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Banach Frames for Conjugate Banach Spaces

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Abstract. Retro Banach frames for conjugate Banach spaces have been introduced and studied. It has been proved that a Banach space E is separable if and only if E^* has a retro Banach frame. Finally, a necessary and sufficient condition for a sequence in a separable Banach space to be a retro Banach frame has been given.

Keywords: *Banach frames, frames*

MSC 2000: Primary 42C15, 42A38, secondary 46B15

1. Introduction

In 1952, Duffin and Schaeffer [5] abstracted the fundamental notion of Gabor for studying signal processing. In the process they defined frames for Hilbert spaces. In 1986, Daubechies, Grossmann and Meyer [4] found a new application to Wavelet and Gabor's transforms in which frames continue to play an important role. Gröchenig [8] generalized frames for Banach spaces and called them *atomic decompositions*. He is also credited for the introduction of a more general concept for Banach spaces called a *Banach frame*. Banach frames were further studied in [1, 2, 3, 7, 9, 10, 11].

In the present paper, we introduce *retro Banach frames* for conjugate Banach spaces and observe that if E^* has a retro Banach frame, then E has a Banach frame. The converse need not be true (Example 3.4). Among other results, it has been proved that the conjugate Banach space E^* of a Banach space E has a retro Banach frame if and only if E is separable (see the Theorem 3.1).

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2. Preliminaries and lemmas

Throughout E will denote a Banach space over the scalar field $\mathbb{K}(\mathbb{R}$ or $\mathbb{C})$, E^* and E^{**}, respectively, the first and the second conjugate spaces of E, π the canonical isomorphism of E into E^{**}, $[x_n]$ the closed linear span of $\{x_n\}$ in the norm topology of E, E_d and $(E^*)_d$, respectively, the associated Banach spaces of the scalar-valued sequences indexed by $\mathbb{N}$.

A sequence $\{x_n\}$ in E is said to be complete if $[x_n] = E$, and a sequence $\{f_n\}$ in E^* is said to be total over E if $\{x \in E : f_n(x) = 0, n \in \mathbb{N}\} = \{0\}$.

Definition. ([8]) Let E be a Banach space and E_d an associated Banach space of scalar-valued sequences, indexed by $\mathbb{N}$. Let $\{f_n\} \subset E^*$ and $S : E_d \to E$ be given. The pair $(\{f_n\}, S)$ is called a *Banach frame for E with respect to E_d* if

(i) $\{f_n(x)\} \in E_d$, for each $x \in E$

(ii) there exist positive constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E\,, \qquad x \in E \tag{2.1}$$

(iii) S is a bounded linear operator such that $(\{f_n(x)\}) = x$ for $x \in E$.

The positive constants A and B, respectively, are called the *lower* and the *upper frame bounds* of the Banach frame $(\{f_n\}, S)$. The operator $S : E_d \to E$ is called the *reconstruction operator* (or, the *pre-frame operator*). The inequality (2.1) is called the *frame inequality.*

The Banach frame $(\{f_n\}, S)$ is called *tight* if $A = B$ and *normalized tight* if $A = B = 1$. If removal of one f_n renders the collection $\{f_n\} \subset E^*$ no longer a Banach frame for E, then $(\{f_n\}, S)$ is called an *exact Banach frame.*

The following results, which are refered in this paper are listed in the form of lemmas.

Lemma 2.1. ([12]) *If E is a Banach space and $\{f_n\} \subset E^*$ is total over E, then E is linearly isometric to the BK-space $\{\{f_n(x)\} : x \in E\}$, where the norm is defined by $\|\{f_n(x)\}\| = \|x\|_E$, $x \in E$.*

Lemma 2.2. ([6]) *If $g, f_1, f_2, \ldots, f_n$ are any $n+1$ linear functionals on a linear space X, and if $f_i(x) = 0$ for $i = 1, 2, \ldots, n$, implies $g(x) = 0$, then g is a linear combination of the f_i.*

