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Hyperbolic Functional Differential Systems with Unbounded Delay

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Abstract. The phase space for quasilinear systems with unbounded delay is constructed. Carathéodory solutions to initial and mixed problems are investigated. Theorems on the local existence and continuous dependence upon initial or initial boundary functions are given. The fixed-point method and integral inequalities are used.

Keywords: *Carathéodory solutions, phase spaces, initial problems, mixed problems*
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1. Introduction

For any metric spaces U and V we denote by $C(U,V)$ the class of all continuous functions from U to V. We use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Let us denote by $M_{k\times n}$ the set of all $k\times n$ matrices with real elements. For $x\in R^n$, $p\in R^k$, $Y\in M_{k\times n}$ where

$$x=(x_1,\dots,x_n),\quad p=(p_1,\dots,p_k),\quad Y=[y_{ij}]_{i=1,\dots,k,\ j=1,\dots,n},$$

we define the norms

$$\|x\|=\sum_{i=1}^{n}|x_i|,\quad \|p\|=\max_{1\le i\le k}\{|p_i|\},\quad \|Y\|=\max\{\sum_{j=1}^{n}|y_{ij}|:1\le i\le k\}.$$

We will denote by $L([0,c],R_+)$, $c>0$, $R_+=[0,+\infty)$, the class of all functions $\gamma:[0,a]\to R_+$, which are integrable on $[0,c]$. Let $B=(-\infty,0]\times[-r,r]$ where $r=(r_1,\dots,r_n)\in R_+^n$, $R_+=[0,+\infty)$. For a function $z:(-\infty,a]\times R^n\to R^k$, $a>0$, and for a point $(t,x)\in(-\infty,a]\times R^n$ we define a function $z_{(t,x)}:B\to R^k$ as follows: $z_{(t,x)}(s,y)=z(t+s,x+y)$, $(s,y)\in B$. Suppose that the functions

$$\psi=(\psi_1,\dots,\psi_k),\quad \psi_i=(\psi_{i.0},\psi_i'),$$

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$$\psi_{i.0} : [0,a] \to R, \quad \psi'_i = (\psi_{i.1}, \dots, \psi_{i.n}) : [0,a] \times R^n \to R^n$$

are given. The requirements on $\psi_{i.0}$, $1 \le i \le k$, are that $\psi_{i.0}(t) \le t$ for $t \in [0,a]$. For $(t,x) \in R^{1+n}$ we write $\psi_i(t,x) = (\psi_{i.0}(t), \psi'_i(t,x))$, $1 \le i \le k$.

The phase space X for equations with unbounded delay is a linear space with the norm $\|\cdot\|_X$ consisting of functions mapping the set B into R^k. Write $\Omega = [0,a] \times R^n \times X$ and suppose that the functions

$$\rho : \Omega \to M_{k\times n}, \quad \rho = [\rho_{ij}]_{i=1,\dots,k,\, j=1,\dots,n}$$
$$f : \Omega \to R^k, \qquad f = (f_1, \dots, f_k)$$

and

$$\varphi : (-\infty, 0] \times R^n \to R^k, \quad \varphi = (\varphi_1, \dots, \varphi_k)$$

are given. We consider the quasilinear functional differential system

$$\partial_t z_i(t,x) + \sum_{j=1}^{n} \rho_{ij}(t,x,z_{\psi_i(t,x)}) \partial_{x_j} z_i(t,x) = f_i(t,x,z_{\psi_i(t,x)}), \ 1 \le i \le k, \tag{1}$$

with the initial condition

$$z(t,x) = \varphi(t,x) \quad \text{for} \quad (t,x) \in (-\infty,0] \times R^n. \tag{2}$$

Note that $z_{\psi_i(t,x)}$ is the restriction of z to the set $(-\infty, \psi_{i.0}(t)] \times [\psi'_i(t,x) - r, \psi'_i(t,x) + r]$, and this restriction is shifted to the set B.

We consider weak solutions of problem (1), (2). A function $\bar{z} : (-\infty, c] \times R^n \to R^k$, $0 < c \le a$, is a solution to the above problem if

(i) $\bar{z}_{\psi_i(t,x)} \in X$ for $(t,x) \in [0,c] \times R^n$, $1 \le i \le k$,
(ii) the derivatives $\partial_t \bar{z}$ and $\partial_x \bar{z} = (\partial_{x_1} \bar{z}, \dots, \partial_{x_n} \bar{z})$ exist almost everywhere on $[0,c] \times R^n$,
(iii) $\bar{z}$ satisfies (1) almost everywhere on $[0,c] \times R^n$ and condition (2) holds.

Recently, numerous papers were published concerning functional differential equations or systems. The following questions were considered: functional differential inequalities, uniqueness and continuous dependence for initial or mixed problems, difference functional inequalities, numerical approximations of classical solutions, existence of classical or generalized solutions to initial or mixed problems. In these considerations, initial or initial boundary functions are defined on bounded domains. Monograph [6] contains an exposition of recent developments of hyperbolic functional differential equations and systems.

Paper [7] initiated the investigations of partial differential equations with unbounded delay. Sufficient conditions for the existence and uniqueness of Carathéodory solutions of initial problems for quasilinear equations were proved.

Functional differential equations in a Banach space were considered. The system of axioms for the phase space is formulated in a form of inequalities for norms in the space $C((-\infty, c] \times R^n, Y)$ and their suitable subspaces, where Y is a Banach space. Methods used in [7] are extended in [3] to initial boundary value problems. Systems of axioms in [7] and [3] are different because domains of solutions for initial problems and mixed problems are different.

The aim of the paper is to propose a new system of axioms for phase spaces. It is important in our considerations that assumptions on phase spaces are generated by differential functional systems and they are the same for initial problems and for mixed problems.

The paper is organized as follows. In Section 2 we formulate a system of axioms and some properties of phase spaces. We give examples of spaces satisfying the main assumption. A theorem on the existence and continuous dependence upon initial data is presented in Section 3. The last part of the paper deals with initial boundary value problems. A result on the existence of Carathéodory solutions is proved. Note that results of the paper are new also in the case when the domain B is a bounded set, see Remark 1, 4 and 5.

In the paper, we use general ideas concerning axiomatic approach to equations with unbounded delay, which were introduced for ordinary differential equations in [5], [8]. We apply a method of bicharacteristics. It was introduced and widely studied in non-functional setting in [1], [2].

2. Definitions and fundamental axioms

Assume that $c > 0$, $w : (-\infty, c] \times [-r, r] \to R^k$ and $t \in (-\infty, c]$. We define a function $w_{(t)} : B \to R^k$ by $w_{(t)}(s, y) = w(t + s, y)$, $(s, y) \in B$. For each $t \in (-\infty, c]$ the function $w_{(t)}$ is the restriction of w to the set $(-\infty, t] \times [-r, r]$, and this restriction is shifted to the set B. If $w : (-\infty, c] \times [-r, r] \to R^k$, $c > 0$, and $w|_{[0,c]\times[-r,r]}$ is continuous, then we write

$$\|w\|_{[0,t]} = \max\{\|w(s, y)\| : (s, y) \in [0, t] \times [-r, r]\},$$

where $t \in [0, c]$.

Assumption $H[X]$. Suppose that $(X, \|\cdot\|_X)$ is a Banach space and

1) there is a constant $\chi \in R_+$ independent of w such that for each function $w \in X$ we have

$$\|w(0, x)\| \le \chi \|w\|_X, \quad x \in [-r, r], \tag{3}$$

2) if $w : (-\infty, c] \times [-r, r] \to R^k$, $c > 0$, is a function such that $w_{(0)} \in X$ and $w|_{[0,c]\times[-r,r]}$ is continuous, then $w_{(t)} \in X$ for $t \in [0, c]$ and

(i) the function $t \to w_{(t)}$ is continuous on $[0, c]$,

(ii) there are $K, K_0 \in R_+$ independent of w such that

$$\|w_{(t)}\|_X \leq K\|w\|_{[0,t]} + K_0\|w_{(0)}\|_X, \quad t \in [0, c]. \tag{4}$$

Now we give examples of phase spaces.

Example 2.1. Let X be the class of all functions $w : B \to R^k$ which are uniformly continuous and bounded on B. For $w \in X$ we put

$$\|w\|_X = \sup\{\|w(s, y)\| : (s, y) \in B\}. \tag{5}$$

Then, Assumption $H[X]$ is satisfied with all the constants equal to 1.

Example 2.2. Let X be the class of all functions $w : B \to R^k_+$ such that $w \in C(B, R^k)$ and there exists the limit $\lim_{t\to-\infty} w(t, x) = w_0(x)$ uniformly with respect to $x \in [-r, r]$. The norm in the space X is defined by (5). Then, Assumption $H[X]$ is satisfied with all the constants equal to 1.

Example 2.3. Let $\gamma : (-\infty, 0] \to (0, \infty)$ be a continuous function. Assume also that γ is nonincreasing on $(-\infty, 0]$. Let X be the space of all continuous functions $w : B \to R^k$ such that

$$\lim_{t\to-\infty} \frac{\|w(t, x)\|}{\gamma(t)} = 0, \quad x \in [-r, r].$$

Write

$$\|w\|_X = \sup\left\{\frac{\|w(t, x)\|}{(\gamma(t)} : (t, x) \in B\right\}.$$

Then, Assumption $H[X]$ is satisfied with $K = \frac{1}{\gamma(0)}$, $K_0 = 1$, $\chi = \gamma(0)$.

Example 2.4. Let $p \geq 1$ be fixed. Denote by Y the class of all $w : B \to R^k$ such that

(i) w is continuous on $\{0\} \times [-r, r]$ and

$$\int_{-\infty}^{0} \|w(\tau, x)\|^p d\tau < +\infty \quad \text{for} \quad x \in [-r, r],$$

(ii) for each $t \in (-\infty, 0]$ the function $w(t, \cdot) : [-r, r] \to R^k$ is continuous.

We define the norm in the space Y by

$$\begin{aligned} \|w\|_Y &= \max\{\|w(t, x)\| : (t, x) \in \{0\} \times [-r, r]\} \\ &\quad + \sup\left\{\left(\int_{-\infty}^{0} \|w(\tau, x)\|^p d\tau\right)^{\frac{1}{p}} : x \in [-r, r]\right\}. \end{aligned}$$

Write $X = \bar{Y}$, where $\bar{Y}$ is the closure of Y with the above given norm. Then, Assumption $H[X]$ is satisfied with $K = 1$, $K_0 = 1 + c^{\frac{1}{p}}$, $\chi = 1$.

Example 2.5. Denote by Y the set of all functions $w : B \to R^k$ which are bounded and which satisfy the following properties:

(i) w is continuous on $\{0\} \times [-r, r]$ and

$$I(x) = \sup\left\{ \int_{-(n+1)}^{-n} \|w(\tau, x)\| d\tau : n \in \mathbf{N} \right\} < +\infty$$

where $x \in [-r, r]$ and $\mathbf{N}$ is the set of natural numbers

(ii) for each $t \in (-\infty, 0]$ the function $w(t, \cdot) : [-r, r] \to R^k$ is continuous.

The norm in the space Y is defined by

$$\|w\|_Y = \max\{|w(t, x)| : (t, x) \in \{0\} \times [-r, r]\} + \sup\{I(x) : x \in [-r, r]\}.$$

Write $X = \bar{Y}$, where $\bar{Y}$ is the closure of Y with the above given norm. Then, Assumption $H[X]$ is satisfied with $K = 1 + c$, $K_0 = 2$, $\chi = 1$.

