Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 23 (2004), No. 3, 589–605

Analysis of the Operator Δ^{-1}div Arising in Magnetic Models

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Abstract. In the context of micromagnetics the partial differential equation

$$\mathrm{div}(-\nabla u + \mathbf{m}) = 0 \text{ in } \mathbb{R}^d$$

has to be solved in the entire space for a given magnetization $\mathbf{m} : \Omega \to \mathbb{R}^d$ and $\Omega \subseteq \mathbb{R}^d$. For an L^p function $\mathbf{m}$ we show that the solution might fail to be in the classical Sobolev space $W^{1,p}(\mathbb{R}^d)$ but has to be in a Beppo-Levi class $W_1^p(\mathbb{R}^d)$. We prove unique solvability in $W_1^p(\mathbb{R}^d)$ and provide a direct ansatz to obtain u via a non-local integral operator $\mathcal{L}_p$ related to the Newtonian potential. A possible discretization to compute $\nabla(\mathcal{L}_2\mathbf{m})$ is mentioned, and it is shown how recently established matrix compression techniques using hierarchical matrices can be applied to the full matrix obtained from the discrete operator.

Keywords: *Laplace equation, integral representation, Calderón-Zygmund kernel, micromagnetics, magnetic potential, panel clustering, hierarchical matrices*
AMS subject classification: Primary 35J05, secondary 31A10, 31B10, 35C15, 65F30, 65F50

1. Motivation and introduction

Let $\Omega \subseteq \mathbb{R}^d$, for $d = 2, 3$, denote the spatial domain of a ferromagnetic body. In the classical model for stationary micromagnetics due to Weiss, Landau, and Lifshitz [3, 11], an energy functional E has to be minimized over an admissible set $\mathcal{A}$ of magnetizations $\mathbf{m} : \Omega \to \mathbb{R}^d$. The functional E comprises four terms, which are known as exchange energy, anisotropic energy, exterior energy, and stray-field (or magnetostatic) energy,

$$E(\mathbf{m}) = \alpha \int_\Omega |\nabla \mathbf{m}|^2 \, dx + \int_\Omega \phi(\mathbf{m}) \, dx - \int_\Omega \mathbf{f} \cdot \mathbf{m} \, dx + \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx. \qquad (1)$$

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ISSN 0232-2064 / $ 2.50

Here, the exchange parameter $\alpha > 0$ and the anisotropy density $\phi \in \mathcal{C}^\infty(\mathbb{R}^d;\mathbb{R}_{\ge 0})$ are given. Frequently, Ω is supposed to be a *large body* so that the exchange contribution can be neglected, i.e. $\alpha = 0$ in (1) [7]. The magnetic potential $u : \mathbb{R}^d \to \mathbb{R}$ and the magnetization $\mathbf{m}$ are linked through Maxwell's equations which imply the partial differential equation

$$\Delta u = \operatorname{div} \mathbf{m} \tag{2}$$

in the entire space $\mathbb{R}^d$ (where $\mathbf{m}$ is extended by zero outside of Ω). As usual, Equation (2) is treated in the sense of distributions. By definition, we are looking for a weakly differentiable function u which satisfies

$$\langle \nabla u \,;\, \nabla v \rangle = \langle \mathbf{m} \,;\, \nabla v \rangle \quad \text{for all } v \in \mathcal{D}(\mathbb{R}^d), \tag{3}$$

where $\langle \cdot \,;\, \cdot \rangle$ denotes the scalar product of $L^2(\mathbb{R}^d;\mathbb{R}^d)$ and $\mathcal{D}(\mathbb{R}^d)$ denotes the vector space of all $\mathcal{C}^\infty$-functions with compact support.

The length $|\mathbf{m}|$ of the vector field depends only on the temperature and is therefore usually assumed to be constant. In particular, one has $\mathbf{m} \in L^\infty(\Omega;\mathbb{R}^d)$. Thus, it seems to be interesting to investigate the solvability of (3) for $\mathbf{m} \in L^p(\Omega;\mathbb{R}^d)$.

For bounded Ω and $d = 3$ it is well known that, given $\mathbf{m} \in L^2(\Omega;\mathbb{R}^d)$, there is a unique solution $u \in H^1(\mathbb{R}^d)$ which solves (3) [12, 14]. But, for unbounded Ω or $d = 2$, the solution $u \in H^1_{\ell oc}(\mathbb{R}^d)$ in general fails to be in $L^2(\mathbb{R}^d;\mathbb{R}^d)$. In particular, we show that, for $d = 2$, this is related to the fundamental solution of the Laplacian.

The paper is organized as follows: Section 2 recalls the necessary definitions and classical results applied in the following sections. In Section 3 the Banach spaces $W^p_1(\mathbb{R}^d)$ are introduced, and it is shown that for a magnetization $\mathbf{m} \in L^2(\mathbb{R}^d;\mathbb{R}^d)$ the Hilbert space $W^2_1(\mathbb{R}^d)$ is the *right* space to be considered; there is a unique solution $u \in W^2_1(\mathbb{R}^d)$ of (3). Section 4 recalls the definition of Calderón-Zygmund kernels and states the main theorem on Calderón-Zygmund convolutions which is applied in Section 5. We show that for $\mathbf{m} \in L^p(\mathbb{R}^d;\mathbb{R}^d)$ and $1 < p < \infty$ the potential equation (3) has a unique solution $u = \mathcal{L}_p\mathbf{m} \in W^p_1(\mathbb{R}^d)$. At least for a magnetization $\mathbf{m} \in L^1(\mathbb{R}^d;\mathbb{R}^d) \cap L^p(\mathbb{R}^d;\mathbb{R}^d)$, the potential $\mathcal{L}_p\mathbf{m}$ is given as a classical convolution

$$\mathcal{L}_p\mathbf{m} = \sum_{j=1}^d \frac{\partial G}{\partial x_j} * m_j,$$

where G denotes the Newtonian kernel and m_j is the j-th component of $\mathbf{m}$. The extension of $\mathcal{L}_p$ defines a continuous linear operator from $L^p(\mathbb{R}^d;\mathbb{R}^d)$ to $W^p_1(\mathbb{R}^d)$. Finally, Section 6 gives the application of the provided results for a

Galerkin discretization with piecewise constant ansatz functions in the context of computational micromagnetics. We show how the theory of $\mathcal{H}^2$-matrices can be applied to the Galerkin elements to decrease computational cost down to (almost) linear.

2. Preliminaries

For functions $u, v : \mathbb{R}^d \to \mathbb{R}$ we define the convolution $u * v$ of u and v by

$$(u * v)(x) := \int_{\mathbb{R}^d} u(x-y)v(y)\,dy \quad \text{for all } x \in \mathbb{R}^d,$$

whenever the integral exists. As usual in the context of convolutions, functions $w : \Omega \to \mathbb{R}$, for $\Omega \subseteq \mathbb{R}^d$, are identified with their trivial extension $w : \mathbb{R}^d \to \mathbb{R}$, i.e. $w(x) := 0$ for $x \in \mathbb{R}^d \backslash \Omega$. We summarize some well-known facts about convolutions [17, 20, 21, 22].

Proposition 2.1.

