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A Global Bifurcation Theorem for Convex-Valued Differential Inclusions

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Abstract. In this paper we prove a global bifurcation theorem for convex-valued completely continuous maps. Basing on this theorem we prove an existence theorem for convex-valued differential inclusions with Sturm-Liouville boundary conditions

$$\left.\begin{array}{l} u''(t) \in \varphi(t, u(t), u'(t)) \quad \text{for a.e. } t \in (a,b) \\ l(u) = 0 \end{array}\right\}.$$

The assumptions refer to the appropriate asymptotic behaviour of $\varphi(t,x,y)$ for $|x|+|y|$ close to 0 and to $+\infty$, and they are independent from the well known Bernstein-type conditions. In the last section we give a set of examples of φ satisfying the assumptions of the given theorem but not satisfying the Bernstein conditions.

Keywords: *Differential inclusions, Sturm-Liouville boundary conditions, global bifurcation*

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1. Global bifurcation theorem

Let E be a real Banach space, $A \subset \mathbb{R}$ an open interval and $cf(E)$ the family of all non-empty, closed, bounded and convex subsets of E. We call a map $F: A \times E \to cf(E)$ completely continuous if F is upper semicontinuous and, for any bounded set $B \subset A \times E$, the set $F(B) \subset E$ is relatively compact.

Let $F: A \times E \to cf(E)$ be a completely continuous map such that $0 \in F(\lambda, 0)$ for $\lambda \in A$ and let $f: A \times E \to cf(E)$ be given by

$$f(\lambda, x) = x - F(\lambda, x). \tag{1.1}$$

We call $(\mu_0, 0) \in A \times E$ a bifurcation point of the map f if for all open subsets $U \subset A \times E$ with $(\mu_0, 0) \in U$ there exists a point $(\lambda, x) \in U$ such that $x \neq 0$ and $0 \in f(\lambda, x)$. Let us denote the set of all bifurcation points of f by $\mathcal{B}_f$.

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Let $\mathcal{R}_f \subset A \times E$ be the closure (in $A \times E$) of the set of non-trivial solutions of the inclusion $0 \in f(\lambda, x)$, i.e.

$$\mathcal{R}_f = \overline{\{(\lambda, x) \in A \times E : x \neq 0 \text{ and } 0 \in f(\lambda, x)\}}.$$

Let us observe that, for each $(\lambda, x) \in \mathcal{R}_f$, $0 \in f(\lambda, x)$.

Let $U \subset E$ be a bounded open subset and let the map $g : \overline{U} \to cf(E)$ be given by $g(x) = x - G(x)$, where $G : \overline{U} \to cf(E)$ is a completely continuous map such that, for $x \in \partial U$, the relation $x \notin G(x)$ holds. It is well known that in such situation we may define the Leray-Schauder degree $\deg(g, U, 0)$ (cf. [2, 3, 8, 17, 19]).

For each λ satisfying $(\lambda, 0) \notin \mathcal{B}_f$ there exists an $r_0 > 0$ such that, for $\|x\| = r \in (0, r_0]$, the relation $x \notin F(\lambda, x)$ holds. So the value $\deg\big(f(\lambda, \cdot), B(0, r), 0\big)$ is defined. Assume that for an interval $[a, b] \subset A$ there exists a $\delta > 0$ such that

$$\Big(\big([a - \delta, a) \cup (b, b + \delta]\big) \times \{0\}\Big) \cap \mathcal{B}_f = \emptyset.$$

Then we may define the bifurcation index $s[f, a, b]$ of the map f with respect to the interval $[a, b]$ as

$$s[f, a, b] = \lim_{\lambda \to b^+} \deg\big(f(\lambda, \cdot), B(0, r), 0\big) - \lim_{\lambda \to a^-} \deg\big(f(\lambda, \cdot), B(0, r), 0\big)$$

where $r = r(\lambda) > 0$ is small enough.

Now we are going to give some auxiliary lemmas, which will be used in the proof of the global bifurcation theorem below. We are going to use a separation lemma for closed subsets of compact Hausdorff spaces given in [9] (see also [24: Section XI]).

Lemma 1. *Assume that X, Y are closed subsets of a compact Hausdorff space K and that there does not exist a connected set $S \subset K$ such that $S \cap X \neq \emptyset$ as well as $S \cap Y \neq \emptyset$. Then there exists a separation $K = K_x \cup K_y$ with $K_x \cap K_y = \emptyset$ such that $X \subset K_x$ and $Y \subset K_y$ and both K_x and K_y are open and closed in K.*

An immediate consequence of Lemma 1 is the following

Proposition 1. *Let the map $f : A \times E \to cf(E)$ be given by* (1.1) *and let $[a, b] \subset A$ be an interval such that $([a, b] \times \{0\}) \cap \mathcal{B}_f \neq \emptyset$. Further, let $\mathcal{C}_0$ be a compact component of the set $\mathcal{R} = \mathcal{R}_f \cup ([a, b] \times \{0\})$ such that $[a, b] \times \{0\} \subset \mathcal{C}_0$. Then there exists an open and closed set $\mathcal{K}_0 \subset \mathcal{R}$ such that*

$$\mathcal{C}_0 \subset \mathcal{K}_0 \subset (c, d) \times B(0, R) \subset [c, d] \times \overline{B(0, R)} \subset A \times E.$$

Now we are going to give a generalization of Ize's lemma (cf. [14] and [20: Lemma 3.4.2]) to convex-valued completely continuous vector fields. For this let the function $\rho(\cdot, [a,b]) : \mathbb{R} \to [0, +\infty)$ be given by

$$\rho(\lambda, [a,b]) = \begin{cases} a - \lambda & \text{for } \lambda < a \\ 0 & \text{for } \lambda \in [a,b] \\ \lambda - b & \text{for } \lambda > b. \end{cases}$$

Lemma 2. *Let the map $f : A \times E \to cf(E)$ be given by* (1.1) *and let $[a,b] \subset A$ be an interval such that $\mathcal{B}_f \subset [a,b] \times \{0\}$. Then there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ there is an $r_0 > 0$ so that the map*

$$\begin{aligned} &f_{r,\varepsilon} : \overline{U_{r,\varepsilon}} \to cf(\mathbb{R} \times E) \\ &f_{r,\varepsilon}(\lambda, x) = \big\{ (\|x\|^2 - r^2, y) : y \in f(\lambda, x) \big\} \end{aligned}$$

with

$$U_{r,\varepsilon} = \Big\{ (\lambda, x) \in \mathbb{R} \times E : \|x\|^2 + \rho^2(\lambda, [a,b]) < r^2 + \varepsilon^2 \Big\}$$

is a completely continuous vector field and

$$\deg(f_{r,\varepsilon}, U_{r,\varepsilon}, 0) = -s[f,a,b] \qquad (r \in (0, r_0]).$$

The proof of the lemma is a modification of that given in [12] for the single-valued case and $[a,b] = \{\lambda_0\}$. It is enough to replace the function $d(\lambda) = |\lambda - \lambda_0|$ by $d(\lambda) = \rho(\lambda, [a,b])$. For an overview of this technique see also [15: Remark 1.5].