If $z : (-\infty, c] \times R^n \to R^k$, $c > 0$, is a function such that $z|_{[0,c]\times R^n}$ is continuous and $(t, x) \in [0, c] \times R^n$, then we put

$$\|z\|_{[0,t;x]} = \max\left\{ \|z(s, y)\| : (s, y) \in [0, t] \times [x - r, x + r] \right\}.$$

Suppose additionally that the function $z|_{[0,c]\times R^n}$ satisfies a Lipschitz condition with respect to x. Then we write

$$\mathrm{Lip}[z]|_{[0,t;R^n]} = \sup\left\{ \frac{\|z(s, y) - z(s, \bar{y})\|}{\|y - \bar{y}\|} : (s, y), (s, \bar{y}) \in [0, t] \times R^n, y \neq \bar{y} \right\}.$$

Lemma 2.1. *Suppose that Assumption $H[X]$ is satisfied and $z : (-\infty, c] \times R^n \to R^k$, $0 < c \leq a$. If $z_{(0,x)} \in X$ for $x \in R^n$ and $z|_{[0,c]\times R^n}$ is continuous, then $z_{(t,x)} \in X$ for $(t, x) \in (0, c] \times R^n$ and*

$$\|z_{(t,x)}\|_X \leq K\|z\|_{[0,t;x]} + K_0\|z_{(0,x)}\|_X. \tag{6}$$

If we assume additionally that the function $z|_{[0,c]\times R^n}$ satisfies the Lipschitz condition with respect to x, then

$$\|z_{(t,x)} - z_{(t,\bar{x})}\|_X \leq K\,\mathrm{Lip}[z]|_{[0,t;R^n]}\|x - \bar{x}\| + K_0\|z_{(0,x)} - z_{(0,\bar{x})}\|_X, \tag{7}$$

where $(t, x), (t, \bar{x}) \in [0, c] \times R^n$.

Proof. Inequality (6) is a consequence of (4) for $w : (-\infty, c] \times [-r, r] \to R^k$ given by $w(s, y) = z(s, x + y)$ with fixed $x \in R^n$.

We prove (7). Suppose that $(t,x),(t,\bar{x}) \in [0,c] \times R^n$ and the function $\tilde{z} : (-\infty,c] \times R^n \to R^k$ is defined by $\tilde{z}(s,y) = z(s,y+\bar{x}-x)$, $(s,y) \in (-\infty,c] \times R^n$. Then $\tilde{z}_{(t,x)} = z_{(t,\bar{x})}$ and

$$\begin{aligned} \|z_{(t,x)} - z_{(t,\bar{x})}\|_X &= \|(z-\tilde{z})_{(t,x)}\|_X \\ &\leq K\|z-\tilde{z}\|_{[0,t;x]} + K_0\|(z-\tilde{z})_{(0,x)}\|_X \\ &\leq K\,\mathrm{Lip}[z]|_{[0,t;R^n]}\|x-\bar{x}\| + K_0\|z_{(0,x)} - z_{(0,\bar{x})}\|_X, \end{aligned}$$

which proves (7). ∎

Our basic assumption on initial functions is the following.

Assumption $H[\varphi]$. Suppose that $\varphi : (-\infty,0] \times R^n \to R^k$, $\varphi_{(0,x)} \in X$ for $x \in R^n$, and there are $b_0, b_1 \in R_+$ such that

$$\|\varphi_{(0,x)}\|_X \leq b_0, \quad \|\varphi_{(0,x)} - \varphi_{(0,\bar{x})}\|_X \leq b_1\|x-\bar{x}\|,$$

where $x,\bar{x} \in R^n$. Let us denote by $I[X]$ the class of all initial functions $\varphi : (-\infty,0] \times R^n \to R^k$ satisfying Assumption $H[\varphi]$. Let be $\varphi \in I[X]$ and let $0 < c \leq a$, $d = (d_0,d_1) \in R_+^2$, $\lambda \in L([0,c],R_+)$. Let us denote by $C_{\varphi.c}[d,\lambda]$ the class of all functions $z : (-\infty,c] \times R^n \to R^k$ such that

(i) $z(t,x) = \varphi(t,x)$ for $(t,x) \in (-\infty,0] \times R^n$,

(ii) the estimates

$$\|z(t,x)\| \leq d_0, \quad \|z(t,x) - z(\bar{t},\bar{x})\| \leq \left| \int_t^{\bar{t}} \lambda(\tau)d\tau \right| + d_1\|x-\bar{x}\|,$$

hold on $[0,c] \times R^n$.

We will prove that under suitable assumptions on f and ψ and for sufficiently small c, $0 < c \leq a$, there exists a solution $\bar{z}$ to problem (1), (2) such that $\bar{z} \in C_{\varphi.c}[d,\lambda]$.

3. Existence of solutions to initial problems

Let us denote by Δ the set of all functions $\alpha : [0,a] \times R_+ \to R_+$ such that $\alpha(\cdot,t) \in L([0,a],R_+)$ for $t \in R_+$ and the function $\alpha(t,\cdot) : R_+ \to R_+$ is continuous and nondecreasing, and $\alpha(t,0) = 0$ for almost all $t \in [0,a]$. Write

$$X[\mu] = \{w \in X : \|w\|_X \leq \mu\}, \quad \mu \in R_+.$$

We will need the following assumptions.

Assumption $H[\rho]$. Suppose that

1) the function $\rho(\cdot, y, w) : [0, a] \to M_{k\times n}$ is measurable for every $(y, w) \in R^n \times X$ and there is a function $\alpha \in \Delta$ such that

$$\|\rho(t, x, w)\| \le \alpha(t, \mu)$$

for $(x, w) \in R^n \times X[\mu]$ almost everywhere on $[0, a]$

2) there is a function $\gamma \in \Delta$ such that

$$\|\rho(t, x, w) - \rho(t, \bar{x}, \bar{w})\| \le \gamma(t, \mu)[\|x - \bar{x}\| + \|w - \bar{w}\|_X]$$

for $(x, w), (\bar{x}, \bar{w}) \in R^n \times X[\mu]$ and for almost $t \in [0, a]$,.

Assumption $H[\psi]$. Suppose that for each i, $1 \le i \le k$, the functions $\psi_{i.0} : [0, a] \to R$ and $\psi_i' = (\psi_{i.1}, \dots, \psi_{i.n}) : [0, a] \times R^n \to R^n$ are continuous and

1) $\psi_{i.0}(t) \le t$ for $t \in (0, a]$
2) there is $s_0 \in R_+$ such that

$$\|\psi_i'(t, x) - \psi_i'(t, \bar{x})\| \le s_0\|x - \bar{x}\|$$

on $[0, a] \times R^n$.

Suppose that Assumptions $H[X]$, $H[\rho]$, $H[\psi]$ are satisfied and $\varphi \in I[X]$, $z \in C_{\varphi.c}[d, \lambda]$. Consider the Cauchy problem

$$\eta'(\tau) = \rho_i(\tau, \eta(\tau), z_{\psi_i(\tau,\eta(\tau))}), \quad \eta(t) = x, \tag{8}$$

where $(t, x) \in [0, c] \times R^n$ and $1 \le i \le k$ is fixed, while $\rho_i = (\rho_{i1}, \dots, \rho_{in})$. Let us denote by $g_i[z](\cdot, t, x)$ the solution to (8). The function $g_i[z]$ is the i-th bicharacteristic of system (1) corresponding to z. For functions $\varphi \in I[X]$ and $z \in C_{\varphi.c}[d, \lambda]$, we write

$$\|\varphi\|_{X,R^n} = \sup\{\|\varphi_{(0,x)}\|_X : x \in R^n\}$$

and

$$\|z\|_t = \sup\{\|z(s, y)\| : (s, y) \in [0, t] \times R^n\},$$

where $t \in [0, c]$.

We first prove a lemma on the existence and regularity of bi-characteristics.

Lemma 3.1. *Suppose that Assumptions* $H[X]$, $H[\rho]$, $H[\psi]$ *are satisfied and* $\varphi, \bar{\varphi} \in I[X]$, $z \in C_{\varphi.c}[d, \lambda]$, $\bar{z} \in C_{\bar{\varphi}.c}[d, \lambda]$, *where* $0 < c \le a$. *Then, for each* $1 \le i \le k$, *the solutions* $g_i[z](\cdot, t, x)$ *and* $g_i[\bar{z}](\cdot, t, x)$ *exist on* $[0, c]$. *They are unique, and we have the estimates*

$$\|g_i[z](\tau, t, x) - g_i[z](\tau, \bar{t}, \bar{x})\| \le L(c)\left[\left|\int_{\bar{t}}^{t} \alpha(\xi) d\xi\right| + \|x - \bar{x}\|\right] \tag{9}$$

for $(t,x),(\bar{t},\bar{x})\in[0,c]\times R^n$, $\tau\in[0,c]$, *where*

$$\begin{aligned} L(\tau) &= \exp\Big[A\int_0^{\tau}\gamma(\xi,\mu_0)d\xi\Big], \quad \tau\in[0,c] \\ A &= 1+s_0(Kd_1+K_0b_1) \\ \mu_0 &= Kd_0+K_0b_0 \end{aligned}$$

and

$$\begin{aligned} &\|g_i[z](\tau,t,x)-g_i[\bar{z}](\tau,t,x)\| \\ &\quad\le L(c)\Big|\int_t^{\tau}\gamma(\xi,\mu_0)\Big[K\|z-\bar{z}\|_{\xi}+K_0\|\varphi-\bar{\varphi}\|_{X,R^n}\Big]d\xi\Big| \end{aligned} \tag{10}$$

where $(\tau,t,x)\in[0,c]\times[0,c]\times R^n$.

Proof. We begin by proving that problem (8) has exactly one solution. It follows from the Assumptions $H[X]$, $H[\psi]$, $H[\rho]$ and Lemma 2.1 that $\|z_{(s,y)}\|_X\le\mu_0$ for $(s,y)\in[0,c]\times R^n$ and that the following Lipschitz condition is satisfied:

$$\|\rho_i(\tau,y,z_{\psi_i(\tau,y)})-\rho_i(\tau,\bar{y},z_{\psi_i(\tau,\bar{y})})\|\le\gamma(\tau,\mu_0)A\|y-\bar{y}\|,$$

where $\tau\in[0,c]$, $y,\bar{y}\in R^n$. It follows that there exists exactly one Carathéodory solution to problem (8), and the solution is defined on the interval $[0,c]$.

Now we prove estimate (9). The function $g_i[z](\cdot,t,x)$ satisfies the integral equation

$$g_i[z](\tau,t,x)=x+\int_t^{\tau}\rho_i(\xi,g_i[z](\xi,t,x),z_{\psi_i(\xi,g_i[z](\xi,t,x))})d\xi.$$

Write

$$P_i[z](\xi,t,x)=(\xi,g_i[z](\xi,t,x),z_{\psi_i(\xi,g_i[z](\xi,t,x))}).$$

It follows from the Assumptions $H[\psi]$, $H[\rho]$ and Lemma 2.1 that

$$\begin{aligned} &\|g_i[z](\tau,t,x)-g_i[z](\tau,\bar{t},\bar{x})\| \\ &\qquad\le \|x-\bar{x}\|+\Big|\int_t^{\bar{t}}\alpha(\xi,\mu_0)d\xi\Big| \\ &\qquad\quad+\Big|\int_{\tau}^{t}\|\rho_i(P_i[z](\xi,t,x))-\rho_i(P_i[z](\xi,\bar{t},\bar{x}))\|d\xi\Big| \\ &\qquad\le \|x-\bar{x}\|+\Big|\int_t^{\bar{t}}\alpha(\xi,\mu_0)d\xi\Big| \\ &\qquad\quad+A\Big|\int_t^{\tau}\gamma(\xi,\mu_0)\|g_i[z](\xi,t,x)-g_i[z](\xi,\bar{t},\bar{x})\|d\xi\Big|, \end{aligned}$$

where $(t,x),(\bar{t},\bar{x}) \in [0,c] \times R^n$, $\tau \in [0,c]$. Now we obtain (9) from the Gronwall inequality.

