On the Basin of Attraction of Limit Cycles in Periodic Differential Equations

Peter Giesl

Abstract. We consider a general system of ordinary differential equations

$$\dot{x} = f(t, x),$$

where $x \in \mathbb{R}^n$, and $f(t + T, x) = f(t, x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ is a periodic function. We give a sufficient and necessary condition for the existence and uniqueness of an exponentially asymptotically stable periodic orbit. Moreover, this condition is sufficient and necessary to prove that a subset belongs to the basin of attraction of the periodic orbit. The condition uses a Riemannian metric, and we present methods to construct such a metric explicitly.

Keywords: *Dynamical system, periodic differential equation, periodic orbit, asymptotic stability, basin of attraction*

AMS subject classification: Primary 37C55, secondary 34C25, 37C75, 34C05

1. Introduction

Many applications lead to differential equations of the form $\dot{x} = f(t, x)$ with a periodic forcing term $f(t, x) = f(t + T, x)$. Natural mathematical questions are existence and uniqueness of periodic solutions, and their stability. Moreover, for an asymptotically stable periodic orbit, one is interested in its basin of attraction, consisting of all points which eventually approach the periodic orbit.

There are many theorems providing conditions for the existence of periodic orbits in dynamical systems given by periodic differential equations (cf. [20]). For the stability analysis the Floquet theory provides necessary and sufficient conditions for exponentially asymptotically stable periodic orbits. To determine the basin of attraction of such a periodic orbit, one can use Lyapunov functions. However, to calculate the Floquet exponents and to find a Lyapunov function one has to determine the periodic orbit explicitly. Even then, one has to solve a differential equation to obtain the Floquet exponents, and there is no general

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way to find a Lyapunov function.

In this paper we give a condition for existence, uniqueness, and exponential asymptotic stability of a periodic orbit which, at the same time, determines a subset of its basin of attraction. The condition does not require the exact position of the periodic orbit, and it is based on a Riemannian metric $M(t,x)$, i.e. a t-periodic function with symmetric, positive definite matrices as values. More precisely, the main condition is $L_M(t,x) < 0$, where

$$L_M(t,x) \quad := \quad \max_{w \in \mathbb{R}^n, w^T M(t,x)w=1} w^T \left[M(t,x)D_x f(t,x) + \frac{1}{2}M'(t,x) \right] w.$$

M' denotes the orbital derivative of M (cf. Theorem 3.1 for details). The meaning of the condition $L_M(t,x) < 0$ is that orbits through adjacent points move towards each other with respect to the Riemannian metric M as time increases. The condition and the corresponding results have been obtained for the autonomous case by Borg [2], Stenström [19] and Hartman/Olech [7]. Stenström considers general manifolds. In our case the manifold is the cylinder $T \cdot S^1 \times \mathbb{R}^n$, and existence and uniqueness of the periodic orbit as well as the statements concerning the basin of attraction follow from Stenström's results, cf. the second remark after Theorem 3.1. In Theorem 3.1 of this paper, additionally to Stenström's results, we derive a bound for the Floquet exponents of the periodic orbit.

A tool to prove existence and stability of periodic orbits in two-dimensional autonomous systems is the Poincaré-Bendixson Theorem. A generalization of the Poincaré-Bendixson Theorem to higher dimensions is the torus principle. It is a tool for proving existence of periodic orbits (cf. the survey [12]). However, because of its complicated geometry it is difficult to apply to concrete examples. A generalization of the stability aspect of the Poincaré-Bendixson Theorem is given in [17] for autonomous systems. For a time-periodic system this is done in [18]. In condition (H3) of that paper a condition similar to our one is stated, where the Riemannian metric is given by a constant symmetric matrix $M(t,x) = P$. If P is positive definite, the existence of a unique periodic orbit is shown, which is a special case of our result. The paper [18], however, deals mainly with matrices P which have a certain number of negative eigenvalues, and thus it obtains no uniqueness but existence and stability results for periodic orbits. Cronin [3] also deals with time-periodic differential equations. Her main assumption is that all eigenvalues of $D_x f(t,x)$ have negative real parts. The idea of Cronin's proof is to search for asymptotically stable solutions and then to find periodic solutions among them with a theorem of Sell [15]. Hence, also that paper focusses on the existence and stability of periodic orbits rather than on uniqueness results.

A generalization of Borg's criterion for unbounded domains was given in [5]. Similar techniques were used in [9] and [11] (cf. also the book [10]). Explicit verification of Borg's criterion is often difficult in concrete problems. However, in [16], Borg's method is applied to a three-dimensional equation describing a nuclear spin generator.

The necessity of Borg's condition in the autonomous case with a Riemannian metric was shown in [6]. In the present paper, besides the sufficiency, we also prove the necessity of the above condition for the periodic case, i.e., given an exponentially asymptotically stable periodic orbit there exists a Riemannian metric M which satisfies $L_M(t, x) < 0$. The proof uses Floquet theory and a Lyapunov function. Thus, the proof does not serve to explicitly find such a metric.

Hence, the main problem in applying this result is to find an appropriate Riemannian metric. In the second part of this paper we will provide methods to explicitly construct such Riemannian metrics and thus to prove existence, uniqueness and stability of a periodic orbit and, moreover, to determine a part of its basin of attraction. We present two methods which are based on eigenvectors, and we apply them to several examples, among them a three-dimensional differential equation and the movement of a swing.

Let us describe how the paper is organized: In Section 2 we give some definitions about time-periodic systems. In Section 3 we state and prove Theorem 3.1 which gives a sufficient condition for existence and uniqueness of exponentially asymptotically stable periodic orbits based on a Riemannian metric. In Section 4 we show that the conditions of Theorem 3.1 are necessary. Section 5 summarizes the results of Sections 3 and 4. In Section 6 we provide useful formulas for the calculation of special Riemannian metrics. They serve in Section 7 for the construction of special Riemannian metrics. We apply these methods to several examples for which we prove the stability of periodic orbits and determine a part of their basin of attraction.

2. Preliminaries

Throughout all the paper we consider the differential equation

$$\dot{x} = f(t, x), \tag{2.1}$$

where $x \in \mathbb{R}^n$, and $f(t, x) = f(t + T, x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ is a time-periodic function with period $T > 0$. Furthermore we assume $f \in C^0(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, and we also assume that the partial derivatives of order one with respect to x exist and are continuous functions of (t, x). Then there exists a unique solution with continuous first order partial derivatives with respect to both t and x (cf. Corollary 3.3, Chapter 5 of [8]).

In this section we give some generalization of definitions for autonomous systems. The main idea is as usual to consider the time as an additional variable in $T \cdot S^1$ which denotes the circle with radius T. Hence, in the following we consider the dynamical system on the cylinder $T \cdot S^1 \times \mathbb{R}^n$. We use the following notations.

Definition 2.1.

1. The *flow* of the system (2.1) is defined by $S_\theta(t_0, x_0) := (t_0 + \theta, x(\theta)) \in T \cdot S^1 \times \mathbb{R}^n$ for $\theta \geq 0$, where $x(\theta)$ is the solution of (2.1) with initial value $x(t_0) = x_0$. We also set $S^x_\theta(t_0, x_0) := x(\theta)$.

2. Let $G \subset T \cdot S^1 \times \mathbb{R}^n$. Define $G_t := \{x \in \mathbb{R}^n \mid (t, x) \in G\}$ for $t \in T \cdot S^1$.

For the definition of flows on manifolds and ω-limit sets cf. for example [1]. In particular, the following lemma holds.

Lemma 2.1. *Let $G \subset T \cdot S^1 \times \mathbb{R}^n$ be positively invariant and compact. Then $\varnothing \neq \omega(t_0, x_0) \subset G$ for arbitrary $(t_0, x_0) \in G$.*

Definition 2.2.

1. A *periodic orbit* Ω of (2.1) is a set $\Omega = \{S_\theta(0, x_0) \mid \theta \in [0, T]\} \subset T \cdot S^1 \times \mathbb{R}^n$ with $S^x_T(0, x_0) = x_0$.

2. A periodic orbit Ω is called *exponentially asymptotically stable*, if it is orbitally stable, i.e. for all $\epsilon > 0$ there is a $\delta > 0$ such that for all points $p \in T \cdot S^1 \times \mathbb{R}^n$ $\mathrm{dist}(p, \Omega) \leq \delta$ implies $\mathrm{dist}\,(S_\theta p, \Omega) \leq \epsilon$ for all $\theta \geq 0$, and there are constants $\mu, \iota > 0$ such that $\mathrm{dist}(p, \Omega) \leq \iota$ implies $\mathrm{dist}\,(S_\theta p, \Omega)\, e^{\mu\theta} \overset{\theta \to \infty}{\longrightarrow} 0$. Here and below, dist denotes the distance which is locally given by Euclidean distance on the standard chart in $\mathbb{R}^{n+1}$.

3. The *basin of attraction* $A(\Omega)$ of an exponentially asymptotically stable periodic orbit Ω is the set

$$A(\Omega) := \left\{ (t, x) \in T \cdot S^1 \times \mathbb{R}^n \mid \mathrm{dist}\,(S_\theta(t, x), \Omega) \overset{\theta \to \infty}{\longrightarrow} 0 \right\}.$$

We will often use the notion of a Riemannian metric given in Definition 2.3. If M is a Riemannian metric in the sense of Definition 2.3, then $v^T M(t, x)\, w$ defines a scalar product in $(v, w) \in \mathbb{R}^n \times \mathbb{R}^n$ for each $(t, x) \in T \cdot S^1 \times \mathbb{R}^n$.

Definition 2.3. The matrix-valued function $M \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^{n \times n})$ will be called a *Riemannian metric*, if $M(t, x)$ is a symmetric and positive definite matrix for each $(t, x) \in G$ and $M(t+T, x) = M(t, x)$ holds for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$.

3. Sufficiency

In Theorem 3.1 we give a sufficient condition for existence and uniqueness of an exponentially asymptotically stable periodic orbit and its basin of attraction. In Section 4 we will show in Theorem 4.1 that the condition is also necessary. Theorems 5.1 and 5.2 of Section 5 will finally summarize our main results.

Theorem 3.1. *Consider the equation $\dot{x} = f(t, x)$, where $x \in \mathbb{R}^n$, and assume that $f \in C^0(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is a t-periodic function with period T, and all partial derivatives of order one with respect to x are continuous functions of (t, x). Let $\varnothing \neq G \subset T \cdot S^1 \times \mathbb{R}^n$ be a connected, compact and positively invariant set. Let M be a Riemannian metric in the sense of Definition 2.3. Moreover, assume $L_M(t, x) < 0$ for all $(t, x) \in G$, where*

$$L_M(t, x) \quad := \quad \max_{w \in \mathbb{R}^n, w^T M(t,x)w = 1} L_M(t, x; w) \tag{3.1}$$

$$L_M(t, x; w) \quad := \quad w^T \left[M(t, x) D_x f(t, x) + \frac{1}{2} M'(t, x) \right] w, \tag{3.2}$$

where $M'(t, x)$ denotes the matrix with entries

$$m_{ij} = \frac{\partial M_{ij}(t, x)}{\partial t} + \sum_{k=1}^{n} \frac{\partial M_{ij}(t, x)}{\partial x_k} f_k(t, x)$$

which is also the orbital derivative of $M(t, x)$, i.e. $M'(t, x) = \frac{d}{d\theta} M(S_\theta(t, x))\big|_{\theta=0}$. Then there exists one and only one periodic orbit $\Omega \subset G$ which is exponentially asymptotically stable. Moreover, for its basin of attraction $G \subset A(\Omega)$ holds, and the largest real part $-\nu_0$ of all Floquet exponents of Ω satisfies

$$-\nu_0 \leq -\nu := \max_{(t,x) \in G} L_M(t, x).$$

Remark. The function $L_M(t, x)$ in (3.1) is continuous. This follows in a similar way as in [4], Proposition A.2.

