \leq 1, \ \sum_{j=1}^{s} y_j \leq 1 \quad \text{and} \quad x_j y_j^a \geq b$$

for every $j \in \{1, \dots, s\}$. Then $b \leq s^{-(a+1)}$.

Proof. From

$$\sqrt[s]{x_1 \cdots x_s} \leq \frac{x_1 + \ldots + x_s}{s} \leq \frac{1}{s} \text{ and } \sqrt[s]{y_1 \cdots y_s} \leq \frac{y_1 + \ldots + y_s}{s} \leq \frac{1}{s}$$

it follows that

$$b^s \leq x_1 \cdots x_n (y_1 \cdots y_s)^a \leq s^{-s} s^{-as} \ . \qquad \blacksquare$$

Theorem 3.1. *Suppose that J is a non-negative lower semi-additive function such that $J(Q^l) \leq 1$. Then for every $n \in \mathbb{N}$ we can find a partition Θ of Q^l into cubes such that $\#\Theta \leq n$ and*

$$G_a(J, \Theta) \leq c_1 n^{-(a+1)} \ \text{where } c_1 = \left(\frac{2^{2m}}{1 - 2^{-\frac{ma}{(a+1)}}} \right)^{a+1} |Q^l|^a \ . \tag{3.1}$$

Proof. We will construct a sequence of partitions Θ_i of Q^l by induction. Set $\Theta_0 = \{Q^l\}$. In the $(i+1)$-th step we will divide each cube $\Delta \in \Theta_i$ satisfying

$$|\Delta|^a J(\Delta) \geq 2^{-ma} G_a(J, \Theta_i) \tag{3.2}$$

into 2^m equal cubes. Now, Θ_{i+1} consists of undivided cubes from Θ_i and of the new cubes which were created by the division.

Let $n_i = \#\Theta_i$, $\delta_i = G_a(J, \Theta_i)$, and by s_i we will denote the number of cubes from Θ_i which satisfy (3.2). Plainly $n_0 = 1$ and

$$n_{i+1} \leq 2^m n_i \ . \tag{3.3}$$

If we divide a cube Δ into 2^m equal cubes Δ_j, then obviously

$$\max_j |\Delta_j|^a J(\Delta_j) \leq 2^{-ma} |\Delta|^a J(\Delta) \ . \tag{3.4}$$

Thanks to (3.2) and (3.4) we obtain

$$\delta_{i+1} \leq 2^{-ma} \delta_i \ . \tag{3.5}$$

Now let us focus on those s_i cubes Δ_j which are divided in the $(i+1)$-th step. Set $x_j = J(\Delta_j)$, $y_j = |\Delta_j|/|Q^l|$ for $j \in \{1, \dots, s_i\}$ and $b = 2^{-ma}\delta_i |Q^l|^{-a}$. Thanks to $\sum_{j=1}^{s_i} J(\Delta_j) \le J(Q^l) \le 1$, $\sum_{j=1}^{s_i} |\Delta_j| \le |Q^l|$ and (3.2) we obtain from Lemma 3.1 that

$$\delta_i \le 2^{ma} s_i^{-(a+1)} |Q^l|^a \ . \tag{3.6}$$

From the construction it follows that

$$n_{j+1} = n_j + s_j(2^m - 1)$$

for every $j \in \mathbb{N}_0$, and therefore

$$n_i \le 2^m \sum_{j=0}^{i-1} s_j \tag{3.7}$$

for every $i \in \mathbb{N}$. For a fixed $t \in \mathbb{N}$ we obtain from (3.5) and (3.6) that for every $0 \le i \le t$ we have

$$s_i \le 2^{-(t-i-1)\frac{ma}{a+1}} \delta_t^{-\frac{1}{a+1}} |Q^l|^{\frac{a}{a+1}} \ .$$

Together with (3.7) this implies

$$n_t \le 2^m \Big(\sum_{j=0}^{t-1} 2^{-(t-j-1)\frac{ma}{a+1}} \Big) \delta_t^{\frac{-1}{a+1}} |Q^l|^{\frac{a}{a+1}} \le 2^m \frac{1}{1 - 2^{\frac{-ma}{a+1}}} \delta_t^{-\frac{1}{a+1}} |Q^l|^{\frac{a}{a+1}} \ .$$

Hence (3.1) gives us

$$\delta_i \le c_1 2^{-m(a+1)} n_i^{-(a+1)} \tag{3.8}$$

for every $i \in \mathbb{N}$. For $i = 0$ this inequality is obvious. Given $n \in \mathbb{N}$, we can find $i \in \mathbb{N}$ such that $n_i \le n < n_{i+1} \le 2^m n_i$ thanks to (3.3). Thus $\#\Theta_i \le n$ and (3.8) gives us

$$G_a(J, \Theta_i) = \delta_i \le c_1 2^{-m(a+1)} n_i^{-(a+1)} \le c_1 n^{-(a+1)} \ . \qquad \blacksquare$$

4. Polynomial approximations

For a cube $\Delta \subset \mathbb{R}^m$ we denote by $P_r(\Delta)$ the set of all polynomials on Δ of degree less or equal to $(r-1)$. By $P_r(\Delta, M)$ we will denote the set of all polynomials $v \in P_r(\Delta)$ such that $||v||_{C(\Delta)} < M$.

