Homogenization Structures and Applications II

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Abstract. In a recent work we presented a mathematical theory of homogenization structures and we subsequently constructed a new homogenization approach that proves highly fitted to systematically tackle nonstochastic homogenization problems beyond usual periodic homogenization theory. In this way, various concrete homogenization problems arising in nonperiodic physical processes can henceforth be considered. Of course, this releases us from the classical periodicity hypothesis to which reference is usually systematically made for lack of a suitable mathematical framework beyond the periodic setting. With a view to pointing out the wide scope of this new homogenization approach, we consider in this paper two classes of homogenization problems of major interest as regards their close connection with practical applications: the so-called discrete problems dealing with differential operators whose coefficients are constant on each cell $\varepsilon(k+Y)$ with $k \in \mathbb{Z}^N$, where $\varepsilon > 0$ and $Y = (0,1)^N$, and the so-called composite homogenization problems arising in the technology of composite materials. The exactness of the homogenization results confirms the essential role the homogenization structures are destined to play in homogenization.

Keywords: *Homogenization structures, discrete homogenization, composite homogenization.*

AMS subject classification: 46J10, 35B40

1. Introduction

In this paper we continue the studies begun in [14] on the homogenization structures and their application to the homogenization of partial differential equations. Let us return to the model boundary value problem considered in [14], viz.

$$-\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}^{\varepsilon}\frac{\partial u_{\varepsilon}}{\partial x_j}\right) = f \quad \text{in } \Omega, \ u_{\varepsilon} \in H_0^1(\Omega) \tag{1.1}$$

where $\varepsilon > 0$, Ω is a bounded open set in $\mathbb{R}_x^N$, $f \in H^{-1}(\Omega)$, $a_{ij}^{\varepsilon}(x) = a_{ij}(\frac{x}{\varepsilon})$ $(x \in \Omega)$ with $a_{ij} \in L^{\infty}(\mathbb{R}_y^N)$ and the ellipticity condition : there exists a constant

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$\alpha > 0$ such that

$$\mathrm{Re} \sum_{i,j=1}^{N} a_{ij}(y)\xi_j \overline{\xi}_i \geq \alpha \left| \xi \right|^2 \quad (\xi \in \mathbb{C}^N, \text{ a.e. in } y \in \mathbb{R}^N). \qquad (1.2)$$

It is well known that u_ε is uniquely determined by (1.1). For convenience we will in the sequel assume the symmetry condition $a_{ji} = \overline{a}_{ij}$ $(1 \leq i, j \leq N)$.

Now, suppose our purpose is to investigate the behaviour of u_ε as $\varepsilon \to 0$. Very likely, this is not possible without requiring the family $\{a_{ij}\}$ to satisfy one further appropriate condition called a *structure hypothesis*, which gives specific information on the way the said family is structured from an algebraic, a topological or a geometric point of view.

The classical structure hypothesis is the so-called *periodicity hypothesis* which states that each a_{ij} is S-periodic, that is, it satisfies for each $k \in S$ the equality $a_{ij}(y + k) = a_{ij}(y)$ a.e. in $y \in \mathbb{R}^N$, where S is a given network in $\mathbb{R}^N$, say $S = \mathbb{Z}^N$ ($\mathbb{Z}$ denotes the integers). The periodicity hypothesis has since long ago led to a powerful periodic homogenization theory for which there is an enormous bibliography; see, e.g., refs. [1, 2, 7-9, 17-20]. However, the periodicity hypothesis is only a particular structure hypothesis. No doubt, there is a wide variety of concrete structure hypotheses, and one major concern in this area is to bridge the gap between periodic and stochastic homogenization (concerning stochastic homogenization, see, e.g., ref. [6]).

In [14] we considered the homogenization problem for (1.1) under an abstract structure hypothesis (see (2.2)) depending on a given suitable homogenization structure, and we obtained a homogenization result which is in all respects similar to that provided by classical periodic homogenization theory, with the same explicitness and the same degree of accuracy. This is a great progress, since we are henceforth in a position to tackle a homogenization problem for (1.1) under any concrete structure hypothesis reducible to the preceding abstract structure hypothesis. Various examples of such concrete structure hypotheses (including the periodicity hypothesis, of course) are given in [14]. The present study is intended to illustrate this still more, for it goes without saying that the scope of our undertaking will depend on the variety of the concrete cases that can be covered by this new homogenization approach. Also, we prove in passing a practical outstanding result that was merely stated in [14: Example 5.5].

In this work we discuss two classes of homogenization problems for (1.1), namely the so-called *discrete homogenization* problem and the *composite homogenization* problem. Let us first state what is meant by a discrete homogenization problem for (1.1). Let

$$Y = \left(-\frac{1}{2}, \frac{1}{2} \right)^N \quad \text{and} \quad S = \mathbb{Z}^N. \qquad (1.3)$$

Suppose a_{ij} $(1 \leq i, j \leq N)$ is constant on each cell $k + Y$ $(k \in S)$. Clearly it amounts to assuming that for each fixed $k \in S$,

$$a_{ij}(y) = r_{ij}(k) \quad \text{a.e. in } y \in k + Y \quad (1 \leq i, j \leq N) \tag{1.4}$$

where

$$r_{ij}(k) = \int_{k+Y} a_{ij}(z)dz. \tag{1.5}$$

Then, it is easily seen that the statement of a structure hypothesis on a_{ij} cannot be made otherwise than by means of the family $\{r_{ij}\}$ (r_{ij} denotes the complex mapping on S defined at each $k \in S$ by (1.5)). For an obvious reason such a structure hypothesis is qualified as *discrete*. As examples of discrete structure hypotheses on a_{ij} $(1 \leq i, j \leq N)$, we have the following three cases :

(1) $r_{ij} \in \mathcal{B}_\infty(S)$
(2) $r_{ij} = c_{ij} + \gamma_{ij}$ with $c_{ij} \in \mathbb{C}$ and $\gamma_{ij} \in \ell^1(S)$
(3) $r_{ij} = t_{ij} + \gamma_{ij}$ with $t_{ij} \in \mathcal{B}_\infty(S)$ and $\gamma_{ij} \in \ell^1(S)$

where $\mathcal{B}_\infty(S)$ denotes the space of all complex mappings on S that converge at infinity, and $\ell^1(S)$ denotes the usual space of all mappings $a : S \to \mathbb{C}$ that are summable, i.e., that they satisfy $\sum_{k \in S} |a(k)| < +\infty$.

This being so, by a discrete homogenization problem for (1.1) we understand the study of the behaviour, as $\varepsilon \to 0$, of u_ε (the solution of (1.1)) under a discrete structure hypothesis on $\{a_{ij}\}$.

Remark. To say that a_{ij} is constant on each cell $k + Y$ $(k \in S)$ again amounts to say that a_{ij} writes as $a_{ij} = \sum_{k \in S} r_{ij}(k)\chi_{k+Y}$ (a locally finite sum), where $r_{ij}(k)$ is given by (1.5) and χ_{k+Y} denotes the characteristic function of $k + Y$.

Remark. The discrete structure hypothesis arises quite naturally in homogenization theory. To see this, let us place ourselves in the general case where we know merely that a_{ij} lies in $L^\infty(\mathbb{R}_y^N)$ (i.e., we do not assume that a_{ij} is constant on each $k + Y$). Then, it is easily verified that a_{ij}, as a function in $L^\infty(\mathbb{R}_y^N)$, is uniquely expressible in the form

$$a_{ij} = \lambda_{ij} + \mu_{ij} \tag{1.6}$$

where λ_{ij} and μ_{ij} are functions in $L^\infty(\mathbb{R}_y^N)$ with

$$\lambda_{ij}(y) = r_{ij}(k) \quad \text{a.e. in } y \in k + Y \tag{1.7}$$

($r_{ij}(k)$ given by rel. (1.5)), and

$$\int_{k+Y} \mu_{ij}(y)dy = 0 \tag{1.8}$$

for each fixed $k \in S$. Thus, a structure hypothesis on a_{ij} $(1 \leq i, j \leq N)$ may be reduced to two complementary structure hypotheses : one on r_{ij} $(1 \leq i, j \leq N)$, the other on μ_{ij} $(1 \leq i, j \leq N)$. The notion of a discrete structure hypothesis falls under the particular case where $\mu_{ij} = 0$ $(1 \leq i, j \leq N)$.

We will next consider a so-called composite homogenization problem for (1.1), that is, roughly speaking, a homogenization problem for (1.1) under a structure hypothesis on $\{a_{ij}\}$ depending on a given suitable partition of $\mathbb{R}_y^N$. From a practical point of view, such a problem is of considerable interest in so far as it is closely connected to the homogenization of composite materials.

The rest of the work is organized as follows. In Section 2 we quickly recall the notion of a homogenization structure together with the manner in which the latter arises in the homogenization of partial differential equations. Section 3 deals with the so-called discrete homogenization structures together with their application to the homogenization of (1.1). Finally, in Section 4 we develop a short theory of composite homogenization for (1.1) and we present two practical examples.

We conclude this section by summarizing some of the basic notation we will be using. Except where otherwise stated, all vector spaces are considered over $\mathbb{C}$ (the complex numbers) and the scalar functions are assumed to take their values in $\mathbb{C}$. If X and F denote a locally compact space and a Banach space, respectively, then we denote by $\mathcal{C}(X; F)$ the space of all continuous mappings of X into F, by $\mathcal{K}(X; F)$ the space of those functions in $\mathcal{C}(X; F)$ having compact supports, and by $\mathcal{B}(X; F)$ the space of those functions in $\mathcal{C}(X; F)$ that are bounded. We will always assume $\mathcal{B}(X; F)$ to be equipped with the supremum norm $\|u\|_\infty = \sup_{x \in X} \|u(x)\|$ $(u \in \mathcal{B}(X; F))$ where $\|\cdot\|$ denotes the norm in F. For shortness we write $\mathcal{C}(X) = \mathcal{C}(X; \mathbb{C})$, $\mathcal{K}(X) = \mathcal{K}(X; \mathbb{C})$ and $\mathcal{B}(X) = \mathcal{B}(X; \mathbb{C})$. Likewise the spaces $L^p(X; F)$ and $L^p_{loc}(X; F)$ (where X is provided with a positive Radon measure) will be denoted by $L^p(X)$ and $L^p_{loc}(X)$, respectively, when $F = \mathbb{C}$. We refer to [3], [4] and [10] for integration theory. Finally, the space $\mathbb{R}^N$ and its open sets are assumed to be provided with the Lebesgue measure denoted by λ or, as usual, $dx = dx_1 \cdots dx_N$.

2. Fundamentals of homogenization structures: Basic notation and results

For the benefit of the reader we summarize below basic notions and results concerning the theory of homogenization structures. We refer to [14] for more details.

We start with a definition which is at the root of the notion of a homogenization structure. A *structural representation* on $\mathbb{R}_y^N$ is defined as being any

subset Γ of $\mathcal{B}(\mathbb{R}_y^N)$ such that

(HS)$_1$ Γ is a group under multiplication in $\mathcal{B}(\mathbb{R}_y^N)$

(HS)$_2$ Γ is countable

(HS)$_3$ if $\gamma \in \Gamma$ then $\overline{\gamma} \in \Gamma$ ($\overline{\gamma}$ the complex conjugate of γ)

(HS)$_4$ $\Gamma \subset \Pi^\infty$,

where Π^∞ denotes the space of all functions $u \in \mathcal{B}(\mathbb{R}_y^N)$ with the property that $u^\varepsilon \to M(u)$ in $L^\infty(\mathbb{R}_x^N)$-weak $*$ as $\varepsilon \to 0$, $M(u)$ being a complex number (depending on u), and

$$u^\varepsilon(x) = u\left(\frac{x}{\varepsilon}\right) \text{ for } x \in \mathbb{R}^N \quad (\varepsilon > 0). \tag{2.1}$$

Let us note in passing that the mapping $u \to M(u)$ of Π^∞ into $\mathbb{C}$ is a positive continuous linear form M with $M(1) = 1$ and $M(\tau_a u) = M(u)$ for $u \in \Pi^\infty$ and $a \in \mathbb{R}^N$, where $\tau_a u(y) = u(y - a)$ for $y \in \mathbb{R}^N$ (it is shown that $\tau_a u \in \Pi^\infty$ for $u \in \Pi^\infty$ and $a \in \mathbb{R}^N$). See [15] for further detail.