Theorem 1. *Let the map $f : A \times E \to cf(E)$ be given by* (1.1) *and assume that there exists an interval $[a,b] \subset A$ such that $\mathcal{B}_f \subset [a,b] \times \{0\}$ and $s[f,a,b] \neq 0$. Then there exists a non-compact component $\mathcal{C} \subset \mathcal{R}_f$ satisfying $\mathcal{C} \cap \mathcal{B}_f \neq \emptyset$.*

Proof. As consequence of the homotopy property of the topological degree and $s[f,a,b] \neq 0$ we have $([a,b] \times \{0\}) \cap \mathcal{B}_f \neq \emptyset$. Let $\mathcal{C}_0$ be a component of the set $\mathcal{R} = \mathcal{R}_f \cup ([a,b] \times \{0\})$ such that $[a,b] \times \{0\} \subset \mathcal{C}_0$. Assume further that $\mathcal{C}_0$ is compact. By Proposition 1 there exists a bounded open and closed set $\mathcal{K} \subset \mathcal{R}$ such that $\mathcal{C}_0 \subset \mathcal{K}$. So there exists a bounded and open set $U \subset A \times E$ satisfying $\mathcal{K} \subset U$ and $(\mathcal{R} \setminus \mathcal{K}) \cap \overline{U} = \emptyset$. Hence for $(\lambda, x) \in \partial U$ and $r > 0$ we have $0 \notin f_r(\lambda, x)$. Moreover, for any $r_1, r_2 > 0$ the maps f_{r_1} and f_{r_2} may be joined by homotopy. We can see as well that for large $R > 0$ the map f_R has no zeroes in $\overline{U}$ so that $\deg(f_r, U, 0) = 0$ for $r > 0$. There exist $\varepsilon > 0$ and $r_1 > 0$ such that $\overline{U_{r_1,\varepsilon}} \subset U$. Further, by Lemma 2 there exists $r' \in (0, r_1]$ such that $\deg(f_{r'}, U_{r',\varepsilon}, 0) = -s[f,a,b]$. Of course, $\overline{U_{r',\varepsilon}} \subset U$.

Because $\mathcal{B}_f \subset [a,b] \times \{0\}$ and U is bounded, there exists a number $r_2 > 0$ such that $0 \notin f(\lambda, x)$ for $(\lambda, x) \in U$ with $0 < \|x\| \leq r_2$ and $\rho(\lambda, [a,b]) \geq \varepsilon$.

Let $r \in (0, \min\{r', r_2\})$. Then $\overline{U_{r,\varepsilon}} \subset U$. Hence, if $0 \in f_r(\lambda, x)$ then $\|x\| = r < r_2$ and $\rho(\lambda, [a,b]) < \varepsilon$. Then we have $\|x\|^2 + \rho^2(\lambda, [a,b]) < r^2 + \varepsilon^2$ and $(\lambda, x) \in U_{r,\varepsilon}$. Consequently, we have the implication

$$(\lambda, x) \in \overline{U} \setminus U_{r,\varepsilon} \quad \Longrightarrow \quad 0 \notin f_r(\lambda, x).$$

That is why we have $\deg(f_r, U_{r,\varepsilon}, 0) = \deg(f_r, U, 0)$ and the contradiction

$$0 = \deg(f_r, U, 0) = \deg(f_r, U_{r,\varepsilon}, 0) = -s[f, a, b] \neq 0.$$

Because of this contradiction there exists a non-compact component $\mathcal{C}_0 \subset \mathcal{R}_f \cup ([a,b] \times \{0\})$. What we are going to prove now is that there exists a non-compact component $\mathcal{C}$ of $\mathcal{R}_f$ such that $\mathcal{C} \cap \mathcal{B}_f \neq \emptyset$. Of course, such component has to satisfy $\mathcal{C} \subset \mathcal{C}_0$.

At the beginning let us denote by Γ the family of all components γ of $\mathcal{R}_f$ such that $\gamma \cap \mathcal{B}_f \neq \emptyset$. Further, let $G = \bigcup_{\gamma \in \Gamma} \gamma$. We can observe that $G \subset \mathcal{C}_0$. We are going to show that there exists a $\gamma \in \Gamma$ such that γ is not compact. But first assume, contrary to our claim, that each $\gamma \in \Gamma$ is compact.

Let us now take $B = (c,d) \times B(0,R)$ such that

$$[a,b] \times \{0\} \subset B \subset \overline{B} \subset A \times E,$$

let us denote by Γ_B the family of all that components γ of $\mathcal{R}_f \cap \overline{B}$ for which $\gamma \cap \mathcal{B}_f \neq \emptyset$ and let us also denote $G_B = \bigcup_{\gamma \in \Gamma_B} \gamma$. We can see that $\mathcal{B}_f \subset G_B$. We are going to show that G_B is a closed subset of $\mathcal{R}_f \cap \overline{B}$. For this let $\{(\lambda_n, x_n)\} \subset G_B$ be a sequence such that $(\lambda_n, x_n) \to (\lambda_0, x_0) \in \mathcal{R}_f \cap \overline{B}$ and let $\gamma_n \in \Gamma_B$ be such that $(\lambda_n, x_n) \in \gamma_n$. Assume, contrary to our claim, that $(\lambda_0, x_0) \notin G_B$. Then $x_0 \neq 0$ and the component γ_0 of $\mathcal{R}_f \cap \overline{B}$ containing (λ_0, x_0) is such that $\gamma_0 \cap \mathcal{B}_f = \emptyset$. In this case we may apply Lemma 1 to the case of $K = \mathcal{R}_f \cap \overline{B}$, $X = \{(\lambda_0, x_0)\}$ and $Y = \mathcal{B}_f$. Then there exist sets $K_x, K_y \subset K$ open and closed in K such that

$$(\lambda_0, x_0) \in K_x, \quad \mathcal{B}_f \subset K_y, \quad K_x \cap K_y = \emptyset, \quad K = K_x \cup K_y.$$

Because for large $n \in \mathbb{N}$ the relation $\gamma_n \cap K_x \neq \emptyset$ holds and $\gamma_n \cap K_y \neq \emptyset$, this contradicts the connectedness of γ_n.

Now we are going to consider the following two situations:

(i) There exists $B_0 = (c,d) \times B(0,R)$ such that $[a,b] \times \{0\} \subset B_0 \subset \overline{B_0} \subset A \times E$ and $G \subset B_0$.

(ii) There exists a sequence $\{\gamma_n\} \subset \Gamma$ such that, for each $B = (c,d) \times B(0,R)$ satisfying $[a,b] \times \{0\} \subset B \subset \overline{B} \subset A \times E$, the relation $\gamma_n \not\subset \overline{B}$ holds for $n \in \mathbb{N}$ large enough.

