Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 23 (2004), No. 2, 303–311

Another Version of Maher's Inequality

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Abstract. Let H be a separable infinite dimensional complex Hilbert space, and let $L(H)$ denote the algebra of bounded linear operators on H into itself. Let $A = (A_1, A_2..., A_n)$, $B = (B_1, B_2..., B_n)$ be n-tuples of operators in $L(H)$. We define the elementary operator $\Delta_{A,B} : L(H) \mapsto L(H)$ by $\Delta_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i - X$. In this paper we minimize the map $F_p(X) = \|T - \Delta_{A,B}(X)\|_p^p$, where $T \in \ker \Delta_{A,B} \cap C_p$, and we classify its critical points.

Keywords: *Orthogonality, derivation, elementary operators*

MSC 2000: Primary 47B47, 47A30, 47B20, secondary 47B10

1. Introduction

Let H be a separable infinite dimensional complex Hilbert space, and let $L(H)$ denote the algebra of bounded linear operators on H into itself. Given $A, B \in L(H)$, we define the generalized derivation $\delta_{A,B} : L(H) \mapsto L(H)$ by $\delta_{A,B}(X) = AX - XB$. Let $A = (A_1, A_2..., A_n)$, $B = (B_1, B_2..., B_n)$ be n-tuples of operators in $L(H)$. We define the elementary operator $\Delta_{A,B} : L(H) \mapsto L(H)$, $\Delta^*_{A,B} : L(H) \mapsto L(H)$ by

$$\Delta_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i - X$$

and

$$\Delta^*_{A,B}(X) = \sum_{i=1}^{n} A_i^* X B_i^* - X$$

respectively. Denote $\delta_{A,A}(X) = \delta_A(X) = AX - XA$ and $\Delta_{A,A} = \Delta_A = \sum_{i=1}^{n} A_i X A_i - X$. A well-known result of J. Anderson [1: p.136-137] says that if A is a normal operator such that $AS = SA$, then for all $X \in L(H)$,

$$\|S - (AX - XA)\| \geq \|S\| . \tag{1.1}$$

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This work was supported by our research center project No. Math/1422/10.

ISSN 0232-2064 / \$ 2.50

The related inequality (1.1) was obtained by P. J. Maher [9: Theorem 3.2]. It shows that, if A is a normal operator and $AS = SA$, where $S \in C_p$, $1 \leq p < \infty$ and $S \in \ker \delta_{A,B} \cap C_p$, then the map F_p defined by

$$F_p(X) = \|S - (AX - XA)\|_p^p$$

has a global minimizer at V if, and for $1 < p < \infty$ only if, $AV - VA = 0$. In other words, we have

$$\|S - (AX - XA)\|_p p \geq \|T\|_p^p, \tag{1.2}$$

where C_p is the von Neumann-Schatten class, $1 \leq p < \infty$ and $\|.\|_p$ its norm. In [6] and [3] the authors generalized P. J. Maher's result, showing that if the pair (A, B) has the property $(FP)_{C_p}$ (i.e. $AT = TB$, where $T \in C_p$ implies $A^*T = TB^*$), $1 \leq p < \infty$, and $S \in \ker \delta_{A,B} \cap C_p$, then the map F_p defined by

$$F_p(X) = \|S - (AX - XB)\|_p^p$$

has a global minimizer at V if, and for $1 < p < \infty$ only if, $AV - VB = 0$. In other words, we have

$$\|S - (AX - XB)\|_p^p \geq \|T\|_p^p \tag{1.3}$$

if, and for $1 < p < \infty$ only if, $AV - VB = 0$. In this paper we obtain an inequality similar to (1.3), where the operator $AX - XB$ is replaced by the operator $\Delta_{A,B}(X) = \sum_{i=1}^n A_i X B_i - X$. We prove that if $\Delta_{A,B}(T) = 0 = \Delta_{A^*,B^*}(T)$ and $T \in \ker \Delta_{A,B} \cap C_p$, then the map F_p defined by

$$F_p(X) = \|T - \Delta_{A,B}(X)\|_p^p \tag{1.4}$$

has a global minimizer at V if, and for $1 < p < \infty$ only if, $\sum_{i=1}^n A_i V B_i - V = 0$. Moreover, we show that if $\Delta_{A,B}(T) = 0 = \Delta_{A^*,B^*}(T)$ and $T \in \ker \Delta_{A,B} \cap C_p, 1 < p < \infty$, then the map F_p has a critical point at W if and only if $\sum_{i=1}^n A_i W B_i - W = 0$, i.e. if $D_W F_p$ is the Frechet derivative at W of F_p, then the set

$$\{W \in L(H)\colon\ D_W F_p = 0\}$$

coincides with $\ker \Delta_{A,B}$ (the kernel of $\Delta_{A,B}$).

