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Weighted Integrals of Holomorphic Functions on the Polydisc

Stevo Stević

Abstract. We show that a holomorphic function on the unit polydisc U^n in $\mathbf{C}^n$ belongs to the weighted Bergman space $\mathcal{A}^p_\alpha(U^n)$, when $p \in (0,1]$, if and only if all weighted derivations of order $|k|$ (with positive orders of derivations) belong to the related weighted Lebesgue space $\mathcal{L}^p_\alpha(U^n)$. This result extends Theorem 1.8 by Benke and Chang in [Nagoya Math. J. 159 (2000), 25 – 43].

Keywords: *Holomorphic function, weighted Bergman space, polydisc*

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1. Introduction

Let $z = (z_1, ..., z_n)$ and $w = (w_1, ..., w_n)$ be points in the complex vector space $\mathbf{C}^n$, U the unit disc in the complex plane $\mathbf{C}$, $D(a, r_0) = \{ z \in \mathbf{C} \,|\, |z - a| < r_0 \}$ the open disk centered at a of radius r_0 and $r, \rho, \delta \in (0, \infty)^n$.

Let $\gamma = (\gamma_1, ..., \gamma_n)$ be a multi-index, γ_k being nonnegative integers, we write

$$|\gamma| = \gamma_1 + \cdots + \gamma_n, \qquad \gamma! = \gamma_1! \cdots \gamma_n!, \qquad z^\gamma = z_1^{\gamma_1} \cdots z_n^{\gamma_n}.$$

For a holomorphic function f we denote

$$D^\gamma f = \frac{\partial^{|\gamma|} f}{\partial z_1^{\gamma_1} \cdots \partial z_n^{\gamma_n}}.$$

Let

$$P^n(w, r) = \{ z \in \mathbf{C}^n \,|\, |z_j - w_j| < r_j, j = 1, ..., n \}$$

be a polydisc in $\mathbf{C}^n$, and let $H(P^n(w, r))$ be the class of all holomorphic functions f defined on $P^n(w, r)$. Let $\alpha = (\alpha_1, ..., \alpha_n)$, with $\alpha_j > -1$ for $j = 1, ..., n$, and $p \in (0, \infty)$. The space $\mathcal{L}^p_\alpha(U^n) = \mathcal{L}^p_\alpha$ denotes the class of all measurable

Stevo Stević: Mathematical Institute of Serbian Academy of Science, Knez Mihailova 35/I, 11000 Beograd, Serbia; e-mail: sstevic@ptt.yu; sstevo@matf.bg.ac.yu

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functions defined on the polydisc U^n such that

$$||f||^p_{\mathcal{L}^p_\alpha} = \int_{U^n} |f(z)|^p \prod_{j=1}^n (1-|z_j|^2)^{\alpha_j} dV(z_j) < \infty,$$

where $dV(z_j)$ is the normalized area measure on the unit disk U. The weighted Bergman space $\mathcal{A}^p_\alpha(U^n) = \mathcal{A}^p_\alpha$ is the intersection of $\mathcal{L}^p_\alpha$ and $H(U^n)$.

Weighted Bergman spaces of analytic functions of one variable have been studied, for example, in [3 - 7], [12 - 14] and [19], while the weighted Bergman spaces of analytic functions on the unit ball $B \subset \mathbf{C}^n$ have been studied, for example, in [1], [8 - 11], [17] and [18] (see, also the references therein).

In [1, p.33] the authors proved the following theorem.