Given partition Θ of Q^l, we denote by $P_r(\Theta)$ the set of functions g such that $g \in P_r(\Delta)$ for every $\Delta \in \Theta$. That is, on every cube from the partition the function is possibly a different polynomial of degree less or equal to $(r-1)$.

Definition 4.1. For any cube $\Delta \subset \mathbb{R}^m$ and for any $u \in W^{r,p}(\Delta)$ we define

$$(P_\Delta u)(x) = \frac{\chi_{\overline{\Delta}}(x)}{|\Delta|} \sum_{|\alpha| \le r-1} \int_{\mathbb{R}^m} \chi_\Delta(y) \frac{(x-y)^\alpha}{\alpha!} D^\alpha u(y) \ dy \ .$$

for every $x \in \overline{\Delta}$. Plainly $P_\Delta u \in P_r(\Delta)$.

Following the ideas from [4, Lemma 6.1, page 289] we compute the constant in the approximation of a function $u \in W^{r,p}(\Delta)$ by $P_\Delta u$.

Lemma 4.1. *Let $r \in \mathbb{N}$, $p \in (1,\infty)$ and $rp > m$. For every cube $\Delta \subset \mathbb{R}^m$ and for every $u \in W^{r,p}(\Delta)$ we have*

$$||u - P_\Delta u||_{C(\overline{\Delta})} \leq c_2 |\Delta|^{\frac{r}{m}-\frac{1}{p}} N_p(u,r,\Delta)$$

where

$$c_2 = \frac{r}{m} \frac{2^{\frac{m}{p^*}}}{\left(p^*(\frac{r}{m}-\frac{1}{p})\right)^{\frac{1}{p^*}}} \Big(\sum_{|\alpha|=r} \frac{1}{(\alpha!)^{p^*}}\Big)^{\frac{1}{p^*}} \tag{4.1}$$

Proof. Transformation of coordinates gives us that, without loss of generality, we can suppose $\Delta = (-\frac{1}{2},\frac{1}{2})^n$ (i.e. $|\Delta| = 1$). Since $C^r(\overline{\Delta})$ is dense in $W^{r,p}(\Delta)$ and

$$u_n \overset{W^{r,p}(\Delta)}{\longrightarrow} u \implies \left\{ \begin{array}{c} u_n \overset{C(\overline{\Delta})}{\longrightarrow} u \\ P_\Delta u_n \overset{C(\overline{\Delta})}{\longrightarrow} P_\Delta u \end{array} \right\}$$

it is sufficient to prove the lemma only for $u \in C^r(\overline{\Delta})$. By Taylor's formula we have for every $x \in \overline{\Delta}$ that

$$\begin{aligned} &u(x) - (P_\Delta u)(x) \qquad (4.2)\\ &= \frac{\chi_{\overline{\Delta}}(x)}{|\Delta|} \sum_{|\alpha|=r} \frac{r}{\alpha!} \int_{\mathbb{R}^m} \chi_\Delta(y) \int_0^1 (1-\tau)^{r-1} (x-y)^\alpha D^\alpha u\,(y + \tau(x-y))\, d\tau dy\ . \end{aligned}$$

We shall write the above expression as $\sum_{|\alpha|=r} \frac{r}{\alpha!} F_\alpha(x)$. The substitution $\tilde{\tau} = (1-\tau)$ and $z = (x-y)\tilde{\tau}$ gives us

$$|F_\alpha(x)| \leq \chi_{\overline{\Delta}}(x) \int_{\mathbb{R}^m} \int_0^1 \chi_\Delta\left(x - \frac{z}{\tilde{\tau}}\right) \frac{|z^\alpha|}{\tilde{\tau}^{m+1}} |D^\alpha u(x-z)|\ d\tilde{\tau}\ dz\ . \tag{4.3}$$

If $x \in \overline{\Delta}$ and $x - \frac{z}{\tilde{\tau}} \in \Delta$, then $x - z = \tilde{\tau}(x - \frac{z}{\tilde{\tau}}) + (1-\tilde{\tau})x \in \Delta$. From $|x_i| \leq \frac{1}{2}$ and $|x_i - \frac{z_i}{\tilde{\tau}}| < \frac{1}{2}$ it follows that $|\frac{z_i}{\tilde{\tau}}| < 1$. Therefore,

$$\chi_{\overline{\Delta}}(x) \chi_\Delta(x - \frac{z}{\tilde{\tau}}) \leq \chi_\Delta(x-z) \chi_{2\Delta}(\frac{z}{\tilde{\tau}}),$$

and hence (4.3) gives us

$$|F_\alpha(x)| \ \leq \ \int_{\mathbb{R}^m} \left(\int_0^1 \frac{|z^\alpha|}{\tilde{\tau}^{m+1}} \chi_{2\Delta}\Big(\frac{z}{\tilde{\tau}}\Big)\ d\tilde{\tau} \right) \chi_\Delta(x-z) |D^\alpha u(x-z)|\ dz\ . \tag{4.4}$$

We shall write the right hand side as $[g_\alpha * (\chi_\Delta |D^\alpha u|)](x)$. Denote $U = (-1,1)^m = 2\Delta$. For any $z \in \mathbb{R}^m \setminus \{0\}$ we can find a unique $s = s(z) \in \partial U$ and $\rho = \rho(z) > 0$ such that $z = \rho s$.

