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A Class of Partial Integro-Differential Equations with Correlation-Convolution Integral II

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Dedicated to Prof. K. Beyer on the occasion of his 65th birthday

Abstract. Three classes of nonlinear integral equations and first order integro-differential equations in two variables are dealt with where the quadratic nonlinearity is given by the correlation-convolution integral. In the case of the quarter plane the equations are reduced to boundary value problems for holomorphic functions which can be solved in closed form. In this way existence and constructive formulas for the solutions to the equations are derived.

Keywords: *Nonlinear integral equations, nonlinear partial integro-differential equations, correlation-convolution integral, boundary value problem for holomorphic function*

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1. Introduction

In the paper the integral and integral-differential equations in two variables with the correlation-convolution integral in the quarter plane are treated. The paper continues the first part of it presented in [11] where the equations in the rectangle and the strip have been dealt with.

In [11] we prove the uniqueness of summable solutions to the equations in any finite interval $(0, T)$ of the convolution variable τ. Hence we have uniqueness of the corresponding solutions to the equations for τ on the semi-axis $\mathbb{R}_+$, too. Therefore, from the very beginning we can concentrate on the construction of one solution of the equations in the quarter plane.

For deriving the solutions to the equations, at first we apply the Laplace transform with respect to the convolution variable τ reducing the equations to autocorrelation equations in the correlation variable t for the Laplace transforms of the solution. In a second step then, applying the Fourier transform with respect to t, the equations are reduced to boundary value problems for a holomorphic function in the upper half-plane as in our papers [8, 9], now

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containing the additional parameter q of the Laplace transform. Taking into account the behaviour of Laplace transforms as Re $q \to \infty$, we construct the unique solution of the equations under suitable general assumptions on the data (and an additional existence condition in two cases).

2. Statement of equations and method of solution

We deal with the following integral and integro-differential equations in two variables:

$$p(t,\tau) + I[p](t,\tau) = h(t,\tau) \tag{1}$$

$$\frac{\partial p}{\partial \tau} + \mu p(t,\tau) + I[p](t,\tau) = h(t,\tau) \tag{2}$$

$$\frac{\partial p}{\partial t} + \lambda \frac{\partial p}{\partial \tau} + \mu p(t,\tau) + I[p](t,\tau) = h(t,\tau) \tag{3}$$

for $t, \tau \in \mathbb{R}_+$. Here $\lambda, \mu \in \mathbb{R}$ are given constant parameters, and I is the *correlation-convolution operator* of p:

$$\begin{aligned} I[p](t,\tau) &= \int_0^\tau \int_0^\infty p(s,\sigma) p(s+t, \tau-\sigma) ds\, d\sigma \\ &= \int_0^\tau \int_0^\infty p(s,\tau-\sigma) p(s+t,\sigma) ds\, d\sigma \,. \end{aligned} \tag{4}$$

In addition, the solution p of eq. (2) has to fulfill the initial condition

$$p(t,0) = f(t) \quad , \qquad t \in \mathbb{R}_+ \tag{5}$$

and the solution p of eq. (3) the conditions (5) and

$$p(0,\tau) = \varphi(\tau) \quad , \qquad \tau \in \mathbb{R}_+ \tag{6}$$

with prescribed functions f and φ. Looking for summable solutions with respect to t we expect that in case of eq. (3) with $\lambda = 0$, $\mu \le 0$ and with $\lambda < 0$, respectively, an existence condition must hold.

In the following, we apply the Laplace transform with respect to τ and the Fourier cosine transform with respect to t to eqs. (1) - (3). As preparation we (formally) calculate the corresponding transforms of the correlation-convolution integral (4) or

$$I[p](t,\tau) = \int_0^\infty p(s,\cdot) * p(s+t,\cdot) ds \tag{7}$$

where $*$ denotes the convolution with respect to τ.

By the convolution theorem for the Laplace transform from (7) we obtain the relation

$$(\mathcal{L}\, I[p](t,\cdot))(q) \equiv \int_0^\infty I[p](t,\tau)e^{-q\tau}d\tau = \int_0^\infty W(s,q)W(s+t,q)ds \tag{8}$$

where

$$W(t,q) = (\mathcal{L}\, p(t,\cdot))(q) \equiv \int_0^\infty p(t,\tau)e^{-q\tau}d\tau\ . \tag{9}$$

Applying to (8) the Fourier cosine transform with respect to t, we have (cf. [7])

$$\begin{aligned}(\mathcal{F}_c\, \mathcal{L}\, I[p])(x,q)& \\ \equiv \int_0^\infty \cos xt \int_0^\infty I[p](t,\tau)e^{-q\tau}d\tau\, dt &= \frac{1}{2}[P^2(x,q)+Q^2(x,q)]\end{aligned} \tag{10}$$

where

$$P(x,q) \quad = \quad (\mathcal{F}_c W(\cdot,q))(x) \equiv \int_0^\infty W(t,q)\cos xt\, dt\ , \tag{11}$$

$$Q(x,q) \quad = \quad (\mathcal{F}_s W(\cdot,q))(x) \equiv \int_0^\infty W(t,q)\sin xt\, dt \tag{12}$$

are the Fourier cosine and Fourier sine transform of $W = \mathcal{L}\, p$, respectively.

We further calculate the integral $J = (\mathcal{F}_c I[p](\cdot,\tau))(x) = \int_0^\infty I[p](t,\tau)\cos xt\, dt\,$. It holds

$$J = \frac{1}{2}\int_0^\tau J_0(x,\sigma)d\sigma$$

where

$$J_0(x,\sigma) \quad = \quad \int_0^\infty\int_0^\infty \cos xt\cdot[p(s,\sigma)p(s+t,\tau-\sigma)+p(s,\tau-\sigma)p(s+t,\tau)]ds\, dt\ .$$

From

$$\begin{aligned}
J_0(x,\sigma) &= \int_0^\infty p(s,\sigma)\cos xs \int_s^\infty p(r,\tau-\sigma)\cos xr\, dr\, d\sigma \\
&\quad + \int_0^\infty p(s,\tau-\sigma)\cos xs \int_s^\infty p(r,\sigma)\cos xr\, dr\, d\sigma \\
&\quad + \int_0^\infty p(s,\sigma)\sin xs \int_s^\infty p(r,\tau-\sigma)\sin xr\, dr\, d\sigma \\
&\quad + \int_0^\infty p(s,\tau-\sigma)\sin xs \int_s^\infty p(r,\sigma)\sin xr\, dr\, d\sigma \\
&= \int_0^\infty p(s,\tau-\sigma)\cos xs\, ds \cdot \int_0^\infty p(r,\sigma)\cos xr\, dr \\
&\quad + \int_0^\infty p(s,\tau-\sigma)\sin xs\, ds \cdot \int_0^\infty p(r,\sigma)\sin xr\, dr\ .
\end{aligned}$$

it follows

$$\begin{aligned}
(\mathcal{F}_c I[p](\cdot,\tau))(x) &= \frac{1}{2}\int_0^\tau [U(x,\sigma)U(x,\tau-\sigma)+V(x,\sigma)V(x,\tau-\sigma)]d\sigma \\
&= \frac{1}{2}\,[U(x,\cdot)*U(x,\cdot)+V(x,\cdot)*V(x,\cdot)]
\end{aligned} \tag{13}$$

with the Fourier cosine and Fourier sine transforms of p

$$\begin{aligned}
U(x,\tau) = (\mathcal{F}_c p(\cdot,\tau))(x) &\equiv \int_0^\infty p(t,\tau)\ \cos xt\, dt\ , \\
V(x,\tau) = (\mathcal{F}_s p(\cdot,\tau))(x) &\equiv \int_0^\infty p(t,\tau)\ \sin xt\, dt\ .
\end{aligned}$$