Remark. The situation of Theorem 3.1 is a special case of the situation in [19]. There a general Riemannian manifold is studied which is in our case the cylinder $T \cdot S^1 \times \mathbb{R}^n$. However, the contraction condition B in [19, Section 2] is stated for all vectors a orthogonal to the vector field which is in our case $(1, f(t, x))$. We, in contrast, use a contraction condition for all vectors orthogonal to $(1, 0)$. The reason is that our contraction condition is easier to compute since the n-dimensional hypersurface at (t, x) of all vectors orthogonal to $(1, 0)$ is $\{(t, v) \mid v \in \mathbb{R}^n\}$. Moreover, the examples for Riemannian metrics in Sections 6 and 7 can be derived from our condition in a natural way.

We show that the results on existence and uniqueness of the periodic orbit and the basin of attraction are a special case of the results in [19], if we choose the Riemannian metric in [19] in the following way: Denote by $M(t, x)$ the Riemannian metric of our approach. The Riemannian metric on the manifold $T \cdot S^1 \times \mathbb{R}^n$ is given by

$$P(t, x) = \begin{pmatrix} d(t, x) & v^T(t, x) \\ v(t, x) & M(t, x) \end{pmatrix}$$

in local coordinates. Here, $d(t, x) \in \mathbb{R}$ and $v(t, x) \in \mathbb{R}^n$.

Denote $F(t, x) = (1, f(t, x))$ and $A = (a^0, a)$ with $a^0 \in \mathbb{R}$ and $a \in \mathbb{R}^n$. Set $v(t, x) = -M(t, x)f(t, x)$. Then $\langle A, F(t, x)\rangle_{P(t,x)} = A^T P(t, x)F(t, x) = 0$ if and only if $a^0[d(t, x) - f(t, x)^T M(t, x)f(t, x)] = 0$. The function $f(t, x)^T M(t, x)f(t, x)$ has a maximum m on the compact set $G \subset T \cdot S^1 \times \mathbb{R}^n$. Set $d(t, x) = 2m$, then $\langle A, F(t, x)\rangle_{P(t,x)} = 0$ if and only if $a^0 = 0$.

Now we calculate the contraction condition in [19], 2.B. We denote $x^0 = t$ (note that $a^0 = 0$) and use that $2\Gamma^i_{jk} = \sum_{h=0}^n p^{hi}\left(\frac{\partial p_{hk}}{\partial x^j} + \frac{\partial p_{jh}}{\partial x^k} - \frac{\partial p_{jk}}{\partial x^h}\right)$, where p^{hi} denotes the entries of $P^{-1}(t, x)$. We get

$$\sum_{i,j,m=0}^n \frac{\partial F^i(t, x)}{\partial x^j}a^j a^m p_{im} + \sum_{i,j,k,m=0}^n \Gamma^i_{jk}F^k a^j a^m p_{im}$$

$$= \sum_{i,j,m=1}^n a^j \frac{\partial f^i(t, x)}{\partial x^j}a^m p_{im} + \frac{1}{2}\sum_{i,j,k,m,h=0}^n a^j F^k\left(\frac{\partial p_{hk}}{\partial x^j} + \frac{\partial p_{jh}}{\partial x^k} - \frac{\partial p_{jk}}{\partial x^h}\right)a^m p^{hi} p_{im}$$

$$= a^T M^T(t, x)D_x f(t, x)a + \frac{1}{2}\sum_{j,k,m=0}^n a^j F^k\left(\frac{\partial p_{mk}}{\partial x^j} + \frac{\partial p_{jm}}{\partial x^k} - \frac{\partial p_{jk}}{\partial x^m}\right)a^m$$

$$= a^T M^T(t, x)D_x f(t, x)a + \frac{1}{2}\sum_{j,m=1}^n a^j \sum_{k=0}^n \frac{\partial p_{jm}}{\partial x^k}F^k a^m$$

$$= a^T M^T(t, x)D_x f(t, x)a + \frac{1}{2}\sum_{j,m=1}^n a^j\left(\frac{\partial p_{jm}}{\partial t} + \sum_{k=1}^n \frac{\partial p_{jm}}{\partial x^k}f^k\right)a^m$$

$$= a^T M^T(t, x)D_x f(t, x)a + \frac{1}{2}a^T M'a$$

$$= L_M(t, x; w)$$

(cf. (3.2)) since $\sum_{j,m=0}^n a^j\left(\frac{\partial p_{mk}}{\partial x^j} - \frac{\partial p_{jk}}{\partial x^m}\right)a^m = 0$ for all k. Our result in Theorem 3.1 gives also an estimate for the Floquet exponents. We give a proof of all statements of Theorem 3.1 in order to make this paper self-consistent. Moreover, the similarities and differences to the autonomous case in [6] become obvious in our proof.

The rest of this section is devoted to the proof of Theorem 3.1 which is split into several propositions corresponding to the following program:

1. Show that the distance between two trajectories with adjacent initial points (t_0, x_0) and (t_0, x_1) is exponentially decreasing and that their ω-limit sets are equal (Proposition 3.1).
2. Show that the ω-limit sets of two adjacent points (t_0, x_0) and (t_1, x_1) are equal (Proposition 3.2).
3. Show that the ω-limit set of all points of G is the same (Proposition 3.3).

Finally, we conclude the proof of Theorem 3.1 by showing that this ω-limit set is an exponentially asymptotically stable periodic orbit with period T and that the bound for the Floquet exponents holds.

Proposition 3.1. *Let the assumptions of Theorem* 3.1 *be satisfied. Then for each $k \in (0,1)$ there are constants $\delta > 0$ and $C \geq 1$ such that for all $(t_0, x_0) \in G$ and for all $\eta \in \mathbb{R}^n$ with $\|\eta\| \leq \frac{\delta}{2}$ we have*

$$\|S_\theta^x(t_0, x_0 + \eta) - S_\theta^x(t_0, x_0)\| \leq Ce^{-\nu(1-k)\theta}\|\eta\| \quad \text{for all } \theta \geq 0 \quad (3.3)$$

and $\omega(t_0, x_0) = \omega(t_0, x_0 + \eta)$.

Proof. The proposition is proven in four steps. We introduce a time-dependent distance between two trajectories with adjacent starting points (t_0, x_0) and $(t_0, x_0 + \eta)$, and we show that this distance decreases exponentially. We define the distance with respect to the Riemannian metric defined by $v^T M(t, x)w$ in G_t, but note that on the compact set G the Riemannian and the Euclidian norms are equivalent. In the first step we give some bounds, in the second step we define the distance $A(\theta)$. In the third step we show that this distance decreases exponentially and in the last step we show that each point $(t_0, x_0) \in G$ has a neighborhood in G_{t_0}, the points of which have the same ω-limit set as (t_0, x_0).

Step I. $M(t, x)$ is symmetric and positive definite for all $(t, x) \in G$. Hence, for the smallest eigenvalue $\lambda_1(t, x) > 0$ holds. Since the eigenvalues depend continuously on (t, x), there are $0 < \lambda_m \leq \lambda_M < \infty$ such that

$$\lambda_m\|\xi\|^2 \leq \xi^T M(t, x)\xi \leq \lambda_M\|\xi\|^2 \quad (3.4)$$
$$\|M(t, x)\xi\| \leq \lambda_M\|\xi\| \quad (3.5)$$

hold for all $\xi \in \mathbb{R}^n$ and all $(t, x) \in G$.

Also, $D_x f(t, x)$ is continuous and thus uniformly continuous on G. Hence, there exists a $\delta_1 > 0$, so that

$$\|D_x f(t, x) - D_x f(t, x + \xi)\| \leq \frac{k\lambda_m}{\lambda_M}\nu \quad (3.6)$$

holds for all $(t, x) \in G$ and all $\xi \in \mathbb{R}^n$ with $\|\xi\| \leq \delta_1$. We set $\delta := \sqrt{\frac{\lambda_m}{\lambda_M}} \, \delta_1$.

Step II. Fix $(t_0, x_0) \in G$ and $\eta \in \mathbb{R}^n$ with $\|\eta\| \leq \frac{\delta}{2}$. For $\theta \in \mathbb{R}_0^+$ we define the distance

$$A(\theta) := \Big\{ [S_\theta^x(t_0, x_0 + \eta) - S_\theta^x(t_0, x_0)]^T \, M(S_\theta(t_0, x_0)) \tag{3.7}$$
$$\cdot [S_\theta^x(t_0, x_0 + \eta) - S_\theta^x(t_0, x_0)] \Big\}^{\frac{1}{2}}.$$

Note that by (3.4) we have

$$\sqrt{\lambda_m} \| S_\theta^x(t_0, x_0 + \eta) - S_\theta^x(t_0, x_0) \| \leq A(\theta) \leq \sqrt{\lambda_M} \| S_\theta^x(t_0, x_0 + \eta) - S_\theta^x(t_0, x_0) \|.$$

We consider only the nontrivial case $\eta \neq 0$. Then we have $A(\theta) \neq 0$ for all $\theta \geq 0$, and we set

$$v(\theta) := \frac{S_\theta^x(t_0, x_0 + \eta) - S_\theta^x(t_0, x_0)}{A(\theta)}.$$

In other words, we have $S_\theta^x(t_0, x_0 + \eta) - S_\theta^x(t_0, x_0) = A(\theta)v(\theta)$. Note that $v(\theta)$ is a vector with $\sqrt{v(\theta)^T M(S_\theta(t_0, x_0))v(\theta)} = 1$, and thus $\frac{1}{\sqrt{\lambda_M}} \leq \|v(\theta)\| \leq \frac{1}{\sqrt{\lambda_m}}$ holds by (3.4).

Step III. We show that $A(\theta)$ tends to zero exponentially. We calculate the temporal derivative of A^2 (cf. (3.7)) and use $M(t, x) = M(t, x)^T$

$$\begin{aligned}
\frac{d}{d\theta} A^2(\theta) &= 2 [S_\theta^x(t_0, x_0 + \eta) - S_\theta^x(t_0, x_0)]^T \, M(S_\theta(t_0, x_0)) \\
&\quad \cdot [f(S_\theta(t_0, x_0 + \eta)) - f(S_\theta(t_0, x_0))] \\
&\quad + [S_\theta^x(t_0, x_0 + \eta) - S_\theta^x(t_0, x_0)]^T \, M'(S_\theta(t_0, x_0)) \\
&\quad \cdot [S_\theta^x(t_0, x_0 + \eta) - S_\theta^x(t_0, x_0)] \\
&= 2A(\theta)v(\theta)^T M(S_\theta(t_0, x_0)) \\
&\quad \cdot [f\{t_0 + \theta, S_\theta^x(t_0, x_0) + A(\theta)v(\theta)\} - f(t_0 + \theta, S_\theta^x(t_0, x_0))] \\
&\quad + A^2(\theta)\, v(\theta)^T M'(S_\theta(t_0, x_0))v(\theta).
\end{aligned}$$