Clearly, for $\tilde{\tau} < \rho(z)$ we have $\frac{z}{\tilde{\tau}} \notin U$. This property and $\chi_U(\frac{z}{\tilde{\tau}}) \le \chi_U(z)$ for $\tilde{\tau} < 1$ give us

$$\begin{aligned}
g_\alpha(z) &= \int_\rho^1 |z^\alpha| \tilde{\tau}^{-m-1} \chi_U\Big(\frac{z}{\tilde{\tau}}\Big)\, d\tilde{\tau} \\
&\le \frac{1}{m}\Big(\Big(\frac{1}{\rho}\Big)^m - 1\Big)\rho^{|\alpha|}|s^\alpha|\chi_U(z) \\
&\le \frac{1}{m}\rho^{r-m}|s^\alpha|\chi_U(z)\ .
\end{aligned}$$

Thanks to (2.1) we obtain

$$\begin{aligned}
||g_\alpha||^{p^*}_{L^{p^*}(\mathbb{R}^m)} &\le m^{-p^*}\int_U \rho^{p^*(r-m)}(z)|s^\alpha(z)|^{p^*}\ dz \\
&\le m^{-p^*}\int_{\partial U}|s^\alpha|^{p^*}\int_0^1 \rho^{(r-m)p^*+m-1}\ d\rho\ d\mathcal{H}^{m-1}(s) \\
&= \frac{m^{-p^*}}{(r-m)p^*+m}\int_{\partial U}|s^\alpha|^2\ d\mathcal{H}^{m-1}(s) \\
&\le \frac{m^{-p^*}}{(r-m)p^*+m}\ \mathcal{H}^{m-1}(\partial U) \\
&= \frac{m^{-p^*}}{mp^*(\frac{r}{m}-1+\frac{1}{p^*})}\ 2m\ 2^{m-1} \\
&= \frac{m^{-p^*}2^m}{p^*(\frac{r}{m}-\frac{1}{p})}\ .
\end{aligned} \tag{4.5}$$

From (4.2), (4.4), Hölder's inequality and (4.5) we have

$$\begin{aligned}
|u(x) - P_\Delta u(x)| &\le \sum_{|\alpha|=r}\frac{r}{\alpha!}|F_\alpha(x)| \\
&\le \sum_{|\alpha|=r}\frac{r}{\alpha!}||g_\alpha||_{L^{p^*}(\mathbb{R}^m)}||\chi_\Delta D^\alpha u||_{L^p(\mathbb{R}^m)} \\
&\le \frac{r}{m}\frac{2^{\frac{m}{p^*}}}{\big(p^*(\frac{r}{m}-\frac{1}{p})\big)^{\frac{1}{p^*}}}\Big(\sum_{|\alpha|=r}\frac{1}{\alpha!}||D^\alpha u||_{L^p(\Delta)}\Big) \\
&\le \frac{r}{m}\frac{2^{\frac{m}{p^*}}}{\big(p^*(\frac{r}{m}-\frac{1}{p})\big)^{\frac{1}{p^*}}}\Big(\sum_{|\alpha|=r}\frac{1}{(\alpha!)^{p^*}}\Big)^{\frac{1}{p^*}}N_p(u,r,\Delta) \\
&= c_2 N_p(u,r,\Delta)\ ,
\end{aligned}$$

which proves the assertion. ∎

Remark 1. It is possible to have a more explicit bound for c_2 since

$$\sum_{|\alpha|=r} \frac{1}{(\alpha!)^{p^*}} \leq \sum_{|\alpha|=r} \frac{1}{\alpha!} = \frac{m^r}{r!} \ .$$

The last equality can be proved by induction with respect to m. Denote $A(r,m) = \sum_{|\alpha|=r} \frac{1}{\alpha!}$. Plainly $P(r,1) = \frac{1}{r!}$. For $m > 1$ we have

$$\begin{aligned} P(r,m) &= \sum_{i=0}^{r} \frac{1}{i!} \ P(r-i, m-1) \\ &= \sum_{i=0}^{r} \frac{r!}{r!} \ \frac{1}{i!(r-i)!} (m-1)^{r-i} \\ &= \frac{1}{r!} \sum_{i=0}^{r} \binom{r}{r-i} (m-1)^{r-i} \\ &= \frac{m^r}{r!} \ . \end{aligned}$$

On the other hand, this bound is not very sharp, and for given values of r, m and p the sum is better to compute .

Definition 4.2. Given $u \in W^{r,p}(Q^l)$ and a partition Θ of Q^l, we define

$$(P_\Theta u)(x) = (P_\Delta)(x)$$

for every $\Delta \subset \Theta$ and $x \in \Delta$. Plainly $P_\Theta u \in P_r(\Theta)$.

