Grand and Small Lebesgue Spaces and Their Analogs

A. Fiorenza and G. E. Karadzhov

Abstract. We give the following, equivalent, explicit expressions for the norms of the small and grand Lebesgue spaces, which depend only on the non-decreasing rearrangement (we assume here that the underlying measure space has measure 1):

$$\|f\|_{L^{(p}} \approx \int_0^1 (1 - \ln t)^{-\frac{1}{p}} \left(\int_0^t [f^*(s)]^p ds \right)^{\frac{1}{p}} dt/t \qquad (1 < p < \infty)$$

$$\|f\|_{L^{p)}} \approx \sup_{0 < t < 1} (1 - \ln t)^{-\frac{1}{p}} \left(\int_t^1 [f^*(s)]^p ds \right)^{\frac{1}{p}} \qquad (1 < p < \infty).$$

Similar results are proved for the generalized small and grand spaces.

Keywords: *extrapolation, interpolation, Lorentz-Karamata spaces*

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1. Introduction

The goal of this paper is mainly to investigate further the properties of the small and grand Lebesgue spaces, using the interpolation-extrapolation theory. We also introduce some generalizations and analogs of these spaces, such as generalized small (grand) Lorentz spaces of functions or sequences, or compact operators. The grand Lebesgue spaces were introduced by Iwaniec and Sbordone ([21]) and they found many application in analysis, see [3, 13, 14, 16, 20, 21, 29, 30]. The small Lebesgue spaces were introduced by A. Fiorenza ([10])

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as associate to grand spaces. They have applications to some boundary value problems, see [12], [28].

The small Lebesgue space ([10]) $L^{(p}$ consists of all measurable functions g on a finite measure space (Ω, μ) which can be represented in the form $g = \sum_{k=1}^{\infty} g_k$ (convergence a.e.) and such that the following norm is finite:

$$\|g\|_{(p} := \inf_{g=\sum g_k} \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p'-1} \varepsilon^{-\frac{1}{p'-\varepsilon}} \|g_k\|_{(p'-\varepsilon)'},$$

where $\|g\|_p$ stands for the normalised norm in L^p space:

$$\|g\|_p = \left(\frac{1}{|\Omega|} \int_\Omega |g(x)|^p dx\right)^{\frac{1}{p}}$$

and $1 < p < \infty$, $\frac{1}{p'} + \frac{1}{p} = 1$. The grand Lebesgue space ([21]) $L^{p)}$, $1 < p < \infty$ is defined by the norm

$$\|g\|_{p)} := \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \|g\|_{p-\varepsilon}.$$

We give the following characterisation of these spaces (if $\mu(\Omega) = 1$):

$$\|f\|_{L^{(p}} \approx \int_0^1 (1-\ln t)^{-\frac{1}{p}} \left(\int_0^t [f^*(s)]^p ds\right)^{\frac{1}{p}} dt/t$$

and

$$\|f\|_{L^{p)}} \approx \sup_{0<t<1} (1-\ln t)^{-\frac{1}{p}} \left(\int_t^1 [f^*(s)]^p ds\right)^{\frac{1}{p}}.$$

Analogous results are proved for the generalized spaces.

2. Background from extrapolation-interpolation theory

Our investigation is based on extrapolation-interpolation of quasi-Banach spaces. Here we recall some definitions and results from this theory. Let $\vec{A} = (A_0, A_1)$ be (a compatible) pair of quasi-Banach spaces, i.e. we suppose that A_0 and A_1 are quasi-Banach spaces continuously embedded in some quasi-Banach space Σ_A. For $0 < \theta < 1, 0 < p \le \infty$, we let $\vec{A}_{\theta,p}$ denote the real interpolation spaces of Lions and Peetre (see [2], [4]), provided with the K-method norm

$$\|f\|_{\vec{A}_{\theta,p}} = \left\{\int_0^\infty [t^{-\theta} K(t, f; A_0, A_1)]^p dt/t\right\}^{\frac{1}{p}},$$

where

$$K(t, f; A_0, A_1) = \inf_{f=f_0+f_1} \{\|f_0\|_{A_0} + t\|f_1\|_{A_1}\} \ .$$

Let $0 \leq \theta_0 < \theta_1 \leq 1$ be fixed, and let Θ denote the interval (θ_0, θ_1). The K and J methods of interpolation give equivalent quasi-norms on $\vec{A}_{\theta,p}$, $\theta \in \Theta$. Moreover, if $0 < \theta_0 < \theta_1 < 1$, the equivalence of the K and J quasi-norms is uniform (see [2]).

It will be convenient to use also the normalised K-spaces ([18]):

$$\langle \vec{A}_{\theta,p} \rangle := [\theta(1-\theta)p]^{\frac{1}{p}} \vec{A}_{\theta,p}.$$

Note that ([2])

$$A_0 \cap A_1 \subset \langle \vec{A}_{\theta,p} \rangle \subset A_0 + A_1$$

and ([18])

$$\langle \vec{A}_{\theta,p} \rangle \subset \langle \vec{A}_{\theta,q} \rangle, \qquad p < q, \tag{1}$$

uniformly with respect to θ. The characterization of extrapolation spaces as interpolation spaces requires spaces that fall outside the classical Lions-Peetre spaces. In particular, it requires the replacement of power weights $t^{-\theta}$ by more general continuous weights w. Note that given a weight w one can define in the familiar way the $\vec{A}_{w,p}$ and $\vec{A}_{w,p,J}$ spaces associated with the K and J methods. The corresponding K and J norms are then given (respectively) by

$$\|f\|_{\vec{A}_{w,p}} = \left\{ \int_0^\infty \left[w(t)K(t,f;A_0,A_1) \right]^p dt/t \right\}^{\frac{1}{p}}$$

and

$$\|f\|_{\vec{A}_{w,p;J}} = \inf \left\{ \left\{ \sum_{\nu=-\infty}^{\infty} \left[w(2^\nu)J(2^\nu, u_\nu; A_0, A_1) \right]^p \right\}^{\frac{1}{p}} : f = \sum_{\nu=-\infty}^{\infty} u_\nu \right\},$$

where

$$J(t,f;A_0,A_1) = \max \left(\|f\|_{A_0}, t\|f\|_{A_1} \right).$$

We assume that the weights $w(t)$ satisfy the following condition: There exist positive constants c_1, c_2, such that

$$c_1 w(2^\nu) \leq w(t) \leq c_2 w(2^\nu) \quad \text{for all} \quad 2^\nu \leq t \leq 2^{\nu+1}, \nu \in Z.$$

Then we can "discretize" the $\vec{A}_{w,p}$ norm (see [2], Lemma 3.1.3) and obtain

$$\|f\|_{\vec{A}_{w,p}} \approx \left\{ \sum_{\nu=-\infty}^{\infty} \left[w(2^\nu)K(2^\nu, f; A_0, A_1) \right]^p \right\}^{\frac{1}{p}}.$$

From the strong fundamental lemma of interpolation (see, for example, [18]) it follows the following relation between K and J spaces. Suppose that $\vec{A}$ is a Banach pair and let $\vec{A}^c$ be its Gagliardo completion. Then (see [2, 4, 18])

$$\vec{A}^c_{w^*,1;J} = \vec{A}^c_{w,1} = \vec{A}_{w,1}, \tag{2}$$

where

$$w^*(u) = \int_0^\infty \min(1, t/u) w(t) dt/t. \qquad (3)$$

The following remark will be useful in the sequel.

Remark 2.1. Let

$$\left\{ \int_0^\infty \big[\min(1, t) w(t) \big]^p dt/t \right\}^{\frac{1}{p}} < \infty, \quad 0 < p \le \infty.$$

If $A_1 \subset A_0$, the quasi-norm of the embedding being 1, then

$$\|f\|_{\vec{A}_{w,p}} \approx \left\{ \int_0^1 \big[w(t) K(t, f; A_0, A_1) \big]^p dt/t \right\}^{\frac{1}{p}}.$$

(Use the formula $K(t, f) = \|f\|_{A_0}$ for $t \ge 1$.) Analogously, if $A_0 \subset A_1$, with the quasi-norm of the embedding being 1, then $K(t, f; A_0, A_1) = t \|f\|_{A_1}$ for $0 < t < 1$, and so now

$$\|f\|_{\vec{A}_{w,p}} \approx \left\{ \int_1^\infty \big[w(t) K(t, f; A_0, A_1) \big]^p dt/t \right\}^{\frac{1}{p}}.$$

Let $M(\theta)$ be a positive continuous function on the interval $\Theta = (\theta_0, \theta_1)$, such that $\frac{1}{M(\theta)}$ is bounded. We define "one sided" $\Sigma^{(p)}$ spaces ([22]):

$$\Sigma^{(p)-}(M(\theta)\vec{A}_{\theta,p}) = \Sigma_{\theta_0,\alpha}^{(p)-}(M(\theta)\vec{A}_{\theta,p})$$

$$= \left\{ f \in \Sigma_A : \begin{array}{c} f = \sum_{\theta \in (\theta_0, \alpha]} g(\theta), \ g(\theta) \in \vec{A}_{\theta,p}, \\ \text{and } \|f\|_{\Sigma^{(p)-}(M(\theta)\vec{A}_{\theta,p})} < \infty \end{array} \right\},$$

where

$$\|f\|_{\Sigma^{(p)-}(M(\theta)\vec{A}_{\theta,p})} = \inf \left\{ \left\{ \sum_{\theta \in (\theta_0, \alpha]} \big[M(\theta) \|g(\theta)\|_{\vec{A}_{\theta,p}} \big]^p \right\}^{\frac{1}{p}} : f = \sum_{\theta \in (\theta_0, \alpha]} g(\theta) \right\}.$$

Remark 2.2. We are using the *notation of summation over uncountable sets.* In this paper this should be understood as follows. Suppose that $N(\theta)$ is a continuous function on $\Theta = (\theta_0, \alpha]$, $0 \le \theta_0 < \alpha < 1$, such that $N(\theta) \to 0$ as $\theta \to \theta_0$. We fix a discretization say $\theta_n = \theta_0 + 2^{-n}$ if $n \ge n_1 > 0$. Then

$$\sum_{\theta \in (\theta_0, \alpha]} N(\theta) := \sum_{n \ge n_1} N(\theta_n), \ \alpha := \theta_0 + 2^{-n_1}.$$

Remark 2.3. In the same fashion the $\sum^{(p)}$ construction can be applied to other compatible scales $\{A_\theta\}_{\theta \in \Theta}$ of quasi-Banach spaces, where by "compatible" we mean scales such that there exists a constant $c > 0$ such that for all $\theta \in \Theta$ we have $\|f\|_{\Sigma_A} \leq c \|f\|_{A_\theta}$.