(i) *For* $1 \leq p, q, r \leq \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, *the convolution* $u * v$ *of* $u \in L^p(\mathbb{R}^d)$ *and* $v \in L^q(\mathbb{R}^d)$ *satisfies*

$$u * v \in L^r(\mathbb{R}^d) \quad \textit{with} \quad \|u * v\|_{L^r(\mathbb{R}^d)} \leq \|u\|_{L^p(\mathbb{R}^d)} \|v\|_{L^q(\mathbb{R}^d)}.$$

(ii) *For* $q = \frac{p}{p-1} =: p'$, *the convolution* $u * v \in \mathcal{C}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ *is uniformly continuous. Further,* $1 < p < \infty$ *and* $q = p'$ *imply* $u * v \in \mathcal{C}_0(\mathbb{R}^d)$, *i.e.* $u * v$ *vanishes at infinity.*

(iii) *For* $k \in \mathbb{N}$, $u \in L^p_{\ell oc}(\mathbb{R}^d)$, $v \in \mathcal{C}^k_c(\mathbb{R}^d)$, *and multi-indices* $\alpha \in \mathbb{N}_0^d$ *with* $|\alpha| \leq k$, *we have*

$$u * v \in \mathcal{C}^k(\mathbb{R}^d) \quad \textit{with} \quad \partial^\alpha(u * v) = u * (\partial^\alpha v).$$

For $d \geq 2$, we define the Newtonian kernel $G : \mathbb{R}^d \backslash \{0\} \to \mathbb{R}$ by

$$G(x) := \begin{cases} \frac{1}{\gamma_d} \log|x| & \text{for } d = 2, \\ \frac{1}{(2-d)\gamma_d} |x|^{2-d} & \text{for } d > 2, \end{cases} \tag{4}$$

where $\gamma_d = |\partial B(0,1)|$ denotes the surface measure of the unit sphere, in particular $\gamma_2 = 2\pi$, $\gamma_3 = 4\pi$. The Newtonian kernel is the fundamental solution of the Laplacian, i.e. we have the following well-known proposition [8].

Proposition 2.2. *For any test function* $f \in \mathcal{D}(\mathbb{R}^d)$ *the Newtonian potential* $w := G * f$ *satisfies* $w \in \mathcal{C}^\infty(\mathbb{R}^d)$ *with* $\Delta w = f$ *in* $\mathbb{R}^d$. *Moreover, for the partial derivatives of* w *it holds that* $\partial w / \partial x_j = (\partial G / \partial x_j) * f = G * (\partial f / \partial x_j)$.

Corollary. *For any smooth magnetization* $\mathbf{m} = (m_1, \dots, m_d) \in \mathcal{D}(\mathbb{R}^d; \mathbb{R}^d)$, *a solution of Equation* (2) *is given by* $u := G * (\operatorname{div} \mathbf{m}) = \sum_{j=1}^d (\partial G/\partial x_j) * m_j$.

For later use, we need the standard notation of Sobolev and Lebesgue spaces. For $1 \le p < \infty$ and $\omega \subseteq \mathbb{R}^d$ an open set, we denote with $L^p(\omega)$ the Banach space of all measurable functions whose absolute value to the power p is integrable. The inner product of the Hilbert space $L^2(\omega)$ is given by

$$\langle u \,;\, v \rangle := \int_\omega u(x)v(x)\,dx \quad \text{for all } u, v \in L^2(\omega). \tag{5}$$

For $p = \infty$, $L^\infty(\omega)$ denotes the Banach space of all measurable functions which are essentially bounded. For $n \in \mathbb{N}_0$, the classical Sobolev space $W^{n,p}(\omega)$ consists of all functions $u : \omega \to \mathbb{R}$ which are n times weakly differentiable and whose (weak) derivatives of order $|\alpha| \le n$ satisfy $\partial^\alpha u \in L^p(\omega)$. The norm on $W^{n,p}(\omega)$ is given by

$$\begin{aligned} \|u\|_{W^{n,p}(\omega)} &:= \Big(\sum_{|\alpha| \le n} \|\partial^\alpha u\|_{L^p(\omega)}^p \Big)^{\frac{1}{p}} \\ \|u\|_{W^{n,\infty}(\omega)} &:= \max_{|\alpha| \le n} \|\partial^\alpha u\|_{L^\infty(\omega)}. \end{aligned} \tag{6}$$

In this sense we have $L^p(\omega) = W^{0,p}(\omega)$ and it is quite common to denote the Hilbert space $W^{n,2}(\omega)$ by $H^n(\omega)$. The subspace $W_0^{n,p}(\omega)$ is the completion of the test functions $\mathcal{D}(\omega)$ with respect to $\|\cdot\|_{W^{n,p}(\omega)}$. The subscript ℓoc, e.g. in $W_{\ell oc}^{n,p}(\omega)$, indicates that $u \in W_{\ell oc}^{n,p}(\omega)$ satisfies $u \in W^{n,p}(K)$ for all compact subsets $K \subseteq \omega$. We write $W^{n,p}(\omega; \mathbb{R}^d)$ whenever we are dealing with vector valued functions. Finally, we point out that, for any $1 \le p \le \infty$, the conjugate index is denoted with $p' := \frac{p}{p-1} \in [1, \infty]$. Further, $B(x, \varepsilon)$ denotes the closed ball with radius $\varepsilon > 0$ and center $x \in \mathbb{R}^d$. As usual, $|\cdot|$ denotes both the absolute value of a scalar $\lambda \in \mathbb{R}$ and the (Euclidean) norm of a vector $x \in \mathbb{R}^d$, respectively. The scalar product of two vectors $x, y \in \mathbb{R}^d$ is written as $x \cdot y$.

3. The Banach spaces $W_1^p(\mathbb{R}^d)$

For $1 \le p \le \infty$, we define the vector space [15, 13]

$$L^p_\nabla(\mathbb{R}^d; \mathbb{R}^d) := \big\{ \mathbf{f} \in L^p(\mathbb{R}^d; \mathbb{R}^d) \,\big|\, \exists u \in W^{1,p}_{\ell oc}(\mathbb{R}^d) \quad \nabla u = \mathbf{f} \big\} \tag{7}$$

of all L^p functions which are weak gradients.

Lemma 3.1. $L^p_\nabla(\mathbb{R}^d; \mathbb{R}^d)$ *is a closed subspace of* $L^p(\mathbb{R}^d; \mathbb{R}^d)$, *whence a Banach space. Furthermore,* $L^p_\nabla(\mathbb{R}^d; \mathbb{R}^d)$ *is reflexive for* $1 < p < \infty$ *and a Hilbert space for* $p = 2$.

Proof. Let $(\mathbf{f}_n)$ be a Cauchy sequence in $L^p_\nabla(\mathbb{R}^d;\mathbb{R}^d)$ with limit $\mathbf{f} \in L^p(\mathbb{R}^d;\mathbb{R}^d)$. Further, let (u_n) be a sequence in $W^{1,p}_{\ell oc}(\mathbb{R}^d)$ with $\mathbf{f}_n = \nabla u_n$. For $k \in \mathbb{N}$, define $v_n^{(k)} := u_n - \fint_{B(0,k)} u_n\,dx$, where $\fint_B dx := \frac{1}{|B|}\int_\omega dx$ denotes the integral mean over $B \subseteq \mathbb{R}^d$. Since $\nabla v_n^{(k)} = \mathbf{f}_n$, a Poincaré inequality on $B(0,k)$ yields that $(v_n^{(k)})_{n\in\mathbb{N}}$ converges to a function $v^{(k)} \in W^{1,p}(B(0,k))$ with $\nabla v^{(k)} = \mathbf{f}|_{B(0,k)}$. Now define $u^{(k)} := v^{(k)} - \fint_{B(0,1)} v^{(k)}\,dx$ and observe $u^{(k)} \in W^{1,p}(B(0,k))$ with $u^{(k)}|_{B(0,k-1)} = u^{(k-1)}$. Thus, $k \to \infty$ gives a function $u \in W^{1,p}_{\ell oc}(\mathbb{R}^d)$ with $\nabla u = \mathbf{f}$. The remaining claims follow from principles of functional analysis [20]. ∎

Now, we consider the vector space

$$\widetilde{W}^p_1(\mathbb{R}^d) := \left\{ u \in W^{1,p}_{\ell oc}(\mathbb{R}^d) \,\middle|\, \nabla u \in L^p(\mathbb{R}^d;\mathbb{R}^d) \right\}. \tag{8}$$

Note that the natural definition

$$\|u\|_{W^p_1(\mathbb{R}^d)} := \|\nabla u\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)} \tag{9}$$

only induces a seminorm on this space. Two functions $u, v \in \widetilde{W}^p_1(\mathbb{R}^d)$ have the same gradient, i.e. $\nabla u = \nabla v$, if and only if $u = v + c$ for a constant $c \in \mathbb{R}$. Factored out the piecewise constants from $\widetilde{W}^p_1(\mathbb{R}^d)$, i.e

$$W^p_1(\mathbb{R}^d) := \widetilde{W}^p_1(\mathbb{R}^d)/\mathbb{R}, \tag{10}$$

the quotient space equipped with $\|\cdot\|_{W^p_1(\mathbb{R}^d)}$ obviously satisfies the following lemma.