Theorem A. *Let* $p \in [1, \infty)$, $\alpha = (\alpha_1, ..., \alpha_n)$, *with* $\alpha_j > -1$ *for* $j = 1, ..., n$, m *be a fixed positive integer and let* $\mathbf{k} = (k_1, ..., k_n) \in (\mathbf{Z}_+)^n$. *Let* $f \in H(U^n)$, *then* $f \in \mathcal{A}^p_\alpha$ *if and only if*

$$\left[\prod_{j=1}^n (1-|z_j|^2)^{k_j}\right] \frac{\partial^{|\mathbf{k}|} f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}(z) \in \mathcal{L}^p_\alpha.$$

Moreover,

$$\begin{aligned} ||f||_{\mathcal{A}^p_\alpha} \asymp & \sum_{|\mathbf{k}|=0}^{m-1} \left| \frac{\partial^{|\mathbf{k}|} f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}(0) \right| \\ & + \sum_{|\mathbf{k}|=m} \left|\left| \left[\prod_{j=1}^n (1-|z_j|^2)^{k_j}\right] \frac{\partial^m f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}} \right|\right|_{\mathcal{L}^p_\alpha}. \end{aligned} \tag{1}$$

The above means that there are finite positive constants C and C' independent of f such that the left and right hand sides $L(f)$ and $R(f)$ satisfy

$$C \cdot R(f) \le L(f) \le C' \cdot R(f)$$

for all holomorphic f.

In the proof of Theorem A the authors used the weighted Bergman projection $\mathbf{B}_\alpha : \mathcal{L}^2_\alpha \to \mathcal{A}^2_\alpha$, which can be extended as a bounded operator from $\mathcal{L}^p_\alpha$ onto $\mathcal{A}^p_\alpha$. Closely related results on the unit disc and the unit ball in $\mathbf{C}^n$ or $\mathbb{R}^n$ can be found in [1 - 3], [11], [13 - 17] and [19].

The main purpose of this paper is to generalize Theorem A in the case $p \in (0, 1)$. We prove the following theorem.

Theorem 1.1. *Let* $p \in (0,1]$, $\alpha = (\alpha_1, ..., \alpha_n)$, *with* $\alpha_j > -1$ *for* $j = 1, ..., n$, m *be a fixed positive integer and let* $\gamma = (\gamma_1, ..., \gamma_n) \in (\mathbf{Z}_+)^n$. *Let* $f \in H(U^n)$, *then* $f \in \mathcal{A}^p_\alpha$ *if and only if*

$$\left[\prod_{j=1}^{n}(1-|z_j|^2)^{\gamma_j}\right] \frac{\partial^{|\gamma|} f}{\partial z_1^{\gamma_1} \cdots \partial z_n^{\gamma_n}}(z) \in \mathcal{L}^p_\alpha \quad \textit{for all} \quad |\gamma| = m.$$

Moreover, the relationship (1) *also holds.*

2. Auxiliary results

In order to prove the main result we need several auxiliary results which are incorporated in the following lemmas. In what follows we will use C to denote a positive constant which may vary from line to line.

Lemma 2.1. *Let* $f \in H(U^n)$, γ *be a multi-index and* $p > 0$. *Then*

$$|D^\gamma f(w)|^p \le \frac{C}{r^{\gamma p} \prod_{j=1}^n r_j^2} \int_{P^n(w,r)} |f|^p \prod_{j=1}^{n} dV(z_j), \tag{2}$$

whenever $P^n(w,r) \subset U^n$, *where* C *is a constant depending only on* p, γ *and* n.

Proof. By the subharmonicity of functions of one variable we obtain

$$|f(w_1, w_2, ..., w_n)|^p \le \frac{1}{r_1^2} \int_{D(w_1,r_1)} |f(z_1, w_2, ..., w_n)|^p \, dV(z_1),$$

whenever $D(w_1, r_1) \subset U$. Further, for a fixed $z_1 \in D(w_1, r_1)$ we have

$$|f(z_1, w_2, ..., w_n)|^p \le \frac{1}{r_2^2} \int_{D(w_2,r_2)} |f(z_1, z_2, w_3, ..., w_n)|^p \, dV(z_2),$$

whenever $D(w_2, r_2) \subset U$. Continuing this process and combining obtained inequalities we obtain

$$|f(w)|^p \le \frac{1}{\prod_{j=1}^n r_j^2} \int_{P^n(w,r)} |f|^p \prod_{j=1}^{n} dV(z_j), \tag{3}$$

whenever $P^n(w,r) \subset U^n$.