Remark 2.4. When dealing with Banach pairs we can replace sums by integrals in the definition of the $\Sigma^{(p)-}$ spaces. This corresponds to the familiar equivalence between the so called "continuous" and "discrete" definitions of the J and K methods of interpolation. For future reference we discuss in more detail a special case of this equivalence. Suppose that $\vec{A} = (A_0, A_1)$ is a Banach pair and moreover suppose that for some small positive ε we have $\{\int_0^\varepsilon [M(\sigma)]^{-p'} d\sigma / \sigma\}^{\frac{1}{p'}} < \infty$, where $\frac{1}{p'} + \frac{1}{p} = 1$, if $1 \leq p < \infty$. Let us say that $f \in \int_{p,0,\varepsilon} (M(\sigma)\vec{A}_{\sigma,p})$ if and only if there exists a representation

$$f = \int_0^\varepsilon g(\sigma) d\sigma / \sigma \quad \text{with } g(\sigma) \in \vec{A}_{\sigma,p},$$

with

$$\int_0^\varepsilon \left[M(\sigma) \|g(\sigma)\|_{\vec{A}_{\sigma,p}} \right]^p d\sigma / \sigma < \infty.$$

Let

$$\|f\|_{\int_{p,0,\varepsilon}(M(\sigma)\vec{A}_{\sigma,p})} = \inf \left\{ \left(\int_0^\varepsilon \left[M(\sigma) \|g(\sigma)\|_{\vec{A}_{\sigma,p}} \right]^p d\sigma / \sigma \right)^{\frac{1}{p}} : f = \int_0^\varepsilon g(\sigma) d\sigma / \sigma \right\}.$$

Suppose that $A_1 \subset A_0$. Suppose in addition that $M(\sigma)$ is a positive, continuous function such that for some $c_1, c_2 > 0$,

$$c_1 M(2^{-n}) \leq M(\sigma) \leq c_2 M(2^{-n}) \quad \text{for all } 2^{-n} \leq \sigma \leq 2^{-n+1}, \ n \geq n_1.$$

Then

$$\int_{p,0,\varepsilon} \left(M(\sigma)\vec{A}_{\sigma,p} \right) = \Sigma_{(0,\varepsilon]}^{(p)-} (M(\sigma)\vec{A}_{\sigma,p}), \quad \varepsilon = 2^{-n_1}.$$

The proof is analogous to the usual proof of the discretization of the $J-$method (see [2]).

In [22] the following characterization of the $\Sigma^{(p)-}$ spaces in terms of J spaces (see [18] for the case $p = 1$) is given.

Theorem 2.5 (Banach case ([22])). *Let $\vec{A} = (A_0, A_1)$ be a Banach pair. Suppose that $0 \leq \theta_0 < \alpha < 1$, let Θ be the interval $(\theta_0, \alpha]$, and furthermore let $1 \leq p \leq \infty$. Define the weight w^* by*

$$\frac{1}{w^*(t)} = \left\{ \sum_{\theta \in \Theta} \left[\frac{t^\theta}{M(\theta)} \right]^{p'} \right\}^{\frac{1}{p'}}, \quad t > 0, \ \frac{1}{p} + \frac{1}{p'} = 1. \tag{4}$$

Then

$$\Sigma_{\theta_0,\alpha}^{(p)-} (M(\theta)\vec{A}_{\theta,p;J}) = \vec{A}_{w^*,p;J}, \quad p > 0.$$

Now we recall the construction of the $\Delta^{(p)}$ methods of extrapolation (see [18],[22]). Let $0 < p \leq \infty$, $0 \leq \theta_0 < \theta_1 \leq 1$, $\Theta = (\theta_0, \theta_1)$, and suppose that $\{\int_\Theta [N(\theta)]^p d\theta\}^{\frac{1}{p}} < \infty$, where $N(\theta)$ is positive and continuous on the interval Θ. Then we let

$$\Delta^{(p)}_{\theta \in \Theta}(N(\theta)\vec{A}_{\theta,p}) = \Delta^{(p)}(N(\theta)\vec{A}_{\theta,p})$$

$$= \left\{ f \in \bigcap_{\theta \in \Theta} \vec{A}_{\theta,p} : \|f\|_{\Delta^{(p)}(N(\theta)\vec{A}_{\theta,p})} < \infty \right\},$$

where

$$\|f\|_{\Delta^{(p)}(N(\theta)\vec{A}_{\theta,p})} := \left\{ \int_\Theta \left[N(\theta)\|f\|_{\vec{A}_{\theta,p}} \right]^p d\theta \right\}^{\frac{1}{p}}.$$

Remark 2.6. We can apply the $\Delta^{(p)}$ construction to any scale $\{A_\theta\}_{\theta \in \Theta}$ of compatible quasi-Banach spaces, i.e., such that there exist quasi-Banach spaces Δ_A and Σ_A such that $\Delta_A \subset A_\theta \subset \Sigma_A$, and the quasi-norms of the embeddings are uniformly bounded with respect to $\theta \in \Theta$. In this fashion it follows that $\Delta^{(p)}(N(\theta)A_\theta) \supset \Delta_A$.

Using Fubini and the definition of the $K-$method of interpolation, it is readily seen that

$$\Delta^{(p)}(N(\theta)\vec{A}_{\theta,p}) = \vec{A}_{W,p}, \tag{5}$$

where the weight function W is defined by the formula

$$W(t) = \begin{cases} \left\{ \int_\Theta [t^{-\theta} N(\theta)]^p d\theta \right\}^{\frac{1}{p}} & \text{for } p < \infty \\ \sup_\theta t^{-\theta} N(\theta) & \text{for } p = \infty \ . \end{cases} \tag{6}$$

3. Small Lebesgue spaces

We start with the basic definition given in [10], Proposition 2.4. The space $L^{(p)}$ consists of all measurable functions g on a finite measure space Ω which can be represented in the form $g = \sum_{k=1}^\infty g_k$ (convergence a.e.) and such that the following norm is finite:

$$\|g\|_{(p} := \inf_{g = \sum g_k} \sum_{k=1}^\infty \inf_{0 < \varepsilon < p'-1} \varepsilon^{-\frac{1}{p'-\varepsilon}} \|g_k\|_{(p'-\varepsilon)'}, \tag{7}$$

where $\|g\|_p$ stands for the normalised norm in L^p space and $1 < p < \infty$, $\frac{1}{p'} + \frac{1}{p} = 1$. Our goal is to characterise these spaces as $\Sigma-$extrapolation and interpolation spaces. First we prove

Lemma 3.1. *It holds*

$$\|g\|_{(p} \approx \inf_{g=\sum g_k} \sum_{k=1}^{\infty} \inf_{0<\varepsilon<\varepsilon_0} \varepsilon^{-\frac{1}{p'}} \|g_k\|_{L^{p+\varepsilon,p}}, \quad 1 < p < \infty. \tag{8}$$

Here $L^{p,r}$, $0 < p < \infty$, $0 < r \le \infty$ are the usual Lorentz spaces with the quasi-norm

$$\|g\|_{L^{p,r}} := \left\{ \int_0^{\infty} \left[t^{\frac{1}{p}} f^*(t) \right]^r dt/t \right\}^{\frac{1}{r}}. \tag{9}$$

Proof. Since $\varepsilon^{-\frac{1}{p'-\varepsilon}} \approx \varepsilon^{-\frac{1}{p'}}$, what means that the quotient of the two positive quantities is bounded from below and from above, uniformly w.r.t. ε, and since $(p'-\varepsilon)' = p + \gamma\varepsilon$ for some $\gamma \approx 1$, we see that the norm (7) is equivalent to

$$\|g\|_{(p} \approx \inf_{g=\sum g_k} \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p'-1} \varepsilon^{-\frac{1}{p'}} \|g_k\|_{p+\varepsilon}. \tag{10}$$

Further, using the fact that $\|g\|_p \le \|g\|_r$ if $p < r$, we can replace the infimum over $0 < \varepsilon < p' - 1$ by the infimum over $0 < \varepsilon < \varepsilon_0$ for any $0 < \varepsilon_0 < p' - 1$. Indeed, we have

$$\inf_{0<\varepsilon<\varepsilon_0} \varepsilon^{-\frac{1}{p'}} \|g\|_{p+\varepsilon} = q^{\frac{1}{p'}} \inf_{0<\sigma<q\varepsilon_0} \sigma^{-\frac{1}{p'}} \|g\|_{p+\frac{\sigma}{q}},$$

where $q\varepsilon_0 = p' - 1$, hence $q > 1$. Therefore, the above quantity is smaller than

$$q^{\frac{1}{p'}} \inf_{0<\sigma<p'-1} \sigma^{-\frac{1}{p'}} \|g\|_{p+\sigma},$$

and the assertion is proven. Thus we have the following equivalent norm:

$$\|g\|_{(p} \approx \inf_{g=\sum g_k} \sum_{k=1}^{\infty} \inf_{0<\varepsilon<\varepsilon_0} \varepsilon^{-\frac{1}{p'}} \|g_k\|_{p+\varepsilon}. \tag{11}$$

Here we can take ε_0 small enough (smaller than $p' - 1$ and 1).