   Now, in the collection of all structural representations on $\mathbb{R}_y^N$ we consider the binary relation : $\Gamma \sim \Gamma'$ *if and only if* $CLS(\Gamma) = CLS(\Gamma')$, where $CLS(\Gamma)$ denotes the closed vector subspace of $\mathcal{B}(\mathbb{R}_y^N)$ spanned by Γ. It is easily checked that this is an equivalence relation. By an *H-structure* on $\mathbb{R}_y^N$ (H stands for *homogenization*) is meant any equivalence class modulo $\sim$. If Σ is a given H-structure on $\mathbb{R}_y^N$, we let $A = CLS(\Gamma)$, where Γ is any equivalence class representative of Σ (such a Γ is termed a *representation* of Σ). A is a so-called H-algebra on $\mathbb{R}_y^N$, that is, a closed subalgebra of $\mathcal{B}(\mathbb{R}_y^N)$ with the propreties :

(HA)$_1$ A with the supremum norm is separable

(HA)$_2$ A contains the constants

(HA)$_3$ if $u \in A$ then $\overline{u} \in A$

(HA)$_4$ $A \subset \Pi^\infty$.

Futhermore, A depends only on Σ and not on the chosen representation Γ of Σ, so that we may set $A = \mathcal{J}(\Sigma)$ (*image of* Σ). The mapping $\Sigma \to \mathcal{J}(\Sigma)$ thus defined carries the collection of all H-structures bijectively over the collection of all H-algebras on $\mathbb{R}^N$ [14: Theorem 3.1].

   An H-algebra A on $\mathbb{R}^N$ is clearly a commutative C^*-algebra with identity. We denote by $\Delta(A)$ the spectrum of A, i.e., $\Delta(A)$ is the set of all nonzero multiplicative linear forms on A, $\Delta(A)$ being endowed with the relative weak $*$ topology on A' (the topological dual of A). The Gelfand transformation on A is denoted by $\mathcal{G}$. We recall that $\mathcal{G}$ is the mapping $u \to \mathcal{G}(u)$ of A into $\mathcal{C}(\Delta(A))$ such that $\mathcal{G}(u)(s) = \langle s, u \rangle$ for $s \in \Delta(A)$, where $\langle \ , \ \rangle$ denotes the duality between A' and A. It is shown that $\Delta(A)$ is a compact metrizable space and $\mathcal{G}$ is an isometric isomorphism of the C^*-algebra A onto the C^*-algebra $\mathcal{C}(\Delta(A))$ (see, e.g., [12:

p. 277]). The space $\Delta(A)$ will be equipped with the so-called *M-measure for A*, that is, the Radon measure β on $\Delta(A)$ such that

$$M(u) = \int_{\Delta(A)} \mathcal{G}(u)(s) d\beta(s) \ (u \in A).$$

On the other hand, by means of the transformation $\mathcal{G}$ we can carry over to $\Delta(A)$ the partial derivatives on $\mathbb{R}_y^N$. Specifically, let $A^1 = \left\{ \Psi \in \mathcal{C}^1(\mathbb{R}_y^N) : \ \Psi, \frac{\partial \Psi}{\partial y_i} \right.$ $\in A \ (1 \le i \le N)\}$. The partial derivative of index i $(1 \le i \le N)$ on $\Delta(A)$ is defined to be the mapping $\partial_i = \mathcal{G} \circ \frac{\partial}{\partial y_i} \circ \mathcal{G}^{-1}$ (usual composition) of $\mathcal{D}^1(\Delta(A)) = \{\varphi \in \mathcal{C}(\Delta(A)) : \ \mathcal{G}^{-1}(\varphi) \in A^1\}$ into $\mathcal{C}(\Delta(A))$, where $\mathcal{G}^{-1}$ (the inverse of $\mathcal{G}$) is viewed as defined on $\mathcal{D}^1(\Delta(A))$. Higher order derivatives are defined analogously (see [14]). Now, let A^∞ be the set of all functions $\Psi \in \mathcal{C}^\infty(\mathbb{R}_y^N)$ (the complex functions of class $\mathcal{C}^\infty$ on $\mathbb{R}_y^N$) such that $D_y^\alpha \Psi = \frac{\partial^{|\alpha|} \Psi}{\partial y_1^{\alpha_1} \cdots \partial y_N^{\alpha_N}} \in A$ for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$. Let $\mathcal{D}(\Delta(A)) = \{\varphi \in \mathcal{C}(\Delta(A)) : \ \mathcal{G}^{-1}(\varphi) \in A^\infty\}$. Endowed with a suitable locally convex topology (see [14]), each of these two spaces is a Fréchet space and, further, $\mathcal{G}$ viewed as defined on A^∞ is a topological isomorphism of A^∞ onto $\mathcal{D}(\Delta(A))$.

Any continuous linear form on $\mathcal{D}(\Delta(A))$ is referred to as a *distribution* on $\Delta(A)$. The space $\mathcal{D}'(\Delta(A))$ (topological dual of $\mathcal{D}(\Delta(A))$) of all distributions on $\Delta(A)$ is endowed with the strong dual topology. If we assume that A^∞ is dense in A, which is equivalent to assuming that $\mathcal{D}(\Delta(A))$ is dense in $\mathcal{C}(\Delta(A))$, then we have $L^p(\Delta(A)) \subset \mathcal{D}'(\Delta(A))$ $(1 \le p \le +\infty)$ with continuous embedding. Hence we may define the Hilbert space (see [14])

$$H^1(\Delta(A)) = \left\{ u \in L^2(\Delta(A)) : \ \partial_i u \in L^2(\Delta(A)) \ (1 \le i \le N) \right\}$$

where the derivative $\partial_i u$ is taken in the distribution sense on $\Delta(A)$. Furthermore, we consider the space

$$H^1(\Delta(A))/\mathbb{C} = \left\{ u \in H^1(\Delta(A)) : \int_{\Delta(A)} u(s) d\beta(s) = 0 \right\}$$

equipped with the seminorm

$$\|u\|_{H^1(\Delta(A))/\mathbb{C}} = \left(\sum_{i=1}^N \|\partial_i u\|_{L^2(\Delta(A))}^2 \right)^{\frac{1}{2}} \quad (u \in H^1(\Delta(A))/\mathbb{C}),$$

which makes it a pre-Hilbert space. In general $H^1(\Delta(A))/\mathbb{C}$ so topologized is non-separated and non-complete. This leads us to introduce the separated completion $H_\#^1(\Delta(A))$ of $H^1(\Delta(A))/\mathbb{C}$, and the canonical mapping J of $H^1(\Delta(A))/\mathbb{C}$ into $H_\#^1(\Delta(A))$ (see [5: Chapter II]). It is shown that the distribution derivative

∂_i viewed as a mapping of $H^1(\Delta(A))/\mathbb{C}$ into $L^2(\Delta(A))$, extends to a unique continuous linear mapping, still denoted by ∂_i, of $H^1_\#(\Delta(A))$ into $L^2(\Delta(A))$ such that $\partial_i J(v) = \partial_i v$ for $v \in H^1(\Delta(A))/\mathbb{C}$. Furthermore,

$$\|u\|_{H^1_\#(\Delta(A))} = \left(\sum_{i=1}^N \|\partial_i u\|^2_{L^2(\Delta(A))} \right)^{\frac{1}{2}} \quad \left(u \in H^1_\#(\Delta(A)) \right).$$

As we mentioned in [14], the role of the proper H-structures is essential in this study. Before we recall the definition of a proper H-structure, let us introduce one simple notion that will be needed. By a *fundamental sequence* is meant any ordinary sequence of real numbers $0 < \varepsilon_n \le 1$ $(n \in \mathbb{N})$ with $\varepsilon_n \to 0$ as $n \to +\infty$.

Now, let Σ be an H-structure on $\mathbb{R}^N$, and let $A = \mathcal{J}(\Sigma)$. We say that Σ is proper if the following conditions hold :

(PR)₁ Σ is of class $\mathcal{C}^\infty$, i.e., $\mathcal{D}(\Delta(A))$ is dense in $\mathcal{C}(\Delta(A))$.

(PR)₂ Σ is total, i.e., $\mathcal{D}(\Delta(A))$ is dense in $H^1(\Delta(A))$.

(PR)₃ For any bounded open set $\Omega \subset \mathbb{R}^N_x$, $H^1(\Omega)$ is Σ-reflexive in the following sense : given a fundamental sequence E and a sequence $(u_\varepsilon)_{\varepsilon \in E}$ which is bounded in $H^1(\Omega)$, a subsequence E' can be extracted from E such that as $E' \ni \varepsilon \to 0$, we have $u_\varepsilon \to u_0$ in $H^1(\Omega)$-weak and $\frac{\partial u_\varepsilon}{\partial x_j} \to \frac{\partial u_0}{\partial x_j} + \partial_j u_1$ in $L^2(\Omega)$-weak Σ $(1 \le j \le N)$, where $u_1 \in L^2(\Omega; H^1_\#(\Delta(A)))$ (see [14: Remark 4.3]).

Remark. We recall that a sequence $(v_\varepsilon)_{\varepsilon > 0} \subset L^p(\Omega)$ $(1 \le p < \infty)$ is said to be weakly Σ-convergent in $L^p(\Omega)$ to some $v_0 \in L^p(\Omega \times \Delta(A))$ as $\varepsilon \to 0$, and we denote $v_\varepsilon \to v_0$ in $L^p(\Omega)$-weak Σ, if as $\varepsilon \to 0$,

$$\int_\Omega v_\varepsilon(x) \Psi(x, \frac{x}{\varepsilon}) \, dx \to \int\!\!\int_{\Omega \times \Delta(A)} v_0(x, s) \widehat{\Psi}(x, s) \, dx \, d\beta(s)$$

for every $\Psi \in L^{p'}(\Omega; A)$ $(\frac{1}{p'} = 1 - \frac{1}{p})$, where $\widehat{\Psi}(x, \cdot) = \mathcal{G}(\Psi(x))$, $x \in \Omega$, and where β denotes the M-measure for A. The concept of Σ-convergence is discussed in the fullest detail in [14] (the case where Σ is a periodic H-structure sends back to two-scale convergence [13]).