Applying to (13) the Laplace transform with respect to τ, we again arrive at the relation (10) for $\mathcal{L}\mathcal{F}_c I[p]$ with $P = \mathcal{L}U = \mathcal{L}\mathcal{F}_c p$ and $Q = \mathcal{L}V = \mathcal{L}\mathcal{F}_s p$ now.

Utilizing relation (10), we reduce eqs. (1) - (3) to boundary value problems for the complex Fourier transform $F(z,q)$ of p with respect to t which is a holomorphic function of z in the upper half-plane (cf. [8, 9]). Now, in addition,

F depends on the parameter q of the Laplace transform which varies in some right half-plane, and we have to choose the solution of the boundary value problem which represents a Laplace transform of a locally summable and at most exponentially growing function of τ.

We remark that by means of relation (13) eqs. (1) - (3) can be reduced to fixed point equations of convolution type for the function U where V is the conjugate function (Hilbert transform) of U with respect to x. In the case of a finite interval $(0,T)$ for τ, the iteration method of weighted norms [3] in Hölder spaces on the real axis can be applied to this fixed point equation. But it is more convenient to use this iteration method in the original equations (1) - (3) as we did it in the first part of this paper [11]. Conversely, in solving for instance the equation

$$U(x,\tau) + \frac{1}{2}\,[U * U + V * V](x,\tau) = H(x,\tau)$$

for $x \in \mathbb{R}_+$ and $\tau \in (0,T)$, we can reduce it to eq. (1) by applying the Fourier cosine transform with respect to x.

3. Solution of equation (1)

For convenience we put $h = \frac{g}{2}$ and write eq. (1) in the form

$$p(t,\tau) + I[p](t,\tau) = \frac{1}{2}\,g(t,\tau)\,, \qquad 0 < t,\tau < \infty\,. \tag{14}$$

Applying to (14) the Laplace transform with respect to τ and the Fourier cosine transform with respect to t, in view of relation (10) we obtain the equation

$$P^2(x,q) + Q^2(x,q) + 2P(x,q) = G(x,q)\,, \qquad -\infty < x < \infty,\ Re\, q > q_0$$

with some $q_0 \in \mathbb{R}$ and $G(x,q) = (\mathcal{F}_c\mathcal{L}q)(x,g)$ or

$$\hat{P}(x,q)^2 + \hat{Q}(x,q)^2 = \hat{G}(x,q)\,, \qquad -\infty < x < \infty,\ Re\, q > q_0 \tag{15}$$

where $\hat{P} = 1 + P$, $\hat{Q} = Q$ and $\hat{G} = 1 + G$.

Dealing with q as a real parameter, we solve eq. (15) by the method used in [8, 9]. We assume that $\hat{G}(x,q) > 0$ for $x \in \mathbb{R},\ q > q_0$ and introduce the complex Fourier transform

$$\hat{F}(z,q) = 1 + \int_0^\infty W(t,q)e^{itz}dt, \quad z = x + iy \tag{16}$$

where $W = \mathcal{L}p$. Looking for a solution p of eq. (1) satisfying the relation $\hat{F}(z,q) \to 1$ as $Re\, q \to \infty,\ Im\ z \geq 0$, we choose the solution of eq. (15) for which the function $\hat{F}(z,q)$ has no zeros on $Im\, z \geq 0$, $q > q_0$. This gives the formal solution

$$P(x,q) = -1 + \hat{G}(x,q)^{\frac{1}{2}} \cos K(x,q) \ , \qquad Re\, q > q_0 \tag{17}$$

where

$$K(x,q) = \frac{x}{\pi} \int\limits_0^\infty \frac{\ln \hat{G}(x,q)}{\xi^2 - x^2}\, d\xi \ , \qquad Re\, q > q_0 \tag{18}$$

with the principal value of logarithm, and we again have extended q to complex values.

It remains to state a group of sufficient conditions under which the function $P = \mathcal{F}_c \mathcal{L} p$ defined by (11) yields an actual solution p to eq. (1). Let us first assume that g fulfils the following conditions:

(i) $\quad g(t,\cdot) \in L(\mathbb{R}_+) \quad$ for a.e. $\quad t > 0$

(ii) $\quad g(\cdot,\tau) \in L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+) \quad$ for $\quad \tau \geq 0$

where

(iii) $\quad \int\limits_0^\infty |g(t,\tau)| dt \leq C_0 \quad$ uniformly for $\quad \tau \geq 0$

with a positive constant C_0 and

(iv) $\quad \int\limits_0^\infty |g(t,\tau)|^2 dt \leq C(\tau) \ , \quad C \in L(\mathbb{R}_+),$

i.e. $\quad g \in L^2(\mathbb{R}_+ \times \mathbb{R}_+)$.

By (i), the Laplace transform $G_0(t,q) = (\mathcal{L} g(t,\cdot))(q)$ of $g(t,\cdot)$ exists in $Re\, q \geq 0$ for a.e. $t > 0$. In view of (iii) and (iv) we further have

$$\int\limits_0^\infty |G_0(t,q)| dt \ \leq \ \int\limits_0^\infty \int\limits_0^\infty |g(t,\tau)| e^{-Re\, q \cdot \tau} d\tau\, dt \ \leq \ \frac{C_0}{Re\, q} \tag{19}$$

for $Re\, q > 0$, and

$$\begin{aligned} \int\limits_0^\infty |G_0(t,q)|^2 dt \ &\leq \ \int\limits_0^\infty \left[\int\limits_0^\infty |g(t,\tau)| e^{-Re\, q \cdot \tau} d\tau \right]^2 dt \\ &\leq \ \frac{1}{2\, Re\, q} \int\limits_0^\infty \int\limits_0^\infty |g(t,\tau)|^2 d\tau\, dt \\ &\leq \ \frac{C_1}{Re\, q} \end{aligned}$$

for $Re\, q > 0$ with the positive constant $C_1 = \frac{1}{2}\int\limits_0^\infty C(\tau)d\tau$ so that $G_0(\cdot, q) \in L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ for $Re\, q > 0$. From (19) and

$$|G(x,q)| = |(\mathcal{F}_c G_0)(x,q)| \le \int\limits_0^\infty |G_0(t,q)|dt$$

it follows that $G(x,q) \to 0$ as $Re\, q \to \infty$ uniformly in $x \in \mathbb{R}$. Hence, there holds

$$\hat{G}(x,q) \ge 1 - \varepsilon > 0 \qquad \text{for} \quad x \in \mathbb{R}\ ,\ q > q_1(\varepsilon)$$

and any $\varepsilon \in (0,1)$.