Theorem 4.1. *Let $r \in \mathbb{N}$, $p \in (1, \infty)$ and $rp > m$. For every $n \in \mathbb{N}$ and for every $u \in W^{r,p}(Q^l)$ such that $N_p(u,r,Q^l) \leq 1$ there is a partition Θ of Q^l such that $\#\Theta \leq n$ and*

$$||u - P_\Theta u||_{L^\infty(\Delta)} \leq c_2 c_1^{\frac{1}{p}} n^{-\frac{r}{m}} \ .$$

Proof. For every partition of Q^l into cubes Δ_j we can use Lemma 4.1 to obtain

$$||u - P_{\Delta_j} u||_{C(\Delta_j)} \leq c_2 |\Delta_j|^{\frac{r}{m} - \frac{1}{p}} N_p(u, r, \Delta_j) \ .$$

The function

$$J(\Delta) = (N_p(u,r,\Delta))^p = \sum_{|\alpha|=r} \int_\Delta |D^\alpha u|^p \tag{4.6}$$

is clearly non-negative and lower semi-additive. Hence, we can use Theorem 3.1 for $a = \frac{pr}{m} - 1$ to obtain the partition Θ such that $\#\Theta \leq n$ and

$$||u - P_\Delta u||_{C(\Delta)} \leq c_2 \left(|\Delta|^a N_p^p(u,r,\Delta) \right)^{\frac{1}{p}} \leq c_2 c_1^{\frac{1}{p}} n^{-\frac{(a+1)}{p}} \tag{4.7}$$

for every $\Delta \in \Theta$. ∎

5. Auxiliary estimate of covering numbers

In this section we prove that it is possible to cover the unit ball of $W^{r,p}(Q^l)$ by a finite number of balls in $C(\overline{Q^l})$.

Lemma 5.1. *Let $Q \subset \mathbb{R}^m$ be a cube and $P = P_r(\overline{Q}, M)$. Let this set be equipped with the supremum norm as in $C(\overline{Q})$. Then, for every $\varepsilon < M$ the following estimate holds for the number of elements in an optimal ε-net:*

$$N_\varepsilon(P) \leq \left\lceil \frac{4M}{\varepsilon} \right\rceil^{c_3} \quad \text{where} \;\; c_3 = \binom{r-1+m}{m} \tag{5.1}$$

Proof. Without loss of generality, we can suppose that $M = 1$. On the Banach space E of polynomials on $\overline{Q}$ of degree less or equal to $r-1$ we use a norm $||.||_{C(\overline{Q})}$. From (2.2) we obtain that the dimension of this space is equal to

$$\#\{\alpha \in {N_0}^m : |\alpha| \leq r-1\} = \#\{\alpha \in {N_0}^{m+1} : |\alpha| = r-1\} = \binom{r-1+m}{m} = c_3$$

Clearly, for $n = \lceil \frac{4}{\varepsilon} \rceil^{c_3}$ we have $4n^{\frac{-1}{c_3}} \leq \varepsilon$ and therefore we can cover the unit ball by n balls of diameter ε in view of Lemma 2.1. ∎

We will need an estimate of the norm of the embedding $I : W^{r,p}(Q^l) \to C(\overline{Q^l})$.

Lemma 5.2. *Suppose that $r \in \mathbb{N}$, $p \in (1, \infty)$ and $rp > m$. Then for every $u \in W^{r,p}(Q^l)$ it holds*

$$||u||_{C(\overline{Q^l})} \leq c_4 ||u||_{W^{r,p}(Q^l)} \tag{5.2}$$

where

$$c_4 = c_2 |Q^l|^{\frac{r}{m} - \frac{1}{p}} + \Big(\sum_{|\alpha| \leq r-1} \frac{1}{(\alpha!)^{p^*}} \Big)^{\frac{1}{p^*}} \max \left(|Q^l|^{-\frac{1}{p}}, |Q^l|^{\frac{(r-1)}{m} - \frac{1}{p}} \right) . \tag{5.3}$$

Proof. By Lemma 4.1 we have

$$\begin{aligned} ||u||_{C(\overline{Q^l})} &\leq ||u - P_{Q^l} u||_{C(\overline{Q^l})} + ||P_{Q^l} u||_{C(\overline{Q^l})} \\ &\leq c_2 |Q^l|^{\frac{r}{m} - \frac{1}{p}} ||u||_{W^{r,p}(Q^l)} + ||P_{Q^l} u||_{C(\overline{Q^l})}. \end{aligned} \tag{5.4}$$

From Definition 4.1 and Hölder's inequality we obtain for every $x \in \overline{Q^l}$ that

$$
\begin{aligned}
|P_{Q^l}u(x)| &\leq \frac{1}{|Q^l|}\sum_{|\alpha|\leq r-1}\left|\int_{Q^l}\frac{(x-y)^\alpha}{\alpha!}D^\alpha u(y)\,dy\right| \\
&\leq \frac{1}{|Q^l|}\sum_{|\alpha|\leq r-1}\frac{(2l)^{|\alpha|}}{\alpha!}|Q^l|^{\frac{1}{p^*}}\left(\int_{Q^l}|D^\alpha u|^p\right)^{\frac{1}{p}} \qquad (5.5)\\
&\leq \max\left(|Q^l|^{-\frac{1}{p}},|Q^l|^{\frac{(r-1)}{m}-\frac{1}{p}}\right)\Big(\sum_{|\alpha|\leq r-1}\frac{1}{(\alpha!)^{p^*}}\Big)^{\frac{1}{p^*}}||u||_{W^{r,p}(Q^l)},
\end{aligned}
$$

which shows the assertion. ∎

Remark 2. Using Remark 1 it is possible to have the more explicit bound

$$
\sum_{|\alpha|\leq r-1}\frac{1}{(\alpha!)^{p^*}}\leq\sum_{i=0}^{r-1}\sum_{|\alpha|=i}\frac{1}{\alpha!}=\sum_{i=0}^{r-1}\frac{m^i}{i!}\leq e^m\ .
$$