On the other hand, by Cauchy's integral formula we obtain

$$|D^\gamma f(w)| \le \frac{\gamma!}{r^\gamma} \max_{y \in P^n(w,r)} |f(y)|, \tag{4}$$

whenever $\overline{P^n(w,r)} \subset U^n$. This inequality is obvious for $\gamma = (0,...,0)$. From (4) we obtain

$$|D^\gamma f(w)|^p \le \frac{2^{|\gamma|p}\gamma!^p}{r^{\gamma p}}\Big(\max_{y\in\overline{P^n(w,\frac{r}{2})}}|f(y)|\Big)^p. \tag{5}$$

From (3), we obtain for $y \in \overline{P^n(w,\frac{r}{2})}$ the estimate

$$|f(y)|^p \le \frac{2^{2n}}{\prod_{j=1}^n r_j^2}\int_{P^n(y,\frac{r}{2})}|f|^p\prod_{j=1}^n dV(z_j)\ .$$

Thus

$$\max_{y\in\overline{P^n(w,\frac{r}{2})}}|f(y)|^p \le \frac{2^{2n}}{\prod_{j=1}^n r_j^2}\int_{P^n(w,r)}|f|^p\prod_{j=1}^n dV(z_j). \tag{6}$$

From (5) and (6) we obtain inequality (2). ∎

Lemma 2.2. *Let β be a multi-index and $a \in U^n$. Then the point evaluations $\Lambda_{a,\beta}(f) = D^\beta f(a)$ are bounded linear functionals on $\mathcal{A}^p_\alpha(U^n)$, for all $p > 0$.*

Proof. Choose $\overline{P^n(a,\delta)} \subset U^n$, and let $m = \min_{z\in\overline{P^n(a,\delta)}}\prod_{j=1}^n(1-|z_j|^2)^{\alpha_j} > 0$. By Lemma 1 there is a constant C independent of f, such that

$$\begin{aligned}|D^\beta f(a)|^p &\le \frac{C}{\delta^{\beta p}\prod_{j=1}^n\delta_j^2}\int_{\overline{P^n(a,\delta)}}|f(z)|^p\prod_{j=1}^n dV(z_j)\\ &\le \frac{C}{m\delta^{\beta p}\prod_{j=1}^n\delta_j^2}\int_{\overline{P^n(a,\delta)}}|f(z)|^p\prod_{j=1}^n\big(1-|z_j|^2\big)^{\alpha_j}dV(z_j)\\ &\le \frac{C}{m\delta^{\beta p}\prod_{j=1}^n\delta_j^2}||f||^p_{\mathcal{A}^p_\alpha}.\end{aligned}$$

which proves the assertion. ∎

Lemma 2.3. *Let $M_p^p(f,r_k) = \frac{1}{2\pi}\int_0^{2\pi}|f(...,r_ke^{i\theta_k},...)|^p\,d\theta_k$ for $0 < p \le 1$ and $f \in H(U^n)$. Then*

$$M_p^p(f,\rho_k) - M_p^p(f,r_k) \le C(\rho_k - r_k)^p M_p^p\Big(\frac{\partial f}{\partial z_k},\rho_k\Big)$$

where C is a positive constant independent of f, z_j, $j \ne k$, ρ_k and r_k.

Proof. We have

$$\begin{aligned}M_p^p(f,\rho_k) - M_p^p(f,r_k) &\le \frac{1}{2\pi}\int_0^{2\pi}|f(...,\rho_ke^{i\theta_k},...) - f(...,r_ke^{i\theta_k},...)|^p\,d\theta_k\\ &\le \frac{(\rho_k-r_k)^p}{2\pi}\int_0^{2\pi}\sup_{r_k<t<\rho_k}\Big|\frac{\partial f}{\partial z_k}(...,te^{i\theta_k},...)\Big|^p\,d\theta_k\\ &\le C(\rho_k-r_k)^pM_p^p\Big(\frac{\partial f}{\partial z_k},\rho_k\Big).\end{aligned}$$

In the last inequality we have used the Hardy-Littlewood maximal theorem (see, for example, [2, Theorem 1.9]). ∎