3. Main results

Definition. Let E be a Banach space and E^* be its conjugate space. Let $(E^*)_d$ be a Banach space of scalar-valued sequences associated with E^* indexed by $\mathbb{N}$. Let $\{x_n\} \subset E$ and $T : (E^*)_d \to E^*$ be given. The pair $(\{x_n\}, T)$ is called a *retro Banach frame* for E^* with respect to $(E^*)_d$ if

(i) $\{f(x_n)\} \in (E^*)_d$ for each $f \in E^*$

(ii) there exist positive constants A and B with $0 < A \leq B < \infty$ such that

$$A\|f\|_{E^*} \leq \|\{f(x_n)\}\|_{(E^*)_d} \leq B\|f\|_{E^*}, \qquad f \in E^* \tag{3.1}$$

(iii) T is a bounded linear operator such that $T(\{f(x_n)\}) = f$, $f \in E^*$.

The positive constants A and B, respectively, are called the *lower* and the *upper frame bounds* of the retro Banach frame $(\{x_n\}, T)$. The operator $T : (E^*)_d \to E^*$ is called the *reconstruction operator* (or, the *pre-frame operator*). The inequality (3.1) is called the *retro frame inequality.*

The retro Banach frame $(\{x_n\}, T)$ is called *tight* if $A = B$ and is called *normalized tight* if $A = B = 1$. If removal of one x_n renders the collection $\{x_n\} \subset E$ no longer a retro Banach frame for E^*, then $(\{x_n\}, T)$ is called an *exact retro Banach frame.*

In the following example, we show that the sequence of unit vectors in $E = \ell^p$ is a retro Banach frame for E^*.

Example 3.1. Let $E = \ell^p$, if $1 \leq p < \infty$ and let $\{e_n\}$ be the sequence of unit vectors in E. Let π be the canonical isomorphism of E into E^{**}. Put $\phi_n = \pi(e_n)$, $n \in \mathbb{N}$. Then $\{\phi_n\} \subset E^{**}$ is such that $\{f \in E^* : \phi_n(f) = 0, n \in \mathbb{N}\} = \{0\}$. Therefore, by Lemma 2.1 there exists a Banach space $(E^*)_d = \{\{f(e_n)\} : f \in E^*\}$ with norm given by $\|\{f(e_n)\}\|_{(E^*)_d} = \|f\|_{E^*}$. Define $T : (E^*)_d \to E^*$ by $T(\{f(e_n)\}) = f$, $f \in E^*$. Then T is bounded linear operator such that $(\{e_n\}, T)$ is a retro Banach frame for E^* with frame bounds $A = B = 1$.

Next we construct a sequence in E which is not a retro Banach frame for E^*.

Example 3.2. Let $\{x_n\} \subset E$ be a Schauder basis for E. Define $\{y_n\} \subset E$ by $y_1 = x_1$, $y_2 = 2x_1$ and $y_n = x_n$, $n \geq 3$. Then there exists no bounded linear operator T such that $(\{y_n\}, T)$ is a retro Banach frame for E^*, since otherwise $[y_n] = E$.

It is easy to observe that if $(\{x_n\}, T)$ $(\{x_n\} \subset E, T : (E^*)_d \to E^*)$ is a retro Banach frame for E^* with respect to $(E^*)_d$, then $(\{\pi(x_n)\}, T)$ is a Banach frame for E^* with respect to $(E^*)_d$. Towards the converse, we see that if $(\{\phi_n\}, S)$ is a Banach frame for E^*, then there exists in general no $\{x_n\} \subset E$ associated with $\{\phi_n\}$ such that $(\{x_n\}, S)$ is a retro Banach frame for E^* (see the following example).

Example 3.3. Let $E = c_0$ and let $\{\phi_n\}$ be a sequence of unit vectors in E^{**}. Define a sequence $\{g_n\} \subset E^{**}$ by

$$\begin{cases} g_1(f) = \phi_1(f) + \displaystyle\sum_{j=2}^{\infty} (-1)^j \phi_j(f), & f \in E^* \\ g_n = \phi_n & n = 2, 3, \ldots . \end{cases}$$

By Lemma 2.1, there exists a bounded linear operator $T : \{\{g_n(f)\} : f \in E^*\} \to E^*$ such that $(\{g_n\}, T)$ is a Banach frame for E^*. But $(\{g_n\}, T)$ is not a retro Banach frame for E^* since $g_1 = (1, 1, -1, 1, -1, 1, \ldots) \notin c_0$.