Lemma 3.2. *$W^p_1(\mathbb{R}^d)$ is a Banach space which is reflexive for $1 < p < \infty$, and $W^2_1(\mathbb{R}^d)$ is a Hilbert space. Moreover, the gradient $\nabla : W^p_1(\mathbb{R}^d) \to L^p_\nabla(\mathbb{R}^d;\mathbb{R}^d)$ is an isometric isomorphism.*

Remark. The inclusion $i_p : W^{1,p}(\mathbb{R}^d) \hookrightarrow W^p_1(\mathbb{R}^d)$ which maps a function $u \in W^{1,p}(\mathbb{R}^d)$ to the corresponding equivalence class in $W^p_1(\mathbb{R}^d)$, is continuous and injective. Thus, $W^{1,p}(\mathbb{R}^d)$ and $\mathcal{D}(\mathbb{R}^d)$ can be treated as subspaces of $W^p_1(\mathbb{R}^d)$.

The following result can be found in [13, Appendix A] or easily be verified by use of the Fourier transform.

Lemma 3.3. *The test functions $\mathcal{D}(\mathbb{R}^d)$ are dense within $W^2_1(\mathbb{R}^d)$.*

Proposition 3.1. *For $\mathbf{m} \in L^2(\mathbb{R}^d;\mathbb{R}^d)$, there is a unique $u = u_\mathbf{m} \in W^2_1(\mathbb{R}^d)$ which satisfies* (3). *The operator $\mathcal{L} : L^2(\mathbb{R}^d;\mathbb{R}^d) \to W^2_1(\mathbb{R}^d), \mathbf{m} \mapsto u_\mathbf{m}$ is linear and bounded with operator norm $\|\mathcal{L}\| = 1$. The composition*

$$\mathcal{P} := \nabla \circ \mathcal{L} \in L(L^2(\mathbb{R}^d;\mathbb{R}^d); L^2(\mathbb{R}^d;\mathbb{R}^d)) \tag{11}$$

is the L^2-orthogonal projection onto $L^2_\nabla(\mathbb{R}^d;\mathbb{R}^d)$, and we have

$$L^2_\nabla(\mathbb{R}^d;\mathbb{R}^d)^\perp = \left\{ \mathbf{m} \in H(\mathrm{div};\mathbb{R}^d) \,\middle|\, \mathrm{div}\,\mathbf{m} = 0 \right\}. \tag{12}$$

Proof. According to the Cauchy inequality and $\|v\|_{W_1^2(\mathbb{R}^d)} = \|\nabla v\|_{L^2(\mathbb{R}^d;\mathbb{R}^d)}$,

$$\Phi(v) := \langle \mathbf{m} \,;\, \nabla v \rangle$$

defines a bounded linear functional $\Phi \in W_1^2(\mathbb{R}^d)^*$ with norm $\|\Phi\| \le \|\mathbf{m}\|_{L^2(\mathbb{R}^d;\mathbb{R}^d)}$. Now, (3) reads

$$\langle u \,;\, v \rangle_{W_1^2(\mathbb{R}^d)} = \Phi(v) \quad \text{for all } v \in \mathcal{D}(\mathbb{R}^d). \tag{13}$$

With respect to Lemma 3.3, $\mathcal{D}(\mathbb{R}^d)$ in (13) can be replaced by $W_1^2(\mathbb{R}^d)$ and Riesz' theorem yields the existence of a unique $u \in W_1^2(\mathbb{R}^d)$ satisfying the equality. The estimation of the norm again follows from the Cauchy inequality,

$$\|u\|^2_{W_1^2(\mathbb{R}^d)} = \langle \nabla u \,;\, \nabla u \rangle = \langle \mathbf{m} \,;\, \nabla u \rangle \le \|\mathbf{m}\|_{L^2(\mathbb{R}^d;\mathbb{R}^d)} \|u\|_{W_1^2(\mathbb{R}^d)},$$

i.e. $\|\mathcal{L}\mathbf{m}\|_{W_1^2(\mathbb{R}^d)} \le \|\mathbf{m}\|_{L^2(\mathbb{R}^d;\mathbb{R}^d)}$. For $\mathbf{m} \in L^2_\nabla(\mathbb{R}^d;\mathbb{R}^d)$, we have $\mathbf{m} = \nabla(\mathcal{L}\mathbf{m})$, whence $\|\mathcal{L}\| = 1$ and $\mathcal{P}$ is a projection onto $L^2_\nabla(\mathbb{R}^d;\mathbb{R}^d)$. From $\|\mathcal{P}\| = \|\mathcal{L}\| = 1$, we derive that $\mathcal{P}$ is orthogonal [20]. Equation (12) follows directly from Lemma 3.3. ∎

Remark. Since the embedding $i_2 : H^1(\mathbb{R}^d) \hookrightarrow W_1^2(\mathbb{R}^d)$ is injective and (3) has a unique solution in $W_1^2(\mathbb{R}^d)$, there is at most one solution in $H^1(\mathbb{R}^d)$. Later we will investigate in which cases the unique solution $u \in W_1^2(\mathbb{R}^d)$ is represented by a function in $H^1(\mathbb{R}^d)$.

Remark. For the numerical treatment of $\mathcal{P}$ the latter proposition is meaningless. However, in Section 5 an analytical representation of $\mathcal{L}$ is introduced which carries over to the case $1 < p < \infty$ instead of $p = 2$.

4. The analytical main result

The theorem we want to prove requires some preliminaries on the Calderón-Zygmund kernels defined below. For any kernel $h : \mathbb{R}^d\backslash\{0\} \to \mathbb{R}$ we make the convention to write $\widetilde{h}$ and h_ε for $\widetilde{h}(x) := h(-x)$ and $h_\varepsilon := h\chi_{\mathbb{R}^d\backslash B(0,\varepsilon)}$ with arbitrary $\varepsilon > 0$, respectively, where $\chi_{\mathbb{R}^d\backslash B(0,\varepsilon)}$ denotes the characteristic function.