Let us first assume that (i) holds and let $C_0^{B_0}$ be a component of $\mathcal{C}_0 \cap \overline{B_0}$ such that $[a,b] \times \{0\} \subset C_0^{B_0}$. Of course, we have $G_{B_0} \subset C_0^{B_0}$. By Lemma 1, in this case $C_0^{B_0} \subset B_0$ and there must be also $\mathcal{C}_0 \subset B_0$, what contradicts that $\mathcal{C}_0$ is not compact. So we can assume that there exists $(\lambda_0, x_0) \in \partial B_0 \cap C_0^{B_0}$. We can apply Lemma 1 for $K = \mathcal{R}_f \cap \overline{B_0}$, $X = \{(\lambda_0, x_0)\}$ and $Y = \mathcal{B}_f$. Because $(\lambda_0, x_0) \notin G_{B_0}$, there does not exist a component γ of K such that $(\lambda_0, x_0) \in \gamma$ and $\gamma \cap \mathcal{B}_f \neq \emptyset$. Then by Lemma 1, there exist open and closed sets $K_x, K_y \subset K$ such that

$$(\lambda_0, x_0) \in K_x, \quad \mathcal{B}_f \subset K_y, \quad K_x \cap K_y = \emptyset, \quad K_x \cup K_y = K.$$

This implies that there exist an $r > 0$ such that $K_x \cap ([a,b] \times \overline{B(0,r)}) = \emptyset$. Hence

$$\begin{aligned} K_x \cap (K_y \cup ([a,b] \times \{0\})) &= \emptyset \\ K_x \cup (K_y \cup ([a,b] \times \{0\})) &= K \cup ([a,b] \times \{0\}) \end{aligned}$$

and both K_x and $K_y \cup ([a,b] \times \{0\})$ are open and closed in $K \cup ([a,b] \times \{0\})$. But the set $C_0^{B_0} \subset K \cup ([a,b] \times \{0\})$ is connected and

$$\begin{aligned} C_0^{B_0} \cap K_x &\neq \emptyset \\ C_0^{B_0} \cap (K_y \cup ([a,b] \times \{0\}) &\neq \emptyset \end{aligned}$$

what gives the contradiction.

In this case the situation (ii) holds true. Let us fix any B as given in (ii) and let $\tilde{\gamma}_n \in \Gamma_B$ be such that $\tilde{\gamma}_n \subset \gamma_n$ and $(\lambda_n, x_n) \in \tilde{\gamma}_n \cap \partial B$. Because $x_n \in F(\lambda_n, x_n)$, we may assume that there exists a subsequence of (λ_n, x_n) converging to (λ_0, x_0). As we observed before, $(\lambda_0, x_0) \in G_B$. So there exists a component $\tilde{\gamma}_0 \in \Gamma_B$ such that $(\lambda_0, x_0) \in \tilde{\gamma}_0$. Let $\gamma_0 \in \Gamma$ be such that $\tilde{\gamma}_0 \subset \gamma_0$. From our general assumption γ_0 is compact. By Proposition 1 there exists an open and closed set $K \subset \mathcal{R}_f$ such that $\gamma_0 \subset K \subset B_0$ for some $B_0 = (c,d) \times B(0, R_0)$ so that $B_0 \subset \overline{B_0} \subset A \times E$. But for $n \in \mathbb{N}$ large enough the relations $K \cap \gamma_n \neq \emptyset$ and $\gamma_n \not\subset B_0$ hold. This gives $\gamma_n \cap K \neq \emptyset$ and $\gamma_n \cap (\mathcal{R}_f \setminus K) \neq \emptyset$, what contradicts the connectedness of γ_n.

So both (i) and (ii) cannot hold what implies that there exists $\gamma \in \Gamma$ which is not compact. ∎

The existence of components (in the single-valued case) emanating from bifurcation points was studied by Krasnoselskii (see [16]). The global bifurcation theorem for the single-valued case was proved by Rabinowitz in [23] (see also [9]) in the following version:

Theorem A. *Let $L : E \to E$ be a compact linear map, let $H : \mathbb{R} \times E \to E$ be a compact and continuous map such that $H(\lambda, u) = o(\|u\|)$ for u near*

0 *uniformly on bounded* λ *intervals, and let the map* $f : \mathbb{R} \times E \to E$ *be given by* $f(\lambda, u) = u - \lambda L(u) - H(\lambda, u)$. *Then, if* μ *is an eigenvalue of* L *of odd multiplicity, then* $\mathcal{R}_f$ *posseses a maximal subcontinuum* $\mathcal{C}_\mu$ *such that* $(\mu, 0) \in \mathcal{C}_\mu$ *and* $\mathcal{C}_\mu$ *either*

(i) *meets infinity in* $\mathbb{R} \times E$

or

(ii) *meets* $(\hat{\mu}, 0)$, *where* $\mu \neq \hat{\mu}$ *and* $\hat{\mu}$ *is an eigenvalue of* L.

The proof of Theorem 1 follows the ideas of complementing the map introduced by Ize (see [14], but also [20: Section 3.4]). The original version of the Rabinowitz theorem found numerous generalizations and modifications (for an overview see [4, 15]). The single-valued version of the global bifurcation theorem is probably most similar to what is proved in [18: Theorem 2.5]. Theorem 1 is not only a generalization of [18: Theorem 2.5] to convex-valued maps, but also gives stronger results (it gives the existence of the component of $\mathcal{R}_f$ instead of the component of $\mathcal{R}_f \cup ([a, b] \times \{0\})$).

The convex-valued case was already considered by the authors in [1] for a much more general situation of parameter space of dimension greater than 1. The authors gave there sufficient conditions for the existence of a global bifurcation branch emanating from $(0, 0)$. In Theorem 1 we focus on the case of scalar parameters but, on the other hand, we do not assume that the bifurcation points are isolated in the set of all bifurcation points.

2. Existence theorem for convex-valued differential inclusion

In this section we need the following notations. For $x = (x_1, ..., x_k) \in \mathbb{R}^k$ we write $|x| = \sum_{i=1}^k |x_i|$ and call x non-negative (and write $x \geq 0$) when $x_1, ..., x_k \geq 0$. Let the map $p : \mathbb{R}^k \to \mathbb{R}^k$ be given by

$$p(x_1, ..., x_k) = (\eta_1 |x_1|, ..., \eta_k |x_k|)$$

where $\eta_1, ..., \eta_k \geq 0$ and $\eta_1^2 + ... + \eta_k^2 > 0$, let $\|\cdot\|_0$ be the supremum norm in $C[a, b]$ and let $\|\cdot\|_k$ be the norm in $C^1([a, b], \mathbb{R}^k)$ given by $\|u\|_k = \sum_{i=1}^k (\|u_i\|_0 + \|u_i'\|_0)$ for $u = (u_1, ..., u_k) \in C^1([a, b], \mathbb{R}^k)$.

Let us recall that a multi-valued map $\varphi : [a, b] \times \mathbb{R}^k \times \mathbb{R}^k \to cf(\mathbb{R}^k)$ is a Carathéodory map if the map $\varphi(\cdot, x, y) : [a, b] \to cf(\mathbb{R}^k)$ is measurable for all $(x, y) \in \mathbb{R}^{2k}$, the map $\varphi(t, \cdot, \cdot) : \mathbb{R}^{2k} \to cf(\mathbb{R}^k)$ is upper semicontinuous for all $t \in [a, b]$, and for each $R > 0$ there exists an integrable function $m_R \in L^1(a, b)$ such that

$$\left\{ \begin{array}{l} \forall\ w \in L^1((a, b), \mathbb{R}^k) \\ \forall\ (x, y) \in \mathbb{R}^{2k} \\ \forall\ t \in [a, b] \end{array} \right\} : \quad \left\{ \begin{array}{l} |x| + |y| \leq R \\ w(t) \in \varphi(t, x, y) \end{array} \right\} \implies |w(t)| \leq m_R(t).$$