Before we can recall how the H-structures arise in the homogenization of partial differential equations, we still need a few preliminaries. To begin, let $1 \le p < +\infty$. Let Ξ^p be the set of all $u \in L^p_{loc}(\mathbb{R}^N_y)$ for which the sequence $(u^\varepsilon)_{0 < \varepsilon \le 1}$ is bounded in $L^p_{loc}(\mathbb{R}^N_x)$ (u^ε defined in (2.1)). This is a Banach space under the norm

$$\|u\|_{\Xi^p} = \sup_{0 < \varepsilon \le 1} \left(\int_{B_N} \left| u\left(\frac{x}{\varepsilon} \right) \right|^p dx \right)^{\frac{1}{p}} \quad (u \in \Xi^p)$$

where B_N denotes the open unit ball of $\mathbb{R}_x^N$. This being so, given an H-structure Σ on $\mathbb{R}^N$, we define $\mathfrak{X}_\Sigma^p(\mathbb{R}_y^N)$ (or $\mathfrak{X}_\Sigma^p$, or simply $\mathfrak{X}^p$ when there is no danger of confusion) to be the closure of $\mathcal{J}(\Sigma)$ in Ξ^p. Equipped with the Ξ^p-norm, $\mathfrak{X}_\Sigma^p$ is a Banach space. Next, let $\mathbb{F}_0^1 = H_0^1(\Omega) \times L^2(\Omega; H_\#^1(\Delta(A)))$ with $A = \mathcal{J}(\Sigma)$. This is a Hilbert space with norm

$$\|V\|_{\mathbb{F}_0^1} = \left[\sum_{i=1}^N \|\mathbb{D}_i V\|_{L^2(\Omega \times \Delta(A))}^2 \right]^{\frac{1}{2}} \qquad (V \in \mathbb{F}_0^1)$$

where

$$\mathbb{D}_i V = \frac{\partial v_0}{\partial x_i} + \partial_i v_1 \qquad \text{for} \quad V = (v_0, v_1) \in \mathbb{F}_0^1.$$

Finally, suppose

$$a_{ij} \in \mathfrak{X}_\Sigma^2 \quad (1 \le i, j \le N). \tag{2.2}$$

Then, according to [14: Proposition 2.4 and Corollary 2.2], we can define $\widehat{a}_{ij} = \mathcal{G}(a_{ij}) \in L^\infty(\Delta(A))$ for $1 \le i, j \le N$, and hence the sesquilinear form $\widehat{a}_\Omega$ on $\mathbb{F}_0^1 \times \mathbb{F}_0^1$ given by

$$\widehat{a}_\Omega(U, V) = \sum_{i,j=1}^N \int \int_{\Omega \times \Delta(A)} \widehat{a}_{ij}(s) \mathbb{D}_j U(x, s) \overline{\mathbb{D}_i V}(x, s) dx d\beta(s) \tag{2.3}$$

for $U, V \in \mathbb{F}_0^1$. This sesquilinear form is continuous and coercive. Consequently, if L denotes the continuous antilinear form on $\mathbb{F}_0^1$ given by $L(V) = \langle f, \overline{v}_0 \rangle$ (the same f as in (1.1)), $V = (v_0, v_1) \in \mathbb{F}_0^1$, then the variational problem

$$\begin{cases} U = (u_0, u_1) \in \mathbb{F}_0^1 \text{ and} \\ \\ \widehat{a}_\Omega(U, V) = L(V) \quad \text{for all } V \in \mathbb{F}_0^1 \end{cases} \tag{2.4}$$

admits a unique solution. We are now in a position to state the following fundamental homogenization result (for the proof see [14]).

Theorem 2.1. *Let Σ be a proper H-structure on $\mathbb{R}^N$. Let $A = \mathcal{J}(\Sigma)$. Suppose (2.2) is satisfied. Let $U = (u_0, u_1)$ be uniquely given by the variational problem (2.4), and for each real $\varepsilon > 0$, let u_ε be the unique solution of the boundary value problem (1.1). Then, as $\varepsilon \to 0$, $u_\varepsilon \to u_0$ in $H_0^1(\Omega)$-weak, $u_\varepsilon \to u_0$ in $L^2(\Omega)$ and $\frac{\partial u_\varepsilon}{\partial x_j} \to \mathbb{D}_j U$ in $L^2(\Omega)$-weak Σ $(1 \le j \le N)$.*

¿From this theorem one deduces local and macroscopic homogenization problems (see [14: Corollary 5.1]) similar to those of the classical periodic homogenization theory. In particular u_0 is here the solution of a boundary value problem

$$-\sum_{i,j=1}^N q_{ij} \frac{\partial^2 u_0}{\partial x_i \partial x_j} = f \quad \text{in } \Omega, \, u_0 \in H_0^1(\Omega)$$

where the coefficients q_{ij} $(1 \leq i, j \leq N)$ satisfy the usual symmetry and ellipticity conditions.

3. Discrete H-structures and application

Now that we have a clear idea of what is termed a *discrete homogenization problem*, it is of interest to note that the mathematical analysis of discrete homogenization problems requires the use of specific H-structures called *discrete H-structures*. In the present section we discuss such tools and show how to handle them to tackle discrete homogenization problems and many more besides.

3.1. Essential algebras. The notion of a discrete H-structure is based on that of an essential algebra. First we recall some fundamentals of the theory of essential functions (for the proofs of the forthcoming results we refer to [15]). In what follows, G denotes either $\mathbb{R}_y^N$ or the network $S = \mathbb{Z}^N$ (endowed with the discrete topology). By a *finite net* in G we mean any family $F = (a_i)_{i \in I}$ of points in G, where the index set I is finite and nonempty. To F we attach the transformation $\mathfrak{M}_F$ in $\mathcal{B}(G)$ defined by $\mathfrak{M}_F(u) = \frac{1}{|I|} \sum_{i \in I} \tau_{a_i} u$ $(u \in \mathcal{B}(G))$, where $|I|$ denotes the cardinality of I, and $\tau_{a_i} u(y) = u(y - a_i)$ for $y \in G$.

Definition 3.1. We say a mapping $u : G \to \mathbb{C}$ is an *essential function* on G if u lies in $\mathcal{B}(G)$ and further if u satisfies: for every real $\eta > 0$, there exists a finite net F in G such that $|\mathfrak{M}_F(u)(y) - \mathfrak{M}_F(u)(z)| \leq \eta$ for all $y, z \in G$.

The set of all essential functions on G is denoted by $ES(G)$. It is easily checked that $ES(G)$ is a closed vector subspace of $\mathcal{B}(G)$. In the sequel $ES(G)$ is assumed to be equipped with the supremum norm. We also note that $ES(G)$ contains the constants, is translation invariant, and finally that if u lies in $ES(G)$ then so also does $\overline{u}$ (the complex conjugate of u). We have

Proposition 3.1. *The following two conditions are equivalent for a function $u \in \mathcal{B}(G)$:*

(i) $u \in ES(G)$.

(ii) *There exists a complex number $\mathfrak{M}(u)$ with the property : given $\eta > 0$, there is a finite net F in G such that $|\mathfrak{M}_F(u)(y) - \mathfrak{M}(u)| \leq \eta$ $(y \in G)$. Furthermore, $\mathfrak{M}(u)$ is unique.*

This yields a unique positive continuous linear form $\mathfrak{M}$ on $ES(G)$ such that

(1) $\mathfrak{M}(\tau_a u) = \mathfrak{M}(u)$ for $u \in ES(G)$ and $a \in G$

(2) $\mathfrak{M}$ attains the value 1 on the constant function 1.

Definition 3.2. The linear form $\mathfrak{M}$ is called the *essential mean value* on G, and $\mathfrak{M}(u)$ is called the *essential mean* of the function $u \in ES(G)$.

The case $G = \mathbb{R}_y^N$ is of particular interest.

Proposition 3.2. $ES(\mathbb{R}_y^N) \subset \Pi^\infty(\mathbb{R}_y^N)$ *and* $\mathfrak{M}(u) = M(u)$ *for all* $u \in ES(\mathbb{R}_y^N)$ *(M was defined in Section 2).*

As mentioned above, the preceding propositions are proved in [15]. Also, several examples of essential functions are given in [15].

We conclude this subsection by discussing the notion of an essential algebra. We purposely restrict ourselves to $G = S$.

Definition 3.3. By an *essential algebra* on S is meant any closed subalgebra $\mathfrak{S}$ of $\mathcal{B}(S) = \ell^\infty(S)$ with the following properties :

$(\mathbf{ESA})_\mathbf{1}$ $\mathfrak{S}$ with the supremum norm is separable.
$(\mathbf{ESA})_\mathbf{2}$ $\mathfrak{S}$ contains the constants.
$(\mathbf{ESA})_\mathbf{3}$ If $a \in \mathfrak{S}$ then $\bar{a} \in \mathfrak{S}$.
$(\mathbf{ESA})_\mathbf{4}$ $\mathfrak{S} \subset ES(S).$

A few practical examples of essential algebras are collected below.

Example 3.1. *The algebra* $\mathcal{B}_\infty(S).$ We recall that $\mathcal{B}_\infty(S)$ denotes the set of all mappings $a : S \to \mathbb{C}$ such that $\lim_{|k| \to +\infty} a(k) = \zeta \in \mathbb{C}$, where ζ depends on a and where $|k|$ is the Euclidean norm (in $\mathbb{R}^N$) of $k \in S$. Clearly $\mathcal{B}_\infty(S)$ is a closed subalgebra of $\ell^\infty(S) = \mathcal{B}(S)$. Furthermore, for $\mathfrak{S} = \mathcal{B}_\infty(S)$, condition $(\text{ESA})_1$ follows by classical arguments (see [5: page 25 of Chapter X and page 18 of Chapter IX]), $(\text{ESA})_2$ and $(\text{ESA})_3$ are evident, and finally $(\text{ESA})_4$ follows by [15: Proposition 3.3]. Therefore, $\mathcal{B}_\infty(S)$ is an essential algebra on S.

Example 3.2. *The algebra* $\ell_0^1(S).$ We define $\ell_0^1(S)$ to be the closure in $\ell^\infty(S)$ of the set of all functions $a \in \ell^\infty(S)$ of the form $a = \xi + b$, $\xi \in \mathbb{C}$, $b \in \ell^1(S)$. The space $\ell_0^1(S)$ is an essential algebra. Indeed, for $\mathfrak{S} = \ell_0^1(S)$, $(\text{ESA})_1$ is a classical property, $(\text{ESA})_2$ and $(\text{ESA})_3$ are evident, and $(\text{ESA})_4$ follows by [15: Example 3.5].

Example 3.3. *The algebra* $\ell_\infty^1(S).$ By this we denote the closure in $\ell^\infty(S)$ of the set of all $a \in \ell^\infty(S)$ of the form $a = \theta + b$ with $\theta \in \mathcal{B}_\infty(S)$ and $b \in \ell^1(S)$. It is clear that $\ell_\infty^1(S)$ is an essential algebra on S. Furthermore, $\ell_0^1(S) \subset \ell_\infty^1(S)$ and $\mathcal{B}_\infty(S) \subset \ell_\infty^1(S).$

3.2. Discrete H-structures. Throughout this subsection, $\mathfrak{S}$ denotes a given essential algebra on $S = \mathbb{Z}^N$. Let $\mathcal{F}$ be the set of all functions $f : \mathbb{R}_y^N \to \mathbb{C}$ of the form

$$f = \sum_{k \in S} a(k)\tau_k\varphi \quad \text{with } a \in \mathfrak{S} \text{ and } \varphi \in \mathcal{K}(Y) \tag{3.1}$$

where, as is customary, $\mathcal{K}(Y)$ is identified with the space of those $\varphi \in \mathcal{K}(\mathbb{R}^N)$ with supports in Y (see (1.3)). Attention must be drawn to the fact that the above sum is actually locally finite.

The following lemma is essential to the notion of a discrete H-structure.

Lemma 3.1. *The following statements are true :*

(i) $\mathcal{F} \subset ES(\mathbb{R}^N_y)$.