Our next aim is to prove (10). For $z \in C_{\varphi.c}[d,\lambda]$ and $\bar{z} \in C_{\bar{\varphi}.c}[d,\lambda]$ we have

$$\begin{aligned} &\|g_i[z](\tau,t,x) - g_i[\bar{z}](\tau,t,x)\| \\ &\quad\leq \left| \int_t^\tau \|\rho_i(P_i[z](\xi,t,x)) - \rho_i(P_i[\bar{z}](\xi,t,x))\| d\xi \right|. \end{aligned} \tag{11}$$

It follows from Assumption $H[X]$ and Lemma 2.1 that

$$\begin{aligned} &\|z_{\psi_i(\xi,g_i[z](\xi,t,x))} - \bar{z}_{\psi_i(\xi,g_i[\bar{z}](\xi,t,x))}\|_X \\ &\quad\leq \|z_{\psi_i(\xi,g_i[z](\xi,t,x))} - z_{\psi_i(\xi,g_i[\bar{z}](\xi,t,x))}\|_X \\ &\qquad + \|z_{\psi_i(\xi,g_i[\bar{z}](\xi,t,x))} - \bar{z}_{\psi_i(\xi,g_i[\bar{z}](\xi,t,x))}\|_X \\ &\quad\leq s_0(Kd_1 + K_0 b_1)\|g_i[z](\xi,t,x) - g_i[\bar{z}](\xi,t,x)\| \\ &\qquad + K\|z-\bar{z}\|_\xi + K_0\|\varphi - \bar{\varphi}\|_{X,R^n} \end{aligned} \tag{12}$$

where $(\xi,t,x) \in [0,c] \times [0,c] \times R^n$. The above estimate and (11) imply the integral inequality

$$\begin{aligned} &\|g_i[z](\tau,t,x) - g_i[\bar{z}](\tau,t,x)\| \\ &\quad\leq \left| \int_t^\tau \gamma(\xi,\mu_0)\Big[K\|z-\bar{z}\|_\xi + K_0\|\varphi-\bar{\varphi}\|_{X,R^n}\Big] d\xi \right| \\ &\qquad + A\left| \int_t^\tau \gamma(\xi,\mu_0)\|g_i[z](\xi,t,x) - g_i[\bar{z}](\xi,\bar{t},\bar{x})\| d\xi \right|, \end{aligned}$$

where $(\xi,t,x) \in [0,c] \times [0,c] \times R^n$. Now we obtain (10) from the Gronwall inequality. ∎

Suppose that $\varphi \in I[X]$, $c \in (0,a]$, $z \in C_{\varphi.c}[d,\lambda]$ and

$$g[z](\cdot,t,x) = (g_1[z](\cdot,t,x),\ldots,g_k[z](\cdot,t,x))$$

are bicharacteristics of system (8). Write

$$\begin{aligned} &f^*(t,g[z](\tau,t,x),z_{\psi(\tau,g[z](\tau,t,x))}) \\ &\quad= \big(f_1(t,g_1[z](\tau,t,x),z_{\psi_1(\tau,g_1[z](\tau,t,x))}),\ldots,f_k(t,g_k[z](\tau,t,x),z_{\psi_k(\tau,g_k[z](\tau,t,x))})\big) \end{aligned}$$

and

$$\varphi^*(\tau,g[z](\tau,t,x)) = (\varphi_1(\tau,g_1[z](\tau,t,x)),\ldots,\varphi_k(\tau,g_k[z](\tau,t,x)))\,.$$

Let us define operator F as

$$F[z](t,x) = \varphi^*(0, g[z](0,t,x)) + \int_0^t f^*(\tau, g[z](\tau,t,x), z_{\psi(\tau,g[z](\tau,t,x))})d\tau$$

on $[0,c] \times R^n$, and let

$$F[z](t,x) = \varphi^*(t,x) \quad \text{on} \quad (-\infty, 0) \times R^n.$$

Assumption $H[f]$. Suppose that

1) the function $f(\cdot, y, w) : [0,a] \to R^k$ is measurable for $(y,w) \in R^n \times X$ and there is $\tilde{\gamma} \in \Delta$ such that

$$\|f(t,x,w)\| \leq \tilde{\gamma}(t,\mu)$$

for $(x,w) \in R^n \times X[\mu]$ and for almost all $t \in [0,a]$,

2) there exists a function $\beta \in \Delta$ such that the Lipschitz condition

$$\|f(t,x,w) - f(t,\bar{x},\bar{w})\| \leq \beta(t,\mu)[\|x-\bar{x}\| + \|w-\bar{w}\|_X]$$

is satisfied for $(x,w), (\bar{x},\bar{w}) \in R^n \times X[\mu]$ and for almost all $t \in [0,a]$.

Theorem 3.1. *Suppose that $\varphi \in I[X]$ and Assumptions $H[\rho]$, $H[f]$, $H[\Psi]$ are satisfied. Then there are $(d_0,d_1) = d \in R_+^2$, $c \in (0,a]$ and $\lambda \in L([0,c],R_+)$ such that there exists exactly one solution $u \in C_{\varphi.c}[d,\lambda]$ to problem* (1), (2).

If $\bar{\varphi} \in I[X]$ and $\bar{u} \in C_{\bar{\varphi}.c}[d,\lambda]$ is a solution to system (1) *with the initial condition $z(t,x) = \bar{\varphi}(t,x)$ on $(-\infty,0] \times R^n$, then there is a function $\Lambda \in C((0,c],R_+)$ such that*

$$\|u-\bar{u}\|_t \leq \Lambda(t)\|\varphi - \bar{\varphi}\|_{X,R^n}, \quad t \in (0,c]. \tag{13}$$

Proof. We have divided the proof into three steps.

`Step I.` We first show that there are $(d_0,d_1) = d \in R_+^2$, $c \in (0,a]$ and $\lambda \in L([0,c],R_+)$ such that

$$F : C_{\varphi.c}[d,\lambda] \to C_{\varphi.c}[d,\lambda].$$

Suppose that the constants $(d_0,d_1) = d \in R_+^2$, $c \in (0,a]$ and the function $\lambda \in L([0,c],R_+)$ satisfy the conditions

$$\begin{aligned} d_0 &\geq \chi b_0 + \int_0^c \tilde{\gamma}(\tau,\mu_0)d\tau \\ d_1 &\geq \left(\chi b_1 + A\int_0^c \beta(\tau,\mu_0)d\tau\right) L(c) \end{aligned}$$

and

$$\lambda(t) \geq \alpha(t,\mu_0)\left(\chi b_1 + A\int_0^c \beta(\tau,\mu_0)d\tau\right)L(c) + \tilde{\gamma}(t,\mu_0) \quad \text{on} \quad t \in [0,c].$$

Suppose that $z \in C_{\varphi.c}[d,\lambda]$. Then we have

$$\|F[z](t,x)\| \leq \chi b_0 + \int_0^c \tilde{\gamma}(\tau,\mu_0)d\tau \leq d_0 \tag{14}$$

It follows from Assumption $H[f]$ and Lemma 2.1 that for

$$\begin{aligned}
&\|F[z](t,x) - F[z](\bar{t},\bar{x})\| \\
&\quad\leq \|\varphi^*(0,g[z](0,t,x)) - \varphi^*(0,g[z](0,\bar{t},\bar{x}))\| \\
&\qquad + \int_0^t \|f^*(\tau,g[z](\tau,t,x),z_{\psi(\tau,g[z](\tau,t,x))}) \\
&\qquad\qquad - f^*(\tau,g[z](\tau,\bar{t},\bar{x}),z_{\psi(\tau,g[z](\tau,\bar{t},\bar{x}))})\|d\tau \\
&\qquad + \Big|\int_t^{\bar{t}} \tilde{\gamma}(\tau,\mu_0)d\tau\Big| \\
&\quad\leq \chi b_1 \max_{1\leq i\leq k} \|g_i[z](0,t,x) - g_i[z](0,\bar{t},\bar{x})\| \\
&\qquad + A\int_0^t \beta(\tau,\mu_0)\max_{1\leq i\leq k}\|g_i[z](\tau,t,x) - g_i[z](\tau,\bar{t},\bar{x})\|d\tau \\
&\qquad + \Big|\int_t^{\bar{t}} \tilde{\gamma}(\tau,\mu_0)d\tau\Big| \\
&\quad\leq \Big(\chi b_1 + A\int_0^c \beta(\tau,\mu_0)d\tau\Big)L(c)\Big[\,\Big|\int_t^{\bar{t}} \alpha(\tau,\mu_0)d\tau\Big| + \|x-\bar{x}\|\Big] \\
&\qquad + \Big|\int_t^{\bar{t}} \tilde{\gamma}(\tau,\mu_0)d\tau\Big|,
\end{aligned}$$

that is

$$\|F[z](t,x) - F[z](\bar{t},\bar{x})\| \leq \left|\int_t^{\bar{t}} \lambda(\tau)d\tau\right| + d_1\|x-\bar{x}\| \tag{15}$$

on $[0,c]\times R^n$. It follows from (14) and (15) that $F[z] \in C_{\varphi.c}[d,\lambda]$.

`Step II.` We shall prove that F is a contraction on $C_{\varphi.c}[d,\lambda]$. For $z, \bar{z} \in$

$C_{\varphi.c}[d,\lambda]$ we have

$$\begin{aligned}\|F[z](t,x)-F[\bar z](t,x)\| \\ &\le \|\varphi^*(0,g[z](0,t,x))-\varphi^*(0,g[\bar z](0,t,x))\| \\ &\quad+\int_0^t \|f^*(\tau,g[z](\tau,t,x),z_{\psi(\tau,g[z](\tau,t,x))}) \\ &\qquad -f^*(\tau,g[\bar z](\tau,t,x),z_{\psi(\tau,g[\bar z](\tau,t,x))})\|d\tau\ .\end{aligned}$$

It follows from (12) and Lemma 3.1 that

$$\begin{aligned}&\|F[z](t,x)-F[\bar z](t,x)\| \\ &\le \chi b_1 \max_{1\le i\le k}\|g_i[z](0,t,x)-g_i[\bar z](0,t,x)\| \\ &\quad+\int_0^t \beta(\tau,\mu_0)\Big[A\max_{1\le i\le k}\|g_i[z](\tau,t,x)-g_i[z](\tau,t,\bar x)\|+K\|z-\bar z\|_\tau\Big]d\tau \\ &\le \chi b_1 KL(c)\int_0^t \gamma(\xi,\mu_0)\|z-\bar z\|_\xi d\xi \\ &\quad+K\int_0^t \beta(\tau,\mu_0)\Big[\|z-\bar z\|_\tau+AL(c)\int_\tau^t \gamma(\xi,\mu_0)\|z-\bar z\|_\xi d\xi\Big]d\tau \\ &\le \int_0^t \|z-\bar z\|_\xi \Psi(\xi)d\xi,\end{aligned}$$

where

$$\Psi(\xi)=KL(c)\Big[\chi b_1+A\int_0^\xi \beta(s,\mu_0)ds\Big]\gamma(\xi,\mu_0)+K\beta(\xi,\mu_0).$$

For functions $z,\bar z\in C_{\varphi.c}[d,\lambda]$ we write

$$[|z-\bar z|]=\sup\Big\{\|z-\bar z\|_t\exp\Big[-2\int_0^t\Psi(\xi)d\xi\Big]: t\in[0,c]\Big\}.$$