In this section we will give sufficient conditions for the existence of the solution of the boundary value problem

$$\left.\begin{aligned} &u''(t) \in \varphi\big(t, u(t), u'(t)\big) \quad \text{for a.e. } t \in (a,b) \\ &l(u) = 0 \end{aligned}\right\} \tag{2.1}$$

where $\varphi : [a,b] \times \mathbb{R}^k \times \mathbb{R}^k \to cf(\mathbb{R}^k)$ is a Carathéodory map and the map $l : C^1([a,b],\mathbb{R}^k) \to \mathbb{R}^k \times \mathbb{R}^k$ is given by

$$l(u_1,...,u_k) = \big(l_1(u_1),...,l_k(u_k)\big) \tag{2.2}$$

where $l_j(u_j) = \Big(u_j(a)\sin\alpha_j - u_j'(a)\cos\alpha_j, u_j(b)\sin\beta_j + u_j'(b)\cos\beta_j\Big)$ with $\alpha_j, \beta_j \in [0,\frac{\pi}{2}]$ and $\alpha_j^2 + \beta_j^2 > 0$ $(j = 1,...,k)$. It is well known (cf. [13: Theorem XI.4.1]) that with the boundary value problem

$$\left.\begin{aligned} &u_i''(t) = h_i(t) \quad \text{for a.e. } t \in (a,b) \\ &l_i(u_i) = 0 \end{aligned}\right\} \tag{2.3}$$

we may associate a continuous map $T_i : L^1(a,b) \to C^1[a,b]$ such that $T_i(h_i) = u_i$ if and only if $u_i \in C^1[a,b]$, $u_i' : [a,b] \to \mathbb{R}^1$ is absolutely continuous and u_i is a solution of problem (2.3).

Consider the map

$$\begin{aligned} &T : L^1((a,b),\mathbb{R}^k) \to C^1([a,b],\mathbb{R}^k) \\ &T(u_1,...,u_k) = (T_1u_1,...,T_ku_k). \end{aligned}$$

We can see that

$$u = Th \quad \Longleftrightarrow \quad \begin{cases} u''(t) = h(t) \quad \text{for a.e. } t \in (a,b) \\ \;l(u) = 0 \end{cases}$$

for $h \in L^1((a,b),\mathbb{R}^k)$. The map T has the following properties:

- For the Niemytzki operator $\Phi : C^1([a,b],\mathbb{R}^k) \to cf(L^1((a,b),\mathbb{R}^k))$ associated with φ and given by

$$\Phi(u) = \Big\{w \in L^1((a,b),\mathbb{R}^k) : w(t) \in \varphi(t,u(t),u'(t))\Big\} \tag{2.4}$$

  the superposition $T \circ \Phi : C^1([a,b],\mathbb{R}^k) \to cf(C^1([a,b],\mathbb{R}^k))$ is completely continuous (cf. [22: Proposition 3.6]).

- For $u, v \in C([a,b], \mathbb{R}^k)$ such that $l(u) = l(v) = 0$ we have
$$\langle Tu, v\rangle = \langle u, Tv\rangle \tag{2.5}$$
where $\langle u, v\rangle = \int_a^b \big(\sum_{i=1}^k u_i(t)v_i(t)\big)dt$ (cf. [13: Theorem XI.4.1]).
- (Maximum principle, cf. [21: Chapter 1/Theorem 2]) If the functions $u \in C^2([a,b], \mathbb{R}^k)$ and $h \in C([a,b], \mathbb{R}^k)$ satisfy
$$\left.\begin{aligned} u''(t) &= h(t) \ \text{ for a.e. } t \in (a,b) \\ l(u) &= 0 \end{aligned}\right\} \tag{2.6}$$
and $h \le 0$, then $u \ge 0$.

Before state the existence theorem we must refer to some spectral properties of the linear single-valued problem
$$\left.\begin{aligned} u''(t) + \lambda u(t) = 0 \ \text{ for } t \in (a,b) \\ l(u) = 0 \end{aligned}\right\}. \tag{2.7}$$
It is obvious that $\mu \in \mathbb{R}$ is an eigenvalue of problem (2.7) if and only if there exists $j \in \{1, ..., k\}$ such that μ is an eigenvalue of the scalar problem
$$\left.\begin{aligned} u_j''(t) + \lambda u_j(t) = 0 \ \text{ for } t \in (a,b) \\ l_j(u_j) = 0 \end{aligned}\right\}. \tag{2.7$_j$}$$
It is well known (cf [13: Theorem XI.4.1]) that there exists exactly one eigenvalue $\mu_j \in \mathbb{R}$ of problem $(2.7)_j$, for which there exists an eigenvector v_{μ_j} such that $v_{\mu_j}(t) > 0$ for $t \in (a,b)$, and then $\mu_j > 0$. Let us observe that then $u_{\mu_j} = (0, ..., v_{\mu_j}, ...0)$ is the eigenvector of problem (2.7) associated with the eigenvalue μ_j.

Lemma 3. *Assume that $(\lambda, u) \in (0, +\infty) \times C^1([a,b], \mathbb{R}^k)$ is a solution of the problem*
$$\left.\begin{aligned} u''(t) + \lambda p(u(t)) = 0 \ \textit{ for } t \in (a,b) \\ l(u) = 0 \end{aligned}\right\} \tag{2.8}$$
and $u \ne 0$. Then $\lambda \in \Lambda = \big\{\frac{\mu_i}{\eta_i} : \eta_i > 0\big\}$.

Proof. Let us first observe that $\Lambda \ne \emptyset$. By the maximum principle, for each $(\lambda, u) \in (0, +\infty) \times C^1([a,b], \mathbb{R}^k)$ being a solution of problem (2.8) we have $u \ge 0$. So, for $i = 1, ..., k$,
$$\left.\begin{aligned} u_i''(t) + \lambda\eta_i u_i(t) = 0 \ \text{ for } t \in (a,b) \\ l_i(u_i) = 0 \\ u_i \ge 0 \end{aligned}\right\}.$$
If $\eta_i = 0$, then there must be $u_i = 0$. On the other hand, for $\eta_i > 0$ the only $\lambda > 0$ for which $u \ne 0$ equals $\lambda = \frac{\mu_i}{\eta_i}$. ∎

Before we state the existence theorem let us assume that a Carathéodory map $\varphi:\ [a,b]\times\mathbb{R}^k\times\mathbb{R}^k\to cf(\mathbb{R}^k)$ satisfies the following two conditions:

$$\left.\begin{array}{l}\forall\ \varepsilon>0\ \exists\ \delta>0 \text{ such that}\\ \left.\begin{array}{l}|x|+|y|\le\delta\\ \forall\ (x,y)\in\mathbb{R}^{2k}\end{array}\right\}\implies\left\{\begin{array}{l}\varphi(t,x,y)\subset\overline{B\big(-m_1p(x),\varepsilon(|x|+|y|)\big)}\\ \forall\ t\in[a,b]\end{array}\right.\end{array}\right\}\tag{2.9}$$

$$\left.\begin{array}{l}\forall\ \varepsilon>0\ \exists\ R>0 \text{ such that}\\ \left.\begin{array}{l}|x|+|y|\ge R\\ \forall\ (x,y)\in\mathbb{R}^{2k}\end{array}\right\}\implies\left\{\begin{array}{l}\varphi(t,x,y)\subset\overline{B\big(-m_2p(x),\varepsilon(|x|+|y|)\big)}\\ \forall\ t\in[a,b].\end{array}\right.\end{array}\right\}\tag{2.10}$$

where $m_1,m_2>0$ are constants.