(ii) $\mathcal{F}$ *is stable under multiplication and under complex conjugation.*

Proof. `Part (i).` Let $f \in \mathcal{F}$ with (3.1). The restriction of f to any arbitrarily fixed compact set $K \subset \mathbb{R}^N_y$ is given by the finite sum $f\,|_K = \sum_{k \in F} a(k)(\tau_k \varphi)\,|_K$ where F denotes the (finite) set of all $k \in S$ such that K intersects $k + \overline{Y}$ ($\overline{Y}$ the closure of Y). This shows that f is continuous on $\mathbb{R}^N_y$. On the other hand, recalling that the family $\left\{k + \overline{Y}\right\}_{k \in S}$ is a covering of $\mathbb{R}^N$, we see that to each given $y \in \mathbb{R}^N$ there can be attached some $k \in S$ such that $y \in k + \overline{Y}$, thus $f(y) = a(k)\varphi(y - k)$. Hence $|f(y)| \leq \|a\|_\infty \|\varphi\|_\infty$, which shows that f is bounded. Therefore $f \in \mathcal{B}(\mathbb{R}^N_y)$. To complete the proof of part (i), let us fix an arbitrary real $\eta > 0$. Let $\theta = \sum_{k \in S} \tau_k |\varphi|$. Clearly $\theta \in \mathcal{B}(\mathbb{R}^N_y)$ and so we may fix a constant $c > 0$ such that $|\theta(y)| \leq c$ for all $y \in \mathbb{R}^N$. Now, consider a finite net F in S such that $|\mathfrak{M}_F(a)(k) - \mathfrak{M}(a)| \leq \frac{\eta}{2c}$ for all $k \in S$ (Proposition 3.1), where $\mathfrak{M}$ denotes the essential mean value on S as well as on $\mathbb{R}^N_y$. On the other hand, let $\Psi = \mathfrak{M}(a) \sum_{k \in S} \tau_k \varphi$, and keep in mind that Ψ lies in $ES(\mathbb{R}^N)$, since it is a periodic continuous function on $\mathbb{R}^N_y$ (see [15: Remark 2.2 and Proposition 3.2]). Using the trivial equality

$$\mathfrak{M}_F(f) - \Psi = \sum_{k \in S} [\mathfrak{M}_F(a)(k) - \mathfrak{M}(a)]\tau_k \varphi,$$

we see that

$$|\mathfrak{M}_F(f) - \Psi| \leq \frac{\eta}{2}. \tag{3.2}$$

Now, fix a finite net R in $\mathbb{R}^N_y$ such that $|\mathfrak{M}_R(\Psi)(y) - \mathfrak{M}(\Psi)| \leq \frac{\eta}{2}$ for all $y \in \mathbb{R}^N$, according to Proposition 3.1. Furthermore, observe that by (3.2) we have

$$|\mathfrak{M}_{F+R}(f)(y) - \mathfrak{M}_R(\Psi)(y)| \leq \frac{\eta}{2}$$

for all $y \in \mathbb{R}^N$. Hence $|\mathfrak{M}_{F+R}(f)(y) - \mathfrak{M}(\Psi)| \leq \eta$ for all $y \in \mathbb{R}^N$, which shows that f lies in $ES(\mathbb{R}^N_y)$ (Proposition 3.1). Therefore (i) follows.

`Part (ii).` The stability under complex conjugation is evident whereas the stability under multiplication is easily checked by using the fact that if $\varphi, \theta \in \mathcal{K}(Y)$, then the product $\tau_k \varphi \tau_l \theta$ $(k, l \in S)$ is zero whenever $k \neq l$. This completes the proof. ■

This being so, let A be the closure in $\mathcal{B}(\mathbb{R}_y^N)$ of the space of all functions of the form

$$\Psi = c + \sum_{i=1}^{m} f_i \quad \text{with } c \in \mathbb{C} \text{ and } f_i \in \mathcal{F}$$

where the integer $m \geq 1$, the constant c and the functions f_i depend on Ψ.

Proposition 3.3. *A is an H-algebra.*

Proof. Thanks to Lemma 3.1, A is a closed subalgebra of $\mathcal{B}(\mathbb{R}_y^N)$ satisfying $(\mathrm{HA})_3$ and $(\mathrm{HA})_4$ (use [15: Theorem 4.2]). On the other hand, $(\mathrm{HA})_2$ is evident, so that only $(\mathrm{HA})_1$ remains to be verified. To this end let D_0 be a dense countable set in $\mathfrak{S}$ (use $(\mathrm{ESA})_1$) and $\mathcal{K}_0$ be a dense countable set in $\mathcal{K}(Y)$ with the supremum norm (indeed, $\mathcal{K}(Y)$ with the supremum norm is separable). Consider the set $\mathcal{Q}_0$ of all functions of the form

$$f = \sum_{k \in S} a(k) \tau_k \varphi \quad \text{with} \quad a \in D_0 \text{ and } \varphi \in \mathcal{K}_0,$$

and put $\mathbb{C}_0 = \mathbb{Q} + i\mathbb{Q}$ ($\mathbb{Q}$ denotes the rationals). Then, clearly the functions of the form

$$\Psi = c + \sum_{i=1}^{m} f_i \quad \text{with } c \in \mathbb{C}_0, \ f_i \in \mathcal{Q}_0 \text{ and } m \in \mathbb{N}, \ m \geq 1$$

form a dense countable subset of A. The proof is complete. ∎

This leads us to the notion of a discrete H-structure.

Definition 3.4. The H-structure on $\mathbb{R}^N$ whose image is the H-algebra A (see [14: Theorem 3.1]) is called the *discrete H-structure on $\mathbb{R}^N$ associated to* $\mathfrak{S}$, and is denoted by $\Sigma_{\mathfrak{S}}^0$.

We will need the following

Proposition 3.4. *Let Σ_S be the periodic H-structure on $\mathbb{R}^N$ represented by the network S [14: Example 3.2]. The pair $\{\Sigma_{\mathfrak{S}}^0, \Sigma_S\}$ is summable [14: Section 3].*

Proof. Let $A = \mathcal{J}(\Sigma_{\mathfrak{S}}^0)$ and $\mathcal{C}_{per}(Y) = \mathcal{J}(\Sigma_S)$ (the space of all S-periodic continuous complex functions on $\mathbb{R}_y^N$; see [14: Example 2.1]). Our purpose is to show that $A + \mathcal{C}_{per}(Y)$ is stable under multiplication. But this is equivalent to showing that the bilinear transformation $(u, \Psi) \rightarrow u\Psi$ maps $A \times \mathcal{C}_{per}(Y)$ into $A + \mathcal{C}_{per}(Y)$. So let $u \in A$ and $\Psi \in \mathcal{C}_{per}(Y)$. The proposition is proved if we can check that $u\Psi$ lies in $A + \mathcal{C}_{per}(Y)$. First of all, we need to express u in the form

$$u = \xi + u_0 \text{ with} \quad \xi \in \mathbb{C} \text{ and } u_0 \in CLS(\mathcal{F}) \tag{3.3}$$

where $\mathcal{F}$ denotes the set of all functions of the form (3.1) and $CLS(\mathcal{F})$ stands for the closed linear span of $\mathcal{F}$. According to the definition of A, there is a sequence $(v_n)_{n \geq 1}$ with

$$\begin{cases} v_n = \xi_n + w_n \quad \text{with} \quad \xi_n \in \mathbb{C} \text{ and } w_n \in \langle \mathcal{F} \rangle \\ \langle \mathcal{F} \rangle \text{ being the space of all finite sums } \sum_{finite} f_i \ (f_i \in \mathcal{F}) \end{cases}$$

such that $v_n \to u$ in $\mathcal{B}(\mathbb{R}^N)$ as $n \to +\infty$. Noticing that each function w_n vanishes on ∂Y (the boundary of Y), we deduce that, as $n \to +\infty$, $\xi_n \to \xi$ and

$$w_n \to u_0 \text{ in } \mathcal{B}(\mathbb{R}^N). \tag{3.4}$$

Hence rel. (3.3) follows. Thus, $u\Psi = \xi\Psi + u_0\Psi$, and it suffices to show that $u_0\Psi \in A$. To this end observe that the periodicity property for Ψ (i.e., $\tau_k\Psi = \Psi$ for all $k \in S$) implies that $f\Psi \in \mathcal{F}$ for any $f \in \mathcal{F}$, hence $w_n\Psi \in \langle \mathcal{F} \rangle$ for any integer $n \geq 1$. Recalling (3.4), it follows that $u_0\Psi \in CLS(\mathcal{F}) \subset A$. This completes the proof. ∎

Remark. The pair $\{\Sigma_1 = \Sigma_{\mathfrak{S}}^0, \Sigma_2 = \Sigma_S\}$ does not satisfy condition (3.12) in [14]. Indeed, if f is as in (3.1) with $a = 1$, φ nonzero and with zero integral, then $f \in (A_1/\mathbb{C}) \cap A_2$, as is easily seen by noticing that $\mathfrak{M}(f) = \mathfrak{M}(a) \int_Y \varphi(y) dy$ (a quick survey of the proof of Lemma 3.1 reveals that $\mathfrak{M}(\Psi) = \mathfrak{M}(f)$) and recalling Proposition 3.2. However f is nonzero.

According to Proposition 3.4, we set $\Sigma = \Sigma_{\mathfrak{S}}^0 + \Sigma_S$. Our main purpose is to show that, under some suitable hypothesis, Σ is proper. To accomplish this, we need a few preliminaries. First, we define

$$\mathfrak{S}_0 = \{a \in \mathfrak{S} : \mathfrak{M}(a) = 0\} \quad (\mathfrak{M} \text{ is the essential mean value on } S).$$

Thus, $\mathfrak{S} = \mathfrak{S}_0 \oplus \mathbb{C}$ (direct sum). For each vector subspace V of $\mathfrak{S}$, we denote by $\mathcal{F}_V$ the set of all functions $f : \mathbb{R}_y^N \to \mathbb{C}$ of the form

$$f = \sum_{k \in S} a(k)\tau_k\varphi \quad \text{with } a \in V \text{ and } \varphi \in \mathcal{K}(Y).$$

However, for simplicity we will write $\mathcal{F} = \mathcal{F}_{\mathfrak{S}}$ and $\mathcal{F}_0 = \mathcal{F}_{\mathfrak{S}_0}$. Clearly $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_{\mathbb{C}}$ (with $\mathcal{F}_{\mathbb{C}} = \mathcal{F}_{V=\mathbb{C}}$). In the sequel $\langle \mathcal{F} \rangle$ (resp. $\langle \mathcal{F}_0 \rangle$) denotes the set of all finite sums

$$\sum_{finite} f_i \text{ with } f_i \in \mathcal{F} \text{ (resp. } \mathcal{F}_0).$$

For the sake of convenience and, above all, in order to be in accordance with [14: Subsection 4.4], we are led to put

$$\Sigma_1 = \Sigma_{\mathfrak{S}}^0, \ \Sigma_2 = \Sigma_S, \ A_1 = \mathcal{J}(\Sigma_1), \ A_2 = \mathcal{J}(\Sigma_2) = \mathcal{C}_{per}(Y) \text{ and}$$

$$A = \mathcal{J}(\Sigma) \text{ (instead of } \mathcal{J}(\Sigma_{\mathfrak{S}}^0) \text{ as before) with } \Sigma = \Sigma_1 + \Sigma_2.$$

Finally, throughout the rest of this subsection we assume that

$$\mathfrak{S}_0 \text{ is a subalgebra of } \mathfrak{S}. \qquad (3.5)$$

We turn now to the proof of a basic lemma.

Lemma 3.2. *The following statements are true :*
(i) $\langle \mathcal{F}_0 \rangle$ *is a subalgebra of* $\mathcal{B}(\mathbb{R}^N)$.
(ii) $\langle \mathcal{F}_0 \rangle$ *is stable under complex conjugation.*
(iii) *If* $\Psi \in \langle \mathcal{F}_0 \rangle$ *and* $g \in A_2$, *then* $\Psi g \in \langle \mathcal{F}_0 \rangle$.
(iv) $\langle \mathcal{F}_0 \rangle \cap A_2 = \{0\}$.
(v) $\langle \mathcal{F}_0 \rangle + A_2$ *is dense in* A.