We have

$$\begin{aligned}&\|F[z](t,x)-F[\bar z](t,x)\| \\ &\le \int_0^t \|z-\bar z\|_\xi \exp\Big[-2\int_0^\xi \Psi(s)ds\Big]\exp\Big[2\int_0^\xi\Psi(s)ds\Big]\Psi(\xi)d\xi \\ &\le [|z-\bar z|]\int_0^t \exp\Big[2\int_0^\xi \Psi(s)ds\Big]\Psi(\xi)d\xi \\ &= \frac{1}{2}[|z-\bar z|]\Big(\exp\Big[2\int_0^t\Psi(\xi)d\xi\Big]-1\Big) \\ &\le \frac{1}{2}[|z-\bar z|]\exp\Big[2\int_0^t\Psi(\xi)d\xi\Big]\end{aligned}$$

for $t \in [0,c]$. From the above inequality we get

$$\|F[z](t,x) - F[\bar{z}](t,x)\|_t \le \frac{1}{2}[|z-\bar{z}|]\exp\Big[2\int_0^t \Psi(\xi)d\xi\Big]$$

for $t \in [0,c]$, and consequently $[|F[z]-F[\bar{z}]|] \le \frac{1}{2}[|z-\bar{z}|]$. By the Banach fixed point theorem there exists a unique solution $u \in C_{\varphi.c}[d,\lambda]$ of the equation $z = F[z]$. Now we prove that u is a solution of equation (1). We have proved that

$$u_i(t,x) = \varphi_i(0, g_i[u](0,t,x)) + \int_0^t f_i(s, g_i[u](s,t,x), u_{\psi_i(s,g_i[u](s,t,x))})ds \qquad (16)$$

on $[0,c] \times R^n$ for $1 \le i \le k$. For given $x \in R^n$, $1 \le i \le k$, let us put $\eta^{(i)} = g_i[u](0,t,x)$. It follows that $g_i[u](\tau,t,x) = g_i[u](\tau,0,\eta^{(i)})$ for $\tau \in [0,c]$ and $x = g_i[u](t,0,\eta^{(i)})$. The relations

$$\eta^{(i)} = g_i[u](0,t,x) \quad \text{and} \quad x = g_i[u](t,0,\eta^{(i)})$$

are equivalent for $x, \eta^{(i)} \in R^n$. It follows from (16) that for $1 \le i \le k$

$$\begin{aligned} &u_i(t, g_i[u](t,0,\eta^{(i)})) \\ &\quad = \varphi_i(0,\eta^{(i)}) + \int_0^t f_i(s, g_i[u](s,0,\eta^{(i)}), u_{\psi_i(s,g_i[u](s,0,\eta^{(i)}))})ds\ . \end{aligned} \qquad (17)$$

By differentiating (17) with respect to t and by using the transformations - $\eta^{(i)} = g_i[u](0,t,x)$ which preserve the sets of measure zero, we obtain that u satisfies (1) for almost all $(t,x) \in [0,c] \times R^n$.

`Step III.` Now we prove relation (13). Let $\bar{F}$ be an operator defined as F but with function $\bar{\varphi}$ instead of φ. If $u = F[u]$ and $\bar{u} = \bar{F}[\bar{u}]$, $u = (u_1,\dots,u_k)$, $\bar{u} = (\bar{u}_1,\dots,\bar{u}_k)$, then it holds for each $i \in \{1,\dots,k\}$

$$\begin{aligned} &|u_i(t,x) - \bar{u}_i(t,x)| \\ &\quad \le |\varphi_i(0,g_i[u](0,t,x)) - \bar{\varphi}_i(0,g_i[\bar{u}](0,t,x))| \\ &\qquad + \int_0^t |f_i\big(s, g_i[u](s,t,x), u_{\psi_i(s,g_i[u](s,t,x)))}\big) \\ &\qquad\quad - f_i\big(s, g_i[\bar{u}](s,t,x), \bar{u}_{\psi_i(s,g[\bar{u}](s,t,x)))}\big)|ds \\ &\quad \le \|\varphi - \bar{\varphi}\|_{X,R^n} + \chi b_1 \max_{1\le i\le k}\|g_i[u](0,t,x) - g_i[\bar{u}](0,t,x)\| \\ &\qquad + \int_0^t \beta(s,\mu_0)\Big[\max_{1\le i\le k}\|g_i[u](s,t,x) - g_i[\bar{u}](s,t,x)\| \\ &\qquad + \max_{1\le i\le k}\|u_{\psi_i(s,g_i[u](s,t,x))} - \bar{u}_{\psi_i(s,g_i[\bar{u}](s,t,x))}\|_X\Big]ds. \end{aligned}$$

It follows from Lemma 3.1 that

$$\|g_i[u](\tau,t,x)-g_i[\bar u](\tau,t,x)\| \\ \le L(0,c)\int_0^t \gamma(s,\mu_0)\,[K\|u-\bar u\|_s+K_0\|\varphi-\bar\varphi\|_{X,R^n}]\,ds$$

for $0\le\tau\le t$, and

$$\begin{aligned}&\|u_{\psi_i(s,g_i[u](s,t,x))}-\bar u_{\psi_i(s,g_i[\bar u](s,t,x))}\|_X\\ &\le s_0(Kd_1+K_0b_1)\|g_i[u](s,t,x)-g_i[\bar u](s,t,x)\|\\ &\quad +K\|u-\bar u\|_s+K_0\|\varphi-\bar\varphi\|_{X,R^n}\\ &\le s_0(Kd_1+K_0b_1)L(c)\int_0^t\gamma(\xi,\mu_0)\,[K\|u-\bar u\|_\xi+K_0\|\varphi-\bar\varphi\|_{X,R^n}]\,d\xi\\ &\quad +K\|u-\bar u\|_s+K_0\|\varphi-\bar\varphi\|_{X,R^n}.\end{aligned}$$

Then we have

$$\begin{aligned}&\|u(t,x)-\bar u(t,x)\|\\ &\le \|\varphi-\bar\varphi\|_{X,R^n}+\chi b_1L(c)\int_0^t\gamma(s,\mu_0)\,[K\|u-\bar u\|_s+K_0\|\varphi-\bar\varphi\|_{X,R^n}]\,ds\\ &\quad+\int_0^t\beta(s,\mu_0)\Big[AL(c)\int_0^t\gamma(\xi,\mu_0)\,[K\|u-\bar u\|_\xi+K_0\|\varphi-\bar\varphi\|_{X,R^n}]\,d\xi\\ &\qquad +K\|u-\bar u\|_s+K_0\|\varphi-\bar\varphi\|_{X,R^n}\Big]ds\\ &\le D_c\|\varphi-\bar\varphi\|_{X,R^n}+\int_0^t\Gamma(s)\|u-\bar u\|_s ds\end{aligned}$$

with

$$D_c=1+K_0\int_0^c\Psi_0(s)ds,\quad \Gamma(s)=K\Psi_0(s),$$

where

$$\Psi_0(s)=L(c)\Big[\chi b_1+A\int_0^c\beta(\xi,\mu_0)d\xi\Big]\gamma(s,\mu_0)+\beta(s,\mu_0).$$

Using the Gronwall inequality we obtain $\|u-\bar u\|_t\le\Lambda(t)\|\varphi-\bar\varphi\|_{X,R^n}$, $t\in[0,c]$, with

$$\Lambda(t)=D_c\exp\Big[\int_0^t\Gamma(s)ds\Big].$$

Then we have shown the estimate (13) with the above given Λ. This proves the theorem. ∎

Remark 1. Suppose that $B = [-r_0, 0] \times [-r, r]$ where $r_0 \in R_+$ and

$$\tilde{\rho} : [0, a] \times R^n \times R^k \to M_{k \times n}, \quad \tilde{\rho} = [\tilde{\rho}_{ij}]_{i=1,\dots,k,\ j=1,\dots,n}$$
$$\tilde{f} : [0, a] \times R^n \times R^k \to R^k, \qquad \tilde{f} = (\tilde{f}_1, \dots, \tilde{f}_k)$$

are given functions. Write

$$D_i[t, x] = \{(s, y) \in R^{1+n} : \psi_{i.0}(t) - r_0 \le s \le \psi_{i.0}(t), \psi_i'(t, x) - r \le y \le \psi_i'(t, x) + r\}$$

for $1 \le i \le k$, and

$$\rho(t, x, w) = \tilde{\rho}\Big(t, x, \int_B w(s, y) ds dy\Big)$$
$$f(t, x, w) = \tilde{f}\Big(t, x, \int_B w(s, y) ds dy\Big)$$

where $\int_B w(s, y) ds dy = \big(\int_B w_1(s, y) ds dy, \dots, \int_B w_k(s, y) ds dy\big)$. Then the Cauchy problem (1), (2) is equivalent to the system of differential integral equations

$$\partial_t z_i(t, x) + \sum_{j=1}^{n} \tilde{\rho}_{ij}\Big(t, x, \int_{D_i[t,x]} z(s, y) ds dy\Big) \partial_{x_j} z_i(t, x) = \tilde{f}_i\Big(t, x, \int_{D_i[t,x]} z(s, y) ds dy\Big) \tag{18}$$

for $1 \le i \le k$, with the initial condition

$$z(t, x) = \varphi(t, x) \tag{19}$$

for $(t, x) \in [-r_0, 0] \times [-r, r]$. It is easy to formulate existence result for problem (18), (19) which is based on Theorem 3.1. Note that the results presented in [6, Chapter 4] concern the case when the sets $D_i[t, x]$ do not depend on (t, x) and i, and therefore they are not applicable to (18), (19).

Remark 2. For the above given $\tilde{\varrho}$ and $\tilde{f}$ we put

$$\varrho(t, x, w) = \tilde{\varrho}(t, x, w(0, 0)), \quad f(t, x, w) = \tilde{f}(t, x, w(0, 0))).$$

Then, the Cauchy problem (1), (2) is equivalent to the system of differential equations with deviated variables

$$\partial_t z_i(t, x) + \sum_{j=1}^{n} \tilde{\varrho}_{ij}(t, x, z(\psi_i(t, x))) \partial_{x_j} z_i(t, x) = \tilde{f}_i(t, x, z(\psi_i(t, x))) \tag{20}$$

for $1 \le i \le k$, with the initial condition (19). It is easy to formulate the existence result for (20), (19) which is based on Theorem 3.1.