Definition 4.1. A measurable function $\kappa : \mathbb{R}^d\backslash\{0\} \to \mathbb{R}$ is called *Calderón-Zygmund kernel* if there is a constant $c_1 > 0$ such that for any $x \ne 0$ and $0 < r < R < \infty$ there holds

$$|\kappa(x)| \le c_1 |x|^{-d} \tag{14}$$

$$\int_{|y|>2|x|} |\kappa(y-x) - \kappa(y)|\, dy \le c_1 \tag{15}$$

$$\int_{r<|y|<R} \kappa(y)\, dy = 0. \tag{16}$$

Theorem 4.1 (Calderón-Zygmund [22]). *For a Calderón-Zygmund kernel κ, $1 < p < \infty$, $\varepsilon > 0$ and $f \in L^p(\mathbb{R}^d)$, the convolution of κ_ε and f satisfies*

$$\kappa_\varepsilon * f \in L^p(\mathbb{R}^d) \quad \textit{with} \quad \|\kappa_\varepsilon * f\|_{L^p(\mathbb{R}^d)} \leq c_p \|f\|_{L^p(\mathbb{R}^d)}, \tag{17}$$

where the constant $c_p > 0$ depends only on p and κ but neither on ε nor f. Further, $\kappa_\varepsilon * f$ converges in $L^p(\mathbb{R}^d)$ for $\varepsilon \to 0$. Consequently,

$$S_p f := \lim_{\varepsilon \to 0} (\kappa_\varepsilon * f) \in L^p(\mathbb{R}^d) \tag{18}$$

defines a bounded operator $S_p \in L\big(L^p(\mathbb{R}^d); L^p(\mathbb{R}^d)\big)$ with norm $\|S_p\| \leq c_p$. Since S_p extends the convolution, we shall write $\kappa \,\widetilde{*}\, f := S_p f$.

Remark. The notation $\kappa \,\widetilde{*}\, f$ is independent of p in the following sense: For $1 < p, q < \infty$ and $f \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$, one has $S_p f = S_q f$ since L^p convergence for $\varepsilon \to 0$ implies pointwise convergence almost everywhere [21].

The partial derivatives of the Newtonian kernel G, $\frac{\partial G}{\partial x_j}(x) = \frac{1}{\gamma_d} \frac{x_j}{|x|^d}$, give rise to the following definition.

Definition 4.2. A kernel $h : \mathbb{R}^d \backslash \{0\} \to \mathbb{R}$ is *homogeneous of degree $\alpha \in \mathbb{R}$* if $h(\lambda x) = \lambda^\alpha h(x)$ for any $\lambda > 0$ and $x \neq 0$.

To give first examples of Calderón-Zygmund kernels and to see that the second order partial derivatives of the Newtonian kernel $\kappa := \partial^2 G/(\partial x_j \partial x_k)$ are of Calderón-Zygmund type, we cite the following lemma from [1].

Lemma 4.1. *For $h \in \mathcal{C}^2(\mathbb{R}^d \backslash \{0\})$ homogeneous of degree $1-d$, any partial derivative $\kappa := \partial h/\partial x_j$ is a Calderón-Zygmund kernel.*

Theorem 4.2. *Let $1 < p < \infty$ and $h \in \mathcal{C}^1(\mathbb{R}^d \backslash \{0\})$ be homogeneous of degree $1-d$ such that the first order partial derivatives of h are Calderón-Zygmund kernels. Then, there is a unique bounded operator $T_p \in L(L^p(\mathbb{R}^d); W_1^p(\mathbb{R}^d))$ with*

$$T_p f = h * f \quad \textit{for all } f \in \mathcal{D}(\mathbb{R}^d). \tag{19}$$

For $f \in L^p(\mathbb{R}^d)$, the weak derivative of $T_p f$ is given by

$$\frac{\partial}{\partial x_j}(T_p f) = \kappa_j \,\widetilde{*}\, f + \lambda_j f, \tag{20}$$

where $\kappa_j := \partial h/\partial x_j$ and $\lambda_j := \int_{\partial B(0,1)} h(x) x_j \, ds_x$. The operator T_p has the following mapping properties:

(a) *$T_p f = h * f$ for $f \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$,*
(b) *$T_p f = h * f$ for $f \in L^q(\mathbb{R}^d) \cap L^r(\mathbb{R}^d)$ with $1 \leq q < d < r \leq \infty$ and $q \leq p \leq r$,*

(c) *$T_p f = h * f \in W^{1,p}(\mathbb{R}^d)$ for $f \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ and $d > p'$ (i.e. $pd > p + d$).*

Further, if Ω is bounded, the restriction to $L^p(\Omega)$ satisfies

(d) *$T_p \in L(L^p(\Omega); W^{1,p}(\mathbb{R}^d))$ for $d > p'$.*

Finally, for $1 < p, q < \infty$ we have

(e) *$T_p f = T_q f$ for all $f \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$.*

Remark. The equalities in Theorem 4.2, e.g. (19), have to be understood in the sense that $h * f \in \widetilde{W}_1^p(\mathbb{R}^d)$ (resp. $h * f \in W^{1,p}(\mathbb{R}^d)$) exists and is a representer of the equivalence class $T_p f$. Since the operator T_p extends the convolution with h, we write $h \mathbin{\widetilde{*}} f := T_p f$ for $f \in L^p(\mathbb{R}^d)$. Due to (e) this notation is independent of p.

Remark. For the first order partial derivatives $h := \partial G / \partial x_k$ of the Newtonian kernel G, λ_j from (20) can be computed,

$$\lambda_j = \begin{cases} 0 & \text{for } j \neq k, \\ \frac{1}{d} & \text{for } j = k. \end{cases} \tag{21}$$

With the unit sphere $\mathbb{S} := \partial B(0,1)$ this follows from $\lambda_j = \frac{1}{|\mathbb{S}|} \int_{\mathbb{S}} x_j x_k \, ds_x$ for $j \neq k$ by symmetry and from $|\mathbb{S}| = \sum_{j=1}^d \int_{\mathbb{S}} x_j^2 \, ds_x = d \int_{\mathbb{S}} x_k^2 \, ds_x$ for $j = k$.

The proof of the theorem needs the following elementary lemma which can be derived directly from Proposition 2.1.

Lemma 4.2. *For a measurable function $h : \mathbb{R}^d \backslash \{0\} \to \mathbb{R}$ which is of degree $1 - d$ and $h_1 := h \chi_{\mathbb{R}^d \backslash B(0,1)}$, the following holds:*

(i) *$h_1 \in L^t(\mathbb{R}^d)$, $h - h_1 \in L^s(\mathbb{R}^d)$ for $1 \le s < d' < t \le \infty$,*

(ii) *$h * f \in L^p_{\ell oc}(\mathbb{R}^d)$ for $f \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$, $1 \le p \le \infty$, and $1 \le q < d$,*

(iii) *$h * f \in L^\infty(\mathbb{R}^d)$ for $f \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ and $1 \le q < d < p \le \infty$,*

(iv) *$\langle h * f \,;\, g \rangle = \langle f \,;\, \widetilde{h} * g \rangle$ for f as in (iii), $g \in L^1(\mathbb{R}^d)$, and $\widetilde{h}(x) := h(-x)$.*

Proof of Theorem 4.2 (main part). For $f \in \mathcal{D}(\mathbb{R}^d)$, we have $h * f \in L^\infty(\mathbb{R}^d)$. To verify $h * f \in \widetilde{W}_1^p(\mathbb{R}^d)$ it remains to show that

$$\langle h * f \,;\, \partial \phi / \partial x_j \rangle = -\langle \kappa_j \mathbin{\widetilde{*}} f + \lambda_j f \,;\, \phi \rangle \quad \text{for all } \phi \in \mathcal{D}(\mathbb{R}^d). \tag{22}$$

For the notation, we use the conventions introduced above. Let $\phi \in \mathcal{D}(\mathbb{R}^d)$ and choose $r > 0$ with $\operatorname{supp}(f) \cup \operatorname{supp}(\phi) \Subset B(0,r)$ and note

$$\begin{aligned} \langle h * f \,;\, \partial \phi / \partial x_j \rangle &= \langle f \,;\, \widetilde{h} * (\partial \phi / \partial x_j) \rangle \\ &= \int_{\operatorname{supp}(f)} f(y) \lim_{\varepsilon \to 0} \int_{B(0,r) \backslash B(y,\varepsilon)} h(x - y) \, \frac{\partial \phi}{\partial x_j}(x) \, dx \, dy. \end{aligned}$$