Lemma 2.4. *Let $f \in H(U^n)$, $p \in (0,\infty)$ and $\beta_{k,j} \in \mathbb{R}$, $k,j = 1,...,n$. Then there is a constant $C = C(p,\beta_{k,j},n)$ such that*

$$\max_{z\in \overline{D^n(0,\frac{1}{2})}} |f(z)|^p \le C\Big(|f(\vec{0})|^p + \sum_{k=1}^{n}\int_{U^n}\Big|\frac{\partial f}{\partial z_k}(z)\Big|^p \prod_{j=1}^{n}\big(1-|z_j|^2\big)^{\beta_{k,j}} dV(z_j)\Big).$$

Proof. Without loss of generality, we may assume that $n = 2$. Since

$$f(z_1,z_2) - f(0,0) = \int_0^1 f'(tz_1,z_2)\,dt + \int_0^1 f'(0,tz_2)\,dt,$$

by some well-known inequalities we obtain

$$\begin{aligned} &|f(z_1,z_2)|^p \\ &\le\ c_p\Big(|f(0,0)|^p + |z_1|^p \max_{|\zeta_1|\le\frac{1}{2}}\Big|\frac{\partial f}{\partial z_1}(\zeta_1,z_2)\Big|^p + |z_2|^p\max_{|\zeta_2|\le\frac{1}{2}}\Big|\frac{\partial f}{\partial z_2}(0,\zeta_2)\Big|^p\Big) \\ &\le\ c_p\Big(|f(0,0)|^p + \max_{|\zeta_1|\le\frac{1}{2}}\Big|\frac{\partial f}{\partial z_1}(\zeta_1,z_2)\Big|^p + \max_{|\zeta_2|\le\frac{1}{2}}\Big|\frac{\partial f}{\partial z_2}(0,\zeta_2)\Big|^p\Big) \end{aligned} \tag{7}$$

for all $z_1, z_2 \in \overline{D(0,\frac{1}{2})}$, where $c_p = 1$ for $0 < p < 1$ and $c_p = 2^{p-1}$ for $p \ge 1$. On the other hand by (3) we obtain

$$\begin{aligned} &|f(z_1,z_2)|^p \\ &\le\ C\Big(|f(0,0)|^p + \sum_{k=1}^{2}\int_{D^2(0,\frac{3}{4})}\Big|\frac{\partial f}{\partial z_k}(\zeta_1,\zeta_2)\Big|^p\prod_{j=1}^{2} dV(\zeta_j)\Big) \\ &\le\ C\Big(|f(0,0)|^p + \sum_{k=1}^{2}\int_{D^2(0,\frac{3}{4})}\Big|\frac{\partial f}{\partial z_k}(\zeta_1,\zeta_2)\Big|^p\prod_{j=1}^{2}\big(1-|z_j|\big)^{\beta_{k,j}}\,dV(\zeta_j)\Big) \end{aligned} \tag{8}$$

for all $z_1, z_2 \in \overline{D(0,\frac{1}{2})}$, as desired. ∎

Using the change $r \to \frac{1+r}{2}$ we can easily prove the following lemma.

Lemma 2.5. *Let $g(r)$ be a nonnegative continuous function on the interval $[0,1)$, and let $a > -1$. Then there is a constant $C = C(a)$ such that*

$$\int_0^1 g(r)(1-r)^a dr \le C\Big(\max_{r\in[0,\frac{1}{2}]} g(r) + \int_0^1 \Big|g\Big(\frac{1+r}{2}\Big) - g(r)\Big|\,(1-r)^a dr\Big).$$

3. Proof of the theorem

In this section we prove the main result in this paper.