Further, in the above definition we can replace the Lebesgue space $L^{p+\varepsilon}$ by the Lorentz space $L^{p+\varepsilon,p}$. Namely, since $\varepsilon > 0$ we have (uniformly w.r.t. ε)

$$L^{p+\varepsilon,p} \subset L^{p+\varepsilon},$$

and using Hölder inequality we see that

$$L^{p+2\varepsilon} \subset L^{p+\varepsilon,p}.$$

Thus the lemma is proved. ∎

Now we can state the main result in this section.

Theorem 3.2. *It holds*

$$L^{(p} = (L^p, L^\infty)_{w^*,1;J} = \Sigma_{0,\varepsilon}^{(1)-} \big(\sigma^{-\frac{1}{p'}} (L^p, L^\infty)_{\sigma,1;J} \big), \quad 1 < p < \infty \, ,$$

where w^* is given by $w^*(t) = \Big(\sup_{0<\varepsilon<\varepsilon_0} t^\varepsilon \varepsilon^{\frac{1}{p'}} \Big)^{-1}$.

Proof. We split the proof into several steps.

Step 1. Analogously to $L^{(p}$ we define the space S_p by the norm

$$\|g\| := \inf_{g=\sum g_k} \sum_{k=1}^\infty \inf_{0<\varepsilon<\varepsilon_0} \varepsilon^{-\frac{1}{p'}} \|g_k\|_{(L^p,L^\infty)_{\varepsilon,1;J}}. \tag{12}$$

We have

$$S_p = L^{(p}. \tag{13}$$

To prove this we need the formula

$$\theta^{\frac{1}{r}} (L^r, L^\infty)_{\theta,r} = L^{q,r}, \quad \frac{1}{q} = \frac{1-\theta}{r}. \tag{14}$$

Indeed, by the Holmsted formula ([2]):

$$K^r(t, f; L^r, L^\infty) \approx \int_0^{t^r} [f^*(s)]^r ds.$$

Hence the definition of the K method (using Fubini and calculating an integral) gives (14). Applying this formula, we get

$$L^{p+\varepsilon,p} = \varepsilon^{\frac{1}{p}} (L^p, L^\infty)_{\gamma\varepsilon,p}, \tag{15}$$

for some $\gamma \approx 1$. According to formula (1)

$$\varepsilon^{\frac{1}{p}} (L^p, L^\infty)_{\gamma\varepsilon,p} \supset \varepsilon(L^p, L^\infty)_{\gamma\varepsilon,1} \, , \tag{16}$$

and according to (2)

$$\varepsilon(L^p, L^\infty)_{\gamma\varepsilon,1} = (L^p, L^\infty)_{\gamma\varepsilon,1;J}. \tag{17}$$

Using (8), (15), (16) and the definition (12), we conclude that $S_p \subset L^{(p}$.

Conversely, let $g \in L^{(p}$. Then $g = \sum g_k$, $g_k \in L^{p+\varepsilon,p}$. On the other hand, analogously to (12) of [22],

$$\varepsilon^{\frac{1}{p}} (L^p, L^\infty)_{\gamma\varepsilon,p} \subset \varepsilon(L^p, L^\infty)_{\frac{\gamma\varepsilon}{2},1} + \varepsilon(L^p, L^\infty)_{2\gamma\varepsilon,1}. \tag{18}$$

Hence, using also (15), we can choose an appropriate decomposition,

$$g_k = g_k^1 + g_k^2, \quad g_k^1 \in (L^p, L^\infty)_{\frac{\gamma\varepsilon}{2},1}, \quad g_k^2 \in (L^p, L^\infty)_{2\gamma\varepsilon,1},$$

so that the series $\sum g_k^j$ are convergent $(j = 1, 2)$. Let $g^j = \sum g_k^j$ $(j = 1, 2)$. Then $g = g^1 + g^2$ and using the definition of S_p we see that $g^j \in S_p$, hence $g \in S_p$. Thus (13) is proved.

   Step 2. Here we prove

$$S_p \subset (L^p, L^\infty)_{w^*,1;J}, \quad \frac{1}{w^*(t)} = \sup_{0 < \varepsilon < \varepsilon_0} t^\varepsilon \varepsilon^{\frac{1}{p'}}. \tag{19}$$

It is not hard to see that

$$w^*(t) \approx (1 - \ln t)^{\frac{1}{p'}} \quad \text{if } 0 < t < 1. \tag{20}$$

To prove (19), we remark that for some ε_k (depending in general on g_k) we have

$$\|g\| \approx \inf_{g = \sum g_k} \sum_{k=1}^\infty \varepsilon_k^{-\frac{1}{p'}} \|g_k\|_{(L^p, L^\infty)_{\varepsilon_k,1;J}}. \tag{21}$$

   Let $g \in S_p$. Then $g = \sum g_k$, $g_k \in (L^p, L^\infty)_{\varepsilon_k,1;J}$. Hence, by definition, we can find $u_{\nu k} \in L^p \cap L^\infty$ such that $g_k = \sum u_{\nu k}$ and

$$\|g_k\|_{(L^p, L^\infty)_{\varepsilon_k,1;J}} \approx \sum 2^{-\nu\varepsilon_k} J(2^\nu, u_{\nu k}).$$

Then $g = \sum u_\nu$, where $u_\nu = \sum u_{\nu k}$. We have

$$J(2^\nu, u_\nu) \le \sum J(2^\nu, u_{\nu k}) \le \sum \varepsilon_k^{-\frac{1}{p'}} 2^{-\nu\varepsilon_k} J(2^\nu, u_{\nu k}) \sup_{1 \le k < \infty} 2^{\nu\varepsilon_k} \varepsilon_k^{\frac{1}{p'}},$$

hence

$$w^*(2^\nu) J(2^\nu, u_\nu) \le \sum \varepsilon_k^{-\frac{1}{p'}} 2^{-\nu\varepsilon_k} J(2^\nu, u_{\nu k}),$$

therefore

$$\sum w^*(2^\nu) J(2^\nu, u_\nu) \le c \sum \varepsilon_k^{-\frac{1}{p'}} \|g_k\|_{(L^p, L^\infty)_{\varepsilon_k,1;J}},$$

or

$$\|g\|_{(L^p, L^\infty)_{w,1;J}} \le c \sum \varepsilon_k^{-\frac{1}{p'}} \|g_k\|_{(L^p, L^\infty)_{\varepsilon_k,1;J}},$$

or

$$\|g\|_{(L^p, L^\infty)_{w^*,1;J}} \le c\|g\|.$$

Formula (19) is proved.

   Step 3. Here we complete the proof of Theorem 3.2. First, from the definitions it follows

$$\Sigma_{0,\varepsilon}^{(1)-} \left(\sigma^{-\frac{1}{p'}} (L^p, L^\infty)_{\sigma,1;J} \right) \subset L^{(p}.$$

The reverse embedding follows from (13), (19) and Theorem 2.5. ∎

As a corollary from Theorem 3.2 and (2), taking into account that $(L^p)^c = L^p$, we get the following characterization of the small Lebesgue spaces (for simplicity we suppose that $\mu(\Omega) = 1$).