For fixed $y \in \mathrm{supp}(f)$ and small $\varepsilon > 0$, the inner integral reads with partial integration

$$\begin{aligned}
&\int_{B(0,r)\setminus B(y,\varepsilon)} h(x-y)\frac{\partial\phi}{\partial x_j}(x)\,dx \\
&\quad = -\int_{B(0,r)\setminus B(y,\varepsilon)} \kappa_j(x-y)\phi(x)\,dx + \int_{\partial(B(0,r)\setminus B(y,\varepsilon))} h(x-y)\phi(x)n_j(x)\,ds_x \\
&\quad = -(\widetilde{\kappa}_{j,\varepsilon} * \phi)(y) - \int_{\partial B(y,\varepsilon)} h(x-y)\phi(x)\frac{x_j-y_j}{\varepsilon}\,ds_x,
\end{aligned}$$

where the Calderón-Zygmund kernel $\widetilde{\kappa}_j$ is defined by $\widetilde{\kappa}_j(x) = \kappa_j(-x)$. Recall that according to Theorem 4.1, $\widetilde{\kappa}_{j,\varepsilon} * \phi$ converges to $\widetilde{\kappa}_j \,\widetilde{*}\, \phi$ in $L^p(\mathbb{R}^d)$ for $\varepsilon \to 0$. (This allows us to exchange the limit and the integration with respect to y.) With transformations, the surface integral reads

$$\begin{aligned}
\int_{\partial B(y,\varepsilon)} h(x-y)\phi(x)\frac{x_j-y_j}{\varepsilon}\,ds_x &= \int_{\partial B(0,1)} h(x)x_j\phi(y+\varepsilon x)\,ds_x \\
&= \lambda_j\phi(y) + \int_{\partial B(0,1)} h(x)x_j\big(\phi(y+\varepsilon x)-\phi(y)\big)\,ds_x,
\end{aligned}$$

and the second term in the sum vanishes for $\varepsilon \to 0$. Combining both equations, we end up with

$$\begin{aligned}
\langle h*f\,;\,\partial\phi/\partial x_j\rangle &= -\lim_{\varepsilon\to 0}\langle f\,;\,\widetilde{\kappa}_{j,\varepsilon}*\phi\rangle - \langle f\,;\,\lambda_j\phi\rangle \\
&= -\lim_{\varepsilon\to 0}\langle \kappa_{j,\varepsilon}*f\,;\,\phi\rangle - \langle f\,;\,\lambda_j\phi\rangle = \langle \kappa_j \,\widetilde{*}\, f - \lambda_j f\,;\,\phi\rangle
\end{aligned}$$

and derive (22). In particular, we obtain with Theorem 4.1

$$\|\nabla(h*f)\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)} \le c_2\|f\|_{L^p(\mathbb{R}^d)},$$

where $c_2 > 0$ only depends on d, p, h, and its partial derivatives. Considering $\mathcal{D}(\mathbb{R}^d)$ as a subspace of $L^p(\mathbb{R}^d)$, we have shown that $T_p f := h * f$ defines a bounded operator $T_p \in L(\mathcal{D}(\mathbb{R}^d); W_1^p(\mathbb{R}^d))$. Density provides a unique extension $T_p \in L(L^p(\mathbb{R}^d); W_1^p(\mathbb{R}^d))$. Equality (20) carries over from $\mathcal{D}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ by continuity. ∎

The remaining claims of the theorem follow from classical density arguments, which can be applied according to the additional assumptions on $f \in L^p(\mathbb{R}^d)$.

Proof of Theorem 4.2 (a–e). `Part (a).` We have to show that $h * f \in L^p_{\ell oc}(\mathbb{R}^d)$ is weakly differentiable with weak derivative $\kappa_j \,\widetilde{*}\, f + \lambda_j f$. To this end, choose a sequence (f_n) in $\mathcal{D}(\mathbb{R}^d)$ that converges to f in $L^1(\mathbb{R}^d)$ and $L^p(\mathbb{R}^d)$

(for instance a sequence of mollifications of f). For all $\phi \in \mathcal{D}(\mathbb{R}^d)$, a Hölder inequality shows

$$|\langle h * f\ ;\ \partial\phi/\partial x_j\rangle - \langle h * f_n\ ;\ \partial\phi/\partial x_j\rangle| \le \|f - f_n\|_{L^1(\mathbb{R}^d)} \|\widetilde{h} * (\partial\phi/\partial x_j)\|_{L^\infty(\mathbb{R}^d)},$$

and the right-hand side vanishes for $n \to \infty$. Hence, we obtain

$$\begin{aligned}\langle h * f\ ;\ \partial\phi/\partial x_j\rangle &= \lim_{n\to\infty} \langle h * f_n\ ;\ \partial\phi/\partial x_j\rangle \\ &= \lim_{n\to\infty} \langle \kappa_j \,\widetilde{*}\, f_n + \lambda_j f_n\ ;\ \phi\rangle = \langle \kappa_j \,\widetilde{*}\, f + \lambda_j f\ ;\ \phi\rangle,\end{aligned}$$

where we have used the convergence in $L^p(\mathbb{R}^d)$.

`Part (b):` W.l.o.g. we may assume $f \ge 0$. Define $f_n := \min(f, n)\chi_{B(0,n)}$ and note that f_n converges to f in $L^s(\mathbb{R}^d)$ for $s = p, q, r$. According to Proposition 2.1,

$$\begin{aligned}\|h * (f - f_n)\|_{L^\infty(\mathbb{R}^d)} &\le \|h_1 * (f - f_n)\|_{L^\infty(\mathbb{R}^d)} + \|(h - h_1) * (f - f_n)\|_{L^\infty(\mathbb{R}^d)} \\ &\le \|h_1\|_{L^{q'}(\mathbb{R}^d)} \|f - f_n\|_{L^q(\mathbb{R}^d)} \\ &\quad + \|h - h_1\|_{L^{r'}(\mathbb{R}^d)} \|f - f_n\|_{L^r(\mathbb{R}^d)},\end{aligned}$$

i.e. $h * f_n$ converges to $h * f$ in $L^\infty(\mathbb{R}^d)$. The application of (a) yields

$$\begin{aligned}\langle h * f\ ;\ \partial\phi/\partial x_j\rangle &= \lim_{n\to\infty} \langle h * f_n\ ;\ \partial\phi/\partial x_j\rangle \\ &= \lim_{n\to\infty} \langle \kappa_j \,\widetilde{*}\, f_n + \lambda_j f_n\ ;\ \phi\rangle = \langle \kappa_j \,\widetilde{*}\, f + \lambda_j f\ ;\ \phi\rangle\end{aligned}$$

for any test function $\phi \in \mathcal{D}(\mathbb{R}^d)$.

`Part (c).` According to (a) it remains to show that $h * f \in L^p(\mathbb{R}^d)$. With Lemma 4.2 we have $(h - h_1) \in L^1(\mathbb{R}^d)$ and $h_1 \in L^p(\mathbb{R}^d)$ since $p > d'$. Proposition 2.1 yields $h_1 * f$, $(h - h_1) * f \in L^p(\mathbb{R}^d)$.

`Part (d).` Assertion (c) yields that the restriction $T : L^p(\Omega) \to W^{1,p}(\mathbb{R}^d)$ from T_p to $L^p(\Omega)$ is well-defined and linear. Since the inclusion $i_p : W^{1,p}(\mathbb{R}^d) \hookrightarrow W^p_1(\mathbb{R}^d)$ and the composition $T_p = i_p \circ T$ are continuous, Banach's closed graph theorem implies that T is also continuous.