Theorem 2. *Let the map* $l:\ C^1([a,b],\mathbb{R}^k)\to\mathbb{R}^k\times\mathbb{R}^k$ *be given by* (2.2), *let* $\Lambda=\left\{\frac{\mu_i}{\eta_i}:\ \eta_i>0\right\}$ *and let* $\varphi:\ [a,b]\times\mathbb{R}^k\times\mathbb{R}^k\to cf(\mathbb{R}^k)$ *be a Carathéodory map satisfying* (2.9) – (2.10) *with constants* $m_1,m_2>0$ *such that*

$$\min\{m_1,m_2\}<\min\Lambda\le\max\Lambda<\max\{m_1,m_2\}.$$

Then there exists a non-trivial solution of the Sturm-Liouville problem (2.1).

Proof. Let us denote $m=\min\{m_1,m_2\}$ and $M=\max\{m_1,m_2\}$, let $\nu>\frac{\max\Lambda}{m}$ be a fixed constant, let $q_1,q_2:\ (0,+\infty)\to[0,+\infty)$ be continuous maps forming a partition of unity associated with the open cover $\{(0,2\nu),(\nu,+\infty)\}$ of the interval $(0,+\infty)$, and let us define the Carathéodory map

$$\begin{aligned}&\psi:\ [a,b]\times\mathbb{R}^k\times\mathbb{R}^k\times(0,+\infty)\to cf(\mathbb{R}^k)\\&\psi(t,x,y,\lambda)=q_1(\lambda)\lambda\varphi(t,x,y)-q_2(\lambda)\lambda m_2p(x).\end{aligned}$$

Let us now consider the differential inclusion

$$\left.\begin{array}{l}u''(t)\in\psi(t,u(t),u'(t),\lambda)\ \text{ a.e. on } (a,b)\\ l(u)=0\end{array}\right\}.\tag{2.11}$$

We can see that $(\lambda,u)\in(0,+\infty)\times C^1([a,b],\mathbb{R}^k)$ is a solution of this problem if and only if $u\in T\Psi(\lambda,u)$, where

$$\begin{aligned}&\Psi:\ (0,+\infty)\times C^1([a,b],\mathbb{R}^k)\to cf(L^1((a,b),\mathbb{R}^k))\\&\Psi(\lambda,u)=\Big\{w\in L^1((a,b),\mathbb{R}^k):\ w(t)\in\psi(t,u(t),u'(t),\lambda)\text{ for a.e. } t\in[a,b].\Big\}\end{aligned}$$

Let us also observe that, because $\nu>1$, a pair $(1,u)$ is a solution of problem (2.11) if and only if u is a solution of problem (2.1). Consider the map

$$\begin{aligned}&f:\ (0,+\infty)\times C^1([a,b],\mathbb{R}^k)\to cf(C^1([a,b],\mathbb{R}^k))\\&f(\lambda,u)=u-T\Psi(\lambda,u)\end{aligned}$$

and let

$$P : C^1([a,b],\mathbb{R}^k) \to L^1((a,b),\mathbb{R}^k)$$
$$P(u)(t) = p(u(t))$$

denote the Niemytzki map for the map p. The proof of Theorem 2 will be given now in three steps.

Step 1. We are going to show that $\mathcal{B}_f \subset \{(\frac{\lambda}{m_1},0) : \lambda \in \Lambda\}$. For this let us take a sequence $\{(\lambda_n,u_n)\} \subset (0,+\infty) \times C^1([a,b],\mathbb{R}^k)$ of non-trivial solutions of problem (2.11) such that $\lambda_n \to \lambda_0 \in [0,+\infty)$ and $u_n \to 0$. We have

$$u_n \in\ q_1(\lambda_n)\lambda_n T\big(\Phi(u_n)+m_1P(u_n)\big) - \lambda_n T\big(m_1q_1(\lambda_n)+m_2q_2(\lambda_n)\big)P(u_n).$$

Let us denote $v_n = \frac{u_n}{\|u_n\|_k}$. Then

$$v_n \in q_1(\lambda_n)\lambda_n T\frac{\Phi(u_n)+m_1P(u_n)}{\|u_n\|_k} - \lambda_n T\big((m_1q_1(\lambda_n)+m_2q_2(\lambda_n)\big)P(v_n)).$$

By (2.9) we have $\frac{\Phi(u_n)+m_1P(u_n)}{\|u_n\|_k} \to \{0\}$ (in the Hausdorff metric). Because the sequence $\{\big(m_1q_1(\lambda_n)+m_2q_2(\lambda_n)\big)P(v_n)\}$ is bounded, there exists a subsequence of $\{v_n\}$ convergent to $v_0 \in C^1([a,b],\mathbb{R}^k)$, where $\|v_0\|_k = 1$. So letting $n \to +\infty$ we get $v_0 = -\lambda_0 T\big(\big(m_1q_1(\lambda_0)+m_2q_2(\lambda_0)\big)P(v_0)\big)$ and

$$\left.\begin{aligned} v_0''(t)+\lambda_0\big(m_1q_1(\lambda_0)+m_2q_2(\lambda_0)\big)p(v_0(t)) = 0 \quad &\text{for a.e. } t\in(a,b)\\ l(u)=0 \quad & \end{aligned}\right\}.$$

So, by Lemma 3, $\big(m_1q_1(\lambda_0)+m_2q_2(\lambda_0)\big)\lambda_0 \in \Lambda$. No matter what is the value of λ_0 we have $m_1q_1(\lambda_0)+m_2q_2(\lambda_0) \in [m,M]$. So $\lambda_0 \le \frac{\max\Lambda}{m} < \nu$ what implies $m_1\lambda_0 \in \Lambda$ and finishes the proof of Step 1.

Step 2. We will now show that $s\big[f, \frac{\min\Lambda}{m_1}, \frac{\max\Lambda}{m_1}\big] = -1$. For this, first let us observe that for $\lambda \notin \{\frac{\lambda}{m_1} : \lambda \in \Lambda\}$ there exists $r>0$ such that by (2.9) the map

$$f(\lambda,\cdot) : \overline{B(0,r)} \to cf(C^1([a,b],\mathbb{R}^k))$$

is homotopic to the map

$$\bar{f}(\lambda,\cdot) : \overline{B(0,r)} \to cf(C^1([a,b],\mathbb{R}^k))$$
$$\bar{f}(\lambda,u) = u + \lambda\big(m_1q_1(\lambda)+m_2q_2(\lambda)\big)TP(u).$$

We can see also that the map

$$\bar{f}(\lambda,\cdot) : \overline{B(0,r)} \to C^1([a,b],\mathbb{R}^k)$$

for $\lambda \geq \nu$ may be joined by homotopy with the map

$$f_0(\lambda, \cdot) : \overline{B(0,r)} \to C^1([a,b], \mathbb{R}^k)$$
$$f_0(\lambda, u) = u + \lambda m_1 TP(u).$$

Let the homotopy

$$h : [0,1] \times \overline{B(0,r)} \to C^1([a,b], \mathbb{R}^k)$$
$$h(\tau, u) = u + \lambda\big(\tau m_1 q_1(\lambda) + m_2 \tau q_2(\lambda) + (1-\tau) m_1\big) TP(u)$$

be given. Similarly to what we showed in Step 1 of this proof, for any non-trivial zero of the homotopy h, there must be

$$\lambda\big(\tau m_1 q_1(\lambda) + m_2 \tau q_2(\lambda) + (1-\tau) m_1\big) \in \Lambda$$

what, having $\big(\tau m_1 q_1(\lambda) + m_2 \tau q_2(\lambda) + (1-\tau) m_1\big) \geq (1-\tau) m_1 + \tau m \geq m$, implies $\lambda \leq \frac{\max \Lambda}{m}$ and contradicts $\lambda \geq \nu$. On the other hand, for $\lambda < \nu$ we have $\bar{f}(\lambda, \cdot) = f_0(\lambda, \cdot)$.