Corollary 3.3.

$$L^{(p} = (L^p, L^\infty)_{w,1}, \ w(t) = (1 - \ln t)^{-\frac{1}{p}}, \quad 0 < t < 1, \tag{22}$$

and

$$\|f\|_{L^{(p}} \approx \int_0^1 (1 - \ln t)^{-\frac{1}{p}} \left(\int_0^t [f^*(s)]^p ds \right)^{\frac{1}{p}} dt/t, \qquad 1 < p < \infty. \tag{23}$$

Moreover, $\|f\|_{L^{(p}}$ *is equivalent to the following norm:*

$$\|f\|_{L^{(p}} \approx \int_0^1 (1 - \ln t)^{-\frac{1}{p}} \left(\int_0^t [f^{**}(s)]^p ds \right)^{\frac{1}{p}} dt/t, \quad 1 < p < \infty, \tag{24}$$

where $f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds.$

Remark 3.4. The result of Theorem 3.2 can be written in the form

$$L^{(p} = \Sigma_{0,\varepsilon}^{(1)-} \left(\sigma^{-\frac{1}{p'}} L^{p+\sigma} \right). \tag{25}$$

Indeed,

$$L^{(p} = \Sigma_{0,\varepsilon}^{(1)-} \left(\sigma^{-\frac{1}{p'}} (L^p, L^\infty)_{\sigma,1;J} \right) = \Sigma_{0,\varepsilon}^{(1)-} \left(\sigma^{-\frac{1}{p'}} (L^p, L^\infty)_{\sigma,1} \sigma \right) \ .$$

According to [22, Theorem 2.10] this equals

$$\Sigma_{0,\varepsilon}^{(1)-} \left(\sigma^{-\frac{1}{p'}} (L^p, L^\infty)_{\sigma,p} \sigma^{\frac{1}{p}} \right) \ ,$$

which is according to [22, Theorem 2.13] equal to

$$\Sigma_{0,\varepsilon}^{(1)-} \left(\sigma^{-\frac{1}{p'}} (L^p, L^\infty)_{\gamma\sigma,p+\sigma} \sigma^{\frac{1}{p}} \right) = \Sigma_{0,\varepsilon}^{(1)-} \left(\sigma^{-\frac{1}{p'}} L^{p+\sigma} \right).$$

Remark 3.5. We can show, by using a direct argument, that if $p > 1$ then $L^{(p} \subset L^{p,1}$. Assume $\mu(\Omega) = 1$. Since

$$\sigma(L^p, L^\infty)_{\sigma,1} \subset \sigma^{1-\frac{1}{p}} L^{p+\sigma,1} \subset \sigma^{\frac{1}{p'}} L^{p,1}$$

we have

$$\Sigma^{(1)} \sigma^{-\frac{1}{p'}} (L^p, L^\infty)_{\sigma,1;J} \subset \Sigma^{(1)} L^{p,1} = L^{p,1}$$

Remark 3.6. We can define $L^{(1}$ using the result of Theorem 3.2:

$$L^{(1} := \Sigma_{0,\varepsilon}^{(1)-} \left((L^1, L^\infty)_{\sigma,1;J} \right)$$

Then we see that $L^{(1} = (L^1, L^\infty)_{0,1;J} = L^1.$

4. Grand Lebesgue spaces

The grand Lebesgue space ([21]) $L^{p)}$, $1 < p < \infty$ is defined with the norm

$$\|g\|_{p)} := \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \|g\|_{p-\varepsilon}. \tag{26}$$

Analogously to Lemma 3.1 we have

Lemma 4.1. *If the measure of Ω is finite, then*

$$\|g\|_{p)} \approx \sup_{0<\varepsilon<\varepsilon_0} \varepsilon^{\frac{1}{p}} \|g\|_{L^{p-\varepsilon}} = \sup_{0<\varepsilon<\varepsilon_0} \varepsilon^{\frac{1}{p}} \|g\|_{L^{p-\varepsilon,p}}. \tag{27}$$

Hence the grand Lebesgue space is a Δ extrapolation space ([22]):

$$L^{p)} = \Delta_{0,\varepsilon}^{(\infty)}(\sigma^{\frac{1}{p}} L^{p-\sigma}) = \Delta_{0,\varepsilon}^{(\infty)}(\sigma^{\frac{1}{p}} L^{p-\sigma,p}), \tag{28}$$

where $\varepsilon > 0$ is small enough. Note that this formula defines the grand Lebesgue space for all $0 < p < \infty$ and for any $\sigma-$finite measure space (Ω, μ).

We can characterise the grand Lebesgue space as an interpolation space.

Theorem 4.2. *Let $0 < p < \infty$ and choose any q, $0 < q < p$. Then*

$$L^{p)} = (L^q, L^p)_{w,\infty}, \; w(t) = t^{-1}(1 - \ln t)^{-\frac{1}{p}}. \tag{29}$$

Moreover, if $\mu(\Omega) = 1$, then

$$\|f\|_{L^{p)}} \approx \sup_{0<t<1} (1 - \ln t)^{-\frac{1}{p}} \left(\int_t^1 \left[f^*(s) \right]^p ds \right)^{\frac{1}{p}}. \tag{30}$$

Finally, for $1 < p < \infty$, $\|f\|_{L^{p)}}$ is equivalent to the norm

$$\|f\|_{L^{p)}} \approx \sup_{0<t<1} (1 - \ln t)^{-\frac{1}{p}} \left(\int_t^1 \left[f^{**}(s) \right]^p ds \right)^{\frac{1}{p}}. \tag{31}$$

Proof. We need the formula

$$(1 - \theta)^{\frac{1}{p}} (L^q, L^p)_{\theta,p} = L^{p_\theta,p}, \quad \frac{1}{p_\theta} = \frac{1 - \theta}{q} + \frac{\theta}{p}, \; 0 < \theta_0 < \theta < 1, \tag{32}$$

uniformly on θ.

Indeed, we write

$$L^p = (L^q, L^\infty)_{\eta,p}, \quad \frac{1}{p} = \frac{1 - \eta}{q}$$

and use the Holmsted formula ([2]):

$$K^p(t, f; L^q, L^p) \approx t^p \int_{t^{\frac{1}{\eta}}}^{\infty} u^{-\eta p} K^p(u, f) du/u, \quad K(u, f) := K(u, f; L^q, L^\infty).$$

Then straightforward calculation shows that

$$\|f\|_{(L^q, L^p)_{\theta, p}} = [\eta(1 - \theta)p]^{-\frac{1}{p}} \|f\|_{(L^q, L^\infty)_{\eta\theta, p}}.$$

Thus formula (32) follows.

Now we can continue the formula (28) as follows

$$\Delta_{0,\varepsilon}^{(\infty)}(\sigma^{\frac{1}{p}} L^{p-\sigma, p}) = \Delta_{0,\varepsilon}^{(\infty)}(\sigma^{\frac{2}{p}} (L^q, L^p)_{1 - \frac{\alpha\sigma}{\eta}, p}), \quad \alpha \approx 1,$$

and according to Theorem 3.3 of [22] this is the same as

$$\Delta_{0,\varepsilon}^{(\infty)}(\sigma^{\frac{1}{p}} (L^q, L^p)_{1-\sigma, \infty}).$$

Hence

$$L^{p)} = \Delta_{0,\varepsilon}^{(\infty)}(\sigma^{\frac{1}{p}} (L^q, L^p)_{1-\sigma, \infty}) \tag{33}$$

and (29) follows using also (5).

To prove (30), we use the Holmstedt formula ([2]):

$$K(t, f; L^q, L^p) \approx \left(\int_0^{t^\alpha} [f^*(s)]^q ds \right)^{\frac{1}{q}} + t \left(\int_{t^\alpha}^1 [f^*(s)]^p ds \right)^{\frac{1}{p}}, \quad \frac{1}{\alpha} = \frac{1}{q} - \frac{1}{p}.$$

We have

$$\|f\|_{(L^q, L^p)_{w, \infty}} \approx \sup_{0 < t < 1} (1 - \ln t)^{-\frac{1}{p}} \left(\int_{t^\alpha}^1 [f^*(s)]^p ds \right)^{\frac{1}{p}}.$$

It suffices to see that

$$w(t) \left(\int_0^{t^\alpha} [f^*(s)]^q ds \right)^{\frac{1}{q}} \le c \sup_{0 < t < 1} t^{\frac{1}{p}} (1 - \ln t)^{-\frac{1}{p}} f^*(t)$$

$$\le c \sup_{0 < t < 1} (1 - \ln t)^{-\frac{1}{p}} \left(\int_{t^\alpha}^{2t^\alpha} [f^*(s)]^p ds \right)^{\frac{1}{p}}.$$

The second inequality follows from the monotonicity of f^*, while the first one can be proved as follows. Let $b(t) := (1 - \ln t)^{-\frac{1}{p}}$. Then for small $\varepsilon > 0$,

$$b(t) t^{-1} \left(\int_0^{t^\alpha} [f^*(s)]^q ds \right)^{\frac{1}{q}} \le c t^{-\varepsilon} \left(\int_0^t \left[s^{\frac{1}{p}} b(s) f^*(s) \right]^q s^{-1+q\varepsilon} ds \right)^{\frac{1}{q}}$$

$$\le c \sup_{0 < t < 1} t^{\frac{1}{p}} b(t) f^*(t),$$

which completes the proof. ∎

Remark 4.3. We have the following embeddings. Let $\mu(\Omega) = 1$ and $0 < p < \infty$. Then

$$L^{p,\infty} \subset L^{p)} \subset L^{p,\infty}(\log L)_{-\frac{1}{p}} \tag{34}$$

$$L^{p}(\log L)_{-\frac{1}{p}} \subset L^{p)} \subset L^{p}(\log L)_{-\frac{1}{p}-\delta}, \quad \delta > 0, \tag{35}$$

where $L^{p,r}(\log L)_a$, $0 < p < \infty$, $0 < r \le \infty$, $-\infty < a < \infty$ are Lorentz logarithmic spaces with the quasinorm

$$\|g\|_{L^{p,r}(\log L)_a} := \left\{ \int_0^\infty \left[(1 + |\ln t|)^a t^{\frac{1}{p}} f^*(t) \right]^r dt/t \right\}^{\frac{1}{r}}. \tag{36}$$