2. Preliminaries

Let $T \in L(H)$ be compact, and let $s_1(X) \geq s_2(X) \geq ... \geq 0$ denote the singular values of T, i.e. the eigenvalues of $|T| = (T^*T)^{\frac{1}{2}}$ are arranged in their decreasing order. The operator T is said to belong to the Schatten $p-$class C_p if

$$\|T\|_p = \Big[\sum_{j=1}^{\infty} s_j(T)^p\Big]^{\frac{1}{p}} = [tr(T)^p]^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

where tr denotes the trace function. Hence C_1 is the trace class, C_2 is the Hilbert-Schmidt class, and C_∞ is the class of compact operators where

$$\|T\|_\infty = s_1(T) = \sup_{\|f\|=1} \|Tf\|$$

denotes the usual operator norm. For the general theory of the Schatten $p-$ classes the reader is referred to [11] and [12]. Let $\Re z$ be the real part of a complex number z, $X = U|X|$ be the polar decomposition of the operator X and let tr denote trace.

Theorem 2.1. [2] *If* $1 < p < \infty$, *then the map* $F_p : C_p \longmapsto \mathbb{R}^+$ *defined by* $X \longmapsto \|X\|_p^p$, *is differentiable at every* $X \in C_p$ *with derivative* $\mathcal{D}_X F_p$ *given by*

$$\mathcal{D}_X F_p(T) = p \cdot \Re tr(|X|^{p-1} U^* T), \tag{2.1}$$

If $\dim \mathrm{H} < \infty$, *then the same result holds for* $0 < p \leq 1$ *at every invertible* X.

Theorem 2.2. [9] *If* U *is a convex subset of* C_p *with* $1 < p < \infty$ *and* $X \in \mathrm{U}$, *then the map* $X \longmapsto \|X\|_p^p$ *has at most one global minimizer.*

Lemma 2.1. [13] *Let* C *denote the n-tuple of operators* $(C_1, C_2 ..., C_n)$ *in* $L(H)$. *Suppose that* $\sum_{i=1}^n C_i C_i^* \leq 1$ *and* $\sum_{i=1}^n C_i^* C_i \leq 1$. *If* $\Delta_C(T) = 0 = \Delta_C^*(T)$ *for some compact operator* T, *then the operator* $|T|$ *commutes with* C_i *for all* $1 \leq i \leq n$.

Definition 2.1. Let F and G be two subspaces of a normed linear space E. If $\|x + y\| \geq \|y\|$ for all $x \in F$ and for all $y \in G$, then F *is said to be orthogonal to* G.

3. Main Results

Let $\mathcal{U}(A, B) = \{X \in L(H)$: $(\sum_{i=1}^n C_i X C_i - X) \in C_p\}$ and $F_p : \mathcal{U} \longmapsto \mathbb{R}^+$ be the map defined by $F_p(X) = \|T - (\sum_{i=1}^n C_i X C_i - X)\|_p^p$, where $T \in \ker \Delta_C \cap C_p, 1 \leq p < \infty$. We start with the following lemma which will be used in the proof of Theorem 3.1.

Lemma 3.1. *Let* C *denote the n-tuple of operators* $(C_1, C_2 ..., C_n)$ *in* $L(H)$ *such that* $\sum_{i=1}^n C_i C_i^* \leq 1$, $\sum_{i=1}^n C_i^* C_i \leq 1$. *Let* S *be compact and* $\Delta_c(S) = 0 = \Delta_c^*(S)$. *If*

$$\sum_{i=1}^n C_i |S|^{p-1} U^* C_i = |S|^{p-1} U^*,$$

where $p > 1$ *and* $S = U|S|$ *is the polar decomposition of* S, *then*

$$\sum_{i=1}^n C_i |S| U^* C_i = |S| U^*.$$

Proof. If $T = |S|^{p-1}$, then

$$\sum_{i=1}^{n} C_i T U^* C_i = T U^*. \tag{3.1}$$

We prove that

$$\sum_{i=1}^{n} C_i T^n U^* C_i = T^n U^*. \tag{3.2}$$

It is known that if $\sum_{i=1}^{n} C_i C_i^* \leq 1$, $\sum_{i=1}^{n} C_i^* C_i \leq 1$ and $\Delta_c(S) = 0 = \Delta_c^*(S)$, then the eigenspaces corresponding to distinct non-zero eigenvalues of the compact positive operator $|S|^2$ reduce each C_i (see [4: Theorem 8], [13: Lemma 2.3]). In particular, $|S|$ commutes with C_i for all $1 \leq i \leq n$. This implies also that $|S|^{p-1} = T$ commutes with each C_i for all $1 \leq i \leq n$. Hence