4. Mixed problems

In this part of the paper we shall consider initial boundary value problems for quasilinear partial functional differential systems with unbounded delay. Let use the symbols B and X to denote the spaces defined in Section 1. Let $a > 0$ and $\bar{b} = (\bar{b}_1, \dots, \bar{b}_n)$ with $\bar{b}_i > 0$ for $1 \le i \le n$ be fixed. We define the sets

$$\begin{aligned} E &= [0,a] \times (-\bar{b}, \bar{b}) \\ E_0 &= (-\infty, 0] \times [-\bar{b} - r, \bar{b} + r] \\ \partial_0 E &= ([0,a] \times [-\bar{b} - r, \bar{b} + r]) \backslash E \\ E^* &= E_0 \cup E \cup \partial_0 E \end{aligned}$$

and

$$\begin{aligned} E[c] &= E \cap ([0,c] \times R^n) \\ \partial_0 E[c] &= \partial_0 E \cap ([0,c] \times R^n) \\ D[c] &= E[c] \cup \partial_0 E[c] \end{aligned}$$

where $0 < c \le a$. Write $\Omega = \bar{E} \times X$ where $\bar{E}$ is the closure of E, and suppose that the functions

$$\begin{aligned} &\rho : \Omega \to M_{k \times n}, \rho = [\rho_{ij}]_{i=1,\dots,k,\ j=1,\dots,n}\,, \\ &f : \Omega \to R^k, f = (f_1, \dots, f_k) \\ &\varphi : E_0 \cup \partial_0 E \to R^k, \varphi = (\varphi_1, \dots, \varphi_k) \end{aligned}$$

and

$$\begin{aligned} &\psi = (\psi_1, \dots, \psi_k), \quad \psi_i = (\psi_{i.0}, \psi_i'), \quad \psi_{i.0} : [0,a] \to R \\ &\psi_i' : \bar{E} \to [-\bar{b}, \bar{b}], \quad \psi_i' = (\psi_{i.1}, \dots, \psi_{i.\kappa}) \end{aligned}$$

are given. The requirements on $\psi_{i.0}$ are such that $\psi_{i.0}(t) \le t$ for $t \in [0,a]$ and $1 \le i \le k$. We consider the quasilinear functional differential system

$$\partial_t z_i(t,x) + \sum_{j=1}^{n} \rho_{ij}(t,x,z_{\psi_i(t,x)}) \partial_{x_j} z_i(t,x) = f_i(t,x,z_{\psi_i(t,x)}), \tag{21}$$

for $1 \le i \le k$, with the initial boundary condition

$$z(t,x) = \varphi(t,x) \tag{22}$$

for $(t,x) \in E_0 \cup \partial_0 E$. We consider weak solutions of problem (21), (22). A function $\bar{z} = (\bar{z}_1, \dots, \bar{z}_k) : E^* \cap ((-\infty, c] \times R^n) \to R^k$, $0 < c \le a$, is a solution to the above problem if the following conditions are satisfied:

(i) $\bar{z}_{\psi_i(t,x)} \in X$ for $(t,x) \in E[c]$, $1 \le i \le k$

(ii) the derivatives $\partial_t \bar{z}_i$ and $\partial_x \bar{z}_i = (\partial_{x_1}\bar{z}_i, \dots, \partial_{x_n}\bar{z}_i)$, $1 \le i \le k$, exist almost everywhere on $E[c]$,

(iii) $\bar{z}$ satisfies system (21) almost everywhere on $E[c]$, and $\bar{z}(t,x) = \varphi(t,x)$ on $(E_0 \cup \partial_0 E) \cap ((-\infty, c] \times R^n)$.

Remark 3. Note that existence results presented in [6], [4], [12] are not applicable to (21), (22) also in the case when B is a bounded set.

Let $\tilde{\Delta}$ be the class of all functions $\gamma \in C(R_+, R_+)$ which are non-decreasing on R_+, and $\gamma(0) = 0$. If $z : E_0 \cup D[c] \to R^k$, $0 < c \le a$, is a function such that $z|_{D[c]}$ is continuous and $(t,x) \in [0,c] \times [-\bar{b}, \bar{b}]$, the we put

$$\|z\|_{[0,t;x]} = \max\{\|z(s,y)\| : (s,y) \in [0,t] \times [x-r, x+r]\}.$$

Suppose additionally that the function $z|_{D[c]}$ satisfies the Lipschitz condition with respect to x. Then we write

$$\mathrm{Lip}[z]|_{D[t]} = \sup\left\{ \frac{\|z(s,y) - z(s,\bar{y})\|}{\|y - \bar{y}\|} : (s,y), (s,\bar{y}) \in D[t] \right\}$$

for $t \in [0,c]$.

Lemma 4.1. *Suppose that Assumption $H[X]$ is satisfied and $z : E_0 \cup D[c] \to R^k$, $0 < c \le a$. If $z_{(0,x)} \in X$ for $x \in [-\bar{b}, \bar{b}]$ and $z|_{D[c]}$ is continuous, then $z_{(t,x)} \in X$ for $(t,x) \in [0,c] \times [-\bar{b}, \bar{b}]$ and*

$$\|z_{(t,x)}\|_X \le K \|z\|_{[0,t;x]} + K_0 \|z_{(0,x)}\|_X. \tag{23}$$

If we assume additionally that the function $z|_{D[c]}$ satisfies the Lipschitz condition with respect to x, then

$$\|z_{(t,x)} - z_{(t,\bar{x})}\|_X \le K\, \mathrm{Lip}[z]|_{D[t]} \|x - \bar{x}\| + K_0 \|z_{(0,x)} - z_{(0,\bar{x})}\|_X, \tag{24}$$

where $(t,x), (t,\bar{x}) \in [0,c] \times [-\bar{b}, \bar{b}]$.

The proof of the above lemma is similar to the proof of Lemma 2.1. We omit the details.

Our basic assumption on initial boundary functions is the following.

Assumption $\tilde{H}[\varphi]$. Suppose that for $\varphi : E_0 \cup \partial_0 E \to R^k$ it holds:

1) $\varphi_{(0,x)} \in X$ for $x \in [-\bar{b}, \bar{b}]$, and there are constants $b_0, b_1 \in R_+$ such that

$$\|\varphi_{(0,x)}\|_X \le b_0, \quad \|\varphi_{(0,x)} - \varphi_{(0,\bar{x})}\|_X \le b_1 \|x - \bar{x}\|$$

where $x, \bar{x} \in [-\bar{b}, \bar{b}]$.

2) $\|\varphi(t,x)\| \le q_0$ on $\partial_0 E$, and there are constants $q_1, q_2 \in R_+$ such that

$$\|\varphi(t,x) - \varphi(\bar{t},\bar{x})\| \le q_1|t-\bar{t}| + q_2\|x-\bar{x}\| \quad \text{on} \quad \partial_0 E.$$

Let us denote by $\tilde{I}[X]$ the class of all initial boundary functions $\varphi : E_0 \cup \partial_0 E \to R^k$ satisfying Assumption $\tilde{H}[\varphi]$. Let $\varphi \in \tilde{I}[X]$ and let $0 < c \le a$, $d = (d_0, d_1, d_2) \in R^3_+$, $d_i \ge q_i$ for $i = 0, 1, 2$. We will denote by $C_{\varphi.c}[d]$ the class of all functions $z : E_0 \cup D[c] \to R^k$ such that

$z(t,x) = \varphi(t,x)$ for $(t,x) \in E_0 \cup \partial_0 E[c]$ and the estimates

$$\|z(t,x)\| \le d_0, \quad \|z(t,x) - z(\bar{t},\bar{x})\| \le d_1|t-\bar{t}| + d_2\|x-\bar{x}\|$$

hold on $D[c]$. We will prove that under suitable assumptions on f, ρ and ψ and for sufficiently small c, $0 < c \le a$, there exists a solution $\bar{z}$ to problem (21), (22) such that $\bar{z} \in C_{\varphi.c}[d]$. We will need the following assumptions on ρ and ψ.

Assumption $H_0[\rho]$. Suppose that

1) the function $\rho(\cdot, x, w) : [0,a] \to M_{k\times n}$ is measurable for every $(x,w) \in [-\bar{b},\bar{b}] \times X$ and $\rho(t,\cdot) : [-\bar{b},\bar{b}] \times X \to M_{k\times n}$ is continuous for almost all $t \in [0,a]$
2) there exist $\alpha, \gamma \in \tilde{\Delta}$ such that $\|\rho(t,x,w)\| \le \alpha(\mu)$ and

$$\|\rho(t,x,w) - \rho(t,\bar{x},\bar{w})\| \le \gamma(\mu)[\|x-\bar{x}\| + \|w-\bar{w}\|_X]$$

for $(x,w), (\bar{x},\bar{w}) \in [-\bar{b},\bar{b}] \times X[\mu]$ and for almost all $t \in [0,a]$.

Assumption $\tilde{H}[\psi]$. Suppose that the functions $\psi_i = (\psi_{i.0}, \psi'_i)$, $1 \le i \le k$, satisfy the conditions:

1) $\psi_{i.0} \in C([0,a], R_+)$, $\psi'_i \in C(\bar{E}, [-\bar{b},\bar{b}])$ and $\psi_{i.0}(t) \le t$ for $t \in (0,a]$, $1 \le i \le k$,
2) there is $s_0 \in R_+$ such that

$$\|\psi'_i(t,x) - \psi'_i(t,\bar{x})\| \le s_0\|x-\bar{x}\| \quad \text{on} \quad E \quad \text{for} \quad 1 \le i \le k.$$

Suppose that Assumptions $H[X]$, $H_0[\rho]$, $\tilde{H}[\psi]$ are satisfied and $\varphi \in \tilde{I}[X]$, $z \in C_{\varphi.c}[d]$. Consider the Cauchy problem

$$\eta'(\tau) = \rho_i(\tau, \eta(\tau), z_{\psi_i(\tau,\eta(\tau))}), \quad \eta(t) = x \tag{25}$$

where $(t,x) \in [0,c] \times [-\bar{b},\bar{b}]$ and $1 \le i \le k$ is fixed, while $\rho_i = (\rho_{i1}, \ldots, \rho_{in})$. Let us denote by $g_i[z](\cdot,t,x) = (g_{i1}[z](\cdot,t,x), \ldots, g_{in}[z](\cdot,t,x))$ the solution to (25). The function $g_i[z](\cdot,t,x)$ is the i-th bicharacteristic of system (21) corresponding to z. Let $\delta_i[z](t,x)$ be the left end of the maximal interval on which the solution $g_i[z](\cdot,t,x)$ is defined. Write

$$\begin{aligned}
\Gamma_{j.+} &= \{(t,x)\in\bar{E} : x_j=\bar{b}_j\}\\
\Gamma_{j.-} &= \{(t,x)\in\bar{E} : x_j=-\bar{b}_j\}\\
\Gamma_0 &= \{0\}\times[-\bar{b},\bar{b}]\\
\Gamma &= \Gamma_0\cup\bigcup_{j=1}^{n}\big(\Gamma_{j.+}\cup\Gamma_{j.-}\big).
\end{aligned}$$

For functions $\varphi\in\tilde{I}[X]$ and $z\in C_{\varphi.c}[d]$, we write

$$\|\varphi\|_{X,\bar{b}}=\sup\{\|\varphi_{(0,x)}\|_X : x\in[-\bar{b},\bar{b}]\},$$

and

$$\|z\|_t=\sup\{\|z(s,y)\| : (s,y)\in D[t]\},\quad 0\le t\le c.$$

Lemma 4.2. *Suppose that Assumptions $H[X]$, $H[\rho]$, $\tilde{H}[\psi]$ are satisfied and $\varphi,\bar{\varphi}\in\tilde{I}[X]$, $z\in C_{\varphi.c}[d]$, $\bar{z}\in C_{\bar{\varphi}.c}[d]$, where $0<c\le a$. Then, for each $1\le i\le k$, the solutions $g_i[z](\cdot,t,x)$ and $g_i[\bar{z}](\cdot,t,x)$ exist on the intervals $I^{(i)}_{(t,x)}$ and $\bar{I}^{(i)}_{(t,x)}$ such that for $\zeta=\delta_i[z](t,x)$ and $\bar{\zeta}=\delta_i[\bar{\zeta}](t,x)$ we have $(\zeta,g_i[z](\zeta,t,x))\in\Gamma$ and $(\bar{\zeta},g_i[\bar{z}](\bar{\zeta},t,x))\in\Gamma$. Solutions of* (25) *are unique, and the following estimates hold:*

$$\|g_i[z](\tau,t,x)-g_i[z](\tau,\bar{t},\bar{x})\|\le\Theta(\tau,\max\{t,\bar{t}\})\,[|t-\bar{t}|+\|x-\bar{x}\|] \tag{26}$$

for $\tau\in I^{(i)}_{(t,x)}\cap I^{(i)}_{(\bar{t},\bar{x})}$, $(t,x),(\bar{t},\bar{x})\in E[c]$, where

$$\begin{aligned}
&\Theta(\tau,t)=\max\{1,\alpha(\mu_0)\}\exp\left[A\gamma(\mu_0)|\tau-t|\right]\\
&\mu_0=Kd_0+K_0b_0,\quad A=1+s_0(Kd_1+K_0b_1)
\end{aligned} \tag{27}$$

and

$$\|g_i[z](\tau,t,x)-g_i[\bar{z}](\tau,t,x)\|\le\tilde{\Theta}(\tau,t)\left|\int_t^\tau\Big[K\|z-\bar{z}\|_\xi+K_0\|\varphi-\bar{\varphi}\|_{X,\bar{b}}\Big]d\xi\right| \tag{28}$$

for $\tau\in I^{(i)}_{(t,x)}\cap\bar{I}^{(i)}_{(t,x)}$, $(t,x)\in E[c]$, where

$$\tilde{\Theta}(\tau,t)=\gamma(\mu_0)\exp\Big[A\gamma(\mu_0)|t-\tau|\Big]. \tag{29}$$