Proof. The first three embeddings follow using the equivalent quasinorm (30). To prove the fourth one, we use (28) and obtain

$$\|f\|_{L^{p)}}^p \approx \sup_{0 < \sigma < \varepsilon} \sigma \|f\|_{L^{p-\sigma,p}}^p$$

$$\ge c \int_0^\varepsilon \sigma^\delta \|f\|_{L^{p-\sigma,p}}^p d\sigma$$

$$\approx \int_0^1 (1 - \ln t)^{-1-\delta} [f^*(t)]^p dt,$$

as desired. ∎

Remark 4.4. We can provide simple examples showing that inclusions (34) and (35) are strict. Moreover, inclusions (34) are optimal in the scale of Marcinkiewicz logarithmic spaces $L^{p,\infty}(\log L)_{-a}$ $(a > 0)$. For simplicity, we consider the spaces on the interval $(0, 1)$. The function $f_1(t) = t^{-1/p} \in L^{p,\infty} \subset L^{p)}$, and $f_1 \notin L^p(\log L)_{-1/p}$. The function $f_2(t) = t^{-1/p}|\ln t|^\alpha$, $0 < \alpha < \delta/p$ is such that $f_2 \in L^p(\log L)_{-1/p-\delta}$ and $f_2 \notin L^{p)}$. In [15] there is an example of a function $f_3 \in L^p(\log L)_{-1/p} \subset L^{p)}$, $f_3 \notin L^{p,\infty}$. Finally, consider $f_4(t) = t^{-1/p}(1 - \ln t)^{\alpha/p}$. It is true that $f_4 \in L^{p,\infty}(\log L)_{-\alpha/p}$, $\alpha > 0$, but $f_4 \notin L^{p)}$.

5. Abstract small and grand spaces

The result of Theorem 3.2 suggests the following

Definition 5.1. Let $\vec{A} = (A_0, A_1)$ be a compatible pair of Banach spaces. By definition, *abstract small spaces* are the Σ-extrapolation spaces

$$\Sigma_{0,\varepsilon}^{(1)-}\big(M(\sigma)\langle\vec{A}_{\sigma,r}\rangle\big), \quad 1 < r \le \infty,$$

where M is tempered on the interval $(0, \varepsilon)$, $0 < \varepsilon \le 1$ such that $1/M$ is bounded. By definition (see [18]), M is tempered on the interval $(0, \varepsilon)$, if it is continuous and

$$M(\sigma) \approx M\left(\frac{\sigma}{2}\right), \quad 0 < \sigma < \varepsilon. \tag{37}$$

We can characterize the abstract small spaces as interpolation spaces.

Theorem 5.2.
$$\Sigma_{0,\varepsilon}^{(1)-}\big(M(\sigma)\langle\vec{A}_{\sigma,r}\rangle\big) = \vec{A^c}_{w^*,1;J} = \vec{A}_{w,1}, \tag{38}$$

where
$$\frac{1}{w^*(t)} = \sup_{0<\sigma<\varepsilon}\frac{t^\sigma}{M(\sigma)} \tag{39}$$

and w is given by (3). (Here we suppose that w^* is sufficiently regular, say, has continuous first derivative and locally integrable second derivative.)

Proof. Using Theorem 2.10 of [22] and Theorem 2.5, we obtain

$$\begin{aligned}
\Sigma_{0,\varepsilon}^{(1)-}\big(M(\sigma)\langle\vec{A}_{\sigma,r}\rangle\big) &= \Sigma_{0,\varepsilon}^{(1)-}\big(M(\sigma)\langle\vec{A}_{\sigma,1}\rangle\big)\\
&= \Sigma_{0,\varepsilon}^{(1)-}\big(M(\sigma)\vec{A^c}_{\sigma,1;J}\big)\\
&= \vec{A^c}_{w^*,1;J} = \vec{A}_{w,1},
\end{aligned}$$

as desired. ∎

Example 5.3. If $M(\sigma) = \sigma^{-a}$, $a > 0$, then $w^*(t) \approx (1-\ln t)^a$ for $0 < t < 1$ and $w(t) \approx (1-\ln t)^{a-1}$ for $0 < t < 1$. Thus the small Lebesgue spaces $L^{(p}$ correspond to $a = \frac{1}{p'}$ and $\vec{A} = (L^p, L^\infty)$.

Now we introduce the abstract grand spaces.

Definition 5.4. Let $\vec{A} = (A_0, A_1)$ be a compatible pair of quasi-Banach spaces. By definition, the Δ-extrapolation spaces

$$\Delta_{0,\varepsilon}^{(\infty)}\big(N(\sigma)\langle\vec{A}_{\sigma,r}\rangle\big), \quad 0 < r < \infty,$$

are called *abstract grand spaces.* Here N is a tempered function on the interval $[0,\varepsilon]$, $0 < \varepsilon \le 1$.

We have the following characterization of the abstract grand spaces as interpolation spaces.

Theorem 5.5.
$$\Delta_{0,\varepsilon}^{(\infty)}\big(N(\sigma)\langle\vec{A}_{\sigma,r}\rangle\big) = \vec{A}_{W,\infty}, \tag{40}$$

where
$$W(t) = \sup_{0<\sigma<\varepsilon} t^{-\sigma}N(\sigma). \tag{41}$$

Proof. Using Theorem 3.3 of [22] and (5), we obtain

$$\Delta_{0,\varepsilon}^{(\infty)}\big(N(\sigma)\langle\vec{A}_{\sigma,r}\rangle\big) = \Delta_{0,\varepsilon}^{(\infty)}(N(\sigma)\vec{A}_{\sigma,\infty}) = \vec{A}_{W,\infty},$$

as desired. ∎

Example 5.6. The grand Lebesgue spaces $L^{p)}$ correspond to $N(\sigma) = \sigma^{\frac{1}{p}}$ and $\vec{A} = (L^p, L^q)$ for any $q < p$.

6. Duality

We start with the duality of abstract small and grand spaces. First, Proposition 3.1 of [22] gives

Theorem 6.1. *Let $\vec{A} = (A_0, A_1)$ be a pair of Banach spaces such that the intersection $A_0 \cap A_1$ is dense in A_j, $j = 0, 1$ and let $\vec{A}' = (A_0', A_1')$ be the dual pair . Then*

$$\left\{ \int_{1,0,\varepsilon} \left(M(\sigma)(A_0, A_1)_{\sigma,1;J} \right) \right\}' = \Delta_{0,\varepsilon}^{(\infty)} \left(\frac{1}{M(\sigma)} (A_0', A_1')_{\sigma,\infty} \right).$$

If $A_0 \cap A_1 \subset A \subset A_0 + A_1$ we denote by A° the completion of the intersection $A_0 \cap A_1$ in the space A.

Theorem 6.2. *Let $\vec{A} = (A_0, A_1)$ be a pair of Banach spaces such that the intersection $A_0 \cap A_1$ is dense in A_j, $j = 0, 1$. Then*

$$\left\{ \left[\Delta_{0,\varepsilon}^{(\infty)} \left(N(\sigma)(A_0, A_1)_{\sigma,\infty} \right) \right]^\circ \right\}' = \int_{1,0,\varepsilon} \left(\frac{1}{N(\sigma)} (A_0', A_1')_{\sigma,1;J} \right).$$

Proof. We argue as in the proof of Proposition 3.1 of [22] and use [4]

$$\{ [A_{h,\infty}]^\circ \}' = A_{w,1;J}',$$

where $h(t) = \frac{1}{w(1/t)}$. ∎

As an application we can give another proof of the following theorem about duality of small and grand Lebesgue spaces.

Theorem 6.3 ([10]). *We have*

$$\{ L^{(p'} \}' = L^{p)} \tag{42}$$

and

$$\{ [L^{p)}]^\circ \}' = L^{(p'}. \tag{43}$$

Proof. We want to apply the above abstract duality results. Let $1 < q < p < \infty$ be fixed. Then we can write

$$L^{p-\sigma} = (L^q, L^\infty)_{\eta - \alpha\sigma, p - \sigma},$$

$$L^{p'+\sigma} = (L^{q'}, L^1)_{\eta - \beta\sigma, p' + \sigma},$$

where

$$\frac{1}{p} = \frac{1 - \eta}{q}, \quad \alpha = \frac{q}{p(p - \sigma)}, \quad \beta = \frac{q}{p'(p' + \sigma)}.$$

Hence, by Remark 3.4,

$$
\begin{aligned}
L^{(p'} &= \Sigma_{0,\varepsilon}^{(1)-}(\sigma^{-\frac{1}{p}}L^{p'+\sigma}) \\
&= \Sigma_{0,\varepsilon}^{(1)-}(\sigma^{-\frac{1}{p}}(L^{q'},L^1)_{\eta-\beta\sigma,p'+\sigma}) \\
&= \Sigma_{0,\varepsilon}^{(1)-}(\sigma^{-\frac{1}{p}}(L^{q'},L^1)_{\eta-\beta\sigma,p'}) \\
&\overset{(32)}{=} \Sigma_{0,\varepsilon}^{(1)-}(\sigma^{-\frac{1}{p}}(L^{q'},L^{p'})_{1-\frac{\beta\sigma}{\eta},p'}\sigma^{\frac{1}{p'}})\ .
\end{aligned}
$$

Using Theorem 2.10 of [22] and (2) we get

$$
\begin{aligned}
\Sigma_{0,\varepsilon}^{(1)-}(\sigma^{-\frac{1}{p}}(L^{q'},L^{p'})_{1-\frac{\beta\sigma}{\eta},p'}\sigma^{\frac{1}{p'}}) &= \Sigma_{0,\varepsilon}^{(1)-}(\sigma^{-\frac{1}{p}}(L^{q'},L^{p'})_{1-\frac{\beta\sigma}{\eta},1}\sigma) \\
&= \Sigma_{0,\varepsilon}^{(1)-}(\sigma^{-\frac{1}{p}}(L^{q'},L^{p'})_{1-\frac{\beta\sigma}{\eta},1;J}),
\end{aligned}
$$

thus

$$
L^{(p'} = \Sigma_{0,\varepsilon}^{(1)-}(\sigma^{-\frac{1}{p}}(L^{q'},L^{p'})_{1-\sigma,1;J})\ , \tag{44}
$$

and using Remark 2.4, we can write

$$
L^{(p'} = \int_{1,0,\epsilon}(\sigma^{-\frac{1}{p}}(L^{q'},L^{p'})_{1-\sigma,1;J}). \tag{45}
$$

Now the first part of the theorem follows from (33) and (45), applying Theorem 6.1. To prove the second part, we apply Theorem 6.2.   ∎

7. Generalized grand spaces

7.1. Generalized grand Lorentz function spaces.
In order to define the generalized grand Lorentz function spaces, we first recall some definitions.