Let $r > 0$ and $\lambda_0 \in (0, \frac{\min \Lambda}{m_1})$ be fixed. We will show that

$$f_0(\lambda_0, \cdot) : \overline{B(0,r)} \to C^1([a,b], \mathbb{R}^k)$$

may be joined by homotopy with the identity map. Let a homotopy be given by $h(\tau, u) = u + \lambda_0 \tau T m_1 P(u)$. We can conclude from Lemma 3 that $(\lambda_0 \tau, 0) \notin \mathcal{B}_f$ for $\tau \in [0,1]$. That is why we have no non-trivial zeros of $h(\tau, u) = 0$. Hence, by homotopy property of topological degree, we have $\deg\big(f_0(\lambda_0, \cdot), B(0,r), 0\big) = 1$.

Assume now that $\lambda_0 \in (\frac{\max \Lambda}{m_1}, +\infty)$ and let $i \in \{1, ..., k\}$ be such that $\eta_i > 0$ and $u_{\mu_i} = -\mu_i T u_{\mu_i}$ with $u_{\mu_i, i}(t) > 0$ for $t \in (a,b)$ where $u_{\mu_i, i}$ is the i-th coordinate of u_{μ_i}. We will show that for λ_0 the map $f_0(\lambda_0, \cdot)$ may be joined by homotopy on $\overline{B(0,r)}$ with the map

$$f_1 : \overline{B(0,r)} \to C^1([a,b], \mathbb{R}^k)$$
$$f_1(u) = f_0(\lambda_0, u) - u_{\mu_i}.$$

A homotopy $h : [0,1] \times \overline{B(0,r)} \to C^1([a,b], \mathbb{R}^k)$ is given by

$$h(\tau, u) = f_0(\lambda_0, u) - \tau u_{\mu_i}.$$

Assume now that for $\|u\|_k \leq r$ and $\tau \in (0,1]$ the equality $h(\tau, u) = 0$ holds and

$$u + \lambda_0 m_1 TP(u) - \tau u_{\mu_i} = 0$$
$$u + T\big(\lambda_0 m_1 P(u) + \tau \mu_i u_{\mu_i}\big) = 0.$$

So we have

$$\left.\begin{aligned} u''(t)+\lambda_0 m_1 p(u(t))+\tau\mu_i u_{\mu_i}(t)=0 \quad \text{for a.e. } t\in(a,b)\\ l(u)=0 \end{aligned}\right\}$$

what, by the maximum principle, gives $u\geq 0$ and, consequently, $p_i(u_i)=\eta_i u_i$. Since $u_i=-\lambda_0 T_i m_1\eta_i u_i+\tau u_{\mu_i,i}$ and also

$$\begin{aligned}\langle u_i,u_{\mu_i,i}\rangle &= -\lambda_0\langle T_i m_1\eta_i u_i,u_{\mu_i,i}\rangle+\tau\langle u_{\mu_i,i},u_{\mu_i,i}\rangle\\ &= -\lambda_0\langle m_1\eta_i u_i,T_i u_{\mu_i,i}\rangle+\tau\langle u_{\mu_i,i},u_{\mu_i,i}\rangle\\ &= \frac{\lambda_0 m_1\eta_i}{\mu_i}\langle u_i,u_{\mu_i,i}\rangle+\tau\langle u_{\mu_i,i},u_{\mu_i,i}\rangle\end{aligned}$$

we have

$$\frac{\mu_i-m_i\eta_i\lambda_0}{\mu_i}\langle u_i,u_{\mu_i,i}\rangle=\tau\langle u_{\mu_i,i},u_{\mu_i,i}\rangle>0.$$

Because $u_{\mu_i,i}\geq 0$ and $u_i\geq 0$, it must be also $\mu_i>m_1\eta_i\lambda_0$ what contradicts the assumption $\lambda_0>\frac{\max\Lambda}{m_1}\geq\frac{\mu_i}{\eta_i m_1}$.

If $\tau=0$, then $h(\tau,\cdot)=f_0(\lambda_0,\cdot)$ and $h(0,u)=0$ if and only if $f_0(\lambda_0,u)=0$. Because $m\lambda_0\notin\Lambda$, $f_0(\lambda_0,u)=0$ implies $u=0$. Hence the homotopy h has no non-trivial zeroes. Also, $h(1,\cdot)$ has no zeroes at all and that is why $\deg\big(f_0(\lambda_0,\cdot),B(0,r),0\big)=0$. So Step 2 is proved.

`Step 3.` Let us observe that by Theorem 1 there exists a non-compact component $\mathcal{C}\subset\mathcal{R}_f$. Now we are going to show that there exists a sequence $\{(\lambda_n,u_n)\}\subset\mathcal{C}$ such that $\|u_n\|_k\to+\infty$ and $\lambda_n\to\lambda_0\in\big\{\frac{\lambda}{m_2}:\lambda\in\Lambda\big\}$.

Because the set $\mathcal{C}$ is not compact, there exists a sequence $\{(\lambda_n,u_n)\}\subset\mathcal{C}$ such that $\lambda_n\to 0$, or $\lambda_n\to+\infty$, or $\|u_n\|_k\to+\infty$. We are going to show that there must be $\|u_n\|_k\to+\infty$.

First, let us assume that $\lambda_n\to 0$ and that $\{\|u_n\|_k\}$ is bounded. In this case, for almost all $n\in\mathbb{N}$, the relation $u_n\in\lambda_n T\Phi(u_n)$ holds and consequently $u_n\to 0$. As we showed in Step 1, $u_n\to 0$ and $\lambda_n\to\lambda_0$ implies that $\lambda_0\in\big\{\frac{\lambda}{m_1}:\lambda\in\Lambda\big\}$ what contradicts $\lambda_n\to 0$.

Now let us consider the case $\lambda_n\to+\infty$. Then, for almost all $n\in\mathbb{N}$, if $u_n\neq 0$, then there must be $q_2(\lambda_n)=1$ and $u_n=\lambda_n Tm_2P(u_n)$. By Lemma 3 there is $\lambda_n\in\big\{\frac{\lambda}{m_2}:\lambda\in\Lambda\big\}$ what contradicts $\lambda_n\to+\infty$.