Further, in accordance with (19) we assume that the representation

$$G(x,q) = \frac{c_0(x)}{q} + \frac{c_1(x,q)}{q^{1+\delta_1}}\ ,\ \ \delta_1 > 0 \tag{20}$$

for $x \in \mathbb{R}$, $Re\, q > q_2$ holds where c_0 and c_1 are bounded and with respect to $x \in \mathbb{R}$ Hölder continuous functions satisfying $c_0, c_1 = O(x^{-\gamma})$, $\gamma > 1$, as $x \to \infty$ uniformly with respect to q in $Re\, q > q_2$ in case of c_1. Assuming that $g(t,0) = \lim\limits_{\tau \to +0} g(t,\tau) \in L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ exists, we have $c_0 = \mathcal{F}_c g(\cdot, 0)$.

From (20) the analogous representations

$$\begin{aligned}
\hat{G}(x,q)^{\frac{1}{2}} &= 1 + \frac{1}{2}\,\frac{c_0(x)}{q} + O(q^{-(1+\delta_2)}),\\
\ln \hat{G}(x,q) &= \frac{c_0(x)}{q} + O(q^{-(1+\delta_2)}),\\
K(x,q) &= \frac{\gamma(x)}{q} + O(q^{-(1+\delta_2)})\\
\cos K(x,q) &= 1 + O(q^{-2})
\end{aligned}$$

follow where $\delta_2 = \min(\delta_1, 1) > 0$ and

$$\gamma(x) = \frac{x}{\pi}\int\limits_0^\infty \frac{c_0(\xi)}{\xi^2 - x^2}\, d\xi\ . \tag{21}$$

This implies the representation

$$P(x,q) = \frac{1}{2}\,\frac{c_0(x)}{q} + \frac{c_2(x,q)}{q^{1+\delta_2}}\ ,\ \ \delta_2 > 0 \tag{22}$$

for $x \in \mathbb{R}$, $Re\, q > q_0$ where c_2 behaves like c_1, and we put $q_0 = \max(0, q_1(\varepsilon), q_2)$ for some $\varepsilon \in (0,1)$.

In view of $G_0(\cdot, q) \in L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ and assumption (20) by Theorem 3.1 of [9] we obtain the existence of a solution $W(\cdot, q) = \frac{2}{\pi}\mathcal{F}_c P(\cdot, q) \in L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ for any real $q > q_0$ to the intermediate equation

$$W(t,q) + \int_0^\infty W(s,q)W(s+t,q)ds = \frac{1}{2}\, G_0(t,q), \quad t > 0\,. \tag{23}$$

On account of (22) the function W has the form

$$W(t,q) = \frac{1}{2}\ \frac{d_0(t)}{q}\ + d_1(t,q)\ \ , \quad d_1(t,q) = \frac{d_2(t,q)}{q^{1+\delta_2}} \tag{24}$$

where $d_0 = g(\cdot, 0)$ and $d_2 = \frac{2}{\pi}\mathcal{F}_c c_2$ is a bounded function in $t \geq 0$, $Re\, q \geq q_0 + \varepsilon_0$ for any $\varepsilon_0 > 0$ lying in $L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ with respect to t, for any q in $Re\, q > q_0$. A known theorem of Laplace transform theory [1, Th. 21.3] then yields the solution $p = \mathcal{L}^{-1}W(t, \cdot)$ to eq. (1) in the form

$$p(t,\tau) = \frac{1}{2}\, d_0(t) + \frac{1}{2\pi i} \int_{\hat{q}-i\infty}^{\hat{q}+i\infty} e^{\tau q} d_1(t,q)dq \qquad (\hat{q} > q_0) \tag{25}$$

where $p(t,0) = \frac{1}{2}\, d_0(t) = \frac{1}{2}\, g(t,0)$ in accordance with eq. (14) for $\tau = 0$. The solution p satisfies the conditions

$$p(\cdot, \tau) \in L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+) \tag{26}$$

for $\tau \geq 0$ and

$$p(t,\cdot) \in C(\mathbb{R}_+)\ , \ \ p(t,\cdot) = O(e^{\hat{q}\tau}), \ \ \hat{q} \in \mathbb{R}_+ \ \text{ as } \ \tau \to \infty \tag{27}$$

for a.e. $t > 0$.

Theorem 1. *Let the assumptions* (i) - (iv) *with* $g(t,0) = \lim_{\tau \to +0} g(t,\tau)$ *for* g *be fulfilled and let the representation* (20) *for* $G = \mathcal{F}_c\mathcal{L}g$ *hold.*

Then equation (1) *with* $h = \frac{g}{2}$ *has the solution* $p = \mathcal{L}^{-1}\mathcal{F}_c P$ *where* P *is defined by* (17). *The solution* p *possesses the properties* (26) *and* (27). *It can be represented in the form* (25) *where* $d_0 = g(\cdot, 0)$, *and the function* d_1 *is given by* $d_1 = \frac{2}{\pi} q^{-(1+\delta_2)}\mathcal{F}_c c_2$ *with the coefficient* c_2 *in the representation* (22) *for* P.

Corollaries.

1. If the condition $G(x,q) \geq 0$ for $x \in \mathbb{R}$ and sufficiently large real q holds, the assumptions on the Hölder continuity of the functions c_0, c_1 in the representation (20)for G can be left out using [8, Theorem 1] instead of [9, Theorem 3.1] for the existence of the solution W to eq. (23). Only the function γ in (21) is assumed as bounded now.

2. Assumption (20) corresponds to the asymptotic relation

$$G_0(t,q) = \frac{g(t,0)}{q} + O(q^{-(1+\delta_1)}) \ , \quad \delta_1 > 0$$

as $Re\, q \to \infty$.

3. Instead of [1, Theorem 21.3] other theorems on the inversion of the Laplace transform can be used. For example, using the theorem by Titchmarsh [6, Chapter 11.7], we obtain a solution p to eq. (1) satisfying $p(t,\cdot) \in L^2(\mathbb{R}_+)$ for $t > 0$ if the function $G(x,q)$ for $x \in \mathbb{R}$ is a holomorphic function of q in $|\arg q| < \pi/2$ with $\hat{G}(x,q) > 0$ for real $q > 0$ and

$$G(x,q) = O(q^{-\alpha}) \ , \quad \alpha > \frac{1}{2}$$

as $Re\, q \to \infty$ uniformly in $x \in \mathbb{R}$.