$$C_i |T| = |T| Ci,$$

and $C_i T^2 = T^2 C_i$. Since C_i commutes with the positive operator T^2, then C_i commutes with its square root, that is

$$C_i T = T C_i \qquad (3.3).$$

By (3.3) and (3.1), we obtain(3.2).

By using an argument similar to the proof of Theorem 3.2 in [9], we can consider the map f defined on $\sigma(T) \subset \mathbb{R}^+$ by $f(t) = t^{\frac{1}{p-1}}$, $1 < p < \infty$. Since f is the uniform limit of a sequence (P_i) of polynomials without constant term (since $f(0) = 0$), it follows from (3.2) that $\sum_{i=1}^{n} C_i P_i(T) U^* C_i = P_i(T) U^*$. Therefore $\sum_{i=1}^{n} C_i T^{\frac{1}{p-1}} U^* C_i = U^* T^{\frac{1}{p-1}}$. ∎

Now we are ready to present our first result on the global minimizer.

Theorem 3.1. *Let $C = (C_1, C_2 ..., C_n)$ be an n-tuple of operators in $L(H)$. If*

$$\sum_{i=1}^{n} C_i C_i^* \leq 1, \quad \sum_{i=1}^{n} C_i^* C_i \leq 1,$$

$$\Delta_c(T) = 0 = \Delta_c^*(T)$$

and $T \in \ker \Delta_{A,B} \cap C_p$, then for $1 \leq p < \infty$, the map F_p has a global minimizer at $W \in L(H)$ if, and for $1 < p < \infty$ only if,

$$\sum_{i=1}^{n} C_i W C_i - W = 0.$$

Proof. If
$$\sum_{i=1}^{n} C_iWC_i - W = 0,$$
then $F_p(W) = \|T\|_p^p$. It follows from [13: Theorem 2.4] that
$$F_p(X) \geq F_p(W).$$
Conversely, if F_p has a minimum then
$$\Big\|T - (\sum_{i=1}^{n} C_iWC_i - W)\Big\|_p^p = \|T\|_p^p .$$
Since $\mathcal{U}$ is convex, the set $\mathcal{V} = \{T - (\sum_{i=1}^{n} C_iXC_i - X);\ X \in \mathcal{U}\}$ is also convex. Thus, Theorem 2.2 implies that $T - (\sum_{i=1}^{n} C_iWC_i - W) = T$. ∎

In the following theorem we will classify the critical points of the map $F_p\,(p > 1)$.

Theorem 3.2. *Let $C = (C_1, C_2..., C_n)$ be an n-tuple of operators in $L(H)$. If*
$$\sum_{i=1}^{n} C_iC_i^* \leq 1, \sum_{i=1}^{n} C_i^*C_i \leq 1,$$
$$\Delta_c(T) = 0 = \Delta_c^*(T)$$
and $T \in \ker \Delta_{A,B} \cap C_p$, then for $1 \leq p < \infty$, the map F_p has a critical point at $W \in L(H)$ if, and for $1 < p < \infty$ only if,
$$\sum_{i=1}^{n} C_iWC_i - W = 0.$$

Proof. Since the Frechet derivative of F_p is given by
$$\mathcal{D}_W F_p(T) = \lim_{h \longrightarrow 0} \frac{F_p(W + hT) - F_p(W)}{h},$$
it follows that
$$\mathcal{D}_W F_p(T) = \Big[\mathcal{D}_{S-(\sum_{i=1}^{n} C_iWC_i - W)}\Big]\Big(\sum_{i=1}^{n} C_iTC_i - T\Big).$$
If W is a critical point of F_p, then $\mathcal{D}_W F_p(T) = 0\ \forall T \in \mathcal{U}$. By applying Theorem 2.1 we get
$$\begin{aligned}
\mathcal{D}_W F_p(T) &= p\ \Re tr\Big[\Big|S - \Big(\sum_{i=1}^{n} C_iWC_i - W\Big)\Big|^{p-1} U_1^*\Big(\sum_{i=1}^{n} C_iTC_i - T\Big)\Big] \\
&= p\ \Re tr\Big[Y\Big(\sum_{i=1}^{n} C_iTC_i - T\Big)\Big] \\
&= 0,
\end{aligned}$$