Let $b(t)$ be a positive continuous function on the interval $[1,\infty)$. We say that b is *slowly varying* on $[1,\infty)$ (in the sense of Karamata) if for all $\epsilon > 0$ the function $t^\epsilon b(t)$ is equivalent to a non-decreasing function and the function $t^{-\epsilon}b(t)$ is equivalent to a non-increasing function. By symmetry, we say that a positive continuous function b on the interval $(0,1]$ is slowly varying on $(0,1]$ if the function $t \to b(1/t)$ is slowly varying on $[1,\infty)$. Finally, a positive continuous function on $(0,\infty)$ is said to be slowly varying on $(0,\infty)$ if it is slowly varying on both $(0,1]$ and $[1,\infty)$. Let (Ω,μ) be a σ-finite measure space and let b be slowly varying on $(0,\infty)$. Then the *Lorentz-Karamata function space* $L_b^{q,r}$, $0 < q \le \infty$, $0 < r \le \infty$ (see [26]) is defined with the quasi-norm

$$
\|f\|_{L_b^{q,r}} = \left(\int_0^\infty \left[t^{\frac{1}{q}}b(t)f^*(t)\right]^r dt/t\right)^{\frac{1}{r}}. \tag{46}
$$

Note two particular cases. If $b = 1$, then we obtain the *Lorentz space $L^{q,r}$*; if $b(t) = (1 + |\ln t|)^a$ then we obtain the *logarithmic Lorentz space $L^{r,q}(\log L)_a$*.

Definition 7.1. By definition, the *generalized grand Lorentz space $L_b^{p),r}$*, $0 < p, r < \infty$, is the extrapolation space

$$L_b^{p),r} := \Delta_{0,\varepsilon}^{(\infty)}(N(\sigma)L^{p-\sigma,r}),$$

where N is tempered on the interval $(0, \varepsilon)$ and

$$b(t) := b_\varepsilon(t) = \sup_{0<\sigma<\varepsilon} N(\sigma)t^\sigma. \tag{47}$$

Remark 7.2. The function b, defined by (47), is slowly varying on the interval $(0, 1)$.

Proof. Note that the function b is increasing. On the other hand, $b_\varepsilon(t) \approx b_{\varepsilon/2}(t)$, $0 < t < 1$, for $b_\varepsilon(t) \approx \sup_{0<\sigma<\varepsilon} N(\sigma/2)t^\sigma = \sup_{0<\sigma<\varepsilon/2} N(\sigma)t^{2\sigma} \leq \sup_{0<\sigma<\varepsilon/2} N(\sigma)t^\sigma = b_{\varepsilon/2}(t) \leq b_\varepsilon(t)$. Let $\alpha > 0$ be arbitrary and choose an integer k so that $2^{-k}\varepsilon < \alpha$. Then $t^{-\alpha}b_\varepsilon(t) \approx c_{\varepsilon,\alpha}(t)$, where, by definition, $c_{\varepsilon,\alpha}(t) := \sup_{0<\sigma<\varepsilon 2^{-k}} N(\sigma)t^{\sigma-\alpha}$ and this function is decreasing. ∎

In the particular case $r = p$ we have the definition of the *generalized grand Lebesgue space $L_b^{p)}$*:

$$L_b^{p)} := \Delta_{0,\varepsilon}^{(\infty)}(N(\sigma)L^{p-\sigma,p}) = \Delta_{0,\varepsilon}^{(\infty)}(N(\sigma)L^{p-\sigma}).$$

Spaces of this type (namely, the case $N(\sigma) = \sigma^\theta$, $\theta > 0$) have been considered in [16].

Analogously to Theorem 4.2, we have the following characterization of the generalized grand Lorentz spaces.

Theorem 7.3. *Let N be tempered and let b be defined by (47). Then*

$$L_b^{p),r} = (L^q, L^{p,r})_{w,\infty}, \ w(t) = t^{-1}b(t). \tag{48}$$

Moreover, if $\mu(\Omega) = 1$, then

$$\|f\|_{L_b^{p),r}} \approx \sup_{0<t<1} b(t) \left(\int_t^1 \left[s^{\frac{1}{p}} f^*(s) \right]^r ds/s \right)^{\frac{1}{r}}. \tag{49}$$

The proof is analogous to that given for the case of the grand Lebesgue spaces, but now we use the formula

$$(1-\theta)^{\frac{1}{r}}(L^q, L^{p,r})_{\theta,r} = L^{p_\theta,r}, \quad \frac{1}{p_\theta} = \frac{1-\theta}{q} + \frac{\theta}{p}, \ 0 < \theta_0 < \theta < 1, \tag{50}$$

uniformly on θ. Thus we obtain (using Theorem 3.3 of [22] and (5))

$$L_b^{p),r} = \Delta_{0,\varepsilon}^{(\infty)}(M(\sigma)(L^q, L^{p,r})_{1-\sigma,\infty}) = (L^q, L^{p,r})_{w,\infty},$$

where $w(t) = t^{-1}b(t)$. In order to simplify this formula, we use the Holmstedt formula ([2])

$$K(t, f; L^q, L^{p,r}) \approx \left(\int_0^{t^\alpha} \left[f^*(s) \right]^q ds \right)^{\frac{1}{q}} + t \left(\int_{t^\alpha}^1 \left[s^{\frac{1}{p}} f^*(s) \right]^r ds/s \right)^{\frac{1}{r}}, \qquad (51)$$

where $\frac{1}{\alpha} = \frac{1}{q} - \frac{1}{p}$. Since b is slowly varying we see as in the proof of Theorem 4.2 that (49) is valid.

Analogously to Remark 4.3 we can prove the following embeddings.

Remark 7.4. Let $\mu(\Omega) = 1$. Then

$$L_c^{q,\infty} \subset L_b^{q)} \subset L_b^{q,\infty}, \qquad (52)$$

where $c(t) := b(t)(1 - \ln t)^{\frac{1}{q}}$, and

$$L_b^q \subset L_b^{q)} \subset L_{b_\delta}^q, \quad \delta > 0, \qquad (53)$$

where $b_\delta(t) := \sup_{0 < \sigma < \epsilon} N(\sigma)\sigma^\delta t^\sigma$.

7.2. Generalized grand Lorentz sequence spaces. Let $b(n), n \geq 1$ be a sequence of positive numbers. We say that b is *slowly varying* (in the sense of Karamata) if for all $\epsilon > 0$ the sequence $n^\epsilon b(n)$ is equivalent to non-decreasing and the sequence $n^{-\epsilon}b(n)$ is equivalent to non-increasing. Let $0 < q, r \leq \infty$. Then the *Lorentz-Karamata space of sequences* $l_b^{q,r}$ is defined with the following quasi-norm:

$$\|f\|_{l_b^{q,r}} = \left\{ \sum_{n \geq 1} \left[n^{\frac{1}{q} - \frac{1}{r}} b(n) f^*(n) \right]^r \right\}^{\frac{1}{r}}.$$

The results and proofs here are similar to those in the previous subsection. The generalized grand Lorentz spaces of sequences $l_b^{p),r}$ $(0 < p, r < \infty)$ are defined by

$$l_b^{p),r} := \Delta_{0,\varepsilon}^{(\infty)}(N(\sigma)l^{p+\sigma,r}),$$

where N is tempered and

$$b(n) := \sup_{0 < \sigma < \varepsilon} N(\sigma)n^{-\sigma}. \qquad (54)$$

We can characterise $l_b^{p),r}$ as an interpolation space.