4. Theorem 1 holds true for eq. (1) in the strip $S_0 = (0,T_0) \times (0,\infty)$ where the correlation-convolution operator is given by (cf. [10])

$$I_0[p](t,\tau) = \int_0^{T_0-t} p(s,\cdot) * p(s+t,\cdot) ds \ , \quad 0 < t < T_0 \ . \tag{28}$$

4. Solution of equation (2)

Applying the Laplace transform with respect to τ, eq. (2) with $h = g/2$ and the initial condition (5) is reduced to the intermediate equation

$$W_1(t,q) + \int_0^\infty W_1(s,q) W_1(s+t,q) ds = \frac{1}{2}\,\hat{G}_1(t,q) \ , \quad t > 0 \tag{29}$$

for $Re(q+\mu) > q_0$ where

$$\begin{aligned} \hat{G}_1(t,q) &= G_1(t,q) \cdot (q+\mu)^{-2} \\ G_1(t,q) &= G_0(t,q) + 2f(t), \\ W_1(t,q) &= W(t,q) \cdot (q+\mu)^{-1} \end{aligned}$$

with $G_0(t,q) = (\mathcal{L}g(t,\cdot))(q)$, $W(t,q) = (\mathcal{L}p(t,\cdot))(q)$ and some $q_0 \geq 0$ as above. Applying further the Fourier cosine transform with respect to t to eq. (29) leads to the equation

$$P_1^2(x,q) + Q_1^2(x,q) + 2P_1(x,q) = H_1(x,q) \ , \quad -\infty < x < \infty \tag{30}$$

for $Re(q+\mu) > q_0$ where

$$\begin{aligned} P_1(x,q) &= (\mathcal{F}_c W_1(\cdot,q))(x) = P(x,q)\cdot(q+\mu)^{-1} \\ Q_1(x,q) &= (\mathcal{F}_s W_1(\cdot,q))(x) = Q(x,q)\cdot(q+\mu)^{-1} \end{aligned} \tag{31}$$

with $P = (\mathcal{F}_c\mathcal{L})p$, $Q = (\mathcal{F}_s\mathcal{L})p$ and

$$H_1(x,q) = (\mathcal{F}_c\hat{G}_1(\cdot,q))(x) = H(x,q)\cdot(q+\mu)^{-2} \tag{32}$$

with $H = \mathcal{F}_c G_0 + 2F$, $F = \mathcal{F}_c f$.

Assuming that $\hat{H}_1(x,q) = 1 + H_1(x,q) > 0$ for $x \in \mathbb{R}$, $q+\mu > q_0$ as above, we obtain the formal solution

$$P_1(x,q) = -1 + \hat{H}_1(x,q)^{\frac{1}{2}}\cos\hat{K}(x,q) \tag{33}$$

or

$$P(x,q) = -(q+\mu) + \sqrt{(q+\mu)^2 + H(x,q)}\cos\hat{K}(x,q) \tag{34}$$

where

$$\hat{K}(x,q) = \frac{x}{\pi}\int_0^\infty \frac{\ln[1 + H(\xi,q)\cdot(q+\mu)^{-2}]}{\xi^2 - x^2}\,d\xi\ .$$

This holds in $Re(q+\mu) > q_0$ with the principal values of logarithm and square root.

We again assume that the function g satisfies the conditions (i) - (iv) above so that $G_0(t,q)$ exists in $Re\,q \geq 0$ for a.e. $t > 0$ and $G_0(\cdot,q) \in L(\mathbb{R}_+)\cap L^2(\mathbb{R}_+)$ for $Re\,q > 0$. Further, we assume that condition

(v) $\quad f \in L(\mathbb{R}_+)\cap L^2(\mathbb{R}_+)$

holds. Hence $\hat{G}_1(\cdot,q) \in L(\mathbb{R}_+)\cap L^2(\mathbb{R}_+)$ for $Re\,q > \max(0,-\mu)$. Moreover,

$$\begin{aligned} |H(x,q)| &\leq |(\mathcal{F}_c G_0)(x,q)| + 2|F(x)| \\ &\leq \int_0^\infty |G_0(t,q)|dt + 2\int_0^\infty |f(t)|dt \leq C_2 < \infty \end{aligned}$$

for $Re\,q > 0$ and $H_1(x,q) = (q+\mu)^{-2}H(x,q) \to 0$ as $Re\,q \to \infty$ uniformly in $x \in \mathbb{R}$. Hence

$$\hat{H}_1(x,q) \geq 1 - \varepsilon > 0 \qquad \text{for } x \in \mathbb{R}\ ,\ \ q+\mu > q_1(\varepsilon)$$

and for any $\varepsilon \in (0,1)$.

Finally, we assume the representation

$$H(x,q) = c_0(x) + \frac{c_1(x, q+\mu)}{(q+\mu)^{\delta_1}}\ ,\ \ \delta_1 > 0 \tag{35}$$

for $x \in \mathbb{R}$, $Re(q+\mu) > q_2$ with functions $c_0 = 2F$ and c_1 which are bounded, Hölder continuous with respect to $x \in \mathbb{R}$ and $O(x^{-\gamma})$, $\gamma > 1$ as $x \to \infty$, uniformly with respect to q in $Re(q+\mu) > q_2$ in case of c_1. This implies the representation for P_1 defined by (33):

$$P_1(x,q) = \frac{F(x)}{(q+\mu)^2} + \frac{c_2(x, q+\mu)}{(q+\mu)^{2+\delta_2}}\quad ,\quad \delta_2 = \min(2, \delta_1) > 0 \tag{36}$$

for $x \in \mathbb{R}$, $Re(q+\mu) > q_0$ where we put $q_0 = \max(0, \mu, q_1(\varepsilon), q_2)$ for some $\varepsilon \in (0,1)$.

Again by Theorem 3.1 of [9] we obtain the existence of a solution $W_1(\cdot, q) = \frac{2}{\pi}\mathcal{F}_c P_1(\cdot, q) \in L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ for any real $q + \mu > q_0$ to the intermediate equation (29). In view of (36) the function W_1 has the form

$$W_1(t,q) = \frac{f(t)}{(q+\mu)^2} + \frac{d_1(t, q+\mu)}{(q+\mu)}\quad ,\quad d_1(t,q) = \frac{d_2(t,q)}{q^{1+\delta_2}} \tag{37}$$

where again $d_2 = \frac{2}{\pi}\mathcal{F}_c c_2$ is a bounded function in $t \geq 0$, $Re\, q \geq q_0 + \varepsilon_0$ for any $\varepsilon_0 > 0$ which is in $L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ with respect to t for any q in $Req > q_0$. By [1, Theorem 21.3] again we have the solution $p = \mathcal{L}^{-1}W$, $W(t,q) = (q+\mu)\cdot W_1(t,q)$ to eq. (2) of the form

$$p(t,\tau) = e^{-\mu\tau}\left[f(t) + \frac{1}{2\pi i}\int\limits_{\hat{q}-i\infty}^{\hat{q}+i\infty} e^{\tau q} d_1(t,q)dq\right] \qquad (\hat{q} > q_0) \tag{38}$$

satisfying (26) and (27) with $\hat{q}$ replaced by $\hat{q} - \mu$.