where $S-(\sum_{i=1}^n C_iWC_i-W)=U_1\left|S-(\sum_{i=1}^n C_iWC_i-W)\right|$ is the polar decomposition of the operator $S-(\sum_{i=1}^n C_iWC_i-W)$ and

$$Y=\Big|S-\Big(\sum_{i=1}^n C_iWC_i-W\Big)\Big|^{p-1}U_1^*.$$

An easy calculation shows that $(\sum_{i=1}^n C_iYC_i-Y)=0$, that is

$$\sum_{i=1}^n C_i\Big|S-\Big(\sum_{i=1}^n C_iWC_i-W\Big)\Big|^{p-1}U_1^*C_i=\Big|S-(\sum_{i=1}^n C_iWC_i-W\Big)\Big|^{p-1}U_1^*.$$

It follows from Lemma 3.1 that

$$\sum_{i=1}^n C_i\Big|S-\Big(\sum_{i=1}^n C_iWC_i-W\Big)\Big|U_1^*C_i=\Big|S-\Big(\sum_{i=1}^n C_iWC_i-W\Big)\Big|U_1^*.$$

By taking adjoints and since $\Delta_C=0=\Delta_{C^*}$, we get

$$\sum_{i=1}^n C_i\Big(T-\Big(\sum_{i=1}^n C_iWC_i-W\Big)\Big)C_i=\Big(T-\Big(\sum_{i=1}^n C_iWC_i-W\Big)\Big).$$

Then

$$\sum_{i=1}^n C_i\Big[\Big(\sum_{i=1}^n C_iWC_i-W\Big)\Big]C_i=\Big(\sum_{i=1}^n C_iWC_i-W\Big).$$

Hence

$$\sum_{i=1}^n C_iWC_i-W\in R(\Delta_C)\cap\ker\Delta_C,$$

where $R(\Delta_C)$ is the range of Δ_C. It is easy to see that (arguing as in the proof of [13: Theorem 2.4]), $\Delta_C(T)=0=\Delta_{C^*}(T)$ and $T\in\ker\Delta_C$, where $T\in L(H)$. Then

$$\|T-\Delta_C(X\|\geq\|T\|$$

holds for all $X\in L(H)$ and for all $T\in\ker\Delta_c$. Hence $\sum_{i=1}^n C_iWC_i-W=0$. Conversely, if $\sum_{i=1}^n C_iWC_i=W$, then W is a minimum of F_p, and since F_p is differentiable, W is a critical point. ∎

In the above theorem we classified the critical points of the map F_p for $p>1$. In the following theorem we consider the case $0<p\leq 1$.

Theorem 3.3. *Let $C=(C_1,C_2...,C_n)$ be an n-tuple of operators in $L(H)$. If*

$$\sum_{i=1}^n C_iC_i^*\leq 1,\quad \sum_{i=1}^n C_i^*C_i\leq 1$$

such that $\Delta_C(S)=0=\Delta_{C^*}(S)$ and $S\in\ker\Delta_C\cap C_p$, $0<p\leq 1$, $\dim H<\infty$ and $S-$ $(\sum_{i=1}^n C_iWC_i-W)$ is invertible, then F_p has a critical point at W, if $\sum_{i=1}^n C_iWC_i-W=0$.

Proof. Let $W, S \in U$ and let ϕ, be the map defined by

$$\phi : X \longmapsto S - \Big(\sum_{i=1}^{n} C_i X C_i - X\Big).$$

Suppose that $\dim H < \infty$. If $\sum_{i=1}^{n} C_i W C_i - W = 0$, then S is invertible by hypothesis. Also $|S|$ is invertible, hence $|S|^{p-1}$ exists for $0 < p \leq 1$. Taking $Y = |S|^{p-1} U^*$, where $S = U|S|$ is the polar decomposition of S. As shown in Lemma 3.1, $|S|$ commutes with C_i for all $1 \leq i \leq n$. Hence

$$C_i |S| = |S| C_i.$$

Since $\sum_{i=1}^{n} C_i S^* C_i = S^*$, i.e.