`Necessity.` Let γ be a multi-index, such that $|\gamma| = m$. Let $f \in H(U^n)$ and $z = (z_1, ..., z_n) \in U^n$. By Lemma 1 we have

$$|D^\gamma f(z)|^p \le \frac{C}{r^{\gamma p} \prod_{j=1}^n r_j^2} \int_{P^n(z,r)} |f|^p \prod_{j=1}^n dV(\omega_j), \tag{9}$$

whenever $P^n(z,r) \subset U^n$, where C is a constant depending only on p, γ and n. Taking $z = r = (r_1, ..., r_n)$ in (9) and replacing r by $\rho - r$, where $\rho_j \in (r_j, 1)$, $j = 1, ..., n$, we obtain

$$|D^\gamma f(r)|^p \le \frac{C}{(\rho - r)^{\gamma p} \prod_{j=1}^n (\rho_j - r_j)^2} \int_{P^n(r,\rho-r)} |f|^p \prod_{j=1}^n dV(\omega_j), \tag{10}$$

Applying (10) to the functions $f(z \cdot e^{i\theta})$, $z \cdot e^{i\theta} = (z_1 e^{i\theta_1}, ..., z_n e^{i\theta_n})$, where $\theta_j \in [0, 2\pi)$, $j = 1, ..., n$, we get

$$|D^\gamma f(r \cdot e^{i\theta})|^p \le \frac{C}{(\rho - r)^{\gamma p} \prod_{j=1}^n (\rho_j - r_j)^2} \int_{P^n(r,\rho-r)} |f(\omega \cdot e^{i\theta})|^p \prod_{j=1}^n dV(\omega_j). \tag{11}$$

Integrating (11) over $[0, 2\pi]^n$ and then using Fubini's theorem, we obtain

$$\begin{aligned}\int_{[0,2\pi]^n} |D^\gamma f(r \cdot e^{i\theta})|^p d\theta \le\ & \frac{C}{(\rho - r)^{\gamma p} \prod_{j=1}^n (\rho_j - r_j)^2} \\ & \times \int_{P^n(r,\rho-r)} \int_{[0,2\pi]^n} |f(\omega \cdot e^{i\theta})|^p d\theta \prod_{j=1}^n dV(\omega_j).\end{aligned} \tag{12}$$

Here $d\theta$ denotes $d\theta_1 \cdots d\theta_n$. Since f is holomorphic,

$$\int_{[0,2\pi]^n} |f(\omega \cdot e^{i\theta})|^p d\theta = \int_{[0,2\pi]^n} |f(|\omega_1| e^{i(\theta_1 + \arg\, \omega_1)}, ..., |\omega_n| e^{i(\theta_n + \arg\, \omega_n)})|^p \, d\theta$$

is nondecreasing in $|\omega_j|$, $j = 1, ..., n$. On the other hand, it is clear that

$$\begin{aligned}&\int_{[0,2\pi]^n} |f(|\omega_1| e^{i(\theta_1 + \arg\, \omega_1)}, ..., |\omega_n| e^{i(\theta_n + \arg\, \omega_n)})|^p \, d\theta \\ &\qquad = \int_{[0,2\pi]^n} |f(|\omega_1| e^{i\theta_1}, ..., |\omega_n| e^{i\theta_n})|^p \, d\theta\end{aligned}$$

Hence, from (12) it follows

$$
\begin{aligned}
(\rho - r)^{\gamma p} \int_{[0,2\pi]^n} |D^\gamma f(r \cdot e^{i\theta})|^p \, d\theta \\
\leq \ C \int_{[0,2\pi]^n} |f(\rho_1 e^{i\theta_1}, ..., \rho_n e^{i\theta_n})|^p \, d\theta.
\end{aligned} \tag{13}
$$

Putting $\rho_j = r_j + \frac{1}{2}(1 - r_j),\ 0 \leq r_j < 1$, in (13), we obtain

$$
\begin{aligned}
\prod_{j=1}^{n} (1 - r_j)^{\gamma_j p} \int_{[0,2\pi]^n} \left| D^\gamma f(r \cdot e^{i\theta}) \right|^p d\theta \\
\leq \ C \int_{[0,2\pi]^n} \left| f\Big(\frac{1 + r_1}{2} e^{i\theta_1}, ..., \frac{1 + r_n}{2} e^{i\theta_n} \Big) \right|^p \, d\theta.
\end{aligned} \tag{14}
$$