Proof. It is clear that $\langle \mathcal{F}_0 \rangle$ is a vector subspace of $\mathcal{B}(\mathbb{R}^N)$. Moreover, thanks to (3.5) (see also the proof of Lemma 3.1), $\langle \mathcal{F}_0 \rangle$ is stable under multiplication. Hence (i) follows. Furthermore, it is clear that $\mathfrak{S}_0$ is stable under complex conjugation, from which (ii) follows. Property (iii) results from the fact that $f \in \mathcal{F}_0$ and $g \in A_2$ imply $fg \in \mathcal{F}_0$ (see the proof of Proposition 3.4). As regards (iv), let us begin by noting that, according to the equality $\mathfrak{M}(f) = \mathfrak{M}(a) \int_Y \varphi(y) dy$, where f is of the form (3.1), we have $M(\Psi) = \mathfrak{M}(\Psi) = 0$ for $\Psi \in \langle \mathcal{F}_0 \rangle$. With this in mind, let $\Psi \in \langle \mathcal{F}_0 \rangle \cap A_2$. Then $M(|\Psi|^2) = 0$, since $|\Psi|^2 = \Psi \overline{\Psi} \in \langle \mathcal{F}_0 \rangle \cap A_2$ (use (i) and (ii)). Hence $\Psi = 0$ (indeed, the H-structure Σ_2 satisfies property (3.14) of [14]), which shows (iv). Finally, (v) is achieved by observing that $\langle \mathcal{F}_0 \rangle + A_2 = \langle \mathcal{F} \rangle + A_2$ and using the fact that the space on the right is dense in A. ∎

We are now in a position to prove the desired result.

Proposition 3.5. *Suppose rel. (3.5) holds. Then* Σ *is proper.*

Proof. If we show that Σ is of class $\mathcal{C}^\infty$ and further that the pair $\{\Sigma, \Sigma_2\}$ satisfies hypothesis (H) of [14: Subsection 4.4], since Σ_2 is proper (see, e.g., [16]), then the proposition will follow by [14: Theorem 4.2]. So we begin by verifying that Σ is of class $\mathcal{C}^\infty$. Since Σ_2 is of class $\mathcal{C}^\infty$, it is enough to check that Σ_1 is of class $\mathcal{C}^\infty$ (see [14: Section 4.4]), i.e., that $A_1^\infty = \{\Psi \in \mathcal{C}^\infty(\mathbb{R}_y^N) : D^\alpha \Psi \in A_1 \ (\alpha \in \mathbb{N}^N)\}$ is dense in A_1. That will certainly be the case if we can show that each f of the form (3.1) is the limit in the $\mathcal{B}(\mathbb{R}^N)$-norm of a sequence of functions of the form

$$g = \sum_{k \in S} a(k) \tau_k \theta \quad \text{with } a \in \mathfrak{S} \text{ and } \theta \in \mathcal{D}(Y) = \mathcal{K}(Y) \cap \mathcal{C}^\infty(Y).$$

This is a direct consequence of two facts :
1) $\mathcal{D}(Y)$ is dense in $\mathcal{K}(Y)$ under the inductive limit topology (this is a classical result)

2) given f as in (3.1) and g as above (with the same $a \in \mathfrak{S}$), we have
$\|f - g\|_\infty \leq \|a\|_\infty \|\varphi - \theta\|_\infty$.

We next show that the pair $\{\Sigma, \Sigma_2\}$ satisfies hypothesis (H) of [14]. Let $\mathcal{V} = \langle \mathcal{F}_0 \rangle \oplus A_2$ (see Lemma 3.2/(iv)). This is a dense vector subspace of A (Lemma 3.2). Now, let $\ell_2 : \mathcal{V} \to \mathcal{V}$ be the projection of $\mathcal{V}$ on A_2 along $\langle \mathcal{F}_0 \rangle$. We denote by ℓ the restriction of ℓ_2 to its range, i.e., ℓ is the mapping of $\mathcal{V}$ into A_2 such that $\ell v = \ell_2 v$ $(v \in \mathcal{V})$. We note in passing that ℓ is surjective. One fundamental property worth pointing out is that each $v \in \mathcal{V}$ is uniquely expressible in the form

$$v = v_0 + \ell v, \quad v_0 \in \langle \mathcal{F}_0 \rangle. \tag{3.6}$$

On the order hand, based on Lemma 3.2, the same procedure as followed in the proof of [14: Proposition 3.6] reveals that the mapping $L_2 : \mathcal{G}(\mathcal{V}) \to \mathcal{C}(\Delta(A_2))$ defined by $L_2(\mathcal{G}(\Psi)) = \mathcal{G}(\ell_2 \Psi)$ $(\Psi \in \mathcal{V})$ extends by continuity to an isometric isomorphism L of $L^2(\Delta(A))$ onto $L^2(\Delta(A_2))$. Finally, let $\mathcal{F}_0^\infty$ be the set of all $f \in \mathcal{F}$ of the form

$$f = \sum_{k \in S} a(k) \tau_k \varphi \text{ with } a \in \mathfrak{S}_0 \text{ and } \varphi \in \mathcal{D}(Y).$$

Define $\mathcal{V}^\infty = \langle \mathcal{F}_0^\infty \rangle + A_2^\infty$. Clearly $\mathcal{V}^\infty \subset (\mathcal{V} \cap A^\infty)$. With all that in mind, our goal is to verify points (4.4)-(4.9) of [14]. Point (4.4) is evident. As regards (4.5), the decomposition (3.6) and use of Lemma 3.2/(iii) reveals that $\ell(v\Psi) = \ell v \ell \Psi$ for $v, \Psi \in \mathcal{V}$. Hence (4.5) follows by (4.4) and use of the density of $\mathcal{G}(\mathcal{V})$ in $L^2(\Delta(A))$. Let us next verify (4.6). Since $v - \ell v \in \langle \mathcal{F}_0 \rangle$ $(v \in \mathcal{V})$, it suffices to check that for each $f \in \mathcal{F}_0$, $f^\varepsilon \to 0$ in $L^2_{loc}(\mathbb{R}^N_x)$ as $\varepsilon \to 0$. For this purpose, let K be a compact set in $\mathbb{R}^N_x$. Consider an open ball $B \subset \mathbb{R}^N$ centred at the origin such that $J^\varepsilon(K) \subset B$ for $0 < \varepsilon \leq 1$, where (for fixed $\varepsilon > 0$) $J^\varepsilon(K)$ denotes the union of all $\varepsilon(k + \overline{Y})$ that intersect K. Now, let $0 < \varepsilon \leq 1$ be fixed. Let $i^\varepsilon(B)$ denote the (finite) set of all $k \in S$ such that $\varepsilon k \in B$, and $I^\varepsilon(B)$ denote the union of all $\varepsilon(k + \overline{Y})$ as k ranges over $i^\varepsilon(B)$. Clearly $K \subset J^\varepsilon(K) \subset I^\varepsilon(B)$. Consequently, given $f \in \mathcal{F}_0$, we may write

$$\int_K \left| f\left(\frac{x}{\varepsilon}\right) \right|^2 dx \leq \int_{I^\varepsilon(B)} \left| f\left(\frac{x}{\varepsilon}\right) \right|^2 dx \leq \sum_{k \in i^\varepsilon(B)} \int_{\varepsilon(k+Y)} \left| f\left(\frac{x}{\varepsilon}\right) \right|^2 dx.$$

Hence, by the change of variable $y = \frac{x}{\varepsilon}$,

$$\int_K \left| f\left(\frac{x}{\varepsilon}\right) \right|^2 dx \leq \varepsilon^N \sum_{k \in i^\varepsilon(B)} \int_{k+Y} |f(y)|^2 dy. \tag{3.7}$$

According to [15: Theorem 4.3], the right-hand side of inequality (3.7) tends to $M(|f|^2)\lambda(B)$ when $\varepsilon \to 0$. But $|f|^2 \in \langle \mathcal{F}_0 \rangle$ (Lemma 3.2), hence $M(|f|^2) = 0$.

Therefore, when $\varepsilon \to 0$, the left-hand side of (3.7) tends to zero. From which (4.6) follows. Finally, the verification of (4.7)-(4.9) is an easy matter and so, by [14: Theorem 4.2], Σ is proper, as claimed. ∎

3.3. Discrete homogenization. We suppose here that a_{ij} $(1 \leq i, j \leq N)$ is constant on each cell $k + Y$ (Y defined in (1.3)), that is, for each fixed $k \in S = \mathbb{Z}^N$, we have (1.4) and (1.5). Then, given an essential algebra $\mathfrak{S}$ on S, our goal is to investigate the behaviour, as $\varepsilon \to 0$, of u_ε (the solution of (1.1)) under the structure hypothesis

$$r_{ij} \in \mathfrak{S} \quad (1 \leq i, j \leq N). \tag{3.8}$$

Such a problem is referred to as a discrete homogenization problem for the boundary value problem (1.1).

We have the following homogenization result.

Theorem 3.1. *Suppose that $\mathfrak{S}$ satisfies (3.5). Then, under the structure hypothesis (3.8), condition (2.2) is satisfied with $\Sigma = \Sigma_{\mathfrak{S}}^0 + \Sigma_S$ (see Subsection 3.2 and in particular Proposition 3.4), so that the conclusions of Theorem 2.1 (or more precisely, of* [14: *Subsection 5.2]) hold.*

Proof. First, we introduce the space $(L^2, \ell^\infty)(\mathbb{R}^N)$ of all functions $u \in L_{loc}^2(\mathbb{R}^N)$ such that

$$\|u\|_{2,\infty} = \sup_{k \in S} \left[\int_{k+Y} |u(y)|^2 \, dy \right]^{\frac{1}{2}} < +\infty.$$

This is a Banach space under the norm $\|\cdot\|_{2,\infty}$ (see [11]). We also recall that $a_{ij} = \sum_{k \in S} r_{ij}(k) \chi_{k+Y}$. This being so, let $1 \leq i, j \leq N$ be fixed. Let $\eta > 0$. Based on the density of $\mathcal{K}(Y)$ in $L^2(Y)$, we consider a function $\varphi \in \mathcal{K}(Y)$ such that $\|1 - \varphi\|_{L^2(Y)} \leq \frac{\eta}{c}$, where 1 denotes the function $u \in L^2(Y)$ such that $u(y) = 1$ a.e. in $y \in Y$, and c denotes a positive constant such that $|r_{ij}(k)| \leq c$ $(k \in S)$. Hence $\|a_{ij} - \Psi\|_{2,\infty} \leq \eta$ with $\Psi = \sum_{k \in S} r_{ij}(k) \tau_k \varphi$. Therefore, since $\Psi \in \mathcal{J}(\Sigma_{\mathfrak{S}}^0)$ (see Subsection 3.2), it follows that a_{ij} lies in the closure of $\mathcal{J}(\Sigma_{\mathfrak{S}}^0)$ in $(L^2, \ell^\infty)(\mathbb{R}^N)$. The latter space being continuously embedded in Ξ^2 (use [16: Lemma 1.3]), we deduce that $a_{ij} \in \mathfrak{X}_{\Sigma_{\mathfrak{S}}^0}^2$. But $\Sigma_{\mathfrak{S}}^0 \preccurlyeq \Sigma = \Sigma_{\mathfrak{S}}^0 + \Sigma_S$, i.e., $\mathcal{J}(\Sigma_{\mathfrak{S}}^0) \subset \mathcal{J}(\Sigma)$ (see [14: Subsection 3.3]), therefore $a_{ij} \in \mathfrak{X}_\Sigma^2$. Since Σ is proper (Proposition 3.5), the theorem follows, as claimed. ∎

This is worth illustrating. In the following examples, the coefficients a_{ij} $(1 \leq i, j \leq N)$ are assumed to be constant on each cell $k + Y$ ($k \in S$), as stated above.

Example 3.4. Suppose $\lim_{|k|\to\infty}\int_{k+Y}a_{ij}(y)dy = \zeta_{ij} \in \mathbb{C}$ $(1 \leq i,j \leq N)$. Then the mathematical analysis of the behaviour, as $\varepsilon \to 0$, of the solution of (1.1) leads to the conclusions of Theorem 2.1 (or more precisely, of [14: Subsection 5.2]). Indeed, the above assumption implies (3.8) with $\mathfrak{S} = \mathcal{B}_\infty(S)$ (Example 3.1). Furthermore, $\mathfrak{S}_0 = \mathcal{B}_0(S)$ (the functions in $\mathcal{B}_\infty(S)$ that vanish at infinity), and it is clear that (3.5) holds. Hence the desired result follows by Theorem 3.1.