Theorem 7.5. *Let N be tempered and let b be defined by* (54)*. Then*

$$l_b^{p),r} = (l^{p,r}, l^\infty)_{b,\infty}. \tag{55}$$

Moreover,

$$\|f\|_{l_b^{p),r}} \approx \sup_{n \geq 1} \ b(n) \left(\sum_{j=1}^n \left[j^{\frac{1}{p} - \frac{1}{r}} f^*(j) \right]^r \right)^{\frac{1}{r}}.$$

Proof. The procedure is similar to that used in the previous subsection. We need the formula

$$\theta^{\frac{1}{r}}(l^{p,r}, l^\infty)_{\theta,r} = l^{p_\theta, r}, \quad \frac{1}{p_\theta} = \frac{1 - \theta}{p}, \quad 0 < \theta < \theta_1 < 1, \tag{56}$$

in the sense of equivalent quasi-norms, the equivalence constants being uniform with respect to θ. To see this, we write

$$l^{p,r} = (l^q, l^\infty)_{\eta,r}, \quad \frac{1}{p} = \frac{1 - \eta}{q},$$

and use the Holmstedt formula

$$K^r(t, f; l^{p,r}, l^\infty) \approx \int_0^{t^{\frac{1}{\alpha}}} u^{-\eta r} K^r(u, f) \, du/u, \quad \alpha = 1 - \eta.$$

where $K(u, f) = K(u, f; l^q, l^\infty)$. Then routine computations show that

$$\|f\|_{(l^{p,r}, l^\infty)_{\theta,r}} \approx \{\theta(1 - \eta)\}^{-\frac{1}{r}} \|f\|_{(l^q, l^\infty)_{(1 - \eta)\theta + \eta, r}},$$

and (56) follows. Now we can write

$$\Delta_{0,\varepsilon}^{(\infty)} \left(N(\sigma) l^{p + \sigma, r} \right) = \Delta_{0,\varepsilon}^{(\infty)} \left(N(\sigma) \sigma^{\frac{1}{r}} (l^{p,r}, l^\infty)_{\frac{\sigma}{p + \sigma}, r} \right),$$

and by Theorem 3.3 of [22] this is the same as

$$\Delta_{0,\varepsilon}^{(\infty)} \left(N(\sigma) (l^{p,r}, l^\infty)_{\sigma,\infty} \right) = (l^{p,r}, l^\infty)_{b,\infty},$$

as desired. ∎

7.3. Grand spaces of compact operators. Let $T : B_1 \to B_2$ be a linear bounded operator between two Banach spaces B_1 and B_2. We denote by $\mathcal{R}$ the space of all such operators with the operator norm $\|T\|$. If $a_n(T)$, $n \geq 1$ are the approximation numbers of T (see [2, 9, 17]), we consider the space $S_b^{p,r}$ with the quasi-norm $\|T\|_{S_b^{p,r}} := \|\{a_n(T)\}\|_{l_b^{p,r}}$. The spaces S^p, $0 < p \leq \infty$ have the quasi-norm $\|T\|_{S^p} := \|\{a_n(T)\}\|_{l^p}$.

We define the *grand spaces of compact operators* $S_b^{p),r}$, $(0 < p, r < \infty)$ by

$$S_b^{p),r} := \Delta_{0,\varepsilon}^{(\infty)}(N(\sigma) S^{p + \sigma, r}),$$

where N is tempered and b is given by (54). We can characterize $S_b^{p),r}$ as an interpolation space. Analogously to Theorem 7.5 we have

Theorem 7.6. *Let N be tempered and let b be defined by* (54). *Then*

$$S_b^{p),r} = (S^{p,r}, S^\infty)_{b,\infty}, \tag{57}$$

and moreover,

$$\|T\|_{S_b^{p),r}} \approx \sup_{n \geq 1} \ b(n) \left(\sum_{j=1}^{n} \left[j^{\frac{1}{p} - \frac{1}{r}} a_j(T) \right]^r \right)^{\frac{1}{r}}.$$

Indeed, we only have to apply the procedure from the previous subsection and use the formulas:

$$K(n, T; S^q, \mathcal{R}) \approx K(n, \{a_n(T)\}; l^q, l^\infty),$$

hence

$$S^{p,r} = (S^q, \mathcal{R})_{\eta,r}, \quad \frac{1}{p} = \frac{1 - \eta}{q}.$$

8. Generalized small spaces

8.1. Generalized small Lorentz function spaces.

Definition 8.1. By definition, the *generalized small Lorentz function space* $L_w^{(p,r}$, $1 < p, r < \infty$, is the extrapolation space

$$L_w^{(p,r} := \Sigma_{0,\varepsilon}^{(1)-} \big(M(\sigma) L^{p+\sigma,r} \big),$$

where M is tempered on $(0, \varepsilon)$ such that $1/M$ is bounded and w is defined by (3) and (39).

In the particular case $r = p$ we have the definition of the *generalized small Lebesgue space* $L_w^{(p}$:

$$L_w^{(p} := \Sigma_{0,\varepsilon}^{(1)-} \big(M(\sigma) L^{p+\sigma,p} \big) = \Sigma_{0,\varepsilon}^{(1)-} \big(M(\sigma) L^{p+\sigma} \big).$$

Analogously to Theorem 3.2, we have the following characterization of the generalized small Lorentz spaces.

Theorem 8.2. *Let M be tempered and let w be defined by* (3) *and* (39). *Then*

$$L_w^{(p,r} = (L^{p,r}, L^\infty)_{w,1}, \tag{58}$$

and

$$\|f\|_{L_w^{(p,r}} \approx \int_0^\infty w(t) \left(\int_0^{t^p} \left[s^{\frac{1}{p}} f^*(s) \right]^r ds/s \right)^{\frac{1}{r}} dt/t. \tag{59}$$

Moreover, if $\mu(\Omega) = 1$, then the above integral can be taken only over the interval $(0, 1)$.

Proof. Analogously to (56) we have

$$\theta^{\frac{1}{r}}(L^{p,r}, L^{\infty})_{\theta,r} = L^{p_\theta,r}, \quad \frac{1}{p_\theta} = \frac{1-\theta}{p}, \quad 0 < \theta < \theta_1 < 1. \tag{60}$$

Hence

$$L_w^{(p,r)} := \Sigma_{0,\varepsilon}^{(1)-}\left(M(\sigma)\langle(L^{p,r}, L^{\infty})_{\sigma,r}\rangle\right),$$

and then (58) follows from Theorem 5.2. Finally, (59) follows using the Holmstedt formula for the $K-$functional ([2]):

$$K(t, f : L^{p,r}, L^{\infty}) \approx \left(\int_0^{t^p} \left[s^{\frac{1}{p}} f^*(s)\right]^r ds/s\right)^{\frac{1}{r}}, \tag{61}$$

which completes the proof. ∎

8.2. Generalized small Lorentz sequence spaces.

Definition 8.3. By definition, the *generalized small Lorentz sequence space* $l_h^{(p,r)}$, $1 < p \le \infty, 1 \le r \le \infty$ is the extrapolation space

$$l_h^{(p,r)} := \Sigma_{0,\varepsilon}^{(1)-}\left(M(\sigma)l^{p-\sigma,r}\right), \quad 1 < p < \infty, \tag{62}$$

and

$$l_h^{(\infty,r)} := \Sigma_{0,\varepsilon}^{(1)-}\left(M(\sigma)l^{\frac{1}{\sigma},r}\right), \tag{63}$$

where M is tempered on $(0, \varepsilon)$ such that $1/M$ is bounded and w is defined by (3) and (39), and

$$h(t) := t^{-1} w\left(\frac{1}{t}\right) \tag{64}$$

In the particular case $r = p$ we have the definition of the generalized small Lebesgue space $l_h^{(p}$:

$$l_h^{(p} := \Sigma_{0,\varepsilon}^{(1)-}\left(M(\sigma)l^{p-\sigma,p}\right) = \Sigma_{0,\varepsilon}^{(1)-}\left(M(\sigma)l^{p-\sigma}\right), \quad 1 < p < \infty$$

and

$$l_h^{(\infty} := \Sigma_{0,\varepsilon}^{(1)-}\left(M(\sigma)l^{\frac{1}{\sigma}}\right).$$

We have the following characterization of the generalized small Lorentz sequence spaces.

Theorem 8.4. *Let M be tempered and let h be defined by (64). Moreover, let*

$$h(t) = t^{-1}b(t), \quad \text{where } b(t) \text{ is slowly varying on } (1, \infty). \tag{65}$$

Then

$$l_h^{(p,r)} = (l^1, l^{p,r})_{h,1} \tag{66}$$

and

$$\|f\|_{l_h^{(p,r)}} \approx \sum_{n=1}^{\infty} h(n)\left(\sum_{j=n}^{\infty} \left[j^{\frac{1}{p}-\frac{1}{r}} f^*(j)\right]^r\right)^{\frac{1}{r}}. \tag{67}$$

Proof. Analogously to (50) we have

$$(1-\theta)^{\frac{1}{r}}(l^1, l^{p,r})_{\theta,r} = l^{p_\theta,r}, \quad \frac{1}{p_\theta} = 1 - \theta + \frac{\theta}{p}, \quad 0 < \theta_0 < \theta < 1. \tag{68}$$

Hence

$$l^{p-\sigma,r} = \sigma^{\frac{1}{r}}(l^{p,r}, l^1)_{\alpha\sigma,r}, \quad 1 < p < \infty,$$

where $\alpha \approx 1$, and

$$l^{\frac{1}{\sigma},r} = \sigma^{\frac{1}{r}}(l^{\infty,r}, l^1)_{\alpha\sigma,r}.$$

Therefore, using Theorem 5.2 we get

$$l_h^{(p,r)} = \Sigma_{0,\varepsilon}^{(1)-}\left(M(\sigma)\langle(l^{p,r}, l^1)_{\sigma,r}\rangle\right) = (l^{p,r}, l^1)_{w,1} = (l^1, l^{p,r})_{h,1}.$$