Multiplying (14) by $\prod_{j=1}^{n} (1 - r_j^2)^{\alpha_j} r_j$ and using the fact that $0 < r_j < 1$, $j = 1, ..., n$, we obtain

$$
\begin{aligned}
&\prod_{j=1}^{n} (1 - r_j^2)^{\gamma_j p + \alpha_j} r_j \ \int_{[0,2\pi]^n} \left| D^\gamma f\big(r \cdot e^{i\theta}\big) \right|^p \, d\theta \\
&\leq \ C \int_{[0,2\pi]^n} \left| f\Big(\frac{1 + r_1}{2} e^{i\theta_1}, ..., \frac{1 + r_n}{2} e^{i\theta_n} \Big) \right|^p \, d\theta \prod_{j=1}^{n} (1 - r_j^2)^{\alpha_j} r_j.
\end{aligned}
$$

Integrating this inequality over $r \in [0, 1)^n$ and making the changes $s_j = \frac{1+r_j}{2}$, $j = 1, ..., n$, we obtain

$$
\begin{aligned}
&\int_{[0,1)^n} \int_{[0,2\pi]^n} |D^\gamma f(r \cdot e^{i\theta})|^p d\theta \prod_{j=1}^{n} (1 - r_j^2)^{\gamma_j p + \alpha_j} r_j \, dr_j \\
&\leq \ C \int_{[0,1)^n} \int_{[0,2\pi]^n} \left| f\Big(\frac{1 + r_1}{2} e^{i\theta_1}, ..., \frac{1 + r_n}{2} e^{i\theta_n} \Big) \right|^p \, d\theta \prod_{j=1}^{n} (1 - r_j^2)^{\alpha_j} r_j \, dr_j \\
&\leq \ C \int_{\prod_{j=1}^{n} [\frac{1}{2}, 1)} \int_{[0,2\pi]^n} \left| f\big(s_1 e^{i\theta_1}, ..., s_n e^{i\theta_n} \big) \right|^p \, d\theta \prod_{j=1}^{n} (1 - s_j^2)^{\alpha_j} \, ds_j \\
&\leq \ C ||f||_{\mathcal{A}_\alpha^p}^p,
\end{aligned}
$$

as desired. Let β be a multi-index. By Lemma 2 we know that the linear functional $L(f) = D^\beta f(0)$ is bounded. Hence $|D^\beta f(0)|^p \leq C \, ||f||_{\mathcal{A}_\alpha^p}^p$ for all $f \in H(U^n)$ and for some $C = C(p, \beta, \alpha) > 0$ from which it follows an inequality in (1).

`Sufficiency.` Without loss of generality, we may assume that $n=2$. Let

$$M_p^p(f,r_1,r_2)=\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\left|f\left(r_1e^{i\theta_1},r_2e^{i\theta_2}\right)\right|^p d\theta_1\,d\theta_2,$$

then

$$||f||^p_{\mathcal{A}^p_\alpha}=\int_0^1(1-r_2^2)^{\alpha_2}\int_0^1 M_p^p(f,r_1,r_2)(1-r_1^2)^{\alpha_1}r_2r_1\,dr_1\,dr_2.$$

It is easy to show that

$$||f||^p_{\mathcal{A}^p_\alpha}\asymp\int_0^1(1-r_2)^{\alpha_2}\int_0^1 M_p^p(f,r_1,r_2)(1-r_1)^{\alpha_1}\,dr_1\,dr_2.$$

By Lemmas 3 and 5, and since $M_p^p(f,r_1,r_2)$ is nondecreasing in r_1, we obtain

$$\begin{aligned}
&\int_0^1 M_p^p(f,r_1,r_2)(1-r_1)^{\alpha_1}\,dr_1\\
&\quad\le C\Bigg(M_p^p\Big(f,\frac{1}{2},r_2\Big)\\
&\qquad+\int_0^1\left|M_p^p\Big(f,\frac{1+r_1}{2},r_2\Big)-M_p^p\Big(f,r_1,r_2\Big)\right|(1-r_1)^{\alpha_1}\,dr_1\Bigg)\\
&\quad\le C\left(M_p^p\Big(f,\frac{1}{2},r_2\Big)+\int_0^1 M_p^p\Big(\frac{\partial f}{\partial z_1},\frac{1+r_1}{2},r_2\Big)(1-r_1)^{\alpha_1+p}\,dr_1\right).
\end{aligned}$$