Example 3.5. Suppose there exists a family $\{c_{ij}\} \subset \mathbb{C}$ such that

$$\sum_{k\in S}\left|\int_{k+Y}a_{ij}(y)dy - c_{ij}\right| < +\infty.$$

Then the conclusion of Example 3.4 again holds. Indeed, the present hypothesis on a_{ij} $(1 \leq i,j \leq N)$ leads to (3.8) with $\mathfrak{S} = \ell_0^1(S)$ (Example 3.2). Furthermore, $\mathfrak{S}_0$ is exactly the closure of $\ell^1(S)$ in $\ell^\infty(S)$ and so (3.5) is immediate. Therefore, Theorem 3.1 applies and the alleged conclusion follows.

Example 3.6. We assume here that a family $\{t_{ij}\} \subset \mathcal{B}_\infty(S)$ exists such that $\sum_{k\in S}\left|\int_{k+Y}a_{ij}(y)dy - t_{ij}(k)\right| < +\infty$ $(1 \leq i,j \leq N)$. Then we arrive at the same conclusion as in Example 3.4. Indeed, according to the present hypothesis on a_{ij} $(1 \leq i,j \leq N)$, we have (3.8) with $\mathfrak{S} = \ell_\infty^1(S)$ (Example 3.3). Furthermore, $\mathfrak{S}_0$ is precisely the closure of $\mathcal{B}_0(S) + \ell^1(S)$ in $\ell^\infty(S)$ and hence it is an easy task to check that (3.5) is fulfilled. Therefore, the result follows by applying Theorem 3.1.

In fact, in the present context we can also work out a nondiscrete homogenization problem for (1.1).

Corollary. *We no longer suppose that the functions a_{ij} are constant on each $k + Y$ $(k \in S)$, so that for each pair of indices $1 \leq i,j \leq N$, (1.6)-(1.8) hold in a unique manner. Suppose μ_{ij} is S-periodic $(1 \leq i,j \leq N)$ and that (3.5) and (3.8) hold. Then the conclusions of Theorem 2.1 (or more precisely, of [14: Subsection 5.2]) hold.*

Proof. A quick survey of the proof of Theorem 3.1 reveals that $\lambda_{ij} \in \mathfrak{X}_{\Sigma_\mathfrak{S}^0}^2$, thus $\lambda_{ij} \in \mathfrak{X}_\Sigma^2$ with $\Sigma = \Sigma_\mathfrak{S}^0 + \Sigma_S$. On the other hand, by hypothesis we have that $\mu_{ij} \in \mathfrak{X}_{\Sigma_S}^2$ (see [14: Example 5.1]), hence $\mu_{ij} \in \mathfrak{X}_\Sigma^2$. Therefore (2.2) holds with $\Sigma = \Sigma_\mathfrak{S}^0 + \Sigma_S$. Hence the conclusion follows as in Theorem 3.1. ∎

4. Composite homogenization

4.1. Statement of the problem and preliminaries. Let X_h $(1 \leq h \leq n)$ be a finite family of nonempty open sets in $\mathbb{R}^N$ with the following properties:

(1) $X_k \cap X_h = \emptyset$ (empty set) whenever $k \neq h$.

(2) Each X_h has a negligible boundary.

(3) $\varepsilon X_h \subset X_h$ for every $\varepsilon > 0$ $(1 \leq h \leq n)$.

(4) $\mathbb{R}^N = \bigcup_{1 \leq h \leq n} \overline{X}_h$ $(\overline{X}_h$ is the closure of $X_h)$.

Remark. Such a family does exist. Note that X_h is none other than a cone in $\mathbb{R}^N$ of vertex O (the origin in $\mathbb{R}^N$).

We set

$$\Omega_h = \Omega \cap X_h , \quad 1 \leq h \leq n,$$

where Ω is the same as in (1.1). Observe that we may have $\Omega_h = \emptyset$ for some indices h. With this in mind, we may assume without loss of generality that the family $\{\Omega_h\}_{1 \leq h \leq n}$ is so arranged that its nonempty members are exactly Ω_h $(1 \leq h \leq m)$, where the positive integer m is less than or equal to n. In the sequel we put

$$X = \bigcup_{1 \leq h \leq m} X_h.$$

Now, for each fixed $1 \leq h \leq m$, we consider a family $\{b_{hij}\}_{1 \leq i,j \leq N}$ with

$$b_{hij} \in L^\infty(\mathbb{R}_y^N) \quad \text{and} \quad b_{hji} = \overline{b}_{hij} \tag{4.1}$$

and the ellipticity condition

$$\begin{cases} \text{there exists a constant } \alpha_h > 0 \text{ such that} \\ \operatorname{Re} \sum_{i,j=1}^N b_{hij}(y)\xi_j\overline{\xi}_i \geq \alpha_h |\xi|^2 \quad (\xi \in \mathbb{C}^N, \text{ a.e. in } y \in \mathbb{R}^N). \end{cases} \tag{4.2}$$

This being so, for each pair of indices $1 \leq i, j \leq N$, we define $a_{ij} \in L^\infty(X)$ as

$$a_{ij}(y) = b_{hij}(y) \quad \text{for } y \in X_h \quad (h = 1, \ldots, m). \tag{4.3}$$

Remark. Let $\varepsilon > 0$. We have $a_{ij}\left(\frac{x}{\varepsilon}\right) = b_{hij}\left(\frac{x}{\varepsilon}\right)$ for $x \in \Omega_h$, $1 \leq h \leq m$ (it is essential to note that $\varepsilon X_h = X_h$, as is easily obtained by (3)). Therefore, since $\Omega = \bigcup_{1 \leq h \leq m}(\Omega \cap \overline{X}_h)$, and since X_h has a negligible boundary (use also (1)), this yields a function $a_{ij}^\varepsilon \in L^\infty(\Omega)$ with $a_{ij}^\varepsilon(x) = a_{ij}\left(\frac{x}{\varepsilon}\right)$ for almost every $x \in \Omega$. Furthermore, thanks to (4.2), the family $\{a_{ij}^\varepsilon\}_{1 \leq i,j \leq N}$ thus defined satisfies the ellipticity condition:

$$\operatorname{Re} \sum_{i,j=1}^N a_{ij}^\varepsilon(x)\xi_j\overline{\xi}_i \geq \alpha |\xi|^2 \quad (\xi \in \mathbb{C}^N, \text{ a.e. in } x \in \Omega)$$

where $\alpha = Min_{1 \leq h \leq m}\alpha_h$. Hence the boundary value problem (1.1) has exactly one solution.

Remark. Recalling that the transformation $x \to \frac{x}{\varepsilon}$ maps X into itself, we see that we may directly define a_{ij}^{ε} as being the function in $L^{\infty}(\mathrm{X})$ given by $a_{ij}^{\varepsilon}(x) = a_{ij}(\frac{x}{\varepsilon})$ for $x \in \mathrm{X}$. It suffices then to "restrict" a_{ij}^{ε} to Ω.

Finally, let $\{\Sigma_h\}_{1 \le h \le m}$ be a family of proper H-structures on $\mathbb{R}^N$. Our goal is to investigate the behaviour, as $\varepsilon \to 0$, of the solution, u_{ε}, of (1.1) [with (4.1)-(4.3)] under the abstract structure hypothesis

$$b_{hij} \in \mathfrak{X}_{\Sigma_h}^2 \quad (1 \le i, j \le N, \ 1 \le h \le m). \tag{4.4}$$

Such a homogenization problem is qualified as *composite* because for any two distinct indices $1 \le k \ne h \le m$, the behaviours of the two functions b_{kij} and b_{hij} may be of completely different natures.

In the sequel are collected most of the basic notation and preliminary results we need for the analysis of the problem under consideration. First of all, we set

$$A_h = \mathcal{J}(\Sigma_h) \quad \text{for } 1 \le h \le m.$$

(see Section 2). Next, we introduce the space

$$\mathbb{F}_0^1 = H_0^1(\Omega) \times \prod_{h=1}^{m} L^2(\Omega_h; H_{\#}^1(\Delta(A_h))).$$

Proceeding as in [14: Subsection 5.1], we see immediately that $\mathbb{F}_0^1$ is a Hilbert space with norm

$$\|V\|_{\mathbb{F}_0^1} = \left[\sum_{h=1}^{m} \sum_{i=1}^{N} \|\mathbb{D}_{hi}V\|_{L^2(\Omega_h \times \Delta(A_h))}^2 \right]^{\frac{1}{2}}$$

where $V = (v_0, v_1, \ldots, v_m) \in \mathbb{F}_0^1$ and

$$\mathbb{D}_{hi}V = \frac{\partial v_0}{\partial x_i}\bigg|_{\Omega_h} + \partial_i v_h \quad (1 \le i \le N, \ 1 \le h \le m).$$

We also need the space

$$F_0^{\infty} = \mathcal{D}(\Omega) \times \prod_{h=1}^{m} [\mathcal{D}(\Omega_h) \otimes J(\mathcal{D}(\Delta(A_h))/\mathbb{C})]$$

where J denotes the canonical mapping of $H^1(\Delta(A_h))/\mathbb{C}$ into its separated completion $H_{\#}^1(\Delta(A_h))$, and $\mathcal{D}(\Delta(A_h))/\mathbb{C}$ denotes the space of all $\varphi \in \mathcal{D}(\Delta(A_h))$ with $\int_{\Delta(A_h)} \varphi(s)d\beta(s) = 0$, β being the M-measure for A_h. Again following the same line of proceeding as in [14: Subsection 5.1], we have that F_0^{∞} is dense in $\mathbb{F}_0^1$.

No doubt, there is much to be gained by exhibiting a clear representation of functions in F_0^∞. To do this, let $A_h^\infty/\mathbb{C}$ denote the space of all $\Psi \in A_h^\infty$ such that $M(\Psi) = 0$ (see Section 2). It is easily seen that the Gelfand transformation $\mathcal{G}$ on A_h carries $A_h^\infty/\mathbb{C}$ bijectively over $\mathcal{D}(\Delta(A_h))/\mathbb{C}$. Consequently, Φ belongs to F_0^∞ if and only if Φ writes as

$$\Phi = (\Psi_0, J_{\Omega_1}(\widehat{\Psi}_1), J_{\Omega_2}(\widehat{\Psi}_2), \cdots, J_{\Omega_m}(\widehat{\Psi}_m)) \tag{4.5}$$

with

$$\Psi_0 \in \mathcal{D}(\Omega) \text{ and } \Psi_h \in \mathcal{D}(\Omega_h) \otimes (A_h^\infty/\mathbb{C}) \quad (1 \le h \le m) \tag{4.6}$$

where $J_{\Omega_h}(\widehat{\Psi}_h)$ denotes the mapping $x \to J(\widehat{\Psi}_h(x, \cdot))$ of Ω_h into $J(\mathcal{D}(\Delta(A_h))/\mathbb{C})$ and $\widehat{\Psi}_h$ denotes the mapping $x \to \mathcal{G}(\Psi_h(x, \cdot))$ of Ω_h into $\mathcal{D}(\Delta(A_h))$.