Thus (66) is proved. To prove (67), we use (51) for $1 < p \le \infty$:

$$K(t, f; l^1, l^{p,r}) \approx \int_0^{t^\alpha} f^*(s)ds + t\left(\int_{t^\alpha}^\infty \left[s^{\frac{1}{p}}f^*(s)\right]^r ds/s\right)^{\frac{1}{r}}, \quad \frac{1}{\alpha} = 1 - \frac{1}{p},$$

where $f^*(t)$ is the step function $f^*(t) = f^*(n)$ for $n - 1 < t \le n$. Hence

$$\|f\|_{l_h^{(p,r)}} \approx I + J,$$

where

$$I = \int_0^\infty t^{-\frac{1}{\alpha}}b(t)\int_0^t f^*(s)\,ds\,dt/t$$

and

$$J = \int_0^\infty b(t)\left(\int_t^\infty \left[s^{\frac{1}{p}}f^*(s)\right]^r ds/s\right)^{\frac{1}{r}} dt/t.$$

It is sufficient to prove

$$I \le cJ. \tag{69}$$

Using Fubini and the fact that b is slowly varying, we get

$$I \le c\int_0^\infty t^{-\frac{1}{\alpha}}b(t)f^*(t)dt. \tag{70}$$

On the other hand, the Minkowski inequality gives

$$J \ge c\left\{\int_1^\infty \left(\int_0^\infty t^{-\frac{1}{\alpha}}b\left(\frac{t}{s}\right)f^*(t)dt\right)^r ds/s\right\}^{\frac{1}{r}}.$$

Since b is slowly varying, we have $b(t/s) > c_\varepsilon s^{-\varepsilon}b(t)$, $s > 1$, therefore

$$J \ge c\int_0^\infty t^{-\frac{1}{\alpha}}b(t)f^*(t)dt.$$

From this and (70) we get (69). The theorem is proved. ∎

8.3. Small spaces of compact operators.

Definition 8.5. By definition, the *generalized small Lorentz spaces of compact operators* $S_h^{(p,r}$, $1 < p \leq \infty, 1 \leq r \leq \infty$ is the extrapolation space

$$S_h^{(p,r} := \Sigma_{0,\varepsilon}^{(1)-} \big(M(\sigma) S^{p-\sigma,r} \big), \quad 1 < p < \infty, \tag{71}$$

and

$$S_h^{(\infty,r} := \Sigma_{0,\varepsilon}^{(1)-} \big(M(\sigma) S^{\frac{1}{\sigma},r} \big), \tag{72}$$

where M is tempered on $(0, \varepsilon)$ such that $1/M$ is bounded and w is defined by (3) and (39), and h is given by (64).

In the particular case $r = p$ we have the definition of the *generalized small Lebesgue spaces of compact operators* $S_h^{(p}$:

$$S_h^{(p} := \Sigma_{0,\varepsilon}^{(1)-} \big(M(\sigma) S^{p-\sigma,p} \big) = \Sigma_{0,\varepsilon}^{(1)-} \big(M(\sigma) S^{p-\sigma} \big), \quad 1 < p < \infty$$

and

$$S_h^{(\infty} := \Sigma_{0,\varepsilon}^{(1)-} \big(M(\sigma) S^{\frac{1}{\sigma}} \big).$$

We have the following characterization of the generalized small Lorentz spaces of compact operators, the proof of which is completely analogous to that of Theorem 8.4.

Theorem 8.6. *Let M be tempered and let h be defined by (64) and satisfies (65). Then*

$$S_h^{(p,r} = (S^1, S^{p,r})_{h,1} \tag{73}$$

and

$$\|T\|_{S_h^{(p,r}} \approx \sum_{n=1}^{\infty} h(n) \bigg(\sum_{j=n}^{\infty} \big[j^{\frac{1}{p} - \frac{1}{r}} a_j(T) \big]^r \bigg)^{\frac{1}{r}}. \tag{74}$$

Note the particular case $M(\sigma) = \sigma^{-1}$. Then $h(t) = t^{-1}$ for $t > 1$, hence if we put $S^{(\infty} := S_h^{(\infty}$ we get

$$\|T\|_{S^{(\infty}} \approx \sum_{n=1}^{\infty} n^{-1} a_n(T),$$

which is the Macaev class (see [17, 19]).

References

[1] C. Bennett and K. Rudnick: *On Lorentz-Zygmund spaces.* Diss. Math. 175 (1980), 1 – 72.

[2] J. Berg and J. Löfström: *Interpolation Spaces.* Berlin: Springer-Verlag 1976.

[3] L. Boccardo: *Quelques problémes de Dirichlet avec donnés dans de grands espaces de Sobolev.* C.R. Acad. Sci. Paris Sér. I Math. 325 (1997)12, 1269 – 1272.

[4] Yu. Brudnyi and N. Kruglijak: *Interpolation Functors and Interpolation Spaces.* Vol. 1. Amsterdam: North-Holland 1991.

[5] M. Carro and J. Martín: *Extrapolation theory for the real interpolation method.* Collect. Math. 53 (2002), 165 – 186.

[6] M. Carro: *New extrapolation estimates.* J. Funct. Anal. 174 (2000), 155 – 166.

[7] D. Cruz-Uribe, SFO and M. Krbec: *Localization and extrapolation in Orlicz-Lorentz spaces.* In: Function Spaces, Interpolation Theory and Related Topics (Lund 2000) (eds. M. Cwikel, A. Kufner, L. Persson, G. Sparr). Berlin: de-Gruyter 2002, pp. 273 – 284. .

[8] D. E. Edmunds and G. E. Karadzhov: *Formulae for the distance in some quasi-Banach spaces, and applications* (submitted).

[9] D. E. Edmunds and H. Triebel: *Function Spaces, Entropy Numbers and Differential Operators.* Cambridge: Cambridge Univ. Press 1996.

[10] A. Fiorenza: *Duality and reflexivity in grand Lebesgue spaces.* Collect. Math. 51 (2000), 131 – 148.

[11] A. Fiorenza and M. Krbec: *On an optimal decomposition in Zygmund spaces.* Georgian Math. J. 9 (2002), 271 – 286.

[12] A. Fiorenza and J.M. Rakotoson: *New properties of small Lebesgue spaces and their applications.* Math. Ann. 326 (2003), 543 – 561.

[13] A. Fiorenza and C. Sbordone: *Existence and uniqueness results for solutions of nonlinear equations with right hand side in L^1.* Studia Math 127 (1998)3, 223 – 231.

[14] N. Fusco, P.L. Lions and C. Sbordone: *Sobolev imbedding theorems in borderline cases.* Proc. Amer. Math. Soc. 124 (1996)2, 561 – 565.

[15] L. Greco: *A remark on the equality detDf=DetDf.* Diff. Int. Equations 6 (1993)5, 1089 – 1100.

[16] L. Greco, T. Iwaniec and C. Sbordone: *Inverting the p-harmonic operator.* Manuscripta Math. 92 (1997)2, 249 – 258.

[17] I. C. Gohberg and M. G. Krein: *Introduction to the theory of nonselfadjoint operators.* In: Transl. of Math. Monographs Vol. 18, AMS 1969.

[18] B. Jawerth and M. Milman: *Extrapolation theory with applications.* Mem. Amer. Math. Soc. 440, 1991.

[19] B. Jawerth and M. Milman: *New results and applications of extrapolation theory.* In: Interpolation Spaces and Related Topics (Haifa 1990). Israel Math. Conf. Proc. 5. Ramat Gan: Bar-Ilan Univ. 1992, pp. 81 – 105.

[20] T. Iwaniec, P. Koskela and J. Onninen: *Mappings of finite distortion: Monotonicity and Continuity.* Invent. Math. 144 (2001), 507 – 531.

[21] T. Iwaniec and C. Sbordone: *On the integrability of the Jacobian under minimal hypotheses.* Arch. Rat. Mech. Anal. 119 (1992), 129 – 143.

[22] G. E. Karadzhov and M. Milman: *Extrapolation theory: new results and applications.* Preprint no. 1. Institute of Mathematics and Informatics, Bulgarian Academy of Science 2002.

[23] M. Krbec and H-J. Schmeisser: *On extrapolation of Sobolev and Morrey type imbeddings.* In: Function Spaces (Poznan 1998). Lecture Notes in Pure and Appl. Math. 213. New York: Dekker 2000, pp. 297 – 321.

[24] M. Milman: *Extrapolation and Optimal Decompositions With Applications to Analysis.* Lecture Notes in Math. 1580. New York: Springer Verlag 1994.

[25] M. Milman: *A note on extrapolation theory.* Journal of Math. Anal. Appl. (to appear).

[26] J. Neves: *On decompositions in generalised Lorentz-Zygmund spaces.* Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. 8 (2001)4, 239 – 267.

[27] J. Neves: *Extrapolation results on general Besov-Hölder-Lipschitz spaces.* Math. Nachr. 230 (2001), 117 – 141.

[28] J.M. Rakotoson: *Some new applications of the pointwise relations for the relative rearrangement.* Adv. in Differ. Equat. 7 (2002), 617 – 640.

[29] C. Sbordone: *Grand Sobolev spaces and their applications to variational problems.* Le Matematiche 51 (1996)2, 335 – 347 (1997).

[30] B. Stroffolini: *A stability result for p-harmonic systems with discontinuous coefficients.* Electronic Journal of Diff. Equations 2 (2001), 1 – 7.

[31] S. Yano: *An extrapolation theorem.* J. Math. Soc. Japan, 3(1951), 296 – 305.