Hence

$$\begin{aligned}
||f||^p_{\mathcal{A}^p_\alpha}\le C\Bigg(&\int_0^1(1-r_2)^{\alpha_2}M_p^p\Big(f,\frac{1}{2},r_2\Big)\,dr_2\\
&+\int_0^1(1-r_2)^{\alpha_2}\int_0^1 M_p^p\Big(\frac{\partial f}{\partial z_1},\frac{1+r_1}{2},r_2\Big)(1-r_1)^{\alpha_1+p}\,dr_1\,dr_2\Bigg).
\end{aligned}\tag{15}$$

Since $M_p^p(\frac{\partial f}{\partial z_1},\frac{1+r_1}{2},r_2)$ is nondecreasing in r_2 and applying the changes $r_i\to\frac{1+r_i}{2}$, $i=1,2$, we obtain

$$\begin{aligned}
&\int_0^1(1-r_2)^{\alpha_2}\int_0^1 M_p^p\Big(\frac{\partial f}{\partial z_1},\frac{1+r_1}{2},r_2\Big)(1-r_1)^{\alpha_1+p}\,dr_1\,dr_2\\
&\quad\le\int_0^1(1-r_2)^{\alpha_2}\int_0^1 M_p^p\Big(\frac{\partial f}{\partial z_1},\frac{1+r_1}{2},\frac{1+r_2}{2}\Big)(1-r_1)^{\alpha_1+p}\,dr_1\,dr_2\\
&\quad= C\int_{\frac12}^1(1-r_2)^{\alpha_2}\int_{\frac12}^1 M_p^p\Big(\frac{\partial f}{\partial z_1},r_1,r_2\Big)(1-r_1)^{\alpha_1+p}\,dr_1\,dr_2\\
&\quad\le C\int_{U^2}\left|\frac{\partial f}{\partial z_1}(z)\right|^p(1-|z_1|^2)^{\alpha_1+p}(1-|z_2|^2)^{\alpha_2}\,dV(z_1)\,dV(z_2).
\end{aligned}\tag{16}$$

Using again Lemmas 3 and 5, and since $M_p^p(f, \frac{1}{2}, r_2)$ is nondecreasing in r_2 we get

$$
\begin{aligned}
&\int_0^1 (1-r_2)^{\alpha_2} M_p^p\Big(f, \frac{1}{2}, r_2\Big) dr_2 \\
&\quad \le C\Bigg(M_p^p\Big(f, \frac{1}{2}, \frac{1}{2}\Big) \\
&\qquad + \int_0^1 (1-r_2)^{\alpha_2} \left| M_p^p\Big(f, \frac{1}{2}, \frac{1+r_2}{2}\Big) - M_p^p\Big(f, \frac{1}{2}, r_2\Big)\right| dr_2 \Bigg) \qquad (17)\\
&\quad \le C\Bigg(\max_{|z_1|\le\frac{1}{2}, |z_2|\le\frac{1}{2}} |f(z_1,z_2)|^p \\
&\qquad + \int_0^1 (1-r_2)^{\alpha_2+p} M_p^p\Big(\frac{\partial f}{\partial z_2}, \frac{1}{2}, \frac{1+r_2}{2}\Big) dr_2 \Bigg).
\end{aligned}
$$

It is clear that there is a constant C independent of f such that

$$
\left| \int_0^{\frac{3}{4}} (1-r_2)^{\alpha_2+p} M_p^p\Big(\frac{\partial f}{\partial z_2}, \frac{1}{2}, \frac{1+r_2}{2}\Big) dr_2 \right| \le C \max_{z\in \overline{D^2(0,\frac{7}{8})}} \left|\frac{\partial f}{\partial z_2}(z_1,z_2)\right|^p. \qquad (18)
$$