Proof. It follows that $\|z_{(t,x)}\|_X \le \mu_0$ for $(t,x) \in E[c]$ and

$$\|\rho_i(\tau, y, z_{\psi_i(\tau,y)}) - \rho_i(\tau, \bar{y}, z_{\psi_i(\tau,\bar{y})})\| \le \gamma(\mu_0) A \|y - \bar{y}\|,$$

where $(\tau, y), (\tau, \bar{y}) \in E[c]$. It follows from classical theorems that there exists exactly one Carathéodory solution to problem (25) and the solution is defined on some interval $I^{(i)}_{(t,x)}$ satisfying the condition of the lemma. An easy computation shows that the integral inequalities

$$\begin{aligned} &\|g_i[z](\tau,t,x) - g_i[z](\tau,\bar{t},\bar{x})\| \\ &\qquad \le \|x - \bar{x}\| + \alpha(\mu_0)|t - \bar{t}| \\ &\qquad\quad + A\gamma(\mu_0)\left| \int_t^\tau \|g_i[z](\xi,t,x) - g_i[z](\xi,\bar{t},\bar{x})\| d\xi \right| \end{aligned}$$

for $\tau \in I^{(i)}_{(t,x)} \cap I^{(i)}_{(\bar{t},\bar{x})}$, and

$$\begin{aligned} &\|g_i[z](\tau,t,x) - g_i[\bar{z}](\tau,t,x)\| \\ &\qquad \le \gamma(\mu_0)\left| \int_t^\tau \Big[K\|z - \bar{z}\|_\xi + K_0\|\varphi - \bar{\varphi}\|_{X,\bar{b}} \Big] d\xi \right| \\ &\qquad\quad + A\gamma(\mu_0)\left| \int_t^\tau \|g_i[z](\xi,t,x) - g_i[\bar{z}](\xi,t,x)\| d\xi \right| \end{aligned}$$

for $\tau \in I^{(i)}_{(t,x)} \cap \bar{I}^{(i)}_{(t,x)}$ are satisfied. Now we obtain (26) and (28) from the Gronwall inequality. ∎

Assumption $\tilde{H}[\rho]$. Suppose that Assumption $H_0[\rho]$ is satisfied, and there is a function $\beta : R_+ \to (0, +\infty)$ such that

$$\begin{aligned} \rho_i(t,x,w) &\le -\beta(\mu) \quad \text{for} \quad (t,x) \in \Gamma_{i.+},\ w \in X[\mu] \\ \rho_i(t,x,w) &\ge \ \ \beta(\mu) \quad \text{for} \quad (t,x) \in \Gamma_{i.-},\ w \in X[\mu] \end{aligned}$$

for $i = 1, \ldots, k$.

Now we prove a lemma on the regularity of the function $\delta_i[z]$, $1 \le i \le k$.

Lemma 4.3. *Suppose that Assumptions $H[X]$, $\tilde{H}[\rho]$, $\tilde{H}[\psi]$ are satisfied and $\varphi, \bar{\varphi} \in \tilde{I}[X]$, $z \in C_{\varphi.c}[d]$, $\bar{z} \in C_{\bar{\varphi}.c}[d]$, where $0 < c \le a$. Then, for each $1 \le i \le k$, the functions $\delta_i[z]$ and $\delta_i[\bar{z}]$ are continuous on $\bar{E}[c]$. Moreover, it holds the estimates*

$$|\delta_i[z](t,x) - \delta_i[z](\bar{t},\bar{x})| \le \frac{2\Theta(0,c)}{\beta(\mu_0)} \Big[\|x - \bar{x}\| + |t - \bar{t}| \Big] \tag{30}$$

$$|\delta_i[z](t,x) - \delta_i[\bar{z}](t,x)| \le \frac{2\tilde{\Theta}(0,c)}{\beta(\mu_0)} \int_0^t \big[K\|z - \bar{z}\|_\tau + K_0\|\varphi - \bar{\varphi}\|_{X,\bar{b}} \big] d\tau \tag{31}$$

on $\bar{E}[c]$, where Θ and $\tilde{\Theta}$ are given by (27) *and* (29), *respectively.*

Proof. Let us fix i, $1 \le i \le k$. The continuity of $\delta_i[z]$ and $\delta_i[\bar{z}]$ on $\bar{E}[c]$ follows from classical theorems on continuous dependence on inital conditions for Carathéodory solutions of differential systems.

Now we prove (30). This estimate is obvious in the case $\delta_i[z](t,x) = \delta_i[z](\bar{t},\bar{x}) = 0$ (i.e. in the case when solutions of problem (25) are defined on $[0,t]$ and $[0,\bar{t}]$, respectively). Suppose now that $0 \le \delta_i[z](t,x) < \delta_i[z](\bar{t},\bar{x})$. Then, for $\bar{\zeta} = \delta_i[z](\bar{t},\bar{x})$ we have

$$(\bar{\zeta}, g_i[z](\bar{\zeta},\bar{t},\bar{x})) \in \bigcup_{j=1}^{n} \Big(\Gamma_{j.+} \cup \Gamma_{j.-} \Big).$$

Consider the case when $(\bar{\zeta}, g_i[z](\bar{\zeta},\bar{t},\bar{x})) \in \Gamma_{j.+}$ for some j, $1 \le j \le n$. Then $g_{ij}[z](\bar{\zeta},\bar{t},\bar{x}) = b_j$. Let

$$\Gamma^{(\varepsilon)}_{j.+} = \{(t,x) \in \bar{E} : x_j \in [\bar{b}_j - \varepsilon, \bar{b}_j]\}.$$

It follows from assumption $\tilde{H}[\rho]$ that there is $\varepsilon > 0$ such that

$$\rho_{ij}(t,x,w) \le -\frac{1}{2}\beta(\mu_0) \quad \text{for} \quad (t,x) \in \Gamma^{(\varepsilon)}_{j.+},\ w \in X[\mu_0].$$

Since the functions $\delta_i[z]$ and $g_i[z]$ are continuous with respect to all variables, there exists $\tilde{\delta} > 0$ such that the following implication remains true:

$$|t-\bar{t}| + \|x-\bar{x}\| < \tilde{\delta} \quad \Rightarrow \quad \left\{ \begin{array}{c} (\tau, g_i[z](\tau,t,x)) \in \Gamma^{(\varepsilon)}_{j.+} \\ \text{for} \\ \tau \in \big[\delta_i[z](t,x), \delta_i[z](\bar{t},\bar{x})\big]. \end{array} \right\} \tag{32}$$

For $(t,x), (\bar{t},\bar{x})$ satisfying (32) we obtain

$$\begin{aligned} -\frac{1}{2}\beta(\mu_0)&[\delta_i[z](\bar{t},\bar{x}) - \delta_i[z](t,x)] \\ &\ge \int_{\delta_i[z](t,x)}^{\delta_i[z](\bar{t},\bar{x})} \rho_{ij}(\tau, g_i[z](\tau,t,x), z_{\psi_i(\tau, g_i[z](\tau,t,x))}) d\tau \\ &= g_{ij}[z](\delta_i[z](\bar{t},\bar{x}),t,x) - g_{ij}[z](\delta_i[z](t,x),t,x) \\ &\ge g_{ij}[z](\delta_i[z](\bar{t},\bar{x}),t,x) - g_{ij}[z](\delta_i[z](\bar{t},\bar{x}),\bar{t},\bar{x}) \\ &\ge -\Theta(0,c)\,[|t-\bar{t}| + \|x-\bar{x}\|] \end{aligned}$$

which is our claim. In the same way we prove (30) in the case when $(\bar{\zeta}, g_i[z](\bar{\zeta}, \bar{t},\bar{x})) \in \Gamma_{j.-}$ for some j, $1 \le j \le n$, and a condition analogous to (32) is satisfied. If $(t,x), (\bar{t},\bar{x})$ do not satisfy (32), then we consider intermediate points

$$(t_0, x^{(0)}), (t_1, x^{(1)}), \ldots, (t_p, x^{(p)})$$

with $t_0 = t$, $x^{(0)} = x$, $t_p = \bar{t}$, $x^{(p)} = \bar{x}$, satisfying the following conditions:

(i) we have

$$|t-\bar{t}| = |t_0-t_1|+|t_1-t_2|+\ldots+|t_{p-1}-t_p|$$
$$\|x-\bar{x}\| = \|x^{(0)}-x^{(1)}\|+\|x^{(1)}-x^{(2)}\|+\ldots+\|x^{(p-1)}-x^{(p)}\|$$

(ii) condition of the form (32) holds for each pair $(t_j, x^{(j)}), (t_{j+1}, x^{(j+1)})$, $j = 0, 1, \ldots, p-1$

(iii) the bicharacteristics $g_i[z](\cdot, t_j, x^{(j)})$ and $g_i[z](\,\cdot\,, t_{j+1}, x^{(j+1)})$ reach the same set, either $\Gamma_{j.+}$ or $\Gamma_{j.-}$, $0 \le j \le p-1$.

Now, an application of the previous reasoning to each pair of the above form concludes the proof of estimate (30).