Lemma 5.3. *Let $r\in\mathbb{N}$, $p\in(1,\infty)$ and $rp>m$. Put $\tilde{\varepsilon}_0=2c_2|Q^l|^{\frac{r}{m}-\frac{1}{p}}$. Then, there are functions $f_i\in C(\overline{Q^l})$, $i\in\{1,\ldots,c_5\}$ such that*

$$
B_{W^{r,p}(Q^l)}\subset\bigcup_{i=1}^{c_5}\left(f_i+\tilde{\varepsilon}_0B_{C(\overline{Q^l})}\right)
$$

where

$$
c_5=\left\lceil 8+\frac{4}{c_2}\Big(\sum_{|\alpha|\leq r-1}\frac{1}{(\alpha!)^{p^*}}\Big)^{\frac{1}{p^*}}\max\left(|Q^l|^{-\frac{r}{m}},|Q^l|^{-\frac{1}{m}}\right)\right\rceil^{\binom{r-1+m}{m}}. \qquad (5.6)
$$

Proof. Suppose that $u\in B_{W^{r,p}(Q^l)}$. From Lemma 4.1 and Lemma 5.2 we have

$$
|P_{Q^l}u|\leq|P_{Q^l}u-u|+|u|\leq c_2|Q^l|^{\frac{r}{m}-\frac{1}{p}}+c_4\ .
$$

Hence, we obtain from Lemma 5.1 that the set

$$
\{P_{Q^l}u:||u||_{W^{r,p}(Q^l)}\leq 1\}
$$

has an $\frac{\tilde{\varepsilon}_0}{2}$-set in $C(\overline{Q^l})$ of cardinality at most

$$
C=\left\lceil\frac{8c_2|Q^l|^{\frac{r}{m}-\frac{1}{p}}+8c_4}{\tilde{\varepsilon}_0}\right\rceil^{c_3}.
$$

By Lemma 4.1

$$
|P_{Q^l}u-u|\leq c_2|Q^l|^{\frac{r}{m}-\frac{1}{p}}=\frac{\tilde{\varepsilon}_0}{2}\ ,
$$

and therefore this set serves as an $\tilde{\varepsilon}_0$-set for $B_{W^{r,p}(Q^l)}$ as well. By (5.3), (5.1) and $\tilde{\varepsilon}_0 = 2c_2|Q^l|^{\frac{r}{m}-\frac{1}{p}}$ we obtain

$$C = \left\lceil 8 + \frac{4}{c_2}\Big(\sum_{|\alpha|\le r-1} \frac{1}{(\alpha!)^{p^*}}\Big)^{\frac{1}{p^*}} \max\big(|Q^l|^{-\frac{r}{m}}, |Q^l|^{-1/m}\big)\right\rceil^{\binom{r-1+m}{m}},$$

which proves the assertion. ∎

6. Main result

In this section we give an explicit upper bound for the entropy numbers of the embedding $I : W^{r,p}(Q^l) \to C(\overline{Q^l})$.

Theorem 6.1. *Let $r \in \mathbb{N}$, $p \in (1,\infty)$, $rp > m$ and $\varepsilon > 0$. There are $N(\varepsilon) \in \mathbb{N}$ and $f_i \in L^\infty(Q^l)$, $i \in \{1,\dots,N(\varepsilon)\}$ such that*

$$\begin{aligned} B_{W^{r,p}(Q^l)} &\subset \bigcup_{i=1}^{N(\varepsilon)} \big(f_i + \varepsilon B_{L^\infty(Q^l)}\big) \\ \log_2 N(\varepsilon) &\le \log_2 c_5 + c_{10}^{\frac{m}{r}} \varepsilon^{-\frac{m}{r}} \end{aligned} \tag{6.1}$$

where

$$c_{10} = 2c_2 \frac{2^{2r}}{\left(1-2^{-m(1-\frac{m}{pr})}\right)^{\frac{2r}{m}}} |Q^l|^{\frac{r}{m}-\frac{1}{p}} \left(2 + (r+4)\binom{r-1+m}{m}\right)^{\frac{r}{m}} . \tag{6.2}$$

Proof. We will use results and notation from the previous sections. Set $a = \frac{pr}{m} - 1$ and define J by formula (4.6). For a given $u \in W^{r,p}(Q^l)$ we construct a sequence of partitions Θ_i of Q^l as described in the proof of Theorem 3.1. Set

$$c_6 = c_1 2^{-m(a+1)} = \left(\frac{2^m}{1-2^{-\frac{ma}{(a+1)}}}\right)^{a+1} |Q^l|^a . \tag{6.3}$$

Along with the characteristic numerical sequences $n_i = \#\Theta_i$ and $\delta_i = G_a(J,\Theta_i) = \max_{\Delta\in\Theta_i} |\Delta|^a J(\Delta)$ we also introduce the sequence

$$\tilde{\delta}_i = c_6 \min_{0\le j\le i} \left(2^{-am(i-j)} n_j^{-(a+1)}\right), \; i \in \mathbb{N}_0 . \tag{6.4}$$

From the definition of the numbers $\tilde{\delta}_i$ it follows that

$$n_j^{a+1} \le c_6 \tilde{\delta}_i^{-1} 2^{-am(i-j)} \quad \text{for} \;\; 0 \le j \le i . \tag{6.5}$$