Theorem 2. *Let the assumptions* (i) - (v) *be fulfilled and let the representation* (35) *for* $H = \mathcal{F}_c\mathcal{L}g + 2\mathcal{F}_c f$ *hold.*

Then equation (2) *with initial condition* (5) *has the mild solution* $p = \mathcal{L}^{-1}\mathcal{F}_c$ *where* P *is defined by* (34). *The solution* p *possesses the properties* (26) *and* (27). *It can be represented in the form* (38) *where the function* d_1 *is given by* $d_1 = \frac{2}{\pi} q^{-(1+\delta_2)}\mathcal{F}_c c_2$ *with the coefficient* c_2 *in the representation* (36) *for the function* P_1 *defined by* (33).

Corollaries.

1. Assumption (35) corresponds to the asymptotic relation

$$G_0(t,q) = \mathcal{L}g(t,\cdot)(q) = O(q^{-\delta_1}),\ \ \delta_1 > 0$$

as $Re\, q \to \infty$. Let now stronger the representation

$$G_0(t,q) = \frac{g(t,0)}{q} + \frac{d_0(t,q)}{q^{1+\varepsilon_0}}\,, \quad \varepsilon_0 > 0 \tag{39}$$

hold where $g(t,0) = \lim_{\tau \to +0} g(t,\tau) \in L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ and d_0 is a bounded function possessing a sufficiently smooth Fourier cosine transform with respect to t as above. Then we have in (35)

$$\frac{c_1(x,q)}{q^{\delta_1}} = \frac{h_0(x)}{q} + \frac{h(x,q)}{q^{1+\varepsilon_1}}\,, \quad \varepsilon_1 > 0$$

and in (36)

$$\frac{c_2(x,q)}{q^{2+\delta_2}} = \frac{h_0(x)}{2q^3} + \frac{h_1(x,q)}{q^{3+\varepsilon_2}}\,, \quad \varepsilon_2 > 0$$

with $h_0 = \mathcal{F}_c g(\cdot,0)$ and corresponding functions h and h_1. This leads to the representation of the solution p

$$p(t,\tau) = e^{-\mu\tau}\left[f(t) + \frac{1}{2}\, g(t,0)\tau + \frac{1}{2\pi i}\int_{\hat{q}-i\infty}^{\hat{q}+i\infty} e^{\tau q} k_1(t,q)\,dq\right] \tag{40}$$

where $k_1(t,q) = q^{-(2+\varepsilon_2)} k_2(t,q)$ with a bounded function k_2. From (40) it follows that p is a strong solution of eq. (2) for which the derivative $\partial p/\partial\tau$ exists and satisfies (26) and (27) with $\lim_{\tau\to+0} \partial p/\partial\tau = \frac{1}{2}\, g(t,0) - \mu f(t)$ which is in accordance with eq. (2) for $\tau = 0$.

2. Theorem 2 holds true for eq. (2) in the strip $S_0 = (0,T_0) \times (0,\infty)$ with the integral I_0 defined by (28).

5. Solution of equation (3) with $\lambda = 0$

Next we deal with eq. (3) for $\lambda = 0$

$$\frac{\partial p}{\partial t} + \mu\, p(t,\tau) + I[p](t,\tau) = \frac{1}{2}\, g(t,\tau)\,, \quad 0 < t, \tau < \infty \tag{41}$$

together with the initial condition (6). Applying again the Laplace transform with respect to τ, we obtain the intermediate equation

$$\frac{\partial W}{\partial t} + \mu\, W(t,q) + \int_0^\infty W(s,q)W(s+t,q)\,ds = \frac{1}{2}\, G_0(t,q), \;\; t > 0 \tag{42}$$

for $Re\, q > q_0$, $q_0 \in \mathbb{R}_+$ and the initial condition

$$W(0,q) = \alpha(q) \quad , \qquad Re\, q > q_0 \tag{43}$$

for $W(t,q) = \mathcal{L}p(t,\cdot)(q)$ where $G_0(t,q) = \mathcal{L}g(t,\cdot)(q)$ and $\alpha = \mathcal{L}\varphi$.

We make the assumptions (i) - (iv) for g and assumption

(vi) $\quad \varphi \in C(\mathbb{R}_+) \quad$ with $\quad \varphi = O(e^{\tilde{q}\tau}),\ \tilde{q} \in \mathbb{R} \quad$ as $\quad \tau \to \infty$.

for φ. Further, we look for a solution p with $p(\cdot,\tau) \in L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ for $\tau \geq 0$, hence $W(\cdot,q) \in L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ for $Re\, q > q_0$.

The solution of eq. (42) with (43) depends on the zeros of the holomorphic function of z (cf. [8, 9])

$$\tilde{F}(z,q) = \mu + iz + \int_0^\infty W(t,q)e^{itz}dt \ , \quad Im\, z > 0 \tag{44}$$

which satisfies the asymptotic relation

$$\tilde{F}(z,q) \sim \mu + iz \qquad \text{as} \quad Re\, q \to \infty \ .$$

Therefore, for sufficiently large Req we have that $\tilde{F}(z,q)$ has no zero in $Imz \geq 0$ for $\mu < 0$, and $\tilde{F}(z,q)$ has a simple zero $z_0(q) \sim i\mu$ in $Im\, z \geq 0$ for $\mu > 0$. In the case $\mu = 0$ both possibilities can occur with $z_0(q) \to 0$ as $Re\, q \to \infty$. In the following we suppose $\mu \neq 0$ and discuss the case $\mu = 0$ only sketchily as a limiting case of $\mu > 0$ and $\mu < 0$, respectively.

We assume that the function $G_0(\cdot,q) \in L(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ has a Hölder-continuous Fourier cosine transform $G(\cdot,q) \in L^2(\mathbb{R}_+)$ satisfying $G(x,q) = O(x^{-\gamma})$, $\gamma > 1$ as $x \to \infty$, uniformly for $Re\ q > q_1$ with some $q_1 \in \mathbb{R}_+$. Further, it holds the inequality

$$\mu^2 + x^2 + G(x,q) + 2\alpha(q) > 0 \tag{45}$$

for real $q > q_2(\geq \tilde{q})$, $x \in \mathbb{R}$.

Then, in case $\mu < 0$ by [8, Theorem 2] or [9, Theorem 7.1] eqs. (42) and (43) have the solution $W(\cdot,q) = \frac{2}{\pi}\mathcal{F}_c P(\cdot,q)$, $Re\, q > q_0$ where $q_0 = \max(q_1,q_2)$ and, choosing the parameter $b = -\mu > 0$ in [8, 9],

$$P(x,q) = -\mu + A_0(x,q)^{\frac{1}{2}}[\mu\ \cos\ K_0(x,q) + x\ \sin\ K_0(x,q)] \tag{46}$$

with

$$A_0(x,q) \ = \ 1 \ + \ \frac{G(x,q) + 2\alpha(q)}{\mu^2 + x^2} \tag{47}$$

$$K_0(x,q) \ = \ \frac{x}{\pi}\int_0^\infty \frac{\ln\ A_0(\xi,q)}{\xi^2 - x^2}\, d\xi\,. \tag{48}$$

Moreover, the relation

$$M_0(q) \equiv \frac{1}{\pi} \int_0^\infty \ln\, A_0(x,q)dx = 0\ , \quad Re q > q_0 \tag{49}$$

must hold. We remark, that from (49) it follows that for $g(t,\tau) \equiv 0$ we also must have $\varphi(\tau) \equiv 0$ for an integrable solution p of eq. (41) with (6) in case $\mu < 0$.