Now we consider estimate (31). The inequality is obvious if $\delta_i[z](t,x) = \delta_i[\bar{z}](t,x) = 0$. Suppose now that $0 \le \delta_i[z](t,x) < \delta_i[\bar{z}](t,x)$. Then, for $\zeta = \delta_i[\bar{z}](t,x)$ we have

$$(\zeta, g_i[\bar{z}](\zeta,t,x)) \in \bigcup_{j=1}^{n}\Big(\Gamma_{j.+}\cup\Gamma_{j.-}\Big).$$

Consider that case when $(\zeta, g_i[\bar{z}](\zeta,t,x)) \in \Gamma_{j.+}$ for some j, $1 \le j \le n$. Then $g_{ij}[\bar{z}](\zeta,t,x) = \bar{b}_j$. Again, consider the set $\Gamma^{(\varepsilon)}_{j.+}$. It follows that there exists a $\tilde{\delta} > 0$ such that the following implication holds:

$$\|z-\bar{z}\|_t < \tilde{\delta} \quad\Rightarrow\quad (\tau, g_i[z](\tau,t,x)) \in \Gamma^{(\varepsilon)}_{j.+}, \tag{33}$$

where $\tau \in (\delta_i[z](t,x), \delta_i[\bar{z}](t,x))$. For $z, \bar{z}$ satisfying (33) we obtain

$$\begin{aligned}
-\frac{1}{2}\beta(\mu_0)&[\delta_i[\bar{z}](t,x)-\delta_i[z](t,x)]\\
&\ge \int_{\delta_i[z](t,x)}^{\delta_i[\bar{z}](t,x)} \rho_{ij}(\tau, g_i[z](\tau,t,x), z_{\psi_i(\tau, g_i[z](\tau,t,x))})d\tau\\
&\ge g_{ij}[z](\delta_i[\bar{z}](t,x),t,x) - g_{ij}[\bar{z}](\delta_i[\bar{z}](t,x),t,x)\\
&\ge -\tilde{\Theta}(0,c)\int_0^t \big[K\|z-\bar{z}\|_\tau + K_0\|\varphi-\bar{\varphi}\|_{X,\bar{b}}\big]\,d\tau,
\end{aligned}$$

which proves (31). In a similar way we prove (31) in the case when $(\zeta, g_i[\bar{z}](\zeta, t,x)) \in \Gamma_{j.-}$ for some j, $1 \le j \le n$, and condition analogous to (33) is satisfied. If $z, \bar{z}$ do not satisfy (33), then we consider intermediate functions $z_0, z_1, \ldots, z_p$ with $z_0 = z$, $z_p = \bar{z}$, $z_j \in C_{\varphi_j.c}[d]$, where $\varphi_j \in \tilde{I}[X]$ for $j = 0, 1, \ldots, p$, and $\varphi_0 = \varphi$, $\varphi_p = \bar{\varphi}$, satisfying

i) a condition of the form (33) holds for each pair z_j, z_{j+1}, and

$$\|z-\bar{z}\|_t = \|z_0-z_1\|_t + \|z_1-z_2\|_t + \ldots + \|z_{p-1}-z_p\|_t,$$

(ii) the bicharacteristics $g_j[z](\cdot,t,x)$ and $g_j[\bar{z}(\cdot,t,x)$ reach the same set, either $\Gamma_{j.+}$ or $\Gamma_{j.-}$, $0 \le j \le p-1$.

Now, the application of the previous reasoning to each pair of the above form gives the estimate (31). This completes the proof of the Lemma. ∎

Suppose that $\varphi \in \tilde{I}[X]$, $c \in (0,a]$, $z \in C_{\varphi.c}[d]$ and $g[z](\cdot,t,x) = (g_1[z](\cdot,t,x), \dots, g_k[z](\cdot,t,x))$ is the family of bicharacteristics for system (21) corresponding to z. Let $I^{(k)}_{(t,x)}$, be the domain of the function $g_i[z](\cdot,t,x)$, $1 \le i \le k$. Let us define operator $F = (F_1,\dots,F_k)$ as follows:

$$F_i[z](t,x) = \varphi_i(\delta_i[z](t,x), g_i[z](\delta_i[z](t,x),t,x)) + \int_{\delta_i[z](t,x)}^{t} f_i(\tau, g_i[z](\tau,t,x), z_{\psi_i(\tau,g_i[z](\tau,t,x))})d\tau$$

for $(t,x) \in E[c]$, and

$$F_i[z](t,x) = \varphi_i(t,x) \quad \text{on} \quad D[c],$$

where $\delta_i[z](t,x)$ is the left end of the interval $I^{(i)}_{(t,x)}$.

Assumption $\tilde{H}[f]$. Suppose that

1) the function $f(\cdot,x,w) : [0,a] \to R^k$ is measurable for $(x,w) \in [-\bar{b},\bar{b}] \times X$ and $f(t,\cdot) : [-\bar{b},\bar{b}] \times X \to R^k$ is continuous for almost all $t \in [0,a]$,
2) there exist $\tilde{\alpha}, \tilde{\gamma} \in \tilde{\Delta}$ such that $\|f(t,x,w)\| \le \tilde{\alpha}(\mu)$ and

$$\|f(t,x,w) - f(t,\bar{x},\bar{w})\| \le \gamma(\mu)[\|x-\bar{x}\| + \|w-\bar{w}\|_X]$$

for $(x,w), (\bar{x},\bar{w}) \in [-\bar{b},\bar{b}] \times X[\mu]$ and for almost all $t \in [0,a]$.

We are able now to state the main result on the existence of solutions of mixed problem (21), (22).

Theorem 4.1. *Suppose that $\varphi \in \tilde{I}[X]$ and Assumptions $H[\rho]$, $\tilde{H}[\rho]$, $\tilde{H}[f]$, $\tilde{H}[\Psi]$ are satisfied. Then there are $(d_0,d_1,d_2) = d \in R^3_+$ and $c \in (0,a]$ such that there exists exactly one solution $u \in C_{\varphi.c}[d]$ to problem* (21), (22)*. If $\bar{\varphi} \in \tilde{I}[X]$ and $\bar{u} \in C_{\bar{\varphi}.c}[d]$ is a solution to system* (21) *with the initial boundary condition $z(t,x) = \bar{\varphi}(t,x)$ on $E_0 \cup \partial_0 E$, then there is a function $\Lambda \in C([0,c],R_+)$ such that*

$$\|u-\bar{u}\|_t \le \Lambda(t)\Big[\|\varphi-\bar{\varphi}\|_{*.t} + \|\varphi-\bar{\varphi}\|_{X,\bar{b}}\Big], \quad t \in [0,c], \tag{34}$$

where

$$\|\varphi-\bar{\varphi}\|_{*.t} = \max\{\|(\varphi-\bar{\varphi})(s,y)\| : (s,y) \in \partial_0 E[t]\}$$

holds.

Proof. We have divided the proof into three steps.

`Step I.` We first show that there are $d \in R^3_+$ and $c \in (0,a]$ such that $F : C_{\varphi.c}[d] \to C_{\varphi.c}[d]$. Suppose that the constants $d \in R^3_+$ and $c \in (0,a]$ satisfy the conditions:

$$\begin{aligned} d_0 &\geq \max\{q_0, \chi b_0\} + c\tilde{\alpha}(\mu_0) \\ d_1 &\geq \widehat{\Theta}(c) \\ d_2 &\geq \widehat{\Theta}(c) + \tilde{\alpha}(\mu_0) \end{aligned}$$

where

$$\begin{aligned} \widehat{\Theta}(t) &= \Theta(0,t)(\widehat{A} + \widehat{B}) \\ \widehat{A} &= \frac{2q_1}{\beta\mu_0} + \max\{\chi b_1, q_2\}\left[1 + \frac{2\alpha(\mu_0)}{\beta(\mu_0)}\right] \\ \widehat{B} &= Ac\tilde{\gamma}(\mu_0) + \frac{2\tilde{\alpha}(\mu_0)}{\beta(\mu_0)}. \end{aligned}$$

Suppose that $z \in C_{\varphi.c}[d]$. Then $\|F[z](t,x)\| \leq d_0$ for $(t,x) \in [0,c] \times [-\bar{b}, \bar{b}]$. Our next goal is to evaluate the number $\|F[z](t,x) - R[z](\bar{t},\bar{x})\|$ for $(t,x), (\bar{t},\bar{x}) \in [0,c] \times [-\bar{b}, \bar{b}]$. Suppose that $1 \leq i \leq k$ is fixed, and that $\delta_i[z](\bar{t},\bar{x}) \leq \delta_i[z](t,x)$. It follows from Assumption $\tilde{H}[f]$ and Lemma 4.2 and 4.3 that

$$\begin{aligned} &|F_i[z](t,x) - F_i[z](\bar{t},\bar{x})| \\ &\quad\leq \Big|\varphi_i(\delta_i[z](t,x), g_i[z](\delta_i[z](t,x),t,x)) \\ &\qquad\qquad -\varphi_i(\delta_i[z](\bar{t},\bar{x}), g_i[z](\delta_i[z](\bar{t},\bar{x}),\bar{t},\bar{x}))\Big| \\ &\qquad + \int_{\delta_i[z](t,x)}^{t} \Big| f_i(\tau, g_i[z](\tau,t,x), z_{\psi_i(\tau, g_i[z](\tau,t,x))}) \\ &\qquad\qquad - f_i(\tau, g_i[z](\tau,\bar{t},\bar{x}), z_{\psi_i(\tau, g_i[z](\tau,\bar{t},\bar{x}))})\Big| d\tau \\ &\qquad + \tilde{\alpha}(\mu_0)\Big[|t-\bar{t}| + \delta_i[z](t,x) - \delta_i[z](\bar{t},\bar{x})\Big] \\ &\quad\leq \Theta(0,c)\widehat{A}\Big[|t-\bar{t}| + \|x-\bar{x}\|\Big] \\ &\qquad + \Theta(0,c)\tilde{\gamma}(\mu_0)Ac\Big[|t-\bar{t}| + \|x-\bar{x}\|\Big] \\ &\qquad + \tilde{\alpha}(\mu_0)\left[|t-\bar{t}| + \frac{2\Theta(0,c)}{\beta(\mu_0)}(|t-\bar{t}| + \|x-\bar{x}\|)\right]. \end{aligned}$$

The result is

$$|F_i[z](t,x) - F_i[z](\bar{t},\bar{x})| \leq \Theta(0,c)(\widehat{A} + \widehat{B})\Big[|t-\bar{t}| + \|x-\bar{x}\|\Big] + \tilde{\alpha}|t-\bar{t}|. \quad (35)$$

In a similar way we prove (35) in the case $\delta_i[z](\bar{t},\bar{x}) > \delta_i[z](t,x)$. This gives

$$\|F[z](t,x) - F[z](t,\bar{x})\| \leq d_1|t-\bar{t}| + d_2\|x-\bar{x}\| \tag{36}$$

on $[0,c]\times[-\bar{b},\bar{b}]$, and consequently $F[z] \in C_{\varphi.c}[d]$.

`Step II.` We shall prove that F is a contraction on $C_{\varphi.c}[d]$. For $z,\bar{z} \in C_{\varphi.c}[d]$ we have

$$\begin{aligned}
&|F_i[z](t,x) - F_i[\bar{z}](t,x)| \\
&\quad\leq |\varphi_i(\delta_i[z](t,x), g[z](\delta_i[z](t,x),t,x)) \\
&\qquad\qquad -\varphi_i(\delta_i[\bar{z}](t,x), g[\bar{z}](\delta_i[\bar{z}],t,x))| \\
&\qquad + \int_{\delta_i[z](t,x)}^{t} \Big| f_i(\tau, g_i[z](\tau,t,x), z_{\psi_i(\tau,g_i[z](\tau,t,x))}) \\
&\qquad\qquad - f_i(\tau, g_i[\bar{z}](\tau,t,x), \bar{z}_{\psi_i(\tau,g_i[\bar{z}](\tau,t,x))}) \Big| d\tau \\
&\qquad + \tilde{\alpha}(\mu_0)[\delta_i[z](t,x) - \delta_i[\bar{z}](t,x)] \\
&\quad\leq K\tilde{\Theta}(0,c)\left[\widehat{A} + \frac{2\tilde{\alpha}(\mu_0)}{\beta(\mu_0)}\right]\int_0^t \|z-\bar{z}\|_\xi d\xi \\
&\qquad + K\tilde{\gamma}(\mu_0)\Big[1 + Ac\tilde{\Theta}(0,c)\Big]\int_0^t \|z-\bar{z}\|_\xi d\xi
\end{aligned}$$

where we assumed, without loss of generality, that $\delta_i[\bar{z}](t,x) \leq \delta_i[z](t,x)$. We thus get

$$\|F[z](t,x) - F[\bar{z}](t,x)\| \leq \tilde{C}\int_0^t \|z-\bar{z}\|_\xi d\xi, \quad (t,x)\in[0,c]\times[-\bar{b},\bar{b}], \tag{37}$$

where

$$\tilde{C} = K\tilde{\Theta}(0,c)\left[\widehat{A} + \frac{2\tilde{\alpha}(\mu_0)}{\beta(\mu_0)}\right] + K\tilde{\gamma}(\mu_0)\Big[1 + Ac\tilde{\Theta}(0,c)\Big].$$

For functions $z,\bar{z}\in C_{\varphi.c}[d]$, we write

$$[|z-\bar{z}|] = \max\{\|z-\bar{z}\|_t \exp[-2\tilde{C}t] : t\in[0,c]\}.$$

We conclude from (37) that

$$\|F[z](t,x) - F[\bar{z}](t,x)\| \leq \tilde{C}[|z-\bar{z}|]\int_0^t \exp[2\tilde{C}\xi]d\xi \leq \frac{1}{2}[|z-\bar{z}|]\exp[2\tilde{C}t],$$

and consequently

$$[|F[z] - F[\bar{z}]|] \leq \frac{1}{2}[|z-\bar{z}|].$$

By the Banach fixed point theorem there exists a unique solution $u \in C_{\varphi.c}[d]$ satisfying equation $z = F[z]$.