Now, for $U, V \in \mathbb{F}_0^1$, let

$$\widehat{a}_\Omega(U, V) = \sum_{h=1}^m \sum_{i,j=1}^N \int\int_{\Omega_h \times \Delta(A_h)} \widehat{b}_{hij}(s) \mathbb{D}_{hj}U(x, s)\overline{\mathbb{D}_{hi}V}(x, s)dxd\beta(s)$$

where β denotes the M-measure for A_h $(1 \le h \le m)$ and $\widehat{b}_{hij} = \mathcal{G}(b_{hij})$, $\mathcal{G}$ being here the canonical mapping of $\mathfrak{X}^2_{\Sigma_h}$ into $L^2(\Delta(A_h))$ [14: Subsection 2.3]. This defines a sesquilinear form $\widehat{a}_\Omega$ on $\mathbb{F}_0^1 \times \mathbb{F}_0^1$. In view of (4.1) and (4.2), the form $\widehat{a}_\Omega$ is continuous, Hermitian and coercive (proceed as in [14: Subsection 5.1]). Therefore, if L denotes the continuous antilinear form on $\mathbb{F}_0^1$ given by $L(V) = \langle f, \overline{v}_0 \rangle$ $(V = (v_0, v_1, v_2, \ldots, v_m) \in \mathbb{F}_0^1)$ then the variational problem

$$\begin{cases} U = (u_0, u_{11}, u_{12}, \ldots, u_{1m}) \in \mathbb{F}_0^1 \quad \text{and} \\ \widehat{a}_\Omega(U, V) = L(V) \text{ for all } V = (v_0, v_1, v_2, \ldots, v_m) \in \mathbb{F}_0^1 \end{cases} \tag{4.7}$$

has one and only one solution.

On the other hand, for fixed $\varepsilon > 0$, let

$$a^\varepsilon(u, v) = \sum_{i,j=1}^N \int_\Omega a_{ij}^\varepsilon \frac{\partial u}{\partial x_j} \frac{\overline{\partial v}}{\partial x_i} dx \quad (u, v \in H_0^1(\Omega))$$

and

$$b_h^\varepsilon(u, v) = \sum_{i,j=1}^N \int_{\Omega_h} b_{hij}^\varepsilon \frac{\partial u}{\partial x_j} \frac{\overline{\partial v}}{\partial x_i} dx \quad (u, v \in H^1(\Omega_h)).$$

Then,

$$a^\varepsilon(u, v) = \sum_{h=1}^m b_h^\varepsilon(u|_{\Omega_h}, v|_{\Omega_h}) \quad (u, v \in H_0^1(\Omega)) \tag{4.8}$$

where, as usual, $u|_{\Omega_h}$ denotes the restriction of u to Ω_h.

We end this subsection with one basic convergence result. Let $1 \leq h \leq m$ be fixed. Let $\mathcal{D}\left(\overline{\Omega}_h\right)$ denote the space of all complex functions θ on Ω_h such that $\theta = \varphi \left.\right|_{\Omega_h}$ for some $\varphi \in \mathcal{D}(\mathbb{R}_x^N)$. For $\theta_0 \in \mathcal{D}\left(\overline{\Omega}_h\right)$ and $\theta_1 \in \mathcal{D}\left(\overline{\Omega}_h\right) \otimes (A_h^\infty/\mathbb{C})$, we put $\Theta = (\theta_0, J_{\Omega_h}(\widehat{\theta}_1))$ and $\Theta_\varepsilon(x) = \theta_0(x) + \varepsilon\theta_1\left(x, \frac{x}{\varepsilon}\right)$ for $x \in \Omega_h$. Of course, $\Theta_\varepsilon \in \mathcal{D}\left(\overline{\Omega}_h\right)$.

Lemma 4.1. *Let* $(v_\varepsilon)_{\varepsilon \in E'} \subset H^1\left(\Omega_h\right)$, *where* E' *is a fundamental sequence. Suppose that as* $E' \ni \varepsilon \to 0$,

$$\frac{\partial v_\varepsilon}{\partial x_j} \to \mathbb{D}_j W = \frac{\partial w_0}{\partial x_j} + \partial_j w_1 \quad \text{in } L^2\left(\Omega_h\right)\text{-weak } \Sigma_h \quad (1 \leq j \leq N)$$

where $W = (w_0, w_1)$ *with* $w_0 \in H^1\left(\Omega_h\right)$ *and* $w_1 \in L^2(\Omega_h; H_\#^1(\Delta(A_h)))$. *Then, in the above notation,* $b_h^\varepsilon(v_\varepsilon, \Theta_\varepsilon) \to \widehat{b}_{\Omega_h}(W, \Theta)$ *as* $E' \ni \varepsilon \to 0$, *where*

$$\widehat{b}_{\Omega_h}(W, \Theta) = \sum_{i,j=1}^N \int\int_{\Omega_h \times \Delta(A_h)} \widehat{b}_{hij}(s)\mathbb{D}_j W(x, s)\overline{\mathbb{D}_i\Theta}(x, s)\,dx\,d\beta(s).$$

Proof. This is a simple adaptation of the proof of [14: Lemma 5.1]. ∎

4.2. Homogenization results. The first point is to prove the following basic theorem.

Theorem 4.1. *Let* U *be the solution of* (4.7). *For each fixed real* $\varepsilon > 0$, *let* u_ε *be the solution of* (1.1) *with* (4.1)-(4.3). *Suppose that* (4.4) *holds. Then, as* $\varepsilon \to 0$,

$$u_\varepsilon \to u_0 \text{ in } H_0^1(\Omega)\text{-weak} \tag{4.9}$$

$$u_\varepsilon \to u_0 \text{ in } L^2(\Omega) \tag{4.10}$$

$$\left.\frac{\partial u_\varepsilon}{\partial x_j}\right|_{\Omega_h} \to \mathbb{D}_{hj}U \text{ in } L^2\left(\Omega_h\right)\text{-weak } \Sigma_h \tag{4.11}$$

for $1 \leq j \leq N$, $1 \leq h \leq m$.

Proof. Clearly $a^\varepsilon(u_\varepsilon, v) = \langle f, \overline{v}\rangle$ for any $v \in H_0^1(\Omega)$. Hence, taking in particular $v = u_\varepsilon$ and using the ellipticity property, we see immediately that the sequence $(u_\varepsilon)_{\varepsilon > 0}$ is bounded in $H_0^1(\Omega)$. Therefore, given an arbitrary fundamental sequence E and recalling that $H^1(\Omega)$ is Σ_h-reflexive (see (4.4)) for $1 \leq h \leq m$, we are led to a subsequence E' extracted from E, and to a family of functions $u_0 \in H_0^1(\Omega)$ and $v_{1h} \in L^2(\Omega; H_\#^1(\Delta(A_h)))$ $(1 \leq h \leq m)$ such that when $E' \ni \varepsilon \to 0$, we have (4.9), (4.10) (by the Rellich Theorem) and

$$\frac{\partial u_\varepsilon}{\partial x_j} \to \frac{\partial u_0}{\partial x_j} + \partial_j v_{1h} \quad \text{in } L^2(\Omega)\text{-weak } \Sigma_h \quad (1 \leq j \leq N, \ 1 \leq h \leq m).$$

Let
$$U = (u_0, u_{11}, \ldots, u_{1m}) \quad \text{with } u_{1h} = v_{1h}|_{\Omega_h} \quad (1 \le h \le m).$$

Clearly $U \in \mathbb{F}_0^1$ and further (4.11) holds when $E' \ni \varepsilon \to 0$. Thus, the theorem is proved if we can check that U satisfies the variational equation in (4.7) (we refer to the line of argument followed in the proof of [14: Theorem 5.1]). For this purpose, fix an arbitrary $\Phi \in F_0^\infty$, i.e., Φ is given by (4.5) and (4.6). Next, let

$$\Phi_\varepsilon = \Psi_0 + \sum_{h=1}^m \varepsilon \Psi_h^\varepsilon \quad (\varepsilon > 0)$$

where $\Psi_h^\varepsilon(x) = \Psi_h\left(x, \frac{x}{\varepsilon}\right)$ for $x \in \Omega$. Evidently $\Phi_\varepsilon \in \mathcal{D}(\Omega)$ and $\Psi_h^\varepsilon \in \mathcal{D}(\Omega_h) \subset \mathcal{D}(\Omega)$. Thus, according to (4.8),

$$\sum_{h=1}^m b_h^\varepsilon(u_\varepsilon|_{\Omega_h}, \Phi_\varepsilon|_{\Omega_h}) = \langle f, \overline{\Phi}_\varepsilon \rangle.$$

Noting that $\Phi_\varepsilon \to \Psi_0$ in $H_0^1(\Omega)$-weak as $\varepsilon \to 0$ (see, e.g., [16: Proposition 5.3]), we deduce by Lemma 4.1 and use of (4.11) that $\widehat{a}_\Omega(U, \Phi) = L(\Phi)$ for all $\Phi \in F_0^\infty$. Hence the theorem follows by the density of F_0^∞ in $\mathbb{F}_0^1$. ∎

The next point deals with the so-called local equations. Let us fix freely an integer γ with $1 \le \gamma \le m$. Let $\widehat{b}_\gamma$ denote the sesquilinear form on $H_\#^1(\Delta(A_\gamma)) \times H_\#^1(\Delta(A_\gamma))$ defined by

$$\begin{cases} \widehat{b}_\gamma(u, v) = \sum_{i,j=1}^N \int_{\Delta(A_\gamma)} \widehat{b}_{\gamma ij}(s)\, \partial_j u(s)\, \overline{\partial_i v}(s)\, d\beta(s) \\ \text{for } u, v \in H_\#^1(\Delta(A_\gamma)). \end{cases}$$

For almost every $x \in \Omega_\gamma$, $u_{1\gamma}(x)$ turns out to be the solution of the coercive variational problem

$$\begin{cases} u_{1\gamma}(x) \in H_\#^1(\Delta(A_\gamma)) \quad \text{and} \\ \widehat{b}_\gamma(u_{1\gamma}(x), v) = -\sum_{k,j=1}^N \frac{\partial u_0}{\partial x_j}(x) \int_{\Delta(A_\gamma)} \widehat{b}_{\gamma kj}(s)\, \overline{\partial_k v}(s)\, d\beta(s) \qquad (4.12) \\ \text{for all } v \in H_\#^1(\Delta(A_\gamma)), \end{cases}$$

as is easily seen by taking in (4.7) the particular V's such that $v_0 = 0$, $v_h = 0$ if $h \ne \gamma$, $v_\gamma(x) = \varphi(x)\theta\ (x \in \Omega_\gamma)$ with $\varphi \in \mathcal{D}(\Omega_\gamma)$ and $\theta \in H_\#^1(\Delta(A_\gamma))$.

More can be said about $u_{1\gamma}(x)$. For fixed $1 \le j \le N$, let χ_γ^j be the (unique) solution of the coercive variational problem

$$\begin{cases} \chi_\gamma^j \in H_\#^1(\Delta(A_\gamma)) \text{ and} \\ \widehat{b}_\gamma(\chi_\gamma^j, v) = \sum_{k=1}^N \int_{\Delta(A_\gamma)} \widehat{b}_{\gamma kj}(s)\, \overline{\partial_k v}(s)\, d\beta(s) \qquad (v \in H_\#^1(\Delta(A_\gamma))). \end{cases}$$

Then, thanks to the uniqueness of the solution of (4.12) (see (4.2)), we see at once that

$$u_{1\gamma}(x) = -\sum_{k=1}^{N} \frac{\partial u_0}{\partial x_k}(x)\chi_{\gamma}^{k} \quad \text{a.e. in } x \in \Omega_h. \tag{4.13}$$

Finally, let us derive the boundary value problem for u_0. To this end, let

$$p_{\gamma ij} = \int_{\Delta(A_\gamma)} \widehat{b}_{\gamma ij}(s)\, d\beta(s) - \sum_{k=1}^{N} \int_{\Delta(A_\gamma)} \widehat{b}_{\gamma ik}(s)\, \partial_k \chi_{\gamma}^{j}(s)\, d\beta(s)$$

where $1 \leq i, j \leq N$. An elementary adaptation of the proof of [16: Lemma 5.3] shows that the family $\{p_{\gamma ij}\}_{1 \leq i,j \leq N}$ satisfies the symmetry property $p_{\gamma ji} = \overline{p}_{\gamma ij}$ $(1 \leq i, j \leq N)$ and an ellipticity condition similar to (4.2). Consequently, if for each pair of indices $1 \leq i, j \leq N$ we define $q_{ij} \in L^{\infty}(\Omega)$ as $q_{ij}(x) = p_{\gamma ij}$ a.e. in $x \in \Omega_\gamma$ $(1 \leq \gamma \leq m)$, then the family $\{q_{ij}\}_{1 \leq i,j \leq N}$ also satisfies similar symmetry and ellipticity conditions.