In case $\mu > 0$ the function $\tilde{F}(z,q)$ has the simple zero $z_0(q) = i\, y_0(q)$ where $y_0(q)$ is real and positive for sufficiently large real q. It satisfies the equation

$$\mu - y + \int_0^\infty W(t,q)e^{-ty}dt = 0 \qquad , \quad y = y_0(q) \tag{50}$$

and hence the relation $y \sim \mu$ for real $q \to \infty$. Again by [8, Theorem 2] or [9, Theorem 7.1] eqs. (42) and (43) have the solution $W(\cdot,q) = \frac{2}{\pi}\mathcal{F}_c P(\cdot,q)$, $Re\, q > q_0$ where, choosing $b = \mu > 0$ in [8, 9],

$$\begin{aligned} P(x,q) = -\mu + A_0(x,q)^{\frac{1}{2}} \big[& Re((ix-\mu)B_0(x,q)) \cos\, K_0(x,q) \\ & + Im((ix-\mu)B_0(x,q)) \sin\, K_0(x,q) \big]\ . \end{aligned} \tag{51}$$

A_0 and K_0 are defined by (47) and (48), respectively, and

$$B_0(x,q) \ = \ \frac{x - i\, y_0(q)}{x + i\, y_0(q)}\ . \tag{52}$$

The analogous relation to (49)

$$M_0(q) \ = \ 2[y_0(q) - \mu]$$

where $M_0(q)$ is again defined by (49) now yields the formula

$$y_0(q) \ = \ \mu \ + \ \frac{1}{2}\, M_0(q) \tag{53}$$

for $y_0(q)$ which is real and positive for real $q > q_3$. Then the solution $W(\cdot,q)$ exists for $Re\, q > q_0$ where $q_0 = \max(q_1, q_2, q_3)$.

Another possibility in case $\mu > 0$ is to choose $b = y_0(q)$ obtaining the function $P(x,q)$ in the form (see [8])

$$P(x,q) = -\mu + \hat{A}_0(x,q)^{\frac{1}{2}} [y_0(q) \cos \hat{K}_0(x,q) + x\, \sin\, \hat{K}_0(x,q)] \tag{54}$$

where

$$\hat{A}_0(x,q) \;=\; +\frac{\mu^2 - y_0^2(q) + G(x,q) + 2\alpha(q)}{x^2 + y_0^2(q)} \tag{55}$$

$$\hat{K}_0(x,q) \;=\; \frac{x}{\pi}\int_0^\infty \frac{\ln \hat{A}_0(\xi,q)}{\xi^2 - x^2}\, d\xi \;\; . \tag{56}$$

The analogous relation to (49) now writes as

$$\hat{M}_0(q) \equiv \frac{1}{\pi}\int_0^\infty \ln \hat{A}_0(x,q)dx = y_0(q) - \mu \tag{57}$$

which constitutes an equation for $y_0(q)$. Of course, this equation must have the solution (53) again.

We remark that by [8, Theorem 2] or [9, Theorem 7.1] for the solution W of eq. (42) with initial condition (43) the function $W(\cdot,q)$ is continuous and possesses a continuous derivative $\partial W/\partial t(\cdot,q) \in L^2(\mathbb{R}_+)$ for $Re\, q > q_0$ where $W(t,q)$ and $\partial W/\partial t(t,q)$ tend to zero as $t \to \infty$.

To perform the inverse Laplace transform of the function $W(\cdot,q)$ we again assume that the function G has the representation (20) for $Re\, q > q_0$ (with an enlarged q_0, eventually), and correspondingly the function α has the representation

$$\alpha(q) \;=\; \frac{\alpha_0}{q} + \frac{\alpha_1(q)}{q^{1+\delta_1}} \;\; , \;\; \delta_1 > 0 \qquad \text{for} \quad Re\, q > q_0 \tag{58}$$

with $\alpha_0 = \varphi(0) \in \mathbb{R}$ and a bounded function α_1. Hence, the function A_0 has the analogous representation

$$A_0(x,q) = 1 + \frac{a_0(x) + a_1(x,q)/q^{\delta_1}}{(\mu^2 + x^2)q} \qquad \text{for} \quad Re\, q > q_0 \tag{59}$$

where $a_0(x) = c_0(x) + 2\alpha_0 = \mathcal{F}_c g(\cdot,0) + 2\varphi(0)$ and $a_1(x,q) = c_1(x,q) + 2\alpha_1(q)$. Related representations hold for the functions M_0 and B_0. Further, from (59) we have

$$M_0(q) \sim \frac{1}{\pi}\int_0^\infty \frac{a_0(x)}{\mu^2 + x^2}\, dx \cdot \frac{1}{q} \qquad \text{as} \quad Re\, q \to \infty \; . \tag{60}$$

Hence, in case $\mu < 0$ relation (49) implies the equality

$$\frac{1}{\pi}\int_0^\infty \frac{a_0(x)}{\mu^2 + x^2}\, dx \;\equiv\; \frac{1}{\pi}\int_0^\infty \frac{c_0(x)}{\mu^2 + x^2}\, dx - \frac{\alpha_0}{\mu} = 0 \tag{61}$$

as a necessary condition for the existence of a solution.

Observing condition (61) in case $\mu < 0$ and for $P(x,q)$ the expressions (46) in case $\mu < 0$ and (51) - (53) in case $\mu > 0$, respectively, from (59) again a representation for $P(x,q)$ of the form (22) follows:

$$P(x,q) = \frac{\tilde{c}_0(x)}{q} + \frac{c_2(x,q)}{q^1 + \delta_2} \;, \quad \delta_2 > 0 \tag{62}$$

where

$$\tilde{c}_0(x) = \frac{\mu\, a_0(x)}{2(\mu^2 + x^2)} + \frac{x^2}{x^2 + \mu^2}\, \frac{1}{\pi} \int_0^\infty \frac{c_0(\xi)d\xi}{\xi^2 - x^2}\,. \tag{63}$$

Then the function $W(t,q)$ has the representation (24) where $d_0(t) = 2p(t,0)$ with

$$p(t,0) = e^{-\mu t}[\varphi(0) + \frac{1}{2}\int_0^t e^{\mu s} g(s,0)ds], \qquad t \geq 0\,, \tag{64}$$

and there exists the solution p of eqs. (41) and (6) of the form (25) which obeys the relations (26) and (27).