`Part (e).` The claim follows directly from (20) since the right-hand side is independent of p, q. It defines a function in $L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$ by Theorem 4.1. ∎

5. Unique solvability of the potential equation (3)

In the subsequent section we show that also for $1 < p < \infty$ and $\mathbf{m} \in L^p(\mathbb{R}^d; \mathbb{R}^d)$ (instead of $p = 2$) the potential equation (3) has a unique solution $u \in W^p_1(\mathbb{R}^d)$. We provide a representation of the operator $\mathcal{L}$ which was introduced for $p = 2$

in Proposition 4.2. Recall from Corollary 2.2 that, for any arbitrary smooth magnetization $\mathbf{m} = (m_1, \ldots, m_d) \in \mathcal{D}(\mathbb{R}^d; \mathbb{R}^d)$,

$$u := \mathcal{L}_0\mathbf{m} := \sum_{j=1}^{d} \frac{\partial G}{\partial x_j} * m_j \tag{23}$$

is a solution of Equation (2). In particular, u solves the weak form (3).

Proposition 5.1. *Given* $1 \le p \le \infty$ *and* $\mathbf{m} \in L^p(\mathbb{R}^d; \mathbb{R}^d)$, *Equation* (3) *has at most one solution* $u \in W_1^p(\mathbb{R}^d)$.

Proof. Assume that $u_1, u_2 \in W_1^p(\mathbb{R}^d)$ solve (3). Then $e := u_2 - u_1 \in W_1^p(\mathbb{R}^d)$ satisfies $\Delta e = 0$ in a weak sense. In particular, any derivative $f := \partial e/\partial x_j$ satisfies $\Delta f = 0$ and $f \in L^p(\mathbb{R}^d)$. For any $\phi \in \mathcal{D}(\mathbb{R}^d)$ it follows that $\Delta(\phi * f) = \phi * \Delta f = 0$ and $\phi * f \in \mathcal{C}^\infty(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ by Proposition 2.1. But then Liouville's theorem implies $\phi * f = 0$. Since this holds for any test function ϕ, Lebesgue's differentiation theorem yields $f = 0$, whence $u_1 = u_2 \in W_1^p(\mathbb{R}^d)$. ∎

Since the kernels $h_j := \partial G/\partial x_j$ satisfy the assumptions of Theorem 4.2, we obtain the following result which states, in particular, the unique solvability of (3) for $\mathbf{m} \in L^p(\mathbb{R}^d; \mathbb{R}^d)$ in $W_1^p(\mathbb{R}^d)$ for $1 < p < \infty$.

Theorem 5.1. *For any* $1 < p < \infty$, *there is a unique bounded operator*

$$\mathcal{L}_p \in L\big(L^p(\mathbb{R}^d; \mathbb{R}^d); W_1^p(\mathbb{R}^d)\big) \tag{24}$$

which extends $\mathcal{L}_0$ *from* $\mathcal{D}(\mathbb{R}^d; \mathbb{R}^d)$ *to* $L^p(\mathbb{R}^d; \mathbb{R}^d)$. *For a magnetization* $\mathbf{m} \in L^p(\mathbb{R}^d; \mathbb{R}^d)$, $u := \mathcal{L}_p\mathbf{m}$ *is the unique solution of* (3). *Further,* $\mathcal{L}_p$ *has the following mapping properties:*

(a) $\mathcal{L}_p\mathbf{m} = \mathcal{L}_0\mathbf{m}$ *for* $\mathbf{m} \in L^1 \cap L^p$

(b) $\mathcal{L}_p\mathbf{m} = \mathcal{L}_0\mathbf{m}$ *for* $\mathbf{m} \in L^q \cap L^r$ *with* $1 \le q < d < r \le \infty$ *and* $q \le p \le r$

(c) $\mathcal{L}_p\mathbf{m} = \mathcal{L}_0\mathbf{m} \in W^{1,p}(\mathbb{R}^d)$ *for* $\mathbf{m} \in L^1 \cap L^p$ *and* $d > p'$

where $\mathcal{L}_p\mathbf{m} = \mathcal{L}_0\mathbf{m}$, *in particular, states that the convolution* $\mathcal{L}_0\mathbf{m}$ *exists in the classical sense. Further, for a bounded open set* $\Omega \subseteq \mathbb{R}^d$, *the restriction of* $\mathcal{L}_p$ *to* $L^p(\Omega; \mathbb{R}^d)$ *satisfies*

(d) $\mathcal{L}_p \in L(L^p(\Omega; \mathbb{R}^d); W^{1,p}(\mathbb{R}^d))$ *for* $d > p'$.

Finally, for $p = 2$, *the extended convolution operator* $\mathcal{L}_2$ *coincides with the operator* $\mathcal{L}$ *introduced in Proposition* 3.1 *and we remark that* (c) *and* (d) *hold for* $d \ge 3$.

Proof. According to Theorem 4.2, $\mathcal{L}_p$ is given by

$$\mathcal{L}\mathbf{m} = \sum_{j=1}^{d} \frac{\partial G}{\partial x_j} \,\widetilde{*}\, m_j \quad \text{for } \mathbf{m} = (m_1, \ldots, m_d) \in L^p(\mathbb{R}^d; \mathbb{R}^d), \tag{25}$$

and this extension is unique since the test functions are dense within L^p. The mapping properties (a)–(d) carry over from the corresponding statements in Theorem 4.2. It remains to show that $u = \mathcal{L}_p\mathbf{m}$ solves (3) for arbitrary $\mathbf{m} \in L^p(\mathbb{R}^d;\mathbb{R}^d)$. Let $(\mathbf{m}_k)_{k\in\mathbb{N}}$ be a sequence of test functions which converges to $\mathbf{m}$ in L^p and recall that (3) has already been shown for all $\mathbf{m}_k$. By definition of $W_1^p(\mathbb{R}^d)$ we infer that $\|\mathcal{L}_p(\mathbf{m}-\mathbf{m}_k)\|_{W_1^p(\mathbb{R}^d)} = \|\nabla(\mathcal{L}_p\mathbf{m}) - \nabla(\mathcal{L}_p\mathbf{m}_k)\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)}$. Combined with the Hölder inequality, this yields for all $v \in \mathcal{D}(\mathbb{R}^d)$

$$\begin{aligned}|\langle -\nabla(\mathcal{L}_p\mathbf{m}) + \mathbf{m}\,;\,\nabla v\rangle| &= |\langle -\nabla(\mathcal{L}_p\mathbf{m}) + \nabla(\mathcal{L}_p\mathbf{m}_k)\,;\,\nabla v\rangle + \langle \mathbf{m}-\mathbf{m}_k\,;\,\nabla v\rangle| \\ &\le \|\nabla v\|_{L^{p'}(\mathbb{R}^d)}\big(\|\mathcal{L}_p(\mathbf{m}-\mathbf{m}_k)\|_{W_1^p(\mathbb{R}^d)} + \|\mathbf{m}-\mathbf{m}_k\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)}\big),\end{aligned}$$

and the right-hand side vanishes for $k\to\infty$ since $\mathcal{L}_p$ is continuous. ∎

Remark. Theorem 5.1 yields a constructive proof of Lemma 3.3: Write $W_1^2(\mathbb{R}^d) = H \oplus H^\perp$ with H the closure of $\mathcal{D}(\mathbb{R}^d)$ in $W_1^2(\mathbb{R}^d)$. For $u \in H^\perp$ we have $\langle \nabla u\,;\,\nabla v\rangle = 0$ for all $v \in \mathcal{D}(\mathbb{R}^d)$, whence u is the potential of the zero magnetization, i.e. $u = 0$.