As $\|A(\theta)v(\theta)\| \leq \frac{2\sqrt{\lambda_M}}{\sqrt{\lambda_m}} \|\eta\| \leq \delta_1$ for small $\theta \geq 0$ we can use (3.6). We will justify below that the argumentation in fact holds for all $\theta \geq 0$. Thus, with $L_M(S_\theta(t_0, x_0)) \leq -\nu$ since G is positively invariant and the mean value theorem

we have

$$\frac{d}{d\theta}A^2(\theta)$$

$$= 2A^2(\theta)v(\theta)^T M(S_\theta(t_0,x_0))\Big(\int_0^1 D_x f\{t_0+\theta, S_\theta^x(t_0,x_0)+\lambda A(\theta)v(\theta)\}\, d\lambda$$

$$+ A^2(\theta)v(\theta)^T M'\big(S_\theta(t_0,x_0)\big)\Big)v(\theta)$$

$$\leq 2A^2(\theta)\Big\{ \underbrace{v(\theta)^T\Big[M(S_\theta(t_0,x_0))D_x f(S_\theta(t_0,x_0))+\frac{1}{2}M'(S_\theta(t_0,x_0))\Big]v(\theta)}_{=L_M(S_\theta(t_0,x_0);v(\theta))}$$

$$+ v(\theta)^T M(S_\theta(t_0,x_0))\cdot\Big(\int_0^1 [D_x f\{t_0+\theta, S_\theta^x(t_0,x_0)+\lambda A(\theta)v(\theta)\}$$

$$- D_x f(t_0+\theta, S_\theta^x(t_0,x_0))]\, d\lambda\Big)\, v(\theta)\Big\}$$

$$\leq -2\nu A^2(\theta)+2\frac{A^2(\theta)\lambda_M}{\lambda_m}\frac{k\lambda_m}{\lambda_M}\nu$$

$$= -2(1-k)\nu A^2(\theta).$$

The last inequality follows by (3.5) and (3.6). Thus we have

$$A(\theta)\leq A(0)\,e^{-\nu(1-k)\theta}. \tag{3.8}$$

In particular, $A(\theta)\leq A(0)\leq\sqrt{\lambda_M}\,\|\eta\|\leq\sqrt{\lambda_m}\frac{\delta_1}{2}$ holds and thus $\|A(\theta)v(\theta)\|\leq\frac{\delta_1}{2}$ for all $\theta\geq 0$. This justifies the above argumentation for all $\theta\geq 0$ by a prolongation argument. We have by (3.4)

$$\sqrt{\lambda_m}\,\|S_\theta^x(t_0,x_0+\eta)-S_\theta^x(t_0,x_0)\| \leq A(\theta)$$
$$\leq A(0)e^{-\nu(1-k)\theta}\ \text{by (3.8)}$$
$$\leq \sqrt{\lambda_M}\,\|\eta\|e^{-\nu(1-k)\theta}.$$

Hence, (3.3) follows with $C:=\sqrt{\frac{\lambda_M}{\lambda_m}}\geq 1$.

Step IV. Now we show that all points $(t_0,x_0+\eta)$ with η as above have the same ω-limit set as (t_0,x_0) itself. Assume $(\bar t,\bar x)\in\omega(t_0,x_0)$. Then we have a strictly increasing sequence $\theta_n\to\infty$ satisfying $\big\|(\bar t,\bar x)-S_{\theta_n}(t_0,x_0)\big\|\to 0$ as $n\to\infty$. Because of (3.3), $\|S_{\theta_n}(t_0,x_0+\eta)-S_{\theta_n}(t_0,x_0)\|=\|S_{\theta_n}^x(t_0,x_0+\eta)-S_{\theta_n}^x(t_0,x_0)\|\leq Ce^{-\nu(1-k)\theta_n}\|\eta\|\to 0$ as $n\to\infty$. This proves $S_{\theta_n}(t_0,x_0+\eta)\to(\bar t,\bar x)$ and hence $(\bar t,\bar x)\in\omega(t_0,x_0+\eta)$. The inclusion $\omega(t_0,x_0+\eta)\subset\omega(t_0,x_0)$ follows similarly. Thus, we have shown Proposition 3.1. ∎

Now we show that all points of a full neighborhood in $T\cdot S^1\times\mathbb{R}^n$ have the same ω-limit set.

Proposition 3.2. *Let the assumptions of Theorem* 3.1 *be satisfied. Then there is a constant $\delta^* > 0$ such that $\omega(t_0, x_0) = \omega(t_1, x_1)$ holds for all $(t_0, x_0) \in G$ and all $(t_1, x_1) \in T \cdot S^1 \times \mathbb{R}^n$ with $\|(t_0, x_0) - (t_1, x_1)\| \leq \delta^*$.*

Proof. Set $k = \frac{1}{2}$ and choose $\delta > 0$ according to Proposition 3.1. Let $f_M := \max_{(t,x) \in G} \|f(t, x)\|$. Since $f(t, x)$ is uniformly continuous on the compact set G, there is a $0 < \delta_2 \leq \delta$ such that for all $(t_0, x_0) \in G$ and all $(t_1, x_1) \in \overline{B_{\delta_2}(t_0, x_0)} := \{(t_1, x_1) \in T \cdot S^1 \times \mathbb{R}^n \mid \|(t_1, x_1) - (t_0, x_0)\| \leq \delta_2\}$

$$\|f(t_1, x_1) - f(t_0, x_0)\| \leq f_M \tag{3.9}$$

holds. Set

$$\delta^* := \frac{\delta_2}{2(2f_M + 2)} \leq \frac{\delta}{2(2f_M + 1)}.$$

The orbit through the point $(t_1, x_1) \in \overline{B_{\delta^*}(t_0, x_0)}$ reaches a point (t_0, x_1'). We show that $\|x_1 - x_1'\| \leq \delta^* f_M$. Let us only consider the case $t_1 < t_0$.

First we show that $S_\tau(t_1, x_1)$ remains in $B_{\delta_2}(t_0, x_0)$ for all $\tau \in [0, t_0 - t_1]$. Assuming the opposite, there is a $\tau_0 \in [0, t_0 - t_1]$ with $\|S_{\tau_0}(t_1, x_1) - (t_0, x_0)\| = \delta_2$ and $\|S_\tau(t_1, x_1) - (t_0, x_0)\| < \delta_2$ for all $\tau \in [0, \tau_0)$.

By (3.9) we have $\|f(t, x)\| \leq \|f(t_0, x_0)\| + \|f(t, x) - f(t_0, x_0)\| \leq 2f_M$ for all $(t, x) \in \overline{B_{\delta_2}(t_0, x_0)}$. This yields with $\tau_0 \leq |t_0 - t_1| \leq \delta^*$

$$\begin{aligned}
\delta_2 &= \|S_{\tau_0}(t_1, x_1) - (t_0, x_0)\| \\
&\leq |t_0 - t_1| + \|S_{\tau_0}^x(t_1, x_1) - x_1\| + \|x_1 - x_0\| \\
&= |t_0 - t_1| + \left\|\int_0^{\tau_0} f(S_\tau(t_1, x_1))\, d\tau\right\| + \|x_1 - x_0\| \\
&\leq \delta^* + 2f_M|t_0 - t_1| + \delta^* \\
&\leq \delta^*(2f_M + 2) \\
&\leq \frac{\delta_2}{2},
\end{aligned}$$

which is a contradiction. Hence, $S_\tau(t_1, x_1)$ remains in $B_{\delta_2}(t_0, x_0)$ for all $\tau \in [0, t_0 - t_1]$. Thus, $\|f(S_\theta(t_1, x_1))\| \leq 2f_M$ for all $\theta \in [0, t_0 - t_1]$,

$$\|x_1' - x_1\| = \left\|\int_0^{t_0 - t_1} f(S_\theta(t_1, x_1))\, d\theta\right\| \leq 2f_M|t_0 - t_1| \leq 2f_M\delta^*$$

and hence

$$\|x_1' - x_0\| \leq \|x_1' - x_1\| + \|x_1 - x_0\| \leq \delta^*(2f_M + 1) \leq \frac{\delta}{2}.$$

Thus, by Proposition 3.1, the points (t_0, x_1') and (t_0, x_0) have the same ω-limit set since $\|x_1' - x_0\| \leq \frac{\delta}{2}$. The points (t_1, x_1) and (t_0, x_1') have the same ω-limit set since they lie on the same trajectory. This proves the proposition. ∎

Using the fact that G is connected we can now prove Proposition 3.3 which shows that all points of G have the same ω-limit set.

Proposition 3.3. *Let the assumptions of Theorem* 3.1 *be satisfied. Then* $\varnothing \neq \omega(s,p) = \omega(t,q) =: \Omega \subset G$ *for all* $(s,p), (t,q) \in G$.

Proof. Fix a point $(t_0, x_0) \in G$. By Lemma 2.1 $\varnothing \neq \omega(t_0, x_0) =: \Omega \subset G$. Now consider an arbitrary point $(s,p) \in G$. By Proposition 3.2, $\omega(s,p) = \omega(t,q)$ holds for all (t,q) in a neighborhood of (s,p). Hence $V_1 := \{(s,p) \in G \mid \omega(s,p) = \omega(t_0, x_0)\}$ and $V_2 := \{(s,p) \in G \mid \omega(s,p) \neq \omega(t_0, x_0)\}$ are open sets in G. Since $G = V_1 \dot{\cup} V_2$, $(t_0, x_0) \in V_1$ and G is connected, V_2 must be empty and $V_1 = G$. ∎

Proof of Theorem 3.1. Fix $k = \frac{1}{2}$. Then, according to Proposition 3.1, there are constants $\delta > 0$ and $C \geq 1$ and, according to Proposition 3.2 and 3.3, there is a constant $\delta^* > 0$ and a set Ω which is the ω-limit set of each point of G. Choose a point $(t_0, p_0) \in \Omega$. We have $(t_0, p_0) \in \omega(t_0, p_0)$. Thus, there is an $N \in \mathbb{N}$ with $N \geq \frac{2}{\nu T} \ln(2C)$, so that $S_{N \cdot T}^x(t_0, p_0) \in B_{\frac{\delta_3}{2}}(p_0) := \{x \in \mathbb{R}^n \mid \|x - p_0\| < \frac{\delta_3}{2}\}$, where $\delta_3 := \frac{\delta}{2}$. Set $U_0 := \overline{B_{\delta_3}(p_0)} \subset \mathbb{R}^n$. We define a continuous Poincaré-like map

$$P \colon \begin{cases} U_0 & \longrightarrow & U_0 \\ q & \longmapsto & S_{N \cdot T}^x(t_0, q). \end{cases} \tag{3.10}$$

To prove $P(U_0) \subset U_0$ we calculate with (3.3) $\|P(q) - P(p_0)\| \leq Ce^{-\nu \frac{N \cdot T}{2}} \|q - p_0\| \leq \frac{\delta_3}{2}$. Iteration yields

$$\|P^j(q) - P^j(p_0)\| \leq Ce^{-\nu \frac{N \cdot T}{2} j} \|q - p_0\| \leq \frac{\delta_3}{2^j} \tag{3.11}$$

for all $q \in U_0$. Now $\|P(q) - p_0\| \leq \|P(q) - P(p_0)\| + \|P(p_0) - p_0\| \leq \frac{\delta_3}{2} + \frac{\delta_3}{2}$, and thus $P(U_0) \subset U_0$. In Lemma 3.1 we will show that the diameter of $P^j(U_0) =: U_j$ decreases.