Theorem 3. *Let the assumptions* (i) - (iv) *with* $g(t,0) = \lim_{\tau\to+0} g(t,\tau)$ *for* g *and the assumption (vi) for* φ *be fulfilled, and let the function* $G = \mathcal{F}_c\mathcal{L}g$ *have a Hölder continuous function* $G(\cdot,q)$ *with* $G(x,q) = O(x^{-\gamma})$, $\gamma > 1$ *for* $x \to \infty$, *uniformly for* $Re\, q > q_0$, $q_0 \in \mathbb{R}_+$. *Further, let the inequality* (45) *for real* $q > q_0$ *be fulfilled and let* G *have the representation* (20) *and* $\alpha = \mathcal{L}\varphi$ *the representation* (58).

Moreover, in case $\mu < 0$ *the relation* (49) *is assumed, and in case* $\mu > 0$ *the function* $y_0(q)$ *defined by* (53) *is real and positive for real* $q > q_0$.

Then equation (41) *for* $\mu \neq 0$ *with initial condition* (6) *has a generalized solution* p *with the properties* (26) *and* (27). *The solution* p *is defined by its Fourier-Laplace transform* $P = \mathcal{F}_c\mathcal{L}p$ *given by the formula* (46) *in case* $\mu < 0$ *and by the formulas* (51) *with* (52), (53) *or* (54) *with* (53) *in case* $\mu > 0$, *respectively.*

Corollaries.

1. From $G(x,q) = O(x^{-\gamma})$, $\gamma > 1$ we have $G(\cdot,q) \in L(\mathbb{R}_+)$ and therefore $G_0(\cdot,q) = \mathcal{L}g(\cdot,q)$ is supposed as a continuous function on $\mathbb{R}_+$. Theorem 3 holds for $\gamma > \frac{1}{2}$, too, where $G_0(\cdot,q)$ (as $g(\cdot,\tau)$) is not necessarily a continuous function on $\mathbb{R}_+$. In this case, also the functions $W(\cdot,q)$ and $\partial W/\partial t(\cdot,q)$ need not be continuous.

2. Condition (61) in case $\mu<0$ is equivalent to the relation

$$\varphi(0) \ + \ \frac{1}{2}\int_0^\infty e^{\mu s} g(s,0)ds = 0\ . \tag{65}$$

Further, from (64) it follows that $\lim\limits_{t\to+0} p(t,0)=\varphi(0)=\lim\limits_{\tau\to+0} p(0,\tau)$. Condition (65) is a necessary condition for the integrability of the function $p(\cdot,0)$ on $\mathbb{R}_+$ for a solution p of eq. (41) with initial condition (6) which satisfies this compatibility condition at $(0,0)$. Moreover, formula (64) implies the stronger condition

$$\varphi(0)+\frac{1}{2}\int_0^t e^{\mu s} g(s,0)ds = o(e^{\mu t}) \qquad \text{as } \ t\to\infty$$

as necessary for an integrable solution on $\mathbb{R}_+$ in case $\mu<0$.

3. In case $\mu=0$, analogously to the case $\mu<0$ we can choose the parameter $b=1$ in [8, 9] and get a corresponding solution p without zeros of $\tilde{F}(z,q)$ if the relation (49) is fulfilled. It has the form $p=\frac{2}{\pi}\mathcal{L}^{-1}\mathcal{F}_c P$ where

$$P(x,q)=\tilde{A}_0(x,q)^{\frac{1}{2}}[x\ \sin\tilde{K}_0(x,q)-\cos\tilde{K}_0(x,q)],\quad Re\, q>q_0 \tag{66}$$

where

$$\begin{aligned}\tilde{A}_0(x,q) &= \frac{x^2+G(x,q)+2\alpha(q)}{x^2+1}\\ \tilde{K}_0(x,q) &= \frac{x}{\pi}\int_0^\infty \frac{\ln\tilde{A}_0(\xi,q)}{\xi^2-x^2}\,d\xi\ .\end{aligned}$$

Analogously to the case $\mu>0$, choosing the parameter $b=y_0(q)$, a solution p with zero $z_0(q)=i\,y_0(q)$ of $\tilde{F}(z,q)$ of the form (54) with (55), (56) for $\mu=0$ exists if $\hat{A}_0(x,q)>0$ for real $q>q_0$ with $\hat{M}_0(q)\to 0$ for real $q\to\infty$ and eq. (57) with $\mu=0$ has a positive solution $y_0(q)$ for real $q>q_0$. The function $y_0(q)$ can be obtained in explicit form by taking the corresponding formula to (53) for $P(x,q)$ with $b=1$ again. Under stronger assumptions of the form

$$G(x,q)\sim\frac{c_0(x)}{q}\ +\ \frac{c_1(x)}{q^2}\quad,\quad \alpha(q)\sim\frac{\alpha_0}{q}\ +\ \frac{\alpha_1}{q^2}\qquad \text{as } \ Re\, q\to\infty$$

a study of the solution formula (54) for $\mu=0$ is possible. We omit the details.

6. Solution of equation (3) with $\lambda \neq 0$

Finally, we treat eq. (3) with $\lambda \neq 0$, i.e. the equation

$$\frac{\partial p}{\partial t} + \lambda \frac{\partial p}{\partial \tau} + \mu\, p(t,\tau) + I[p](t,\tau) = \frac{1}{2}\, g(t,\tau)\ , \quad 0 < t, \tau < \infty \tag{67}$$

together with the initial conditions (5) and (6). Applying again the Laplace transform with respect to τ, we get the intermediate equation for $W(t,q) = \mathcal{L}p(t,\cdot)(q)$

$$\frac{\partial W}{\partial t} + (\lambda q + \mu)W + \int_0^\infty W(s,q)W(s+t,q)ds = \frac{1}{2}\, G_\lambda(t,q), \ \ t \geq 0 \tag{68}$$

for $Re q > q_0$, $q_0 \in \mathbb{R}_+$ and the initial condition (43) where $G_\lambda(t,q) = G_0(t,q) + 2\lambda f(t)$ with $G_0(t,q) = \mathcal{L}g(t,\cdot)(q)$. We assume that g fulfils conditions (i) - (iv), f condition (v), and φ condition (vi). In addition, f should be continuous and should have a summable derivative.

Eq. (68) has the same form as eq. (42) with μ replaced by $\mu_\lambda(q) = \mu + \lambda q$ and G_0 replaced by G_λ. We assume that the functions $G = \mathcal{F}_c G_0$, $\alpha = \mathcal{L}\varphi$ and $F = \mathcal{F}_c f$ are bounded where $G(\cdot,q)$ and F are Hölder-continuous functions of $x \in \mathbb{R}$ with $G(x,q), F(x) = O(x^{-\gamma})$, $\gamma > \frac{1}{2}$ as $x \to \infty$, uniformly for $Re\, q > q_1$ with some $q_1 \in \mathbb{R}_+$. Further, now the inequality

$$\mu_\lambda^2(q) + x^2 + G(x,q) + 2\alpha(q) + 2\lambda\, F(x) > 0 \tag{69}$$

for real $q > q_2$, $x \in \mathbb{R}$ with some $q_2 \in \mathbb{R}$ holds, and we have $\mu_\lambda(q) < 0$ for real $q > \tilde{q}_0$ if $\lambda < 0$ and $\mu_\lambda(q) > 0$ for real $q > \tilde{q}_0$ if $\lambda > 0$ with some $\tilde{q}_0 \in \mathbb{R}$.