6. Application to computational micromagnetics

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. For a magnetization $\mathbf{m} \in L^2(\Omega;\mathbb{R}^d)$, let $u := \mathcal{L}_2\mathbf{m}$ denote the corresponding (unique) magnetic potential. According to (3) and Proposition 3.1, the stray-field energy from (1) reads

$$\int_{\mathbb{R}^d} |\nabla u|^2\,dx = \int_\Omega \mathbf{m}\cdot\nabla u\,dx\,. \tag{26}$$

On the right-hand side, the continuous bilinear form

$$a : L^2(\mathbb{R}^d;\mathbb{R}^d)\times L^2(\mathbb{R}^d;\mathbb{R}^d)\to\mathbb{R},\quad a(\mathbf{m},\widetilde{\mathbf{m}}) := \langle \nabla(\mathcal{L}_2\mathbf{m})\,;\,\widetilde{\mathbf{m}}\rangle \tag{27}$$

appears. For the discretization of which, let $\mathcal{T} = \{T_1,\dots,T_N\}$ be a triangulation of Ω, i.e.

(i) every $T\in\mathcal{T}$ is a (bounded) Lipschitz domain which satisfies $T\subseteq\Omega$
(ii) $\overline{\Omega} = \bigcup\big\{\overline{T}\,\big|\,T\in\mathcal{T}\big\}$, where $\overline{T}$ denotes the closure of $T\subseteq\mathbb{R}^d$
(iii) for different $T_j, T_k\in\mathcal{T}$, we have $T_j\cap T_k=\emptyset$.

Further, let $\mathcal{S}^0(\mathcal{T})$ denote the vector space of all $\mathcal{T}$-piecewise constant functions. Then, for piecewise constant magnetizations $\mathbf{m},\widetilde{\mathbf{m}}\in\mathcal{S}^0(\mathcal{T})^d$, the following proposition gives a formula to compute $a(\mathbf{m},\widetilde{\mathbf{m}})$ analytically.

Remark. At least for the large body model of micromagnetics due to DeSimone [7] the consideration of piecewise constant functions is reasonable, cf. [5] for a discrete relaxed model and the corresponding numerical analysis.

Proposition 6.1. *For bounded Lipschitz domains $\omega, \widetilde{\omega} \subseteq \mathbb{R}^d$ and vectors $\mathbf{m}, \widetilde{\mathbf{m}} \in \mathbb{R}^d$, we have*

$$a(\chi_\omega \mathbf{m}, \widetilde{\chi}_{\widetilde{\omega}} \widetilde{\mathbf{m}}) = -\int_{\partial\omega} \int_{\partial\widetilde{\omega}} G(x-y)(\mathbf{n}(x) \cdot \mathbf{m})(\widetilde{\mathbf{n}}(y) \cdot \widetilde{\mathbf{m}})\, ds_y ds_x, \tag{28}$$

where χ_ω and $\chi_{\widetilde{\omega}}$ denote the corresponding characteristic functions and $\mathbf{n}$, $\widetilde{\mathbf{n}}$ denote the outer normal vectors on $\partial\omega$ and $\partial\widetilde{\omega}$, respectively. Further, we have the symmetry properties

$$a(\chi_\omega \mathbf{m}, \chi_{\widetilde{\omega}} \widetilde{\mathbf{m}}) = a(\chi_{\widetilde{\omega}} \widetilde{\mathbf{m}}, \chi_\omega \mathbf{m}) = a(\chi_\omega \widetilde{\mathbf{m}}, \chi_{\widetilde{\omega}} \mathbf{m}). \tag{29}$$

In particular, $B_{jk} := a(\chi_\omega \mathbf{e}_j, \chi_{\widetilde{\omega}} \mathbf{e}_k)$ defines a symmetric matrix $B \in \mathbb{R}^{d\times d}_{sym}$ such that

$$a(\chi_\omega \mathbf{m}, \chi_{\widetilde{\omega}} \widetilde{\mathbf{m}}) = \mathbf{m} \cdot B\widetilde{\mathbf{m}}. \tag{30}$$

In the case $\mathrm{dist}(\omega, \widetilde{\omega}) > 0$, *the coefficients of B can be computed by*

$$B_{jk} = \int_\omega \int_{\widetilde{\omega}} \frac{\partial^2 G}{\partial x_j \partial x_k}(x-y)\, dy\, dx. \tag{31}$$

Proof. Since $\nabla \circ \mathcal{L}_2$ is an orthogonal projection, the bilinear form $a(\cdot,\cdot)$ is symmetric [20]. This shows the first equality in (29). To obtain the other claims of the proposition, note that the bilinearity of $a(\cdot,\cdot)$ leads to

$$a(\chi_\omega \mathbf{m}, \chi_{\widetilde{\omega}} \widetilde{\mathbf{m}}) = \sum_{j,k=1}^{d} m_j \widetilde{m}_k\, a(\chi_\omega \mathbf{e}_j, \chi_{\widetilde{\omega}} \mathbf{e}_k) = \mathbf{m} \cdot B\widetilde{\mathbf{m}}.$$

Therefore only the special case $\mathbf{m} = \mathbf{e}_j$ and $\widetilde{\mathbf{m}} = \mathbf{e}_k$ has to be treated. To abbreviate notation, we write $h_\ell := \partial G/\partial x_\ell$ and $\kappa_{jk} := \partial^2 G/(\partial x_j \partial x_k)$. Theorem 4.2 and Remark 4 yield

$$\frac{\partial(h_j * \chi_\omega)}{\partial x_k} = \kappa_{jk} \,\widetilde{*}\, \chi_\omega + \frac{\delta_{jk}}{d}\chi_\omega = \frac{\partial(h_k * \chi_\omega)}{\partial x_j}$$

with Kronecker's δ_{jk}. Further, we have $\mathcal{L}_2(\chi_\omega \mathbf{e}_j) = \mathcal{L}_0(\chi_\omega \mathbf{e}_j) = h_j * \chi_\omega$. With the definition of the Calderón-Zygmund convolution $\kappa_{jk} \,\widetilde{*}\, \chi_\omega$, we obtain

$$B_{jk} = \langle \nabla \circ \mathcal{L}(\chi_\omega \mathbf{e}_j)\,;\, \chi_{\widetilde{\omega}} \mathbf{e}_k \rangle = \langle \kappa_{jk} \,\widetilde{*}\, \chi_\omega \,;\, \chi_{\widetilde{\omega}} \rangle + \frac{\delta_{jk}}{d} \langle \chi_\omega \,;\, \chi_{\widetilde{\omega}} \rangle$$

The symmetry $\kappa_{jk}(x) = \kappa_{jk}(-x)$ shows $\langle \kappa_{jk} \,\widetilde{*}\, \chi_\omega \,;\, \chi_{\widetilde{\omega}} \rangle = \langle \chi_\omega \,;\, \kappa_{jk} \,\widetilde{*}\, \chi_{\widetilde{\omega}} \rangle$ and therefore $B_{jk} = B_{kj}$, i.e. we obtain the second equality in (29). To prove (28) note that Theorem 4.2 in particular states $h_k * \chi_{\widetilde{\omega}} \in W^{1,p}_{\ell oc}(\mathbb{R}^d)$. Thus, partial integration on the bounded Lipschitz domain ω yields

$$\int_\omega \frac{\partial(h_k * \chi_{\widetilde{\omega}})}{\partial x_j}\,dx = \int_{\partial\omega} (h_k * \chi_{\widetilde{\omega}})(x) n_j(x)\,ds_x.$$