$$\sum_{i=1}^{n} C_i |S| U^* C_i = |S| U^*,$$

we find

$$|S| \Big(\sum_{i=1}^{n} C_i U^* C_i - U^*\Big) = 0 \ ,$$

and since

$$A |S|^{p-1} = |S|^{p-1} A,$$

we have

$$\begin{aligned} \sum_{i=1}^{n} C_i Y C_i - Y &= \sum_{i=1}^{n} C_i |S|^{p-1} U^* C_i - |S|^{p-1} U^* \\ &= |S|^{p-1} \Big(\sum_{i=1}^{n} C_i U^* C_i - U^*\Big), \end{aligned}$$

so that $\sum_{i=1}^{n} C_i Y C_i - Y = 0$ and $tr[(\sum_{i=1}^{n} C_i Y C_i - Y)T] = 0$ for all $T \in L(H)$. Since

$$S = S - \Big(\sum_{i=1}^{n} C_i W C_i - W\Big),$$

we have

$$\begin{aligned} 0 &= tr\Big[Y\Big(\sum_{i=1}^{n} C_i T C_i - T\Big)\Big] \\ &= p\, \Re tr\Big[Y\Big(\sum_{i=1}^{n} C_i T C_i - T\Big)\Big] \\ &= p\, \Re tr\Big[\,|S|^{p-1} U^*\Big(\sum_{i=1}^{n} C_i T C_i - T\Big)\Big] \\ &= (\mathcal{D}_S \phi)\Big(\sum_{i=1}^{n} C_i T C_i - T\Big) \end{aligned}$$

$$= (\mathcal{D}_W F_p)(T),$$

which proofs the assertion. ∎

At the end we use a familar device of considering 2x2 operator matrices to extend the previous theorems to the elementary operator $\sum_{i=1}^n A_i X B_i - X$.

Theorem 3.4. *Let $A = (A_1, A_2..., A_n)$, $B = (B_1, B_2..., B_n)$ be n-tuples of operators in $L(H)$ such that*

$$\sum_{i=1}^n A_i A_i^* \leq 1, \sum_{i=1}^n A_i^* A_i \leq 1, \sum_{i=1}^n B_i B_i^* \leq 1, \sum_{i=1}^n B_i^* B_i \leq 1.$$

If $\Delta_{A,B}(T) = 0 = \Delta_{A,B}^*(T)$ and $T \in \ker \Delta_{A,B} \cap C_p$, then it holds for $1 \leq p < \infty$:

- **(i)** *the map F_p has a global minimizer at W if, and for $1 < p < \infty$ only if, $\sum_{i=1}^n A_i W B_i - W = 0$*
- **(ii)** *the map F_p has a critical point at W if, and for $1 < p < \infty$ only if, $\sum_{i=1}^n A_i W B_i - W = 0$*
- **(iii)** *the map F_p, $0 < p \leq 1$, has a critical point at W if $\sum_{i=1}^n A_i W B_i - W = 0$ provided $\dim H < \infty$ and $S - (\sum_{i=1}^n A_i W B_i - W)$ is invertible.*

Proof. It suffices to take the Hilbert space $H \oplus H$, and operators

$$C_i = \begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix} \; S = \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}, X = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$$

and apply Theorem 3.1, Theorem 3.2 and Theorem 3.3. These arguments use operator matrices as in Bouali and Cherki [3] and Mecheri [7]. ∎

Remark.

1. In Theorem 3.2, the implication

$$W \text{ is a critical point} \Longrightarrow \; \sum_{i=1}^n A_i W B_i - W = 0$$

   does not hold in the case $0 < p \leq 1$ (cf. Maher [8]).
2. Theorems 3.1, 3.2 , 3.3 and 3.4 hold in particular if A and B are contractions. Indeed, it is known from [4] that if A and B are contractions and $\Delta_{A,B}(S) = ASB - S = 0$, where $S \in C_p$, then

$$\Delta_{A^*,B^*}(S) = \delta_{A^*,B}(S) = \delta_{A,B^*}(S) = 0.$$

3. If $A \in C_p$, the conclusions of Theorems 3.1, 3.2, 3.3 and 3.4 hold for all $X \in L(H)$ (cf. Maher [9]).

Acknowledgment. The author would like to thank the referee for his careful reading of the paper. His valuable suggestions, critical remarks, and pertinent comments made numerous improvements throughout.

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Received 10.10.2002; in revised form 13.01.2004