So we may assume that $\|u_n\|_k\to+\infty$ and $\lambda_n\to\lambda_0\in(0,+\infty)$. Now we are going to prove that in such situation $\lambda_0\in\big\{\frac{\lambda}{m_2}:\lambda\in\Lambda\big\}$. Indeed, we can see that

$$u_n\in\begin{cases}\lambda_n q_1(\lambda_n)T\big(\Phi(u_n)+m_2P(u_n)\big)-\lambda_n Tm_2P(u_n)\\ \lambda_n q_1(\lambda_n)T\frac{\Phi(u_n)+m_2P(u_n)}{\|u_n\|_k}-\lambda_n Tm_2P(v_n)\end{cases}$$

where $v_n = \frac{u_n}{\|u_n\|_k}$. We are going to show that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$T\frac{\Phi(u_n) + m_2 P(u_n)}{\|u_n\|_k} \subset \overline{B(0,\varepsilon)} \qquad (n > N).$$

For this, let $\varepsilon > 0$ be fixed. By (2.10) there exists $R > 0$ such that for $|u_n(t)| + |u_n'(t)| \geq R$ the relation

$$\frac{\varphi(t, u(t), u'(t)) + m_2 p(u(t))}{|u(t)| + |u'(t)|} \subset B(0,\varepsilon)$$

holds. Let $m_R \in L^1(a,b)$ be an integrable function such that

$$\left\{\begin{array}{l} \forall w \in L^1((a,b),\mathbb{R}^k) \\ \forall x \in \mathbb{R}^k \\ \forall y \in \mathbb{R}^k \\ \forall\, t \in [a,b] \end{array}\right\} : \quad \left\{\begin{array}{l} |x| + |y| \leq R \\ \\ w(t) \in \varphi(t,x,y) \end{array}\right\} \implies |w(t)| \leq m_R(t).$$

Let us now take any $w \in L^1((a,b),\mathbb{R}^k)$ such that

$$w(t) \in \frac{\varphi\big(t, u_n(t), u_n'(t)\big) + m_2 p(u_n(t))}{\|u_n\|_k} \qquad (t \in [a,b])$$

and consider the two situations

$$|u_n(t)| + |u_n'(t)| \leq R$$
$$|u_n(t)| + |u_n'(t)| > R.$$

For them we have respectively $|w(t)| \leq \frac{m_R(t)}{\|u_n\|_k}$ and

$$\begin{aligned} &\frac{\varphi\big(t, u_n(t), u_n'(t)\big) + m_2 p(u_n(t))}{\|u_n\|_k} \\ &\qquad = \frac{\varphi\big(t, u_n(t), u_n'(t)\big) + m_2 p(u_n(t))}{|u_n(t)| + |u_n'(t)|} \cdot \frac{|u_n(t)| + |u_n'(t)|}{\|u_n\|_k} \\ |w(t)| &\in \frac{\varphi\big(t, u_n(t), u_n'(t)\big) + m_2 p(u_n(t))}{|u_n(t)| + |u_n'(t)|} \cdot \frac{|u_n(t)| + |u_n'(t)|}{\|u_n\|_k} \\ &\subset B(0,\varepsilon). \end{aligned}$$

So for $n \in \mathbb{N}$ big enough and any $t \in [a,b]$ we have $|w(t)| < \max\big\{\varepsilon, \frac{m_R(t)}{\|u_n\|_k}\big\}$ what shows that

$$T\frac{\Phi(u_n) + m_2 P(u_n)}{\|u_n\|_k} \subset B\big(0, \varepsilon\|T\|(b-a)\big)$$

with $\|T\|$ denoting the norm of the map $T: L^1((a,b),\mathbb{R}^k) \to C^1([a,b],\mathbb{R}^k)$.

Let us observe that, because of the compactness of T, we may assume that $v_n \to v_0$, where $v_0 \neq 0$. Hence, letting $n \to +\infty$ we get $v_0 = -\lambda_0 T m_2 P(v_0)$ what results in $\lambda_0 \in \{\frac{\lambda}{m_2} : \lambda \in \Lambda\}$. Further, let us observe that the assumptions of the theorem imply that $\{\frac{\lambda}{m} : \lambda \in \Lambda\} \subset (1,+\infty)$ and $\{\frac{\lambda}{M} : \lambda \in \Lambda\} \subset (0,1)$. As a consequence of Steps 1 and 3 of this proof we can see that the connected set $\mathcal{C}$ contains pairs (λ_1, u) and (λ_2, u) with $\lambda_1 < 1$ and $\lambda_2 > 1$. That is why we can conclude that there exists $(1,u) \in \mathcal{C}$. For such a solution of the inclusion $0 \in f(\lambda, u)$ there must be $u \neq 0$ because $(1,0) \notin \mathcal{R}_f$. ∎

3. Examples and remarks

In this section we will give some applications of Theorem 2 to the convex-valued boundary value problems

$$\left.\begin{array}{l} u''(t) \in \varphi(t,u(t),u'(t)) \quad \text{for a.e. } t \in (0,1) \\ u(0) = u(1) = 0 \end{array}\right\} \tag{3.1}$$

$$\left.\begin{array}{l} u''(t) \in \varphi(t,u(t),u'(t)) \quad \text{for a.e. } t \in (0,1) \\ u(0) = u'(1) = 0 \end{array}\right\}. \tag{3.2}$$

Let us remind that the topological transversality method of Granas and a priori bounds technique have been used to existence theorems for the above second order differential equations (inclusions) [6, 7, 10, 11]. The fundamental assumption there, which guaranteed the bound of zeros of the homotopy joining suitable vector fields associated with the boundary value problem, were the following Bernstein conditions:

(H1) There exists a constant $R > 0$ such that if $|x_0| > R$ and $y_0 \in \mathbb{R}^k$, then there is a $\delta > 0$ such that

$$\operatorname*{ess\,inf}_{t\in[a,b]} \inf \left\{ \langle x, w\rangle + |y|^2 : w \in \varphi(t,x,y),\ (x,y) \in B((x_0,y_0),\delta) \right\} > 0$$

where $B((x_0,y_0),\delta) = \left\{ (x,y) \in \mathbb{R}^k \times \mathbb{R}^k : |x - x_0| + |y - y_0| < \delta \right\}$.

(H2) There is a function $\Phi : [0,+\infty) \to [0,+\infty)$ such that the function $s \to \frac{s}{\Phi(s)}$ is in $L^\infty_{loc}[0,+\infty)$, $\int_0^{+\infty} \frac{s}{\Phi(s)}\, ds = +\infty$, $|\varphi(t,x,y)| \le \Phi(y)$ for a.e. $t \in [a,b]$ and all (x,y) with $|x| + |y| \le R$ where R is given in condition (H1).

(H3) There exist constants $k, \alpha > 0$ such that $|\varphi(t,x,y)| \le 2\alpha(\langle x, w\rangle + |y|^2) + k$ for a.e. $t \in [a,b]$, all (x,y) with $|x| + |y| \le R$ and $w \in \varphi(t,x,y)$.

Below we will give some ordinary differential inclusions, for which the orientors $\varphi(t,x,y)$ locally have linear asymptotics "at zero and at infinity" (also all assumptions of Theorem 2 are satisfied), but they do not satisfy the above Bernstein conditions (H1) - (H3).