Hence, in case $\lambda < 0$ we obtain the analogue expression to (46) for $P(x,q) = \mathcal{F}_c W(\cdot,q)(x)$:

$$P(x,q) = -\mu_\lambda(q) + A_1(x,q)^{\frac{1}{2}}[\mu_\lambda(q)\cos K_1(x,q) + x \sin\ K_1(x,q)] \tag{70}$$

with

$$A_1(x,q) = 1 + \frac{G(x,q) + 2\alpha(q) + 2\lambda\, F(x)}{\mu_\lambda^2(q) + x^2}\ , \tag{71}$$

$$K_1(x,q) = \frac{x}{\pi}\int_0^\infty \frac{\ln A_1(x,q)}{\xi^2 - x^2}\, d\xi \tag{72}$$

for $Re\, q > q_0$, $q_0 = \max(\tilde{q}_0, q_1, q_2)$. The solution $W(\cdot,q) = \frac{2}{\pi}\mathcal{F}_c P(\cdot,q)$ of (68) with (43) exists if the condition

$$M_1(q) \equiv \frac{1}{\pi}\int_0^\infty \ln\ A_1(x,q)dx = 0\ , \qquad Re\, q > q_0 \tag{73}$$

holds. We remark that we now have the asymptotic relation $A_1(x,q) = 1 + O(1/q^2)$ as $Re\,q \to \infty$ and we need the representations (20) and (58) for G and α, respectively, only for the higher terms in the asymptotics for P.

In case $\lambda > 0$, analogously to (53) we define

$$y_1(q) = \mu_\lambda(q) + \frac{1}{2}\,M_1(q) \tag{74}$$

which is real and positive for real $q > q_3$ with some $q_3 \in \mathbb{R}$, and we get the analogous expression to (51) for $P(x,q)$:

$$\begin{aligned} P(x,q) = -\mu_\lambda(q) + A_1(x,q)^{\frac{1}{2}}\big[&Re((ix-\mu_\lambda(q))B_1(x,q))\cos K_1(x,q) \\ &+Im((ix-\mu_\lambda(q))B_1(x,q))\sin K_1(x,q)\big] \end{aligned} \tag{75}$$

where A_1 and K_1 are defined by (71) and (72), respectively, and $B_1(x,q) = [x - iy_1(q)]/[x + iy_1(q)]$. The solution $W(\cdot,q)$ of (68) with (43) exists for $Re\,q > q_0$ where $q_0 = \max(\tilde{q}_0, q_1, q_2, q_3)$. Also we can take for $P(x,q)$ the analogous expression to (54) with (55) and (56).

From the expressions for $P(x,q)$ the asymptotic relations

$$P(x,q) \;\sim\; \frac{\lambda\,F(x)\hat{q}}{\hat{q}^2 + x^2} \quad , \qquad \hat{q} = \mu_\lambda(q) = \lambda q + \mu \;\;,\;\; x \in \mathbb{R}$$

as $Re\,q \to \infty$ and hence

$$W(t,q) \;\sim\; \frac{\lambda}{\hat{q}}\,f(t) \;\sim\; \frac{f(t)}{q} \qquad , \quad t > 0$$

as $Re\,q \to \infty$ with remainders of order $O(q^{-(1+\delta)})$, $\delta > 0$, follow.

Therefore, eqs. (67) and (5), (6) have a solution p which satisfies the relations (26) and (27).

Theorem 4. *Let the assumptions* (i) - (iv) *with* $g(t,0) = \lim_{\tau \to +0} g(t,\tau)$ *for* g *and the assumptions* (v) *for* f *and* (vi) *for* φ *be fulfilled where, in addition,* f *has a summable derivative. Further, let the functions* $G = \mathcal{F}_c\mathcal{L}g$, $\alpha = \mathcal{L}\varphi$ *and* $F = \mathcal{F}_c f$ *be bounded and* $G(\cdot,q)$ *and* F *are Hölder continuous functions with* $G(x,q), F(x) = O(x^{-\gamma})$, $\gamma > \frac{1}{2}$ *as* $x \to \infty$ *uniformly for* $Re\,q > q_0$, $q_0 \in \mathbb{R}_+$. *For real* $q > q_0$ *also the inequalities* (69) *and relation* $sign(\lambda q + \mu) = sign\lambda$ *should be satisfied.*

Moreover, in case $\lambda < 0$ *the relation* (73) *is assumed to be fulfilled, and in case* $\lambda > 0$ *the function* $y_1(q)$ *defined by* (74) *is real and positive for real* $q > q_0$.

Then equation (67) *for* $\lambda \neq 0$ *with initial conditions* (5) *and* (6) *has a generalized solution* p *with the properties* (26) *and* (27). *The solution* p *is defined* *by its Fourier-Laplace transform* $P = \mathcal{F}_c\mathcal{L}p$ *given by the formula* (70) *in case* $\lambda < 0$ *and by* (75) *with* (74) *in case* $\lambda > 0$.

Corollaries.

1. If $G(x, q),\ F(x) = O(x^{-\gamma}),\ \gamma > 1$ in view of [8, Theorem 2] or [9, Theorem 7.1], the functions $W(\cdot, q)$ and $\partial W/\partial t(\cdot, q)$ are continuous functions from $L^2(\mathbb{R}_+)$ for $Re\, q > q_0$ tending to zero as $t \to \infty$.

2. If the functions G and α have the representations (20) and (58), then the function P has the asymptotic expansion

$$P(x, q) \sim \frac{\lambda\, F(x)\hat{q}}{\hat{q}^2 + x^2} + \frac{\lambda\, a_0(x)}{2[\hat{q}^2 + x^2]} + \frac{2\lambda}{\pi}\ \frac{x^2}{x^2 + \hat{q}^2} \int_0^\infty \frac{F(\xi)}{\xi^2 - x^2}\, d\xi$$

as $Re\, q \to \infty$ where again $\hat{q} = \lambda q + \mu$, $a_0(x) = \mathcal{F}_c g(\cdot, 0) + 2\varphi(0)$, and in case $\lambda < 0$ condition (73) has to be observed. Hence, in case $\lambda > 0$ it holds

$$W(t, q) \sim \lambda\, e^{-\hat{q}t} \left(\frac{\varphi(0)}{\hat{q}} + \int_0^t f(s) e^{\hat{q}s} ds \right) \sim \frac{f(t)}{q}$$

for $t > 0$ and also for $t = 0$ if we assume the compatibility condition $\varphi(0) = f(0)$. The function $g(\cdot, 0)$ in $a_0(x)$ leads to higher order terms. Analogous statements hold in case $\lambda < 0$.

3. The existence condition (73) in case $\lambda < 0$ can be looked on as an equation for the function $\varphi = p(0, \cdot)$ if p denotes the solution of eq. (67) with initial condition (5) only (cf. [11]).

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