In addition, from (3.8), (3.5), (6.3) and (6.4) we have

$$\delta_i \le \tilde{\delta}_i \le c_6 n_i^{-(a+1)} \ . \tag{6.6}$$

The sequence $\tilde{\delta}_i$ is more regular than δ_i since

$$2^{-(a+1)m}\tilde{\delta}_i \le \tilde{\delta}_{i+1} \le 2^{-am}\tilde{\delta}_i \ . \tag{6.7}$$

From (6.4) it follows that

$$\tilde{\delta}_{i+1} = \min\left(2^{-am}\tilde{\delta}_i, c_6 n_{i+1}^{-(a+1)}\right) \ .$$

The right-hand side inequality in (6.7) follows immediately. The left-hand side inequality has to be verified only in the case

$$\tilde{\delta}_{i+1} = c_6 n_{i+1}^{-(a+1)} \ge 2^{-(a+1)m}\tilde{\delta}_i$$

which follows easily from (3.3) and (6.6).

Now let ν be a fixed number such that $0 < \nu < c_6$. To each semi additive function J we associate the number $k = k(J, a, \nu) \in \mathbb{N}$ determined by the condition

$$\tilde{\delta}_k < \nu \le \tilde{\delta}_{k-1} \ . \tag{6.8}$$

Let $T_a(J, \nu)$ denote the segment $\{\Theta_i\}_{i=1}^k$. We unite in one class those (and only those) functions J for which the sequences $T_a(J, \nu)$ coincide. The number of classes into which we divide the set of all semi-additive functions will be denoted by $N(a, \nu)$.

Lemma 6.1. *For* $\nu \in (0, c_6)$

$$\log_2 N(a, \nu) \le c_7 \nu^{-(a+1)^{-1}}$$

where

$$c_7 = \frac{2^m}{1 - 2^{-\frac{ma}{(a+1)}}}\left(2^m + \frac{1}{1 - 2^{-\frac{ma}{(a+1)}}}\right)|Q^l|^{\frac{a}{a+1}} \ . \tag{6.9}$$

Proof. First we estimate the number $N_1(a, \nu)$ of more 'extensive' classes which unite in one those functions J for which the actual sequences of partitions $T_a(J, \nu)$ do not necessarily coincide but for which the corresponding numerical sequences $\{n_i\}_{i=1}^k$, $k = k(J, a, \nu)$ do coincide. If we apply (3.3), (6.5) with $i = j = k - 1$ and (6.8) we obtain

$$n_k^{a+1} \le 2^{m(a+1)} n_{k-1}^{a+1} \le 2^{m(a+1)} c_6 \tilde{\delta}_{k-1}^{-1} \le 2^{m(a+1)} c_6 \nu^{-1} \ . \tag{6.10}$$

Let $n^\star$ denote the integer part of the number $2^m (c_6 \nu^{-1})^{(a+1)^{-1}}$. The equality

$$n^\star + 1 = 1 + \sum_{i=1}^k (n_i - n_{i-1}) + (n^\star + 1 - n_k)$$

associates to each sequence $T_a(J,\nu)$ a representation of the number $n^\star$ as a sum of $k+1$ positive integer terms. By (2.3) the number of such representations is $\binom{n^\star-1}{k} < 2^{n^\star-1}$. Thus

$$\log_2 N_1(a,\nu) < n^\star \le 2^m (c_6\nu^{-1})^{(a+1)^{-1}} \; . \tag{6.11}$$

Now let the sequence $\{n_i\}_{i=1}^k$ be fixed. We find an estimate of the number of distinct sequences of partitions $\{\Theta_i\}_{i=1}^k = T_a(J,\nu)$ for which $\#\Theta_i = n_i$ for every $i \in \{0,\dots,k\}$. To do this we note that for a given partition Θ_i the partition Θ_{i+1} is uniquely determined by knowing which s_i cubes are further divided upon formation of Θ_{i+1}. The number of possibilities here is $\binom{n_i}{s_i} < 2^{n_i}$. After k steps this gives us a total range of possibilities numbering less than $2^{n_0+\dots+n_{k-1}}$.

From (6.5) for $i = k-1$ and (6.8) we obtain for $j = 0,\dots,k-1$ that

$$n_j \le (c_6\nu^{-1})^{(a+1)^{-1}} 2^{-am(k-1-j)/(a+1)}$$

which implies the inequality

$$n_0 + \dots + n_{k-1} \le (c_6\nu^{-1})^{(a+1)^{-1}} [1 - 2^{-am/(a+1)}]^{-1} = \hat{n} \; . \tag{6.12}$$

Therefore, $N(a,\nu) \le N_1(a,\nu) 2^{\hat{n}}$, (6.11), (6.12) and (6.3) give us

$$\begin{aligned} \log_2 N(a,\nu) &\le \log_2 N_1(a,\nu) + \hat{n} \\ &\le c_6^{(a+1)^{-1}} \left(2^m + \frac{1}{1 - 2^{-\frac{am}{a+1}}} \right) \nu^{-(a+1)^{-1}} \\ &= c_7 \nu^{-(a+1)^{-1}} \; , \end{aligned}$$

which finishes the proof of Lemma 6.1. ∎

Continuation of the proof of Theorem 6.1. Recall that $a = \frac{pr}{m} - 1$ and that J is defined by (4.6). Since we are interested only in $u \in B_{W^{r,p}(Q^l)}$ we have $N_p(u,r,Q^l) \le 1$, and therefore we can use the inequality (4.7) from the proof of Theorem 4.1. From the left-hand side inequality in (4.7), the definition of δ_i and (6.6) we have

$$||u - P_{\Theta_i} u||_{L^\infty(Q^l)} \le c_2 \delta_i^{\frac{1}{p}} \le c_2 \tilde{\delta}_i^{\frac{1}{p}} \tag{6.13}$$

for every $i \in \mathbb{N}_0$.