For fixed $x \in \partial\omega$ another partial integration for $G \in W^{1,1}_{\ell oc}(\mathbb{R}^d)$ gives

$$\begin{aligned}(h_k * \chi_{\widetilde{\omega}})(x) &= \int_{\widetilde{\omega}} \frac{\partial G}{\partial x_k}(x-y)\,dy \\ &= -\int_{\widetilde{\omega}} \frac{\partial G}{\partial y_k}(x-y)\,dy \\ &= -\int_{\partial\widetilde{\omega}} G(x-y)\widetilde{n}_k(y)\,ds_y.\end{aligned}$$

Combining this with $\mathcal{L}(\chi_{\widetilde{\omega}}\mathbf{e}_k) = h_k * \chi_{\widetilde{\omega}}$ we infer

$$a(\chi_{\widetilde{\omega}}\mathbf{e}_k, \chi_\omega\mathbf{e}_j) = \int_\omega \frac{\partial(h_k * \chi_{\widetilde{\omega}})}{\partial x_j}\,dx = -\int_{\partial\omega}\int_{\partial\widetilde{\omega}} G(x-y)n_j(x)\widetilde{n}_k(y)\,ds_y ds_x.$$

Finally, (31) follows by simple convolution properties. We have $h_k * \chi_{\widetilde{\omega}} \in \mathcal{C}^1(\mathbb{R}^d\backslash\overline{\widetilde{\omega}})$ with $\partial(h_k * \chi_{\widetilde{\omega}}) = (\partial h_k/\partial x_j) * \chi_{\widetilde{\omega}}$, whence

$$a(\chi_{\widetilde{\omega}}\mathbf{e}_k, \chi_\omega\mathbf{e}_j) = \int_\omega \frac{\partial(h_k * \chi_{\widetilde{\omega}})}{\partial x_j}\,dx = \int_\omega \kappa_{jk} * \chi_{\widetilde{\omega}}\,dx = \int_\omega\int_{\widetilde{\omega}} \kappa_{jk}(x-y)\,dy\,dx.$$

This concludes the proof. ∎

Remark. Equation (28) was also proved by Hackbusch and Melenk [10] for $d = 3$ by direct calculation. Although their proof does not use the result due to Calderón and Zygmund, this is what is behind when they use the Fourier transform to show the continuity of the bilinear form $a(\cdot,\cdot)$.

Remark. Obviously, the given proof of Equation (29) carries over to arbitrary functions $\varphi, \widetilde{\varphi} \in L^2(\mathbb{R}^d)$, i.e. the characteristic functions $\chi_\omega, \chi_{\widetilde{\omega}}$ can be replaced by $\varphi, \widetilde{\varphi}$.

Computing the stiffness matrix A for $a(\cdot,\cdot)$. For a Galerkin discretization of (26) with piecewise constant ansatz and test functions, one has to compute the matrix

$$\mathbf{A} \in \mathbb{R}^{dN\times dN}_{sym} \quad \text{with } \mathbf{A}_{jk} := a(\varphi_j, \varphi_k) \tag{32}$$

and a fixed basis $\{\varphi_1, \dots, \varphi_{dN}\}$ of $\mathcal{S}^0(\mathcal{T})^d$. A reasonable choice for a basis is

$$\varphi_j := \chi_{T_j}\mathbf{e}_1, \quad \varphi_{N+j} := \chi_{T_j}\mathbf{e}_2 \quad \text{etc. for } 1 \le j \le N, \tag{33}$$

as is shown in the following: This basis gives rise to the definition of the matrices

$$\mathbf{A}^{\alpha\beta} \in \mathbb{R}^{N\times N}_{sym} \quad \text{for fixed } 1 \le \alpha, \beta \le d, \quad \mathbf{A}^{\alpha\beta}_{jk} := a(\chi_{T_j}\mathbf{e}_\alpha, \chi_{T_k}\mathbf{e}_\beta), \tag{34}$$

where the symmetry of $\mathbf{A}^{\alpha\beta}$ (i.e. an additional symmetry of $\mathbf{A}$) follows from (29). Note that — again by Equation (29) — we have $\mathbf{A}^{\alpha\beta} = \mathbf{A}^{\beta\alpha}$. Therefore, $\mathbf{A}$ is a symmetric $(d \times d)$-block matrix with the symmetric blocks $\mathbf{A}^{\alpha\beta} = \mathbf{A}^{\beta\alpha}$ of dimension $N \times N$,

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}^{11} & \mathbf{A}^{12} \\ \mathbf{A}^{12} & \mathbf{A}^{22} \end{pmatrix} \qquad \text{and} \qquad \mathbf{A} = \begin{pmatrix} \mathbf{A}^{11} & \mathbf{A}^{12} & \mathbf{A}^{13} \\ \mathbf{A}^{12} & \mathbf{A}^{22} & \mathbf{A}^{23} \\ \mathbf{A}^{13} & \mathbf{A}^{23} & \mathbf{A}^{33} \end{pmatrix}$$

for $d = 2$ and $d = 3$, resp. As a first consequence we obtain that one has only to compute and store $\frac{1}{4}d(d+1)N(N+1)$ instead of $(dN)^2$ coefficients of the fully populated matrix $\mathbf{A}$. Provided the geometry of the elements $T_j \in \mathcal{T}$ is simple, the entries $\mathbf{A}^{\alpha\beta}_{jk}$ can be computed exactly: Assume that the boundaries of T_j and T_k are finite unions of pairwise disjoint affine boundary pieces $\Gamma_1, \ldots, \Gamma_\ell$ and $\widetilde{\Gamma}_1, \ldots, \widetilde{\Gamma}_{\widetilde{\ell}}$, respectively. Then, Equation (28) reads, for $\mathbf{m} = \mathbf{e}_\alpha, \widetilde{\mathbf{m}} = \mathbf{e}_\beta$,

$$\mathbf{A}^{\alpha\beta}_{jk} = -\sum_{\mu=1}^{\ell}\sum_{\nu=1}^{\widetilde{\ell}} (n_\alpha|_{\Gamma_\mu})(\widetilde{n}_\beta|_{\widetilde{\Gamma}_\nu}) \int_{\Gamma_\mu}\int_{\widetilde{\Gamma}_\nu} G(x-y)\, ds_y ds_x. \tag{35}$$

The double boundary integrals are well-known in the context of boundary integral methods being the Galerkin elements of Symm's integral equation discretized by piecewise constant functions. Note that analytic formulae are known, cf. [4, 16] for $d = 2$ and [9, 16] for $d = 3$, respectively.

Remark. Equation (31) of Proposition 6.1 motivates panel clustering techniques to obtain an approximation $\widetilde{\mathbf{A}}$ of $\mathbf{A}$ such that assembling, storage, and matrix-vector multiplication of $\widetilde{\mathbf{A}}$ are of (almost) linear complexity although the error, for instance, in the Frobenius norm can be controlled [2]. To apply these techniques to each of the matrices $\mathbf{A}^{\alpha\beta} \in \mathbb{R}^{N\times N}$, note that the kernel

$$g_{\alpha\beta}(x,y) := \frac{\partial^2 G}{\partial x_\alpha \partial x_\beta}(x-y)$$

is asymptotically smooth and use the representation (31) for the entries $\mathbf{A}^{\alpha\beta}_{jk}$ on admissible blocks. Numerical experiments for a blockwise $\mathcal{H}^2$-matrix approach will appear in [18].

Acknowledgement. Parts of the paper have been written when the author was a member of the DFG Graduiertenkolleg 357 "*Effiziente Algorithmen und Mehrskalenmethoden*" at the Christian-Albrecht-Universität zu Kiel, Germany. He thanks W. Hackbusch (Max Planck Institute, Leipzig) and C. Carstensen (Humboldt-Universität Berlin, Germany) for valuable discussions. Financial support by the FWF project P15274 "*Reliable and Efficient Numerical Algorithms in Computational Micromagnetics*" is thankfully acknowledged.

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Received 11.11.2003; in revised form 13.04.2004