Lemma 3.1. *We define the compact sets* $U_j \subset \mathbb{R}^n$ *for all* $j \in \mathbb{N}$ *by* $U_j := P^j(U_0)$. *Then the following statements hold for all* $j \in \mathbb{N}$:

$$U_j \subset U_{j-1} \tag{3.12}$$

$$\operatorname{diam} U_j \leq \frac{\delta_3}{2^{j-1}} \tag{3.13}$$

Proof. The sets U_j are compact by induction, because they are images of the compact set U_{j-1} under the continuous map P. The inclusion (3.12)

follows easily from $U_1 = P(U_0) \subset U_0$. Indeed, we have $P^j(U_0) = P^{j-1}P(U_0) \subset P^{j-1}(U_0)$ for all $j \in \mathbb{N}$. In order to prove (3.13) we have with (3.11)

$$
\begin{aligned}
\operatorname{diam} U_j &= \max_{q',q'' \in U_0} \|P^j(q') - P^j(q'')\| \\
&\leq \max_{q' \in U_0} \|P^j(q') - P^j(p_0)\| + \max_{q'' \in U_0} \|P^j(p_0) - P^j(q'')\| \\
&\leq 2\frac{\delta_3}{2^j}.
\end{aligned}
$$

This proves (3.13). ∎

In Lemma 3.1 we have constructed a sequence of compact sets U_j with decreasing diameter, so we know that there is one and only one point $\tilde{p}$ which lies in all U_j, $j \in \mathbb{N}$. Since $P(\tilde{p})$ lies in all U_j as well, $\tilde{p}$ is a fixed point of P. Hence, $S^x_{N \cdot T}(t_0, \tilde{p}) = \tilde{p}$, and thus $(t_0, \tilde{p})$ is a point of a periodic orbit Ω with period $N \cdot T$. We will show that, in fact, Ω is a periodic orbit with period T later. Since $\tilde{p} = p_0 + \eta$ with $\|\eta\| \leq \frac{\delta}{2}$, the ω-limit sets of (t_0, p_0) and $(t_0, \tilde{p})$ are equal by Proposition 3.1. Thus $(t_0, p_0) \in \omega(t_0, p_0) = \omega(t_0, \tilde{p}) = \Omega$ and (t_0, p_0) is a point of the periodic orbit Ω. Thus, $\Omega \subset G$.

We still have to prove that the period of the periodic orbit is T. We have shown that $S^x_{N \cdot T}(t_0, p_0) = p_0$. Now we claim that $S^x_T(t_0, p_0) = p_0$.

Set $y_0 := S^x_T(t_0, p_0)$. Then $S_{N \cdot T}(t_0, y_0) = (t_0, y_0)$. Note that f is also t-periodic with period $N \cdot T$. Hence, we can apply Proposition 3.1 to $NT \cdot S^1 \times \mathbb{R}^n$ which shows that $\omega_N(t_0, p_0) = \omega_N(t_0, y_0)$, where ω_N denotes the ω-limit set with respect to the cylinder $NT \cdot S^1 \times \mathbb{R}^n$. But as $\Omega_1 := \{S_t(t_0, p_0) \mid t \in [0, N \cdot T]\}$ is a periodic orbit, we have $\omega_N(t_0, p_0) = \Omega_1$ and $\omega_N(t_0, y_0) = \Omega_2 := \{S_t(t_0, y_0) \mid t \in [0, N \cdot T]\}$. So the two periodic orbits coincide which implies $p_0 = y_0$. Thus, Ω is a periodic orbit with period T.

In the last part of the proof we show that the maximal real part $-\nu_0$ of the Floquet exponents of Ω satisfies $-\nu_0 \leq -\nu$. We assume the opposite, i.e. there is a Floquet exponent with real part $-\nu_0 > -\nu$. Assume first that this Floquet exponent is real. Then by the Floquet Theorem there is a solution $y(t) = p(t)e^{-\nu_0 t}$ with $p(0) = p(T)$ of the first variation equation $\dot{y} = D_x f(t, \tilde{x}(t))y$, where $\tilde{x}(t) := S^x_t(0, p_0)$ denotes the periodic orbit. Thus,

$$
\dot{p}(t)e^{-\nu_0 t} - \nu_0 p(t)e^{-\nu_0 t} = D_x f(t, \tilde{x}(t))p(t)e^{-\nu_0 t}
$$

that is

$$
\dot{p}(t) - \nu_0 p(t) = D_x f(t, \tilde{x}(t))p(t). \tag{3.14}
$$

We have

$$
\frac{1}{2}\frac{d}{dt}\left[p(t)^T M(t, \tilde{x}(t))p(t)\right] = p(t)^T M(t, \tilde{x}(t))\dot{p}(t) + \frac{1}{2}p(t)^T M'(t, \tilde{x}(t))p(t). \tag{3.15}
$$

Hence, by our assumption, $L_M(t, \tilde{x}(t)) \leq -\nu$, (3.14) and (3.15) we have

$$
\begin{aligned}
-\nu_0 \; &> \; -\nu \\
&\geq \; \frac{1}{T} \int_0^T \frac{1}{p(t)^T M(t, \tilde{x}(t))p(t)} \Big(p(t)^T M(t, \tilde{x}(t)) D_x f(t, \tilde{x}(t))p(t) \\
&\qquad + \frac{1}{2} p(t)^T M'(t, \tilde{x}(t))p(t) \Big) \, dt \\
&= \; \frac{1}{T} \int_0^T \frac{1}{p(t)^T M(t, \tilde{x}(t))p(t)} \Big(p(t)^T M(t, \tilde{x}(t))\dot{p}(t) \\
&\qquad -\nu_0 p(t)^T M\big(t, \tilde{x}(t)\big)p(t) + \frac{1}{2} p(t)^T M'(t, \tilde{x}(t))p(t) \Big) \, dt \\
&= \; \frac{1}{T} \int_0^T \frac{1}{2} \frac{\frac{d}{dt}\big[p(t)^T M(t, \tilde{x}(t))p(t)\big]}{p(t)^T M(t, \tilde{x}(t))p(t)} \, dt - \nu_0 \\
&= \; \frac{1}{2T} \ln \frac{p(T)^T M(T, \tilde{x}(T))p(T)}{p(0)^T M(0, \tilde{x}(0))p(0)} - \nu_0 \; = \; -\nu_0
\end{aligned}
$$

since M, $\tilde{x}$ and p are t-periodic with period T. This is a contradiction.

Now consider the case, where $-\nu_0 + i\mu_0$ is a complex Floquet exponent with $-\nu_0 > \nu$. Then by the Floquet Theorem there is a complex-valued solution $p(t)e^{(-\nu_0 + i\mu_0)t}$ of $\dot{y} = D_x f(t, \tilde{x}(t))y$ with $p(0) = p(T) =: p_1 + ip_2$. Also $\overline{p(t)}e^{(-\nu_0 - i\mu_0)t}$ is a solution and so are the real functions

$$
\begin{aligned}
q_1(t) \; &:= \; \frac{p(t)e^{i\mu_0 t} + \overline{p(t)}e^{-i\mu_0 t}}{2} e^{-\nu_0 t} \\
q_2(t) \; &:= \; \frac{p(t)e^{i\mu_0 t} - \overline{p(t)}e^{-i\mu_0 t}}{2i} e^{-\nu_0 t}.
\end{aligned}
$$

With $\dot{q}_i = D_x f(t, \tilde{x}(t))q_i$ for $i = 1, 2$ we have for $q = q_i$ in a similar way as above

$$
\begin{aligned}
0 \; &> \; \ln \frac{q_1(T)^T M(T, \tilde{x}(T))q_1(T)}{q_1(0)^T M(0, \tilde{x}(0))q_1(0)} \\
&= \; \ln \frac{(p_1 \cos(\mu_0 T) - p_2 \sin(\mu_0 T))^T M(T, \tilde{x}(T))(p_1 \cos(\mu_0 T) - p_2 \sin(\mu_0 T))}{p_1^T M(T, \tilde{x}(T))p_1}
\end{aligned}
$$

Note that $M(0, \tilde{x}(0)) = M(T, \tilde{x}(T))$. This implies

$$
\begin{aligned}
p_1^T M(T, \tilde{x}(T))p_1 \; &> \; \cos^2(\mu_0 T)p_1^T M(T, \tilde{x}(T))p_1 + \sin^2(\mu_0 T)p_2^T M(T, \tilde{x}(T))p_2 \\
&\quad -2\cos(\mu_0 T)\sin(\mu_0 T)p_1^T M(T, \tilde{x}(T))p_2
\end{aligned}
$$

that is

$$
\begin{aligned}
0 \; &> \; -\sin^2(\mu_0 T)\left[p_1^T M(T, \tilde{x}(T))p_1 - p_2^T M(T, \tilde{x}(T))p_2 \right] \\
&\quad -2\cos(\mu_0 T)\sin(\mu_0 T)p_1^T M(T, \tilde{x}(T))p_2.
\end{aligned}
$$

For $q_2(t)$ we obtain in a similar way

$$
\begin{aligned}
0 \quad > \quad & \sin^2(\mu_0 T) \left[p_1^T M(T, \tilde{x}(T)) p_1 - p_2^T M(T, \tilde{x}(T)) p_2 \right] \\
& + 2 \cos(\mu_0 T) \sin(\mu_0 T) p_1^T M(T, \tilde{x}(T)) p_2.
\end{aligned}
$$

Adding both inequalities we obtain $0 > 0$ which is a contradiction. This completes the proof of Theorem 3.1. ∎

4. Necessity

We assume Ω to be an exponentially asymptotically stable periodic orbit. In order to prove the necessity of the assumptions of Theorem 3.1 we construct a Riemannian metric on Ω using Floquet theory in Theorem 4.1. Then we use a Lyapunov function to prolongate the Riemannian metric to a compact subset of the basin of attraction in Theorem 4.2.

Theorem 4.1 (Local necessity). *Assume that $f \in C^0(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is a t-periodic function with period T, and all partial derivatives of order one with respect to x are continuous functions of (t, x). Let $\Omega := \{(t, \tilde{x}(t)) \subset T \cdot S^1 \times \mathbb{R}^n\}$ be an exponentially asymptotically stable periodic orbit and let the maximal real part of the Floquet exponents be $-\nu < 0$. Then for all $\epsilon > 0$ there exists a Riemannian metric M in the sense of Definition 2.3, such that $L_M(t, x) \leq -\nu + \epsilon$ holds for all $(t, x) \in \Omega$.*

Proof. Fix the point $(0, x_0) \in \Omega$. We consider the first variation equation along the periodic orbit, namely

$$
\dot{y} \quad = \quad D_x f(S_\theta(0, x_0)) y. \tag{4.1}
$$

By Floquet theory, every fundamental matrix $W(\theta)$ of the linear system (4.1) can be expressed as $W(\theta) = P(\theta) e^{B\theta}$, where $P(\theta)$ is a θ-periodic matrix with period T and B is a constant $(n \times n)$-matrix. $P(\theta)$ is C^1 with respect to θ. The real parts of the Floquet exponents, which are the eigenvalues of B, are uniquely defined and their real part is $\leq -\nu$ by assumption.

We fix $\epsilon > 0$. In Lemma B.1 of [6] we transform the matrix B into a special normal form $S^{-1} B S = A$ where S depends on ϵ such that

$$
w^T A w \leq (-\nu + \epsilon) \|w\|^2 \tag{4.2}
$$

holds for all $w \in \mathbb{R}^n$. This is needed in the sequel to ensure that L_M attains only negative values. The matrix S is constructed in the following way: for real eigenvalues with same algebraic and geometric multiplicities write the eigenvectors in the columns of S. For a sequence of generalized eigenvectors satisfying

$(B - \lambda I)v_{k+1} = v_k$ for $k = 0, \ldots, N$ and $(B - \lambda I)v_0 = 0$ write $\epsilon^k v_k$ in the columns. For complex eigenvectors write their real and imaginary part in the columns; for details cf. [6].

We define the Riemannian metric for all points of the periodic orbit by

$$M(S_\theta(0, x_0)) = (P^{-1}(\theta))^T (S^{-1})^T S^{-1} P^{-1}(\theta). \tag{4.3}$$

M is C^1 and θ-periodic with period T. $M(S_\theta(0, x_0))$ is clearly symmetric and positive definite. Indeed, $x^T M(S_\theta(0, x_0))x = \|S^{-1}P^{-1}(\theta)x\|^2 \geq 0$ and $x^T M(S_\theta(0, x_0))x = 0$ if and only if $x = 0$ since S and $P(\theta)$ have full rank. We check now that $L_M(S_\theta(0, x_0))$ is strictly negative for all $\theta \in [0, T]$, i.e., for all points of the periodic orbit.