Now we prove that u is the Carathéodory solution of equation (21). For each i, $1 \le i \le k$, we write $\delta_i(t,x)$ and $g_i(\cdot,t,x) = (g_{i.1}(\cdot,t,x),\dots,g_{i.n}(\cdot,t,x))$ instead of $\delta_i[u](t,x)$ and $g_i[u](\cdot,t,x)$. For each i, $1 \le i \le k$ put

$$E_i^{(0)} = \{(t,x) \in E[c] : \delta_i(t,x) = 0\}$$

and

$$E_i^{(j)} = \{(t,x) \in E[c] : g_{ij}(\delta_i(t,x),t,x) = \bar{b}_j \quad \text{or} \quad g_{ij}(\delta_i(t,x),t,x) = -\bar{b}_j\}$$

for $j = 1,\dots,n$. At first we prove that u satisfies (21) almost everywhere on $E_i^{(0)}$, $1 \le i \le k$. For a fixed t, $(t,x) \in E_i^{(0)}$, we put $\eta_i = g_i(0,t,x)$. Let $I_{(t,x)}^{(i)}$ be the domain of the bicharacteristic $g_i(\cdot,t,x)$. It follows that $g_i(\tau,t,x) = g_i(\tau,0,\eta_i)$ for $t \in I_{(t,x)}^{(i)}$ and $x = g_i(t,0,\eta_i)$. The relations $\eta_i = g_i(0,t,x)$ and $x = g_i(t,0,\eta_i)$, where $(t,x) \in E_i^{(0)}$, $\eta_i \in (-\bar{b},\bar{b})$, are equivalent. Then, the relations

$$u_i(t,x) = F_i[u](t,x), \quad (t,x) \in E_i^{(0)},$$

and

$$\begin{aligned} &u_i(t, g_i(t,0,\eta_i)) \\ &= \varphi_i(0,\eta_i) + \int_0^t f_i(\tau, g_i(\tau,0,\eta_i), u_{\psi_i(\tau,g_i(\tau,0,\eta_i))})d\tau, \quad \eta_i \in (-\bar{b},\bar{b}). \end{aligned} \tag{38}$$

are equivalent. By differentiating (38) with respect to t we get

$$\begin{aligned} \partial_t u(t,g_i(t,0,\eta_i)) + \sum_{j=1}^n \partial_{x_j} u(t,g_i(t,0,\eta_i)) \frac{d}{dt} g_{ij}(t,0,\eta_i) \\ = f_i(t, g_i(t,0,\eta_i), u_{\psi_i(t,g_i(t,0,\eta_i))}) \end{aligned}$$

for almost all $t \in I_{(0,\eta_i)}^{(i)}$. Making use of the transformation $x = g_i(t,0,\eta_k)$ and (25) we get (21) almost everywhere on $E_i^{(0)}$. Now we prove that u satisfies system (21) on $E_i^{(1)} \cup \ldots \cup E_i^{(n)}$. It is enough to show this on any set of the form $E_i^{(j_1)} \cap E_i^{(j_2)} \cap \ldots \cap E_i^{(j_r)}$, where $1 \le r \le n$ and $j_l \in \{1,\dots,n\}$, $l = 1,\dots,r$. Let $(t,x) \in E_i^{(j_1)} \cap E_i^{(j_2)} \cap \ldots \cap E_i^{(j_r)}$, and

$$\begin{aligned} &\lambda_i = \delta_i(t,x) \\ &\eta_i = (\eta_{i.1},\dots,\eta_{i.n}) = (g_{i.1}(\delta_i(t,x),t,x),\dots,g_{i.n}(\delta_i(t,x),t,x)). \end{aligned} \tag{39}$$

Note that $g_{i.q}(\delta_i(t,x),t,x) = \bar{b}_q$ or $g_{i.q}(\delta_i(t,x),t,x) = -\bar{b}_q$ if $q = j_l$ for some $l \in \{1,\dots,r\}$. Consider the family of bicharacteristics $g_i(\cdot,\lambda_i,\eta_i)$ with

$$\lambda_i \in (0,c), \quad \eta_{i.q} \in (-\bar{b}_q,\bar{b}_q), \quad q \notin \{j_1,\dots,j_r\}.$$

The relations (39) and $x = g_i(t,\lambda_i,\eta_i)$ are equivalent. Then, the relations

$$u_i(t,x) = F_i[u](t,x), \quad (t,x) \in E_i^{(j_1)} \cap E_i^{(j_2)} \cap \ldots \cap E_i^{(j_r)},$$

and

$$u_i(t,g_i(t,\lambda_i,\eta_i)) = F_i[u](t,g_i(t,\lambda_i,\eta_i)), \tag{40}$$

where $\lambda_i \in (0,c)$, $x \in I^{(i)}_{(\lambda_i,\eta_i)}$, and $\eta_{i.q} \in (-\bar{b}_q,\bar{b}_q)$, $q \notin \{j_1,\dots,j_r\}$, are equivalent. It follows from (40) that

$$u_i(t,g_i(t,\lambda_i,\eta_i)) = \varphi_i(\lambda_i,\eta_i) + \int_{\lambda_i}^t f_i(\tau,g_i(\tau,\lambda_i,\eta_i),u_{\psi_i(\tau,g_i(\tau,\lambda_i,\eta_i))})d\tau. \tag{41}$$

By differentiating (41) with respect to t we get

$$\begin{aligned}\partial_t u(t,g_i(t,\lambda_i,\eta_i)) + \sum_{j=1}^n \partial_{x_j} u(t,g_i(t,\lambda_i,\eta_i))\frac{d}{dt}g_{ij}(t,\lambda_i,\eta_i)& \\ = \; f_i(t,g_i(t,\lambda_i,\eta_i),u_{\psi_i(t,g_k(t,\lambda_i,\eta_i))})&\end{aligned}$$

for almost all $t \in I^{(i)}_{(\lambda_i,\eta_i)}$. Making use of the transformation $x = g_i(t,\lambda_i,\eta_i)$ and (25) we get (21) almost everywhere on $E_i^{(j_1)} \cap E_i^{(j_2)} \cap \ldots \cap E_i^{(j_r)}$. Since

$$E_c = E_i^{(0)} \cup E_i^{(1)} \cup \ldots \cup E_i^{(n)}$$

for any $i, 1 \le i \le k$ holds, it follows that u is the Carathéodory solution of (21), (22) on $(E_0 \cup \partial_0 E) \cap ((-\infty,c] \times R^n)$.

`Step III.` Now we prove relation (34). Let $\bar{F}$ be operator defined as F but with function $\bar{\varphi}$ instead of φ. Since $u = F[u]$ and $\bar{u} = \bar{F}[\bar{u}]$, where $u = (u_1,\dots,u_k)$, $\bar{u} = (\bar{u}_1,\dots,\bar{u}_k)$, it follows that

$$\begin{aligned}&|u_i(t,x) - \bar{u}_i(t,x)| \\ &\quad \le \; |\varphi_i(\delta_i[u](t,x),g_i[u](\delta_i[u](t,x),t,x)) \\ &\qquad\qquad -\bar{\varphi}_i(\delta_i[\bar{u}](t,x),g_i[\bar{u}](\delta_i[\bar{u}](t,x),t,x))| \\ &\qquad + \int_{\delta_i[u](t,x)}^t \Big| f_i(\tau,g_i[u](\tau,t,x),u_{\psi_i(\tau,g_i[u](\tau,t,x))})) \\ &\qquad\qquad -f_i(\tau,g_i[\bar{u}](\tau,t,x),\bar{u}_{\psi_i(\tau,g[\bar{u}](\tau,t,x))}))\Big| d\tau \\ &\qquad +\tilde{\alpha}(\mu_0)\Big[\delta_i[u](t,x) - \delta_i[\bar{u}](t,x)\Big]\end{aligned}$$

for $(t,x) \in E[c]$ holds, where we have assumed without loss of generality that $\delta_i[\bar{u}](t,x) \le \delta_i[u](t,x)$. It follows from Assumption $\tilde{H}[f]$ and Lemma 4.1, 4.2 and 4.3 that

$$\begin{aligned} &|u_i(t,x) - \bar{u}_i(t,x)| \\ &\quad \le \max\{\|\varphi - \bar{\varphi}\|_{X.\bar{b}}, \|\varphi - \bar{\varphi}_{*.t}\} \\ &\qquad + \widehat{A}\tilde{\Theta}(0,c) \int_0^t \Big[K\|u-\bar{u}\|_\tau + K_0\|\varphi - \bar{\varphi}\|_{X.\bar{b}}\Big] d\tau \\ &\qquad + \Big[\tilde{\Theta}(0,c)\widehat{B} + \tilde{\gamma}(\mu_0)\Big] \int_0^t \Big[K\|u-\bar{u}\|_\tau + K_0\|\varphi - \bar{\varphi}\|_{X.\bar{b}}\Big] d\tau \end{aligned}$$

for $1 \le i \le k$, $(t,x) \in E[c]$ holds. Then we have the integral inequality

$$\|u-\bar{u}\|_t \le \|\varphi - \bar{\varphi}\|_{*.t} + \tilde{d}\|\varphi - \bar{\varphi}\|_{X.\bar{b}} + K\Big[\tilde{\Theta}(c) + \tilde{\gamma}(\mu_0)\Big] \int_0^t \|u-\bar{u}\|_\tau d\tau$$

for $t \in [0,c]$, where $\tilde{d} = 1 + cK_0(\widehat{\Theta}(c) + \tilde{\gamma}(\mu_0))$. Using the Gronwall inequality we obtain (34) for

$$\Lambda(t) = \tilde{d} \exp\Big[K(\widehat{\Theta}(c) + \tilde{\gamma}(\mu_0)t\Big].$$

This proves the theorem. ∎

Remark 4. Existence results for quasilinear functional differential equations with initial boundary conditions are presented in [6], Chapter 5, see also [4] where classical solutions are considered. It is easy to see that Theorem 5.35 from [6] can be extended to quasilinear functional differential systems. The following condition is important in these considerations. Write

$$sign\rho_i(t,x,w) = (sign\rho_{i1}(t,x,w), \ldots, sign\rho_{in}(t,x,w)), \quad 1 \le i \le k. \tag{42}$$

It is assumed in [6] (see also [4]) that the function (42) is constant. Note that we have omitted this assumption in Theorem 4.1.

Remark 5. Suppose that $B = [-r_0, 0] \times [-r, r]$. Consider differential integral system (18) with the initial boundary condition

$$z(t,x) = \varphi(t,x) \quad \text{for} \quad (t,x) \in E_0 \cup \partial_0 E. \tag{43}$$

It is easy to formulate existence result for problem (18), (43) which is based on Theorem 4.1. Note that the results presented in [4], [6], [9]-[12] concern the case when the sets $D_i[t,x]$ do not depend on (t,x) and i, and therefore they are not applicable to (18), (43).

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