Further, if $(\{\phi_n\}, T)$ $(\{\phi_n\} \subset E^{**}, T : (E^*)_d \to E^*)$ is a Banach frame for E^* with respect to $(E^*)_d$, then E^* has a retro Banach frame with same reconstruction operator T if each ϕ_n is weak*-continuous.

Remark. We may observe that if E^* has a Retro Banach frame, then E has a Banach frame. The converse need not be true (see the following example).

Example 3.4. Let $E = \ell^\infty$. Define $\{f_n\} \subset E^*$ by $f_n(x) = \xi_n$, $x = \{\xi_j\} \in E$. Then, there is a bounded linear operator $U : \{\{f_n(x)\} : x \in E\} \to E$ such that $(\{f_n\}, U)$ is a Banach frame for E with respect to $\{\{f_n(x)\} : x \in E\}$. But E^* has no retro Banach frame (Theorem 3.1).

We now give the following characterization of retro Banach frames.

Theorem 3.1. *Let E be a Banach space. Then E^* has a retro Banach frame if and only if E is separable.*

Proof. Let $(\{x_n\}, T)(\{x_n\} \subset E, T : (E^*)_d \to E^*)$ be a retro Banach frame for E^* with respect to $(E^*)_d$ and with frame bounds A and B. Then, for each $f \in E^*$,

$$A\|f\|_{E^*} \leq \|\{f(x_n)\}\|_{(E^*)_d} \leq B\|f\|_{E^*} . \tag{3.2}$$

Suppose E is not separable. Then $[x_n] \neq E$. Therefore there exists a non-zero functional $g \in E^*$ such that $g(x_n) = 0$, $n \in \mathbb{N}$. Then, retro frame inequality (3.2) gives $g = 0$. This is a contradiction.

Conversely, let $\{x_n\} \subset E$ be a sequence such that $[x_n] = E$. Put $\phi_n = \pi(x_n)$, $n \in \mathbb{N}$. Then $\{\phi_n\} \subset E^{**}$ is total over E^*. Therefore, by Lemma 2.1, there exists a bounded linear operator $T : \{\{\phi_n(f)\} : f \in E^*\} \to E^*$ such that $(\{\phi_n\}, T)$ is a Banach frame for E^*. Hence $(\{x_n\}, T)$ is a retro Banach frame for E^*. ∎

Remark. In view of Theorem 3.1, one may observe that if E^* has an atomic decomposition, then E is separable and hence E^* has a retro Banach frame. The converse need not be true. Indeed, if $E = \ell^1$, then E^* has a retro Banach frame but E^* has no atomic decomposition.

The theorem below proves that the linear homeomorphic image of a retro Banach frame is a retro Banach frame.

Theorem 3.2. *Let $(\{x_n\}, T)(\{x_n\} \subset E, T : (E^*)_d \to E^*)$ be a retro Banach frame for E^* with respect to $(E^*)_d$ and with best bounds A_1 and B_1.*

Let F be a Banach space and $U : E \to F$ be a linear homeomorphism, then $(\{U(x_n)\}, (U^{-1})^*T)$ is retro Banach frame for F^* with respect to $(E^*)_d$ and with best bounds A_2, B_2, satisfying

$$\begin{aligned} A_1\|(U^{-1})^*\|^{-1} &\leq A_2 \leq A_1\|U^*\| \\ B_1\|(U^{-1})^*\|^{-1} &\leq B_2 \leq B_1\|U^*\| \,. \end{aligned}$$

Proof. Since $(\{x_n\}, T)$ is a retro Banach frame with bounds A_1 and B_1; for each $f \in E^*$, we have

$$A_1\|f\|_{E^*} \leq \|\{f(x_n)\}\|_{(E^*)_d} \leq B_1\|f\|_{E^*} \,. \tag{3.3}$$