Let us calculate L_M. Since $M = M^T$ we have

$$M'(S_\theta(0, x_0)) = 2(P^{-1}(\theta))^T (S^{-1})^T S^{-1} \dot{P}^{-1}(\theta).$$

By using $\frac{d}{d\theta}[P^{-1}(\theta)P(\theta)] = 0$ we get $\dot{P}^{-1}(\theta) = -P^{-1}(\theta)\dot{P}(\theta)P^{-1}(\theta)$. Also, since $P(\theta)e^{B\theta}$ is a solution of (4.1) we have $\dot{P}(\theta) = D_x f(S_\theta(0, x_0))P(\theta) - P(\theta)B$. Altogether, we get

$$M'(S_\theta(0, x_0)) = -2M(S_\theta(0, x_0))D_x f(S_\theta(0, x_0)) + 2(P^{-1}(\theta))^T (S^{-1})^T S^{-1} B P^{-1}(\theta)$$

and hence we have

$$
\begin{aligned}
L_M(S_\theta(0, x_0); v) &= v^T\left[M(S_\theta(0, x_0))D_x f(S_\theta(0, x_0)) + \frac{1}{2}M'(S_\theta(0, x_0)) \right]v \\
&= v^T (P^{-1}(\theta))^T (S^{-1})^T S^{-1} B P^{-1}(\theta)v.
\end{aligned}
$$

We define $w := S^{-1}P^{-1}(\theta)v$. We have $\|w\| = 1$, if and only if $v^T M(S_\theta(0, x_0))v = 1$. Thus, $L_M(S_\theta(0, x_0); v) = w^T S^{-1} B S w = w^T A w$ and hence

$$
\begin{aligned}
L_M(S_\theta(0, x_0)) &= \max_{v \in \mathbb{R}^n, v^T M(S_\theta(0,x_0))v=1} L_M(S_\theta(0, x_0); v) \\
&= \max_{w \in \mathbb{R}^n, \|w\|=1} w^T A w \\
&\leq -\nu + \epsilon
\end{aligned}
$$

by (4.2). Up to now we have defined $M(t, x)$ for points (t, x) of the periodic orbit. Now we prolongate M in a C^1-way to $T \cdot S^1 \times \mathbb{R}^n$. We set

$$M(\theta, x) := M(S_\theta(0, x_0)),$$

i.e., M only depends on θ and not on x. This is clearly a C^1-prolongation. ∎

Now we prove the global necessity, i.e. the existence of a Riemannian metric M which satisfies $L_M(t, x) < 0$ not only for all points of the periodic orbit, but moreover, for all points of any given compact set $K \subset A(\Omega)$. In the proof we use the local Riemannian metric of Theorem 4.1 and a Lyapunov function.

Theorem 4.2 (Global necessity). *Assume that $f \in C^0(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is a t-periodic function with period T, and all partial derivatives of order one with respect to x are continuous functions of (t, x). Let $\Omega := \{(t, \tilde{x}(t)) \in T \cdot S^1 \times \mathbb{R}^n\}$ be an exponentially asymptotically stable periodic orbit, $A(\Omega)$ be its basin of attraction, and let the maximal real part of the Floquet exponents be $-\nu < 0$. Then for all $\epsilon > 0$ and all compact sets K with $\Omega \subset K \subset A(\Omega)$ there exists a Riemannian metric M, such that $L_M(t, x) \leq -\nu + \epsilon$ holds for all $(t, x) \in K$.*

Proof. Fix a set K and $\epsilon > 0$. Denote the Riemannian metric of Theorem 4.1 for $\frac{1}{2}\epsilon$ by $\tilde{M}(t, x)$, which is defined for all $(t, x) \in K$. Since $L_{\tilde{M}}$ is continuous and Ω is compact, there is a neighborhood Ω_δ such that $L_{\tilde{M}}(t, x) \leq -\nu + \epsilon$ holds for all $(t, x) \in \Omega_\delta$. Set $\mu := \max_{(t,x)\in K} L_{\tilde{M}}(t, x)$. If $\mu \leq -\nu + \epsilon$, then choose $\tilde{M}(t, x)$ as the Riemannian metric $M(t, x)$.

Now let us assume $\mu + \nu - \epsilon > 0$. There is a strict Lyapunov function $V \colon K \to \mathbb{R}_0^+$, i.e. $V \in C^1(K, \mathbb{R})$, $V(t, x) > 0$ for $(t, x) \in K \setminus \Omega$ and $V(t, x) = 0$ for $(t, x) \in \Omega$, $V'(t, x) < 0$ for $(t, x) \in K \setminus \Omega$ and $V'(t, x) = 0$ for $(t, x) \in \Omega$. Here V' denotes the orbital derivative, i.e. $V' = \frac{\partial V}{\partial t} + \langle \nabla_x V, f \rangle$. Thus, $-\nu' := \max_{(t,x)\in K \cap (T \cdot S^1 \times \mathbb{R}^n \setminus \Omega_\delta)} V'(t, x) < 0$ holds. The existence of such a Lyapunov function can be shown similarly to the case of an equilibrium cf. [14], Theorem 8. Set $c := \frac{\mu + \nu - \epsilon}{\nu'} > 0$ and $W(t, x) := cV(t, x)$. Define the Riemannian metric by

$$M(t, x) := e^{2W(t,x)} \tilde{M}(t, x).$$

We show that $L_M(t, x) = L_{\tilde{M}}(t, x) + W'(t, x)$. We set $w := e^{W(t,x)}v$ and verify

$$
\begin{aligned}
&L_M(t, x) \\
&= \max_{v \in \mathbb{R}^n, v^T M(t,x)v=1} v^T \Big[e^{2W(t,x)} \tilde{M}(t, x) D_x f(t, x) \\
&\qquad\qquad\qquad + W'(t, x) e^{2W(t,x)} \tilde{M}(t, x) + \frac{1}{2} e^{2W(t,x)} \tilde{M}'(t, x) \Big] v \\
&= \max_{\substack{v \in \mathbb{R}^n, \\ v^T \tilde{M}(t,x)v = e^{-2W(t,x)}}} e^{2W(t,x)} v^T \Big[\tilde{M}(t, x) D_x f(t, x) + \frac{1}{2} \tilde{M}'(t, x) \Big] v + W'(t, x) \\
&= \max_{w \in \mathbb{R}^n, w^T \tilde{M}(t,x)w=1} w^T \Big[\tilde{M}(t, x) D_x f(t, x) + \frac{1}{2} \tilde{M}'(t, x) \Big] w + W'(t, x) \\
&= L_{\tilde{M}}(t, x) + W'(t, x).
\end{aligned}
$$

Now we calculate $L_M(t, x)$ and distinguish between the cases $(t, x) \in \Omega_\delta$ and $(t, x) \in K \cap (T \cdot S^1 \times \mathbb{R}^n \setminus \Omega_\delta)$. In the first case we have $L_{\tilde{M}}(t, x) \leq -\nu + \epsilon$ and $W'(t, x) = cV'(t, x) \leq 0$. In the second case we have $L_{\tilde{M}}(t, x) \leq \mu$ and $W'(t, x) \leq -c\nu' = -(\mu + \nu - \epsilon)$. Thus, in both cases $L_M(t, x) = L_{\tilde{M}}(t, x) + W'(t, x) \leq -\nu + \epsilon$. ∎

5. Main results

In this section we summarize the results of Sections 3 and 4 by formulating Theorem 5.1, which gives equivalent conditions for existence and uniqueness of an exponentially asymptotically stable periodic orbit, and Theorem 5.2, which provides a characterization of the basin of attraction of such a periodic orbit.

Theorem 5.1. *Assume that $f \in C^0(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is a t-periodic function with period T, and all partial derivatives of order one with respect to x are continuous functions of (t, x). Then the following two conditions are equivalent.*

(i) *The system has an exponentially asymptotically stable periodic orbit.*

(ii) *There is a set $\varnothing \neq G \subset T \cdot S^1 \times \mathbb{R}^n$ and a Riemannian metric M such that G is connected, compact, positively invariant, and $L_M(t, x) < 0$ for all $(t, x) \in G$.*

Proof. (ii)$\Rightarrow$(i) follows by Theorem 3.1. To prove (i)$\Rightarrow$(ii) we fix $(0, x_0) \in \Omega$, where Ω denotes the periodic orbit. Let $-\nu$ be the largest real part of all Floquet exponents of Ω. By Theorem 4.1 with $\epsilon = \frac{\nu}{2}$ we can choose a δ-neighborhood of Ω

$$\Omega_\delta^M := \{(t, x) \in T \cdot S^1 \times \mathbb{R}^n \mid [S_t^x(0, x_0) - x]^T M(S_t(0, x_0)) [S_t^x(0, x_0) - x] < \delta^2\}$$

such that $L_M(t, x) \leq -\frac{\nu}{2}$ holds for all $(t, x) \in \Omega_\delta^M$. By the orbital stability of Ω there is a $\delta' > 0$ such that $(t, x) \in \Omega_{\delta'}^M$ implies $S_\theta(t, x) \in \Omega_\delta^M$ for all $\theta \geq 0$. Set $H := \overline{\{x \in \mathbb{R}^n \mid (0, x) \in \Omega_{\delta'}^M\}}$. Then

$$T \cdot S^1 \times \mathbb{R}^n \supset K_0 := \{S_\theta(0, x) \mid x \in H, \theta \geq 0\} = \{S_\theta(0, x) \mid x \in H, \theta \in [0, T]\}$$

by (3.8) of Proposition 3.1. Clearly, K_0 is connected, compact and positively invariant, and $L_M(t, x) < 0$ holds for all points of K_0 since $K_0 \subset \Omega_\delta^M$. This proves the theorem with $G := K_0$. Note that, moreover, the inclusion $\Omega \subset K_0$ holds for this set. ∎

Theorem 5.2. *Assume that $f \in C^0(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is a t-periodic function with period T, and all partial derivatives of order one with respect to x are continuous functions of (t, x). Moreover, let Ω be an exponentially asymptotically stable periodic orbit. Then its basin of attraction satisfies*

$$A(\Omega) = \bigcup_{K \in \tilde{A}} K$$

where

$$\tilde{A} := \left\{ K \subset T \cdot S^1 \times \mathbb{R}^n \;\middle|\; \begin{array}{l} \Omega \subset K \text{ connected, compact, positively invariant,} \\ \text{and there is a Riemannian metric } M \text{ such that} \\ L_M(t, x) < 0 \text{ holds for all } (t, x) \in K \end{array} \right\}.$$

Proof. The inclusion $\supset$ was proven in Theorem 3.1. We now prove the other inclusion. Let $(t_0, x_0) \in A(\Omega)$. There is a $T_0 > 0$, so that $S_{T_0}(t_0, x_0) \in K_0$, where K_0 is the set defined in the proof of Theorem 5.1. Hence, $(t_0, x_0) \in K := S_{-T_0} K_0$. The set K is connected, compact and positively invariant, and furthermore, $\Omega \subset K \subset A(\Omega)$. By Theorem 4.2 with $\epsilon = \frac{\nu}{2}$ there exists a Riemannian metric M such that $L_M(t, x) < 0$ holds for all $(t, x) \in K$. Hence, $(t_0, x_0) \in K \subset \tilde{A}$, and the theorem is proven. ∎

6. Special Riemannian metrics – calculation

In this section we prove three results for special Riemannian metrics. In Proposition 6.1 we assume that we know the periodic orbit explicitly and that the Riemannian metric $M(t, x) = M(t)$ only depends on t. In this case the positively invariant set can be obtained in an easy way. In Lemma 6.1 we give a formula for L_M for Riemannian metrics of the form $M(t, x) = (S^{-1}(t, x))^T S^{-1}(t, x)$. This special form is motivated by the proof of Theorem 4.1, cf. (4.3). Finally, we give in Lemma 6.2 a formula to calculate L_M via the eigenvalues of a symmetric matrix. A corollary deals with the two-dimensional case.