For a fixed $\varepsilon > 0$ we set

$$\nu = \left(\frac{\varepsilon}{2c_2} \right)^p \; . \tag{6.14}$$

First let us suppose that $\nu < c_6$ and define $k = k(J,a,\nu)$ by (6.8). We unite in one class those functions u for which the sequences $T_a(J,\nu)$ coincide. The number of such classes is estimated by Lemma 6.1 :

$$\log_2 N(a,\nu) \le c_7 \left(\frac{2c_2}{\varepsilon} \right)^{\frac{m}{r}} \; . \tag{6.15}$$

We select one of those classes (denoting it by K) and construct an ε-net for it. First we note that the numbers n_i and $\tilde{\delta}_i$, $i = 0, \ldots, k$ (but not the numbers δ_i) coincide for all functions in K.

We will construct the ε-net in $L^\infty(Q^l)$ by induction. We set $\varepsilon_i = 2c_2\tilde{\delta}_i^{\frac{1}{p}}$ for $i = 0, \ldots, k$. Suppose that in the i-th step we have constructed for the set K an ε_i-net of cardinality N_i contained in $P_r(\Theta_i)$. Let $u \in K$ and let $w \in P_r(\Theta_i)$ be the element of the ε_i-net for which $||u - w||_{L^\infty(Q^l)} \le \varepsilon_i$. Then (6.13) gives us

$$||P_{\Theta_{i+1}}u - w||_{L^\infty(Q^l)} \le ||P_{\Theta_{i+1}}u - u||_{L^\infty(Q^l)} + ||u - w||_{L^\infty(Q^l)} \le c_2\tilde{\delta}_{i+1}^{\frac{1}{p}} + \varepsilon_i \le 2\varepsilon_i \ .$$

Thus, in each cube $\Delta \in \Theta_{i+1}$ the function $P_{\Theta_{i+1}}u - w$ belongs to the set $P_r(\Delta, 2\varepsilon_i)$. By virtue of Lemma 5.1, definition of ε_i and the left-hand side inequality in (6.7) this set has an $(\frac{\varepsilon_{i+1}}{2})$-net whose cardinality does not exceed the number

$$\left\lceil \frac{16\varepsilon_i}{\varepsilon_{i+1}} \right\rceil^{c_3} \le 2^{(r+4)c_3} = c_8 \tag{6.16}$$

where $c_3 = \binom{r-1+m}{m}$. Therefore, the set $\{P_{\Theta_{i+1}}u : \ u \in K\}$ has an $(\frac{\varepsilon_{i+1}}{2})$-net which is contained in $P_r(\Theta_{i+1})$ and whose cardinality satisfies

$$N_{i+1} \le c_8^{n_{i+1}} N_i \ . \tag{6.17}$$

By (6.13) the same net serves as an ε_{i+1}-net for the set K. In this way, starting from ε_0-net, we construct an ε_k-net with cardinality N_k for the set K. By the definition of ε_k, (6.8) and (6.14) this is also an ε-net. By (3.3) and (6.12) it holds

$$N_k \le N_0 c_8^{n_1+\ldots+n_k} \le N_0 c_8^{2^m(n_0+\ldots+n_{k-1})} \le N_0 c_8^{2^m\hat{n}} \ .$$

Clearly, (6.12), (6.14) and (6.3) give us $2^m\hat{n} = c_9\varepsilon^{-\frac{m}{r}}$ where

$$\begin{aligned} c_9 &= 2^m c_6^{(a+1)^{-1}} \frac{1}{1 - 2^{-\frac{am}{a+1}}} (2c_2)^{\frac{p}{a+1}} \\ &= \left(\frac{2^m}{1 - 2^{-\frac{am}{a+1}}}\right)^2 |Q^l|^{\frac{a}{a+1}} (2c_2)^{\frac{m}{r}} \ . \end{aligned} \tag{6.18}$$

From definition of ε_0, (6.4), $n_0 = 1$ and (6.3) we obtain

$$\varepsilon_0 = 2c_2\tilde{\delta}_0^{\frac{1}{p}} = 2c_2 c_6^{\frac{1}{p}} \ge 2c_2|Q^l|^{\frac{r}{m} - \frac{1}{p}} \ .$$

Therefore, $N_0 \le c_5$ in view of Lemma 5.3. Now we combine the estimate

$$N_k \le c_5 c_8^{c_9\varepsilon^{-\frac{m}{r}}}$$

which we have obtained for each of the classes with the estimate (6.15) for the number of classes to obtain

$$\log_2 N(\varepsilon) \le \log_2 c_5 + (\log_2 c_8)c_9\varepsilon^{-\frac{m}{r}} + c_7(2c_2)^{\frac{m}{r}}\varepsilon^{-\frac{m}{r}} \ . \tag{6.19}$$

Let us return to the case $\nu \geq c_6$. Then (6.14) and (6.3) give us

$$\varepsilon \geq 2c_2 c_6^{\frac{1}{p}} \geq 2c_2 |Q^l|^{\frac{r}{m}-\frac{1}{p}} .$$

Due to Lemma 5.3 we have $N(\varepsilon) \leq c_5$, and hence (6.19) is valid also if $\nu \geq c_6$.