By (3) we have

$$
\left|\frac{\partial f}{\partial z_2}(\zeta_1,\zeta_2)\right|^p \le C \int_{P^2((\zeta_1,\zeta_2),(\frac{1}{16},\frac{1}{16}))} \left|\frac{\partial f}{\partial z_2}(\zeta_1,\zeta_2)\right|^p \prod_{j=1}^2 dV(\zeta_j)
$$

for some C independent of f, and for all $\zeta_1, \zeta_2 \in \overline{D(0,\frac{7}{8})}$. Hence

$$
\begin{aligned}
&\max_{z\in \overline{D^2(0,\frac{7}{8})}} \left|\frac{\partial f}{\partial z_2}(z_1,z_2)\right|^p \\
&\quad \le C \int_{D^2(0,\frac{15}{16})} \left|\frac{\partial f}{\partial z_2}(\zeta_1,\zeta_2)\right|^p \prod_{j=1}^2 dV(\zeta_j) \qquad (19)\\
&\quad \le C \int_{U^2} \left|\frac{\partial f}{\partial z_2}(z)\right|^p \big(1-|z_1|^2\big)^{\alpha_1} \big(1-|z_2|^2\big)^{\alpha_2+p} dV(z_1)\, dV(z_2).
\end{aligned}
$$

On the other hand, using the change $\frac{1+r_2}{2} \to r_2$ we obtain

$$
\int_{\frac{3}{4}}^1 (1-r_2)^{\alpha_2+p} M_p^p\Big(\frac{\partial f}{\partial z_2}, \frac{1}{2}, \frac{1+r_2}{2}\Big) dr_2 = C \int_{\frac{7}{8}}^1 (1-r_2)^{\alpha_2+p} M_p^p\Big(\frac{\partial f}{\partial z_2}, \frac{1}{2}, r_2\Big) dr_2
$$

for some $C > 0$. Since

$$
M_p^p\Big(\frac{\partial f}{\partial z_2}, \frac{1}{2}, r_2\Big) \le M_p^p\Big(\frac{\partial f}{\partial z_2}, r_1, r_2\Big),
$$

for $r_1 \in [\frac{1}{2}, 1)$ we obtain that there is a constant C independent of f such that

$$\begin{aligned} &\frac{1}{(\alpha_1+1)2^{\alpha_1+1}} \int_{\frac{7}{8}}^{1} (1-r_2)^{\alpha_2+p} M_p^p\Big(\frac{\partial f}{\partial z_2}, \frac{1}{2}, r_2\Big)\, dr_2 \\ &\quad = \int_{\frac{1}{2}}^{1} (1-r_1)^{\alpha_1} dr_1 \int_{\frac{7}{8}}^{1} (1-r_2)^{\alpha_2+p} M_p^p\Big(\frac{\partial f}{\partial z_2}, \frac{1}{2}, r_2\Big)\, dr_2 \\ &\quad \le \int_{\frac{1}{2}}^{1} (1-r_1)^{\alpha_1} \int_{\frac{7}{8}}^{1} (1-r_2)^{\alpha_2+p} M_p^p\Big(\frac{\partial f}{\partial z_2}, r_1, r_2\Big)\, dr_2\, dr_1 \\ &\quad \le C \int_{U^2} \Big|\frac{\partial f}{\partial z_2}(z)\Big|^p \big(1-|z_1|^2\big)^{\alpha_1} \big(1-|z_2|^2\big)^{\alpha_2+p}\, dV(z_1)\, dV(z_2). \end{aligned} \tag{20}$$

From (15) - (20) and Lemma 4 the result and an inequality in (1) follow, for $m = 1$. Using induction we obtain the result for $m \ge 2$. ∎

Remark. Note that we did not use the condition $p \in (0, 1]$ in the proof of the necessity, and so we proved the result also for $p > 1$.

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