Proposition 6.1. *Let* $\Omega = \{(t, \tilde{x}(t)) \in T \cdot S^1 \times \mathbb{R}^n\}$ *be a periodic orbit and let* $M(t)$ *be a Riemannian metric which only depends on* t. *Moreover assume* $L_M(t, x) < 0$ *for all* $(t, x) \in G^r$ *where* $r > 0$ *and*

$$G^r := \{(t, x) \in T \cdot S^1 \times \mathbb{R}^n \mid [x - \tilde{x}(t)]^T M(t)[x - \tilde{x}(t)] \leq r^2\}.$$

Then Ω *is exponentially asymptotically stable, and for its basin of attraction* $A(\Omega)$ *the following inclusion holds:*

$$G^r \subset A(\Omega).$$

Proof. We show that G^r satisfies the conditions of Theorem 3.1. G^r is connected and compact, and $L_M(t, x) < 0$ holds for all $(t, x) \in G^r$. It rests to show that G^r is positively invariant. We consider a point (t_0, x_0) with $(x_0 - \tilde{x}_0)^T M(t_0)(x_0 - \tilde{x}_0) = r^2$, where $\tilde{x}_0 := \tilde{x}(t_0)$ is the corresponding point of the periodic orbit, and show that the orbital derivative of the following expression is not positive:

$$\frac{d}{dt} \left[[S_t^x(t_0, x_0) - S_t^x(t_0, \tilde{x}_0)]^T M(t_0 + t) [S_t^x(t_0, x_0) - S_t^x(t_0, \tilde{x}_0)] \right]$$

$$= 2 [S_t^x(t_0, x_0) - S_t^x(t_0, \tilde{x}_0)]^T M(t_0 + t) [f(S_t(t_0, x_0)) - f(S_t(t_0, \tilde{x}_0))]$$
$$+ [S_t^x(t_0, x_0) - S_t^x(t_0, \tilde{x}_0)]^T M'(t_0 + t) [S_t^x(t_0, x_0) - S_t^x(t_0, \tilde{x}_0)]$$

and hence

$$\frac{d}{dt}\left[[S_t^x(t_0, x_0) - S_t^x(t_0, \tilde{x}_0)]^T M(t_0 + t)[S_t^x(t_0, x_0) - S_t^x(t_0, \tilde{x}_0)]\right]$$

$$= 2\int_0^1 \left[[S_t^x(t_0, x_0) - S_t^x(t_0, \tilde{x}_0)]^T\right.$$

$$\cdot\left[M(t_0 + t)D_xf\left(t + t_0, S_t^x(t_0, \tilde{x}_0) + [S_t^x(t_0, x_0) - S_t^x(t_0, \tilde{x}_0)]\eta\right)\right.$$

$$\left.\left. + \frac{1}{2}M'(t_0 + t)\right] \cdot [S_t^x(t_0, x_0) - S_t^x(t_0, \tilde{x}_0)]\right] d\eta$$

Evaluated at $t = 0$ this derivative gives

$$2\int_0^1 \left[(x_0 - \tilde{x}_0)^T\left(M(t_0)D_xf\left(t_0, \tilde{x}_0 + [x_0 - \tilde{x}_0]\cdot\eta\right) + \frac{1}{2}M'(t_0)\right)(x_0 - \tilde{x}_0)\right] d\eta.$$

Note that $x^* = x^*(\eta) := \tilde{x}_0 + [x_0 - \tilde{x}_0]\eta$ with $\eta \in [0, 1]$ satisfies $(t_0, x^*) \in G^r$ since $(t_0, x_0) \in G^r$ and

$$(x^* - \tilde{x}_0)^T M(t_0)(x^* - \tilde{x}_0) = \eta^2(x_0 - \tilde{x}_0)^T M(t_0)(x_0 - \tilde{x}_0) \le \eta^2 r^2 \le r^2.$$

Thus,

$$\frac{d}{dt}\left[[S_t^x(t_0, x_0) - S_t^x(t_0, \tilde{x}_0)]^T M(t_0 + t)[S_t^x(t_0, x_0) - S_t^x(t_0, \tilde{x}_0)]\right]\Bigg|_{t=0}$$

$$= 2\int_0^1 L_M\left(t_0, x^*(\eta); x_0 - \tilde{x}_0\right) d\eta \le 0.$$

This proves the proposition. ∎

In order to evaluate L_M we have Lemmas 6.1 and 6.2. Lemma 6.1 deals with the case that $M = (S^{-1}(t, x))^T S^{-1}(t, x)$ and is motivated by the use of a diffeomorphism. Let $\phi: G \to G'$ be a C^2-diffeomorphism and $D_x\phi(t, x) = S^{-1}(t, x)$. Then L_M is L_I of the transformed system $\dot{y} = g(t, y)$ with $y = \phi(x)$. An easy example which we will use in the next section is $\phi(x) = S^{-1}(t)x$.

Lemma 6.1. *Let $M(t, x) = (S^{-1}(t, x))^T S^{-1}(t, x)$ and let $S(t, x)$ have full rank for all (t, x). Then*

$$L_M(t, x) = \max_{v \in \mathbb{R}^n, \|v\|=1} v^T\left[S^{-1}(t, x)D_xf(t, x)S(t, x) + (S^{-1})'(t, x)S(t, x)\right]v$$

where $(S^{-1})'(t, x) = \frac{\partial S^{-1}}{\partial t}(t, x) + \langle\nabla_x S^{-1}(t, x), f(t, x)\rangle$.

Proof. We set $v := S^{-1}w$. Then we have, dropping the dependencies on (t, x),

$$
\begin{aligned}
L_M &= \max_{w \in \mathbb{R}^n, w^T M w = 1} w^T \left[M D_x f + \frac{1}{2} M' \right] w \\
&= \max_{v \in \mathbb{R}^n, v^T v = 1} \left[v^T S^{-1} D_x f S v + \frac{1}{2} v^T S^T ((S^{-1})^T)' v + \frac{1}{2} v^T (S^{-1})' S v \right] \\
&= \max_{v \in \mathbb{R}^n, v^T v = 1} v^T \left[S^{-1} D_x f S + (S^{-1})' S \right] v
\end{aligned}
$$

which proves the assertion. ∎

In order to calculate the maximum of $w^T B w$ for all w with $\|w\| = 1$ we use the eigenvalues of the symmetric matrix $B^s = \frac{1}{2}(B + B^T)$.

Lemma 6.2. *Let B be an $(n \times n)$-matrix. Let λ be the largest eigenvalue of $B^s = \frac{1}{2}(B + B^T)$. Then $\max_{\|v\|=1} v^T B v = \lambda$.*

Proof. Since B^s is a symmetric matrix, it has real eigenvalues $\lambda_1, \ldots, \lambda_n$ with corresponding real eigenvectors $w_1, \ldots, w_n$, which can be chosen to be orthonormal. Write $v \in \mathbb{R}^n$ with $\|v\| = 1$ as $v = \sum_{i=1}^n \alpha_i w_i$ with $\sum_{i=1}^n \alpha_i^2 = 1$. Denoting $B^a = \frac{1}{2}(B - B^T)$, we have $b_{ij}^a = -b_{ji}^a$ since B^a is antisymmetric. In particular, $b_{ii}^a = 0$. Denoting $v = (v_1, \ldots, v_n)$, we get

$$
\begin{aligned}
v^T B v &= \frac{1}{2} v^T (B + B^T) v + \frac{1}{2} v^T (B - B^T) v \\
&= \sum_{i,j=1}^n \alpha_i w_i^T B^s \alpha_j w_j + \sum_{i,j=1}^n v_i b_{ij}^a v_j \\
&= \sum_{i,j=1}^n \alpha_i \alpha_j \lambda_j w_i^T w_j + \sum_{i=1}^n \sum_{1 \le j < i} v_i (b_{ij}^a + b_{ji}^a) v_j
\end{aligned}
$$

and hence

$$
v^T B v = \sum_{i=1}^n \alpha_i^2 \lambda_i \le \lambda \sum_{i=1}^n \alpha_i^2 = \lambda.
$$

The maximum is attained for $v = w_J$, where w_J is the eigenvector with eigenvalue $\lambda_J = \lambda$. ∎

The following corollary to Lemma 6.2 deals with the two-dimensional case.

Corollary. *Let $B := \begin{pmatrix} a & d_1 \\ d_2 & b \end{pmatrix}$. Then*

$$
\max_{\|v\|=1} v^T B v = \frac{1}{2} \left[a + b + \sqrt{(d_1 + d_2)^2 + (a - b)^2} \right].
$$

7. Special Riemannian metrics – construction

In the proof of Theorem 4.2 we have shown the existence of a Riemannian metric, but we did not explicitly construct one. Instead, we proved its existence using Floquet theory, which is often not analytically accessible, and a Lyapunov function, which is not known in general. In this section we will present two methods to construct Riemannian metrics explicitly. However, even though an asymptotically stable periodic orbit exists, they might not lead to a suitable Riemannian metric. Both methods require a known periodic orbit and are based on the use of eigenvectors. They will be used to prove the exponential asymptotic stability of the periodic orbit and to determine a part of their basin of attraction. For the first approach we calculate for a given periodic orbit $\tilde{x}(t)$ the eigenvectors of the matrices $D_x f(t, \tilde{x}(t))$ for all $t \in [0, T]$ and use this information to construct a Riemannian metric $M(t)$ which only depends on t. The second method uses a Riemannian metric M which is constant. It requires a periodic system with a small parameter γ which becomes an autonomous system for $\gamma = 0$. If the periodic orbit becomes an equilibrium point for $\gamma = 0$, then we can calculate the eigenvectors of this equilibrium point and construct a suitable Riemannian metric.

7.1. Eigenvectors of $D_x f(t, \tilde{x}(t))$. We consider a system of autonomous ordinary differential equations $\dot{x} = g(x)$. If x_0 is an equilibrium, i.e. $g(x_0) = 0$, one can determine its stability – if it is hyperbolic – by calculating the eigenvalues of $Dg(x_0)$. If the real parts of all of them are negative, the equilibrium is exponentially asymptotically stable.

One might think of a generalization of this method to time-periodic systems. Consider the system $\dot{x} = f(t, x)$, where f is t-periodic with period T. Assume that $\tilde{x}(t)$ is a periodic orbit. One might expect the condition that the real parts of the eigenvalues of $D_x f(t, \tilde{x}(t))$ are strictly negative for all $t \in [0, T]$ to be sufficient for the asymptotic stability of the periodic orbit. But in fact this is not the case (cf. the counterexample in [13]).

However, this gives an idea for a Riemannian metric $M(t) = (S^{-1}(t))^T S^{-1}(t)$ where the columns of $S(t)$ are the eigenvectors of $D_x f(t, \tilde{x}(t))$ for each $t \in [0, T]$. Note that this corresponds to a transformation with the eigenvectors as new coordinates. Assume for simplicity that all the eigenvalues $-\nu_i(t) < 0$, $i = 1, \ldots, n$, are real, distinct and negative for all $t \geq 0$. Then $M(t)$ is a C^1-function. By Lemma 6.1 we have

$$L_M(t, x) = \max_{v \in \mathbb{R}^n, \|v\|=1} v^T \left[S^{-1}(t) D_x f(t, x) S(t) + (S^{-1})'(t) S(t) \right] v. \qquad (7.1)$$

The first matrix in (7.1) is $D(t) = \mathrm{diag}(-\nu_1(t), \ldots, -\nu_n(t))$ for points $(t, \tilde{x}(t))$ on the periodic orbit. This matrix clearly fulfills $v^T D(t)\, v < 0$ for all $t \geq 0$

and all $v \in \mathbb{R}^n \setminus \{0\}$ by our assumption. But we have to deal with the second matrix, too, which corresponds to the change of the eigenvectors in time. But possibly this term is small compared to the first one and L_M is negative. If L_M is negative for all points of the periodic orbit, then Theorem 3.1 implies that the periodic orbit is exponentially asymptotically stable. Since then L_M is negative also in a neighborhood of the periodic orbit, we can determine a subset of its basin of attraction using Proposition 6.1. For the case of imaginary eigenvectors and multiple eigenvalues we choose the matrix S as discussed in the proof of Theorem 4.1 (cf. also Example 7.1).