This being so, by considering in (4.7) the V's characterized by $v_h = 0$ $(1 \leq h \leq m)$ and then using (4.13), we obtain

$$\sum_{i,j=1}^{N} \int_{\Omega} q_{ij}(x) \frac{\partial u_0}{\partial x_j}(x) \frac{\overline{\partial v}}{\partial x_i}(x)\, dx = \langle f, \overline{v} \rangle$$

for all $v \in H_0^1(\Omega)$. Hence the boundary value problem for u_0 follows :

$$-\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(q_{ij}\frac{\partial u_0}{\partial x_j}\right) = f \quad \text{in } \Omega,\ u_0 \in H_0^1(\Omega). \tag{4.14}$$

Problem (4.7) is referred to as the *global homogenized problem* for (1.1) [with (4.1)-(4.4)], whereas (4.12) and (4.14) are called the local problem at $x \in \Omega_h$ and the macroscopic homogenized problem, respectively, for (1.1). The behaviour (as $\varepsilon \to 0$) of u_ε has two fundamental aspects: the *macroscopic behaviour* and the *microscopic behaviour*. The macroscopic behaviour is described by the solution u_0 of the macroscopic homogenized problem. The microscopic behaviour depends on the observation point in Ω, and is characterized by means of the m-tuple $U_1 = (u_{11}, \ldots, u_{1\gamma}, \ldots, u_{1m})$. Specifically, the microscopic behaviour (as $\varepsilon \to 0$) of u_ε at point $x \in \Omega_\gamma$ is described by the solution $u_{1\gamma}(x)$ of (4.12).

Problem (4.7) is qualified as *global* because it involves both the macroscopic and microscopic descriptions. From a physical point of view, problem (4.7) lays emphasis on the fact that the macroscopic and microscopic effects appear concomitantly.

4.3. Applications. The present subsection deals with two practical applications of Theorem 4.1.

Application 1. Let the basic notation be as in Subsection 4.1. Let b_{hij} ($1 \leq i, j \leq N$, $1 \leq h \leq m$) with (4.1) and (4.2). We assume moreover that, for fixed $1 \leq i, j \leq N$, the family $\{b_{hij}\}_{1 \leq h \leq m}$ is structured as follows :

(i) $b_{1ij} \in \mathcal{B}_\infty(\mathbb{R}; L^2_{per}(Y'))$ where $L^2_{per}(Y')$ denotes the usual Hilbert space of all functions $u \in L^2_{loc}(\mathbb{R}^{N-1})$ ($N \geq 2$) that are periodic in the sense that $u(y' + k) = u(y')$ a.e. in $y' = (y_1, \cdots, y_{N-1}) \in \mathbb{R}^{N-1}$ for each fixed $k \in \mathbb{Z}^{N-1}$, and $\mathcal{B}_\infty(\mathbb{R}; L^2_{per}(Y'))$ denotes the space of all bounded continuous functions $u : \mathbb{R} \to L^2_{per}(Y')$ such that $u(y_N)$ converges in $L^2_{per}(Y')$ as $|y_N| \to \infty$ (Y' is a copy of the open unit cube in $\mathbb{R}^{N-1}$)

(ii) $b_{2ij} \in L^2(\mathbb{R}^N_y) + L^2_{per}(Y)$ (Y defined in (1.3))

(iii) $b_{hij} \in L^2_{AP}(\mathbb{R}^N_y)$ ($3 \leq h \leq m$), where $L^2_{AP}(\mathbb{R}^N_y)$ denotes the space of all functions in $L^2_{loc}(\mathbb{R}^N_y)$ that are almost periodic in Stepanoff sense [16].

Under the preceding hypotheses, our goal is to investigate the behaviour, as $\varepsilon \to 0$, of u_ε given by (1.1) with $a^\varepsilon_{ij}(x) = a_{ij}\left(\frac{x}{\varepsilon}\right)$ ($x \in \Omega$) where a_{ij} is defined in (4.3). The desired result will follow by Theorem 4.1 if we can verify that there exists a family of proper H-structures Σ_h ($1 \leq h \leq m$) on $\mathbb{R}^N$ such that (4.4) holds. According to [14: Examples 5.1, 5.3 and 5.4], we have indeed (4.4) with $\Sigma_1 = \Sigma_{\mathcal{R}'} \times \Sigma_\infty$ (where $\mathcal{R}' = \mathbb{Z}^{N-1}$), $\Sigma_2 = \Sigma_{\infty,S}$ (use [14: Corollary 5.2]) and $\Sigma_h = \Sigma_{\mathcal{R}}$ [14: Example 3.3] for $3 \leq h \leq m$, where $\mathcal{R}$ is a suitable countable subgroup of $\mathbb{R}^N_y$.

Application 2. Let $a_{ij} \in L^\infty(\mathbb{R}^N_y)$ ($1 \leq i, j \leq N$) with the symmetry property $a_{ji} = \bar{a}_{ij}$ and the ellipticity property (1.2). We consider the boundary value problem (1.1) in which the bounded open set $\Omega \subset \mathbb{R}^N$ is assumed to intersect the hyperplane $\{x \in \mathbb{R}^N : x_N = 0\}$. Our purpose is then to investigate the behaviour, as $\varepsilon \to 0$, of u_ε (the solution of (1.1)) under the following structure hypothesis:

$$\begin{cases} a_{ij} \text{ lies in } \mathcal{C}(\mathbb{R}; L^2_{per}(Y')) \text{ and further,} \\ \lim_{y_N \to +\infty} a_{ij}(\cdot, y_N) = \zeta^1_{ij}, \; \lim_{y_N \to -\infty} a_{ij}(\cdot, y_N) = \zeta^2_{ij}, \\ \text{in the } L^2_{per}(Y')\text{-norm, where } \zeta^h_{ij} \in L^2_{per}(Y') \; (h = 1, 2). \end{cases} \tag{4.15}$$

for $1 \leq i, j \leq N$. To this end, fix freely two integers $1 \leq i, j \leq N$ and let

$$b_{1ij}(y', y_N) = \begin{cases} a_{ij}(y', y_N) \text{ for } y' \in \mathbb{R}^{N-1} \text{ and } y_N \geq 0 \\ a_{ij}(y', -y_N) \text{ for } y' \in \mathbb{R}^{N-1} \text{ and } y_N \leq 0 \end{cases}$$

and

$$b_{2ij}(y', y_N) = \begin{cases} a_{ij}(y', -y_N) \text{ for } y' \in \mathbb{R}^{N-1} \text{ and } y_N \geq 0 \\ a_{ij}(y', y_N) \text{ for } y' \in \mathbb{R}^{N-1} \text{ and } y_N \leq 0. \end{cases}$$

Clearly this gives two functions $b_{hij} \in \mathcal{B}_\infty(\mathbb{R}; L^2_{per}(Y'))$ $(h = 1, 2)$ with (in the $L^2_{per}(Y')$-norm)

$$\lim_{|y_N| \to \infty} b_{hij}(\cdot, y_N) = \zeta^h_{ij} \quad (h = 1, 2).$$

However, $b_{hij} \in \mathfrak{X}^2_\Sigma$ $(1 \le i, j \le N, h = 1, 2)$ with $\Sigma = \Sigma_{\mathcal{R}'} \times \Sigma_\infty$ exactly as in Application 1. Hence it follows that the problem under consideration falls within the scope of Subsection 4.1 with

$$m = n = 2, \quad \Sigma_h = \Sigma \ (h = 1, 2)$$
$$\mathrm{X}_1 = \left\{ y \in \mathbb{R}^N : y_N > 0 \right\} \text{ and } \mathrm{X}_2 = \left\{ y \in \mathbb{R}^N : y_N < 0 \right\},$$

and therefore Theorem 4.1 is applicable to the present case.

Remark. If in (4.15) we replace $L^2_{per}(Y')$ by the space $L^2_{AP}(\mathbb{R}^{N-1})$ of all functions in $L^2_{loc}(\mathbb{R}^{N-1})$ that are almost periodic in Stepanoff sense (see [11], [16]), then the preceding results are still valid provided some minor modification is made (see [14: Remark 5.2]).

Remark. If $\zeta^1_{ij} = \zeta^2_{ij}$ $(1 \le i, j \le N)$, then the present study reduces to [14: Example 5.3].

References

[1] Bakhvalov, N. and G. Panasenko: *Homogenization: Averaging Processes in Periodic Media* (Mathematics and its applications: BVil 36). Dordrecht: Kluwer Acad. Publ. 1989.

[2] Bensoussan, A., Lions, J. L. and G. Papanicolaou: *Asymptotic Analysis for Periodic Structures.* Amsterdam: North-Holland 1978.

[3] Bourbaki, N.: *Integration*, Chap. 1-4. Paris: Hermann 1966.

[4] Bourbaki, N.: *Integration*, Chap. 5. Paris: Hermann 1967.

[5] Bourbaki, N.: *Topologie générale*, Chap. I-IV. Paris: Hermann 1971.

[6] Bourgeat, A., Mikelic, A. and S. Wright: *Stochastic two-scale convergence in the mean and applications.* J. Reine Angew. Math. 456 (1994), 19 - 51.

[7] Dal Maso, G.: *An Introduction to Gamma-Convergence.* Boston: Birkhäuser 1993.

[8] De Giorgi, E.: *Sulla convergenza di alcune successioni di integrali del tipo dell' area.* Rend. Mat. 8 (1975), 277 - 294.

[9] De Giorgi, E.: *G-operators and Γ-convergence.* In: Proceedings of the International Congress of Mathematicians (Warsaw, August 1983). Amsterdam: PWN Polish Scientific Publishers and North-Holland 1984, pp. 1175 - 1191.

[10] Edwards, R. E.: *Functional Analysis.* New York et al.: Holt-Rinehart-Winston 1965.

[11] Fournier, J. J. F. and J. Stewart: *Amalgams of L^p and ℓ^q.* Bull. Amer. Math. Soc. 13 (1985), 1 - 21.

[12] Larsen, R.: *Banach Algebras.* New York: Marcel Dekker 1973.

[13] Lukkassen, D., Nguetseng, G. and P. Wall: *Two-scale convergence.* Int. J. Pure Appl. Math. 2 (2002), 35 - 86

[14] Nguetseng, G.: *Homogenization structures and applications I.* Z. Anal. Anw. 22 (2003), 73 - 107

[15] Nguetseng, G.: *Mean value on locally compact abelian groups.* Acta Sci. Math. 69 (2003), 203 - 221.

[16] Nguetseng, G.: *Almost periodic homogenization: Asymptotic analysis of a second order elliptic equation.* (Preprint).

[17] Sanchez-Palencia, E.: *Nonhomogeneous Media and Vibration Theory.* Berlin et al: Springer-Verlag 1980.

[18] Spagnolo, S.: *Sulla convergenza di soluzioni di equazioni paraboliche e ellitiche.* Ann. Sc. Norm. Sup. Pisa 22 (1968), 571 - 597.

[19] Spagnolo, S.: *Convergence in energy for elliptic operators.* Proc. Third symp. Numerical Solutions of Partial Differential Equations (College Park, 1975). San Diego : Acad. Press 1976, 469 - 498.

[20] Tartar, L.: *Cours Peccot.* Paris : Collège de France 1977.