Let $g \in F^*$. Since $U : E \to F$ is a linear homeomorphism, $g = (U^{-1})^*f$, for some $f \in E^*$. Then $g \in F^*$ such that

$$\{f(x_n)\} = \{f(U^{-1}U(x_n))\} = \{(U^{-1})^*f(U(x_n))\} = \{g(U(x_n))\} \,. \tag{3.4}$$

So $\{g(Ux_n)\} \in (E^*)_d$, $g \in F^*$. Also

$$\|g\|_{F^*} = \|(U^{-1})^*f\|_{F^*} \leq \|(U^{-1})^*\| \, \|f\|_{E^*} \,. \tag{3.5}$$

Therefore, by (3.3), (3.4) and (3.5), we have

$$A_1\|(U^{-1})^*\|^{-1}\|g\|_{F^*} \leq \|\{g(U(x_n))\}\|_{(E^*)_d} \leq B_1\|U^*\| \, \|g\|_{F^*}, \quad g \in F^* \,.$$

Further, the operator $(U^{-1})^*T : (E^*)_d \to F^*$ defined by $(U^{-1})^*T(\{g(Ux_n)\}) = g$, $g \in F^*$, is a bounded linear operator. Hence $(\{U(x_n)\}, (U^{-1})^*T)$ is a retro Banach frame for F^* with respect to $(E^*)_d$ and with frame bounds $A_1\|(U^{-1})^*\|^{-1}$ and $B_1\|U^*\|$.

Since the constants A_2 and B_2 are the best bounds for the retro Banach frame $(\{U(x_n)\}, (U^{-1})^*T)$, we have

$$A_1\|(U^{-1})^*\|^{-1} \leq A_2, \quad B_2 \leq B_1\|U^*\| \,. \tag{3.6}$$

Now

$$\|f\|_{E^*} = \|U^*g\|_{E^*} \leq \|U^*\| \, \|g\|_{F^*} \,.$$

So,

$$\begin{aligned} A_2\|U^*\|^{-1}\|f\|_{E^*} &\leq A_2\|g\|_{F^*} \\ &\leq \|\{g(U(x_n))\}\|_{(E^*)_d} \qquad (= \|\{f(x_n)\}\|_{(E^*)_d}) \\ &\leq B_2\|g\|_{F^*} \\ &\leq \big(B_2\|(U^{-1})^*\|\big) \, \|f\|_{E^*} \,. \end{aligned}$$

This gives

$$A_1 \geq A_2\|U^*\|^{-1} \qquad \text{and} \qquad B_1 \leq B_2\|(U^{-1})^*\| \,. \tag{3.7}$$

Hence, by (3.6) and (3.7), the result follows. ∎

Next, we show that the coefficient mapping associated with a retro Banach frame for E^* is a topological isomorphism onto a closed subspace of $(E^*)_d$.

Theorem 3.3. *Let $(\{x_n\}, T)(\{x_n\} \subset E, T : (E^*)_d \to E^*)$ be a retro Banach frame for E^* with respect to $(E^*)_d$ and with frame bounds A and B. Then, the coefficient mapping $S : E^* \to (E^*)_d$ defined by $S(f) = \{f(x_n)\}$, $f \in E^*$ is a topological isomorphism onto a closed subspace of $(E^*)_d$ with $\|S\| \leq B$ and $\|S^{-1}\| \leq \frac{1}{A}$, where S^{-1} is defined on the range $R(S)$.*