Let us give an explicit example to illustrate this procedure. In Example 7.1 we will prove the stability of a given periodic orbit and determine a part of the basin of attraction using Proposition 6.1.

Example 7.1. Consider

$$\begin{cases} \dot{x} & = & y \\ \dot{y} & = & (\sin t - 3)\, x - 2y + y^2 \; . \end{cases} \tag{7.2}$$

The zero solution $\Omega = \{(t, 0, 0) \mid t \in [0, 2\pi]\} \subset 2\pi \cdot S^1 \times \mathbb{R}^2$ of (7.2) is exponentially asymptotically stable and the real parts of all Floquet exponents are smaller or equal to $-\frac{1}{2}\left[2 - \frac{1}{\sqrt{3}}\right] \approx -0.711$. The set $G^r = \{(t, x, y) \mid (x, y)M(t)(x, y)^T \le r^2\}$ with $0 < r < r_0 := -1 + \sqrt{\frac{3}{2} - \frac{1}{4\sqrt{3}}} \approx 0.164$ satisfies $G^r \subset A(\Omega)$.

Proof. The Jacobian is given by

$$D_{(x,y)}f(t, x, y) = \begin{pmatrix} 0 & 1 \\ \sin t - 3 & -2 + 2y \end{pmatrix} \; .$$

The eigenvalues of $D_{(x,y)}f(t, 0, 0)$ are $\lambda_{1,2}(t) = \left(-1 \pm i\sqrt{2 - \sin t}\right)$. Since $2 > \sin t$, the eigenvalues are imaginary and distinct. The corresponding complex eigenvectors are $(1, \lambda_1)^T$, $(1, \lambda_2)^T$, respectively. We define the matrix

$$S(t) := \begin{pmatrix} 1 & 0 \\ -1 & \sqrt{2 - \sin t} \end{pmatrix} \; .$$

Then we have

$$S^{-1}(t) = \begin{pmatrix} 1 & 0 \\ \frac{1}{\sqrt{2-\sin t}} & \frac{1}{\sqrt{2-\sin t}} \end{pmatrix} \; , \quad (S^{-1})'(t) = \begin{pmatrix} 0 & 0 \\ \frac{1}{2}\frac{\cos t}{(2-\sin t)^{3/2}} & \frac{1}{2}\frac{\cos t}{(2-\sin t)^{3/2}} \end{pmatrix} \; .$$

We set $M(t) = (S^{-1}(t))^T S^{-1}(t)$ and calculate

$$S^{-1}(t)D_{(x,y)}f(t, x, y)S(t) + (S^{-1})'(t)S(t) = \begin{pmatrix} -1 & \sqrt{2 - \sin t} \\ -\frac{2+2y-\sin t}{\sqrt{2-\sin t}} & -1 + 2y + \frac{\cos t}{2[2-\sin t]} \end{pmatrix} \; .$$

For the stability of the periodic orbit, for which we have $y = 0$, we calculate with Lemma 6.1 and the corollary to Lemma 6.2

$$
\begin{aligned}
L_M(t,0,0) &= \frac{1}{2}\left[-2 + \frac{\cos t}{2[2-\sin t]} + \frac{|\cos t|}{2[2-\sin t]}\right] \\
&\leq \frac{1}{2}\left[-2 + \frac{|\cos t|}{2-\sin t}\right] \\
&\leq -\frac{1}{2}\left[2 - \frac{1}{\sqrt{3}}\right].
\end{aligned}
$$

We used that the maximal value of $\frac{|\cos t|}{2-\sin t}$ is $\frac{1}{\sqrt{3}}$. Hence, $-\frac{1}{2}\left[2-\frac{1}{\sqrt{3}}\right]$ is an upper bound for the largest real part of the Floquet exponents.

Now we want to determine a part of its basin of attraction. We calculate

$$
\begin{aligned}
&L_M(t,x,y) \\
&= \frac{1}{2}\left[-2 + 2y + \frac{\cos t}{2[2-\sin t]} + \sqrt{\frac{4y^2}{2-\sin t} + \left(2y + \frac{\cos t}{2[2-\sin t]}\right)^2}\right] \quad (7.3)
\end{aligned}
$$

for arbitrary (x,y) using Lemma 6.1 and the corollary to Lemma 6.2. The condition $L_M(t,x,y) < 0$ is equivalent to

$$
-2 + 2y + \frac{\cos t}{2[2-\sin t]} < 0 \qquad (7.4)
$$

and

$$
\frac{4y^2}{2-\sin t} + \left(2y + \frac{\cos t}{2[2-\sin t]}\right)^2 < \left(-2 + 2y + \frac{\cos t}{2[2-\sin t]}\right)^2. \qquad (7.5)
$$

Inequality (7.5) is equivalent to

$$
\frac{2y^2}{2-\sin t} + 4y + \frac{\cos t}{2-\sin t} < 2. \qquad (7.6)
$$

If (7.4) and (7.6) are fulfilled for $y^* > 0$, then they also hold for all $y \in [-y^*, y^*]$. Using Proposition 6.1 we determine a set $G^r = \{(t,x,y) \mid (x,y)M(t)(x,y)^T \leq r^2\}$ such that $L_M(t,x,y) < 0$ holds for all $(t,x,y) \in G^r$. Note that

$$
M(t) = (S^{-1}(t))^T S^{-1}(t) = \begin{pmatrix} 1 + \frac{1}{2-\sin t} & \frac{1}{2-\sin t} \\ \frac{1}{2-\sin t} & \frac{1}{2-\sin t} \end{pmatrix}.
$$

Hence, $(t,x,y) \in G^r$ if and only if $(3-\sin t)x^2 + 2xy + y^2 \leq r^2(2-\sin t)$. For fixed t and r the maximal y is given by $y^* = r\sqrt{3-\sin t}$. Setting thus $y = y^*$ in (7.6) we obtain

$$
2r^2 + \frac{2r^2 + \cos t}{2-\sin t} + 4r\sqrt{3-\sin t} < 2.
$$

In order to show this inequality we use $|\sin t| \leq 1$ and $\frac{\cos t}{2 - \sin t} \leq \frac{1}{\sqrt{3}}$. Then we have for $0 < r < r_0 = -1 + \sqrt{\frac{3}{2} - \frac{1}{4\sqrt{3}}}$ the inequality

$$2r^2 + \frac{2r^2 + \cos t}{2 - \sin t} + 4r\sqrt{3 - \sin t} < 2r_0^2 + 2r_0^2 + \frac{1}{\sqrt{3}} + 8r_0 = 2.$$

With $y < r_0\sqrt{3 - \sin t}$ also (7.4) holds. ∎

In a three-dimensional example we give a bound on the largest real part of the Floquet exponents.

Example 7.2. Consider

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= (\sin t - 3)\,x + (\sin t - 5)y - 3z \end{cases} \tag{7.7}$$

We can add higher order terms. The zero solution $\Omega = \{(t, 0, 0, 0) \mid t \in [0, 2\pi]\} \subset 2\pi \cdot S^1 \times \mathbb{R}^3$ of (7.7) is exponentially asymptotically stable and the real parts of all Floquet exponents are smaller or equal to $-1 + \frac{1+\sqrt{2}}{\sqrt{12}} \approx -0.303$.

Proof. The Jacobian is given by

$$D_{(x,y,z)}f(t, x, y, z) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \sin t - 3 & \sin t - 5 & -3 \end{pmatrix}.$$

The eigenvalues of $D_{(x,y,z)}f(t, 0, 0, 0)$ are $\lambda_{1,2}(t) = -1 \pm i\sqrt{2 - \sin t}$ and $\lambda_3 = -1$. Since $2 > \sin t$, the eigenvalues λ_1 and λ_2 are not real. All eigenvalues are pairwise distinct. The corresponding (complex) eigenvectors are $(1, \lambda_1, \lambda_1^2)^T$, $(1, \lambda_2, \lambda_2^2)^T$, $(1, \lambda_3, \lambda_3^2)^T = (1, -1, 1)^T$, respectively. We define

$$S(t) := \begin{pmatrix} 1 & 0 & 1 \\ -1 & \sqrt{2 - \sin t} & -1 \\ -1 + \sin t & -2\sqrt{2 - \sin t} & 1 \end{pmatrix}.$$

Then we have

$$S^{-1}(t) = \begin{pmatrix} \frac{-1}{2 - \sin t} & \frac{-2}{2 - \sin t} & \frac{-1}{2 - \sin t} \\ \frac{1}{\sqrt{2 - \sin t}} & \frac{1}{\sqrt{2 - \sin t}} & 0 \\ 1 + \frac{1}{2 - \sin t} & \frac{2}{2 - \sin t} & \frac{1}{2 - \sin t} \end{pmatrix}$$

and for the orbital derivative

$$(S^{-1})'(t) = \frac{\cos t}{(2 - \sin t)^2} \begin{pmatrix} -1 & -2 & -1 \\ \frac{\sqrt{2 - \sin t}}{2} & \frac{\sqrt{2 - \sin t}}{2} & 0 \\ 1 & 2 & 1 \end{pmatrix}.$$

Now we calculate

$$B(t) = S^{-1}(t)D_{(x,y,z)}f(t,x,y,z)S(t) + (S^{-1})'(t)S(t)$$

$$= \begin{pmatrix} -1 + \frac{\cos t}{2-\sin t} & \sqrt{2-\sin t} & 0 \\ -\sqrt{2-\sin t} & -1 + \frac{1}{2}\frac{\cos t}{2-\sin t} & 0 \\ -\frac{\cos t}{2-\sin t} & 0 & -1 \end{pmatrix}.$$

For the stability of the periodic orbit, we calculate the eigenvalues of

$$B^s(t) = \frac{1}{2}(B(t) + B(t)^T) = \begin{pmatrix} -1 + \frac{\cos t}{2-\sin t} & 0 & -\frac{1}{2}\frac{\cos t}{2-\sin t} \\ 0 & -1 + \frac{1}{2}\frac{\cos t}{2-\sin t} & 0 \\ -\frac{1}{2}\frac{\cos t}{2-\sin t} & 0 & -1 \end{pmatrix}$$

for each $t \in [0, 2\pi]$ which are

$$\mu_{1,2} = -1 + \frac{1}{2}\frac{\cos t}{2-\sin t} \pm \frac{1}{\sqrt{2}}\frac{|\cos t|}{2-\sin t}$$

$$\mu_3 = -1 + \frac{1}{2}\frac{\cos t}{2-\sin t} .$$

We have $\mu_i(t) \leq -1 + \frac{1+\sqrt{2}}{2}\frac{|\cos t|}{2-\sin t} \leq -1 + \frac{1+\sqrt{2}}{\sqrt{12}} \approx -0.303$. Using Lemma 6.2 we obtain the above result. ∎

7.2. Constant M. Another possible choice for a Riemannian metric is the following: Consider a differential equation with a parameter γ such that $\tilde{x}(t) = 0$ is a solution for all $\gamma \in (-\gamma_0, \gamma_0)$. Assume that for $\gamma = 0$ the system is autonomous and the origin is an exponentially asymptotically stable equilibrium point. Let S be the matrix of eigenvectors corresponding to Theorem 4.1 and set $M(t,x) = M := (S^{-1})^T S^{-1}$. For small values of γ the periodic orbit is stable for reasons of continuity, but it is not obvious how large the parameter range for γ is. With Theorem 3.1 and the above Riemannian metric we have a sufficient condition for the stability. We give two examples.