Corollary 1. *Let* $\varphi : [0,1]\times\mathbb{R}^k\times\mathbb{R}^k \to cf((-\infty,0]^k)$ *be a Carathéodory map satisfying* (2.9) – (2.10) *with constants* $m_1, m_2 > 0$ *such that*

$$\min\{m_1,m_2\} < \min\left\{\frac{\pi^2}{\eta_i} : \eta_i > 0\right\} \le \max\left\{\frac{\pi^2}{\eta_i} : \eta_i > 0\right\} < \max\{m_1,m_2\}.$$

Then there exists a non-trivial solution of problem (3.1).

Proof. Let us observe that the only eigenvalue of the problem

$$\left.\begin{aligned} u''(t) + \lambda u(t) = 0 \\ u(0) = u(1) = 0 \end{aligned}\right\},$$

for which there exists a non-negative eigenvector, is $\mu_0 = \pi^2$. Then φ satisfies all assumptions of Theorem 2. So there exists a non-trivial solution of problem (3.1). ∎

Remark 2. The multi-valued map φ given in Corollary 1 does not satisfy condition (H1). Indeed, let us take large $x_0 \in [0,+\infty)^k$ and $y_0 = 0$. Then, if $w \in \varphi(t,x,y)$ then $w < 0$. So $\langle x, w\rangle + |y|^2 < 0$ and condition (H1) is not satisfied.

Corollary 2. *Let* $\varphi : [0,1]\times\mathbb{R}^k\times\mathbb{R}^k \to cf(\mathbb{R}^k)$ *be a Carathéodory map satisfying* (2.9) – (2.10) *with constants* $m_1, m_2 > 0$ *such that*

$$\min\{m_1,m_2\} < \min\left\{\frac{\pi^2}{\eta_i} : \eta_i > 0\right\} \le \max\left\{\frac{\pi^2}{\eta_i} : \eta_i > 0\right\} < \max\{m_1,m_2\}.$$

Assume additionally that, for each $M > 0$, $\mu(\{t : |\varphi(t,0,y)| > M\}) > 0$ (μ *denotes the Lebesgue measure*) *where* $k < |y| < K$ *for* $k, K > 0$. *Then there exists a non-trivial solution of problem* (3.1).

Proof. Let us observe that the map φ satisfies all assumptions of Theorem 2. So there exists a non-trivial solution of problem (3.1). ∎

Remark 3. The multi-valued map φ given in Corollary 2 does not satisfy condition (H2). Indeed, let us observe that there is no function Φ such that $|\varphi(t,x,y)| \le \Phi(y)$ for a.e. $t \in [0,1]$ and all (x,y) such that $|x| + |y| \le R$. So condition (H2) is not satisfied.

Corollary 3. *Let* $\varphi : [0,1]\times\mathbb{R}^k\times\mathbb{R}^k \to cf((-\infty,0]^k)$ *be a Carathéodory map satisfying* (2.9) – (2.10) *with constants* $m_1, m_2 > 0$ *such that*

$$\min\{m_1,m_2\} < \min\left\{\frac{\pi^2}{4\eta_i} : \eta_i > 0\right\} \le \max\left\{\frac{\pi^2}{4\eta_i} : \eta_i > 0\right\} < \max\{m_1,m_2\}.$$

Then there exists a non-trivial solution of problem (3.2).

Proof. Let us observe that the only eigenvalue of the problem

$$\left.\begin{array}{r} u''(t)+\lambda u(t)=0 \\ u(0)=u'(1)=0 \end{array}\right\},$$

for which there exists a non-negative eigenvector, is $\mu_0 = \frac{\pi^2}{4}$. Then the map φ satisfies all assumptions of Theorem 2. So there exists a non-trivial solution of problem (3.1). ∎

Remark 4. The multi-valued map φ given in Corollary 3 does not satisfy condition (H1). Indeed, let us take large $x_0 \in [0,+\infty)^k$ and $y_0 = 0$. Then, if $w \in \varphi(t,x,y)$ then $w < 0$. So $\langle x, w\rangle + |y|^2 < 0$ and condition (H1) is not satisfied.

Corollary 4. *Let* $\varphi : [0,1]\times\mathbb{R}^k\times\mathbb{R}^k \to cf(\mathbb{R}^k)$ *be a Carathéodory map satisfying* (2.9) – (2.10) *with constants* $m_1, m_2 > 0$ *such that*

$$\min\{m_1,m_2\} < \min\left\{\frac{\pi^2}{4\eta_i} : \eta_i > 0\right\} \le \max\left\{\frac{\pi^2}{4\eta_i} : \eta_i > 0\right\} < \max\{m_1,m_2\}.$$

Additionally, assume that, for each $M > 0$, $\mu(\{t : |\varphi(t,0,y)| > M\}) > 0$ (*Lebesgue measure*), *where* $k < |y| < K$ *for* $k, K > 0$. *Then there exists a non-trivial solution of problem* (3.2).

Proof. Let us observe that the map φ satisfies all assumptions of Theorem 2. So there exists a non-trivial solution of problem (3.2). ∎

Remark 5. The multi-valued map φ given in Corollary 4 does not satisfy condition (H2). Indeed, let us observe that there is no function Φ such that $|\varphi(t,x,y)| \le \Phi(y)$ for a.e. $t \in [0,1]$ and all (x,y) such that $|x|+|y| \le R$. So condition (H2) is not satisfied.

Remark 6. In [5] a special case of problem (2.1) was considered where α_i and β_i are constant (do not depend on $i \in \{1,...,k\}$) and $\varphi : [a,b]\times\mathbb{R}^k\times\mathbb{R}^k \to cl(\mathbb{R}^k)$ is a Carathéodory map satisfying the linear growth condition

$$|\varphi(t,x,y)| \le w_0(t) + w_1(t)|x| + w_2(t)|y| \tag{3.3}$$

for integrable functions $w_0, w_1, w_2 \in L^1(a,b)$. Let us now denote by $G : [a,b]^2 \to \mathbb{R}$ the Green function related with the linear problem (2.7). In [5] it is proved that if w_1, w_2 in (3.3) are integrable functions and the map

$$L : C([a,b],\mathbb{R}^k) \times C([a,b],\mathbb{R}^k) \to C([a,b],\mathbb{R}^k) \times C([a,b],\mathbb{R}^k)$$

$$L(\xi,\eta) = \left(\int_a^b |G(\cdot,s)| \big[w_1(s)\xi(s) + w_2(s)\eta(s)\big]\, ds, \int_a^b |G_t(\cdot,s)| \big[w_1(s)\xi(s) + w_2(s)\eta(s)\big]\, ds \right)$$

has spectral radius $r(L) < 1$, then problem (2.1) has a solution.

In the special case of $w_2 = 0$, w_1 constant and Dirichlet boundary conditions $l(u) = (u(a), u(b))$, condition $r(L) < 1$ is equivalent to $w_1 < \frac{\pi^2}{(b-a)^2}$ (see [5: Example 12.2]). Let us now again consider $\varphi : [0,1] \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ given in Corollary 1, with $\eta_i = 1 \quad (i = 1,...,k)$, satisfying additionally $|\varphi(t,x,y)| \leq w_0 + w_1|x|$ with $w_0, w_1 \in (0,+\infty)$. In this case, because of (2.9) - (2.10), there must be $w_1 > \pi^2$. So the condition $w_1 < \frac{\pi^2}{(b-a)^2}$ is not satisfied and the mentioned theorem given in [5] cannot be applied.

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