Now we can use the definition of the constants involved, namely (6.9), (6.16), (6.18), (5.1), $a = \frac{pr}{m} - 1$ and (6.2) to obtain

$$\begin{aligned} &\left(c_7 (2c_2)^{\frac{m}{r}} + (\log_2 c_8) c_9\right)^{\frac{r}{m}} \\ &\quad \leq \; 2c_2 |Q^l|^{\frac{ar}{(a+1)m}} \left(2 \left(\frac{2^m}{1 - 2^{-\frac{ma}{(a+1)}}} \right)^2 + (r+4) c_3 \left(\frac{2^m}{1 - 2^{-\frac{ma}{(a+1)}}} \right)^2 \right)^{\frac{r}{m}} \\ &\quad = \; c_{10} \, , \end{aligned}$$

which completes the proof of Theorem 6.1. ∎

Finally we can state our main result. The values of constants c_{10}, c_5 and c_2 are given by (6.2), (5.6) and (4.1). Recall that $\lceil x \rceil$ denotes the upper integer part of a number $x > 0$. By Remark 1 and Remark 2 it is possible to have bounds not involving sums.

Theorem 6.2. *Let $r \in \mathbb{N}$, $p \in (1, \infty)$, $rp > m$ and $k > 1 + \log_2 c_5$. Then*

$$e_k\big(I : W^{r,p}(Q^l) \to C(\overline{Q^l})\big) \leq \frac{2c_{10}}{(k - 1 - \log_2 c_5)^{\frac{r}{m}}} .$$

Proof. Given $\varepsilon > 0$, fix $f_i \in L^\infty(Q^l)$, $i \in \{1, \ldots, N(\varepsilon)\}$ as in (6.1) and choose

$$\tilde{f}_i \in A_i := (f_i + \varepsilon B_{L^\infty(Q^l)}) \cap C(\overline{Q^l}) .$$

If $A_i = \emptyset$, then set $f_i \equiv 0$. Since every function from $W^{r,p}(Q^l)$ is in $C(\overline{Q^l})$ we obtain from (6.1) that

$$B_{W^{r,p}(Q^l)} \subset \bigcup_{i=1}^{N(\varepsilon)} \left(\tilde{f}_i + 2\varepsilon B_{C(\overline{Q^l})} \right) \tag{6.20}$$

Now the definition of dyadic entropy numbers and the relation $k > 1 + \log_2 c_5$ give us

$$\begin{aligned} e_k(I : W^{r,p}(Q^l) \to C(\overline{Q^l})) &\leq \inf\{\varepsilon > 0 : N(\varepsilon/2) \leq 2^{k-1}\} \\ &\leq \inf\{\varepsilon > 0 : \log_2 c_5 + c_{10}^{\frac{m}{r}} \varepsilon^{-\frac{m}{r}} 2^{\frac{m}{r}} \leq k - 1\} \\ &= \frac{2c_{10}}{(k - 1 - \log_2 c_5)^{\frac{r}{m}}} \end{aligned}$$

and proves the assertion. ∎

There are several known ways how to extend a function from a domain with Lipschitz boundary, and the explicit norm of the extension is known in some cases (see [5]). The next theorem gives us a bound for the entropy numbers of the embedding $I : W^{r,p}(\Omega) \to C(\overline{\Omega})$ if we know the norm of the extension operator.

Theorem 6.3. *Let $r \in \mathbb{N}$, $p \in (1,\infty)$, $rp > m$, $k > 1 + \log_2 c_5$ and $\Omega \subset Q_l \subset \mathbb{R}^m$. Suppose that there exists a linear extension operator $E : W^{r,p}(\Omega) \to W^{r,p}(Q_l)$ of norm c_{11}, i.e.*

$$||Ef||_{W^{r,p}(Q^l)} \leq c_{11}||f||_{W^{r,p}(\Omega)}$$

for every $f \in W^{r,p}(\Omega)$, and moreover $Ef = f$ on Ω. Then

$$e_k(I : W^{r,p}(\Omega) \to C(\overline{\Omega})) \leq c_{11}\frac{2c_{10}}{(k-1-\log_2 c_5)^{\frac{r}{m}}} \ .$$

Proof. For every function $f \in B_{W^{r,p}(\Omega)}$ we have $Ef \in c_{11}B_{W^{r,p}(Q^l)}$. By multiplying both sides of (6.20) by c_{11} we see that there is an $i \in \{1,\ldots,N(\varepsilon)\}$ such that

$$||f - c_{11}\tilde{f}_i||_{C(\overline{\Omega})} \leq ||Ef - c_{11}\tilde{f}_i||_{C(\overline{Q^l})} < c_{11}2\varepsilon \ .$$

Therefore, analogously to the proof of the previous theorem we obtain the desired bound for entropy numbers. ∎

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