Example 7.3. Consider the eystem

$$\ddot{x} + \dot{x} + kx = \gamma \sin t\, x. \tag{7.8}$$

The zero solution of (7.8) is exponentially asymptotically stable if

either $0 < k \leq \frac{1}{4}$ and $|\gamma| < \frac{\sqrt{1-4k}-(1-4k)}{2}$

or $k > \frac{1}{4}$ and $|\gamma| < \sqrt{k - \frac{1}{4}}$.

Proof. We transform (7.8) into the first-order system

$$\begin{cases} \dot{x} &=& y \\ \dot{y} &=& -kx + \gamma \sin t\, x - y \end{cases}$$

For $\gamma = 0$ we have the following eigenvalues of $D_{(x,y)} f(t, 0, 0) = \begin{pmatrix} 0 & 1 \\ -k & -1 \end{pmatrix}$:

$$\lambda_{1,2} = \begin{cases} \frac{-1 \pm \sqrt{1-4k}}{2} & \text{for } k \leq \frac{1}{4} \\ \frac{-1 \pm i\sqrt{4k-1}}{2} & \text{for } k > \frac{1}{4}\,. \end{cases}$$

The matrix S is given by

$$\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{\sqrt{4k-1}}{2} \end{pmatrix},$$

respectively. First we consider the real case. We calculate

$$S^{-1} D_{(x,y)} f(t, x, y) S$$

$$= \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_1 \lambda_2 + k - \gamma \sin t + \lambda_1 & \lambda_2^2 + k - \gamma \sin t + \lambda_2 \\ -\lambda_1^2 - k + \gamma \sin t - \lambda_1 & -\lambda_1 \lambda_2 - k + \gamma \sin t - \lambda_2 \end{pmatrix}.$$

Note that $(S^{-1})' = 0$ and $(\lambda_2 - \lambda_1)(\lambda_2 + \lambda_1 + 1) = 0$. Thus, with Lemma 6.1 and the corollary to Lemma 6.2 we have

$$2 \cdot L_M(t, 0, 0)$$

$$= -1 + \frac{\sqrt{[\lambda_2^2 - \lambda_1^2 + \lambda_2 - \lambda_1]^2 + [2(\lambda_1 \lambda_2 + k - \gamma \sin t) + \lambda_1 + \lambda_2]^2}}{|\lambda_2 - \lambda_1|}$$

$$= -1 + \frac{|4k - 2\gamma \sin t - 1|}{\sqrt{1 - 4k}}$$

$$\leq -1 + \sqrt{1 - 4k} + \frac{2|\gamma|}{\sqrt{1 - 4k}}\,.$$

Thus, if $|\gamma| < \frac{\sqrt{1-4k} - (1-4k)}{2}$, then the periodic orbit is asymptotically stable. Now we consider the complex case, i.e. $k > \frac{1}{4}$. Here we have

$$S^{-1} D_{(x,y)} f(t, x, y) S = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{4k-1}}{2} \\ \frac{2(\frac{1}{4} - k + \gamma \sin t)}{\sqrt{4k-1}} & -\frac{1}{2} \end{pmatrix}.$$

Hence, with Lemma 6.1 and the corollary to Lemma 6.2 we have

$$2 \cdot L_M(t, 0, 0) = -1 + \frac{2|\gamma \sin t|}{\sqrt{4k-1}} \leq -1 + \frac{2|\gamma|}{\sqrt{4k-1}}.$$

Thus, if $|\gamma| < \sqrt{k - \frac{1}{4}}$, then the periodic orbit is asymptotically stable. ∎

Example 7.4 (A swing). We consider a swing on a rope with length l_0. A person of mass m is sitting on the swing and moving its center of gravity up and down following a t-periodic function $u(t)$ with period T satisfying $|u(t)| \leq l_1 < l_0$ in order to move the swing. Thus, the equation of motion is given by $\ddot{x} = \frac{1}{J}T(t, x)$ where x denotes the angle, T the torque and $J = ml^2(t)$ the angular momentum. Here, $T = l(t)F_o$ where $l(t) = l_0 + u(t)$ denotes the moment arm and F_o the component of the force orthogonal to the rope, which is given by $-gm\sin x$. Altogether, including the orthogonal component of the force of the friction given by $-kl(t)\dot{x}$ with $k > 0$, we have

$$\ddot{x} = -\frac{g}{l(t)}\sin x - \frac{k}{m}\dot{x}. \tag{7.9}$$

For $l_1 = 0$, $(x, \dot{x}) = (0, 0)$ is an asymptotically stable equilibrium due to the friction. We want to derive a sufficient condition in terms of l_1 such that $(x, \dot{x}) = (0, 0)$, which is always a solution, is asymptotically stable. This means that for these values of l_1, starting near $(0, 0)$, we stay near this zero solution and approach it. In order to really move the swing starting near $(0, 0)$, a nontrivial asymptotically stable periodic orbit has to exist and the zero solution has to be unstable. Hence, a necessary condition to move the swing is that l_1 is larger than the values given in Lemma 7.1.

Lemma 7.1. *Assume that either*

$$\frac{k^2}{m^2} \geq 4\frac{g}{l_0} \quad and \quad l_1 < l_0 - \frac{1}{\frac{k^2}{2gm^2} - \frac{1}{l_0} + \sqrt{\frac{k^4}{4g^2m^4} - \frac{k^2}{gm^2l_0}}} \tag{7.10}$$

or

$$\frac{k^2}{m^2} < 4\frac{g}{l_0} \quad and \quad l_1 < \frac{l_0^2}{\frac{2mg}{k\sqrt{4\frac{g}{l_0} - \frac{k^2}{m^2}}} + l_0} \tag{7.11}$$

hold. Then the solution $(x(t), \dot{x}(t)) = (0, 0)$ of (7.9) is exponentially asymptotically stable.

Proof. We first transform (7.9) into a first-order system

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -\frac{g}{l(t)}\sin x - \frac{k}{m}y \end{cases} \tag{7.12}$$

$(x(t), y(t)) = (0, 0)$ is a solution of (7.12). We have

$$D_{(x,y)}f(t, x, y) = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l(t)}\cos x & -\frac{k}{m} \end{pmatrix}.$$

For $u(t) \equiv 0$ we have $l(t) \equiv l_0$ and (7.12) becomes an autonomous system. Its Jacobian at $(x, y) = (0, 0)$ then reads

$$D_{(x,y)}f(t, 0, 0) = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l_0} & -\frac{k}{m} \end{pmatrix}.$$

We distinguish between real and complex eigenvalues. In the first case we have the real eigenvalues

$$\lambda_{1,2} = \frac{1}{2}\left(-\frac{k}{m} \pm \sqrt{\frac{k^2}{m^2} - 4\frac{g}{l_0}} \right).$$

The corresponding eigenvectors are given by $(1, \lambda_1)^T$ and $(1, \lambda_2)^T$, respectively. We define

$$S := \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}$$

and obtain

$$S^{-1}D_x f(t, x, y)S = \frac{1}{\lambda_2 - \lambda_1}\begin{pmatrix} \lambda_1\lambda_2 + \frac{g}{l(t)} + \lambda_1\frac{k}{m} & \lambda_2^2 + \frac{g}{l(t)} + \lambda_2\frac{k}{m} \\ -\lambda_1^2 - \frac{g}{l(t)} - \lambda_1\frac{k}{m} & -\lambda_1\lambda_2 - \frac{g}{l(t)} - \lambda_2\frac{k}{m} \end{pmatrix}$$

with $M := (S^{-1})^T S^{-1}$. By Lemma 6.1 and the corollary to Lemma 6.2 we have

$$2 \cdot L_M(t, 0, 0) = -\frac{k}{m} + \frac{\left| \frac{k^2}{m^2} - 2g\left(\frac{1}{l_0} + \frac{1}{l(t)} \right) \right|}{\sqrt{\frac{k^2}{m^2} - 4\frac{g}{l_0}}}.$$

Note that $(S^{-1})' = 0$, $\lambda_1 + \lambda_2 + \frac{k}{m} = 0$ and thus $\lambda_2^2 - \lambda_1^2 + (\lambda_2 - \lambda_1)\frac{k}{m} = 0$. In order to have $L_M(t, 0, 0) < 0$ we need

$$\left(\frac{k^2}{m^2} - 2g\left(\frac{1}{l_0} + \frac{1}{l(t)} \right) \right)^2 < \frac{k^4}{m^4} - 4\frac{gk^2}{l_0 m^2}$$

$$\left(\frac{1}{l_0} + \frac{1}{l(t)} \right)^2 < \frac{k^2}{gm^2 l(t)}$$

that is $\frac{1}{l(t)} \in \left(\frac{k^2}{2gm^2} - \frac{1}{l_0} - \sqrt{\frac{k^4}{4g^2m^4} - \frac{k^2}{gm^2 l_0}}, \frac{k^2}{2gm^2} - \frac{1}{l_0} + \sqrt{\frac{k^4}{4g^2m^4} - \frac{k^2}{gm^2 l_0}} \right)$. Since the minimal and maximal value for $l(t)$ is $l_0 - l_1$ and $l_0 + l_1$, respectively, this yields

$$l_1 < -l_0 + \frac{1}{\frac{k^2}{2gm^2} - \frac{1}{l_0} - \sqrt{\frac{k^4}{4g^2m^4} - \frac{k^2}{gm^2 l_0}}} \tag{7.13}$$

$$l_1 < l_0 - \frac{1}{\frac{k^2}{2gm^2} - \frac{1}{l_0} + \sqrt{\frac{k^4}{4g^2m^4} - \frac{k^2}{gm^2 l_0}}}. \tag{7.14}$$

Note that by $\frac{k^2}{m^2} \geq 4\frac{g}{l_0}$, inequality (7.14) implies (7.13). This proves (7.10).

Now we will consider the complex case $\frac{k^2}{m^2} < 4\frac{g}{l_0}$. The eigenvalues are given by

$$\lambda_{1,2} = \frac{1}{2}\left(-\frac{k}{m} \pm i\sqrt{4\frac{g}{l_0} - \frac{k^2}{m^2}}\right).$$

The corresponding complex eigenvectors are given by $(1, \lambda_1)^T$ and $(1, \lambda_2)^T$, respectively. We define

$$S := \begin{pmatrix} 1 & 0 \\ -\frac{k}{2m} & \frac{1}{2}\sqrt{4\frac{g}{l_0} - \frac{k^2}{m^2}} \end{pmatrix}$$

and obtain with $M := (S^{-1})^T S^{-1}$ the relation

$$S^{-1} D_{(x,y)} f(t,x,y) S = \begin{pmatrix} -\frac{k}{2m} & \frac{1}{2}\sqrt{4\frac{g}{l_0} - \frac{k^2}{m^2}} \\ \frac{2}{\sqrt{4\frac{g}{l_0} - \frac{k^2}{m^2}}}\left(\frac{k^2}{4m^2} - \frac{g}{l(t)}\right) & -\frac{k}{2m} \end{pmatrix}.$$

By Lemma 6.1 and the corollary to Lemma 6.2 we have

$$
\begin{aligned}
2 \cdot L_M(t,0,0) &= -\frac{k}{m} + \frac{2g}{\sqrt{4\frac{g}{l_0} - \frac{k^2}{m^2}}}\left|\frac{1}{l_0} - \frac{1}{l(t)}\right| \\
&\leq -\frac{k}{m} + \frac{2g}{\sqrt{4\frac{g}{l_0} - \frac{k^2}{m^2}}}\frac{l_1}{l_0(l_0 - l_1)}
\end{aligned}
$$

and hence (7.11) implies $L_M(t,0,0) < 0$. ∎

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