Proof. Since $(\{x_n\}, T)$ is a retro Banach frame for E^* with respect to $(E^*)_d$ and with frame bounds A and B, $\|S\| \leq B$. Let $f \in \ker S$. Then $S(f) = 0$. This gives $f(x_n) = 0$, $n \in \mathbb{N}$. Then, by retro frame inequality, $f = 0$. Thus S is an injective bounded linear mapping from E^* onto $R(S)$. Therefore S^{-1} exists on $R(S)$ and $\|S^{-1}\| \leq \frac{1}{A}$. In order to show that $R(S)$ is closed, let $\{\alpha_n\} \subset R(S)$ be a sequence converging to say α in $(E^*)_d$. Let $\{g_n\} \subset E^*$ be such that $S(g_n) = \alpha_n$, $n \in \mathbb{N}$. Then $\{S(g_n)\}$ is Cauchy sequence in $(E^*)_d$ and so by continuity of S^{-1}, $\{g_n\}$ is a Cauchy sequence in E^*. Then $\lim_{n\to\infty} g_n = g$ exists in E^*. Therefore, by the continuity of S, $\lim_{n\to\infty} S(g_n) = S(g)$. Hence $\alpha = S(g) \in R(S)$. ∎

The following theorem gives a necessary and sufficient condition for a sequence in a separable Banach space to be a retro Banach frame.

Theorem 3.4. *Let $(\{x_n\}, T)(\{x_n\} \subset E, T : (E^*)_d \to E^*)$ be a retro Banach frame for E^* with respect to $(E^*)_d$ and with frame bounds A_x and B_x. Let $\{y_n\} \subset E$. Then, there is a reconstruction operator U such that $(\{y_n\}, U)$ is a retro Banach frame for E^* with respect to $(E^*)_d$ if and only if there exists a constant $\lambda > 0$ such that*

$$\|L(\{f(x_n)\})\|_{(E^*)_d} \geq \lambda \|\{f(x_n)\}\|_{(E^*)_d} ,$$

where $L : (E^*)_d \to (E^*)_d$ be a bounded linear operator given by $L(\{f(x_n)\}) = \{f(y_n)\}$, $f \in E^*$.

Proof. If $(\{y_n\}, U)$ is a retro Banach frame for E^* with respect to $(E^*)_d$ and with frame bounds A_y and B_y, then

$$A_y \|f\|_{E^*} \leq \|\{f(y_n)\}\|_{(E^*)_d} \leq B_y \|f\|_{E^*}, \qquad f \in E^* .$$

This gives $\|L(\{f(x_n)\})\|_{(E^*)_d} \geq \lambda \|\{f(x_n)\}\|_{(E^*)_d}$, where $\lambda = \frac{A_y}{B_x}$.

Conversely, for each $f \in E^*$,

$$\|\{f(y_n)\}\|_{(E^*)_d} \geq \lambda \|\{f(x_n)\}\|_{(E^*)_d} \geq \lambda A_x \|f\|_{E^*}$$

and

$$\|\{f(y_n)\}\|_{(E^*)_d} = \|L(\{f(x_n)\})\|_{(E^*)_d} \le \|L\|\, B_x\, \|f\|_{E^*}\,.$$

Define $U : (E^*)_d \to E^*$ by $U(\{f(y_n)\}) = f$, $f \in E^*$. Then U is a bounded linear operator such that $(\{y_n\}, U)$ is a retro Banach frame for E^* with respect to $(E^*)_d$. ∎

In the concluding result of the paper, we prove that the conjugate Banach space of a separable Banach space always have a normalized tight and exact retro Banach frame.

Theorem 3.5. *If E^* has a retro Banach frame, then E^* has a normalized tight retro Banach frame as well as a normalized tight and exact retro Banach frame.*

To prove the theorem we need the following lemma.

Lemma 3.1. *Let $(\{x_n\}, T)$ $(\{x_n\} \subset E, T : (E^*)_d \to E^*)$ be a retro Banach frame for E^* with respect to $(E^*)_d$. Then $(\{x_n\}, T)$is exact if and only if $x_n \notin [x_i]_{i \neq n}$.*

Proof. Suppose that $(\{x_n\}, T)$ is exact. Then for each $n \in \mathbb{N}$, there exists no bounded linear operator T_0 such that $(\{x_n\}_{i\neq n}, T_0)$ is a retro Banach frame for E^*. Therefore, by retro frame inequality $[x_i]_{i\neq n} \neq E$. Hence $x_n \notin [x_i]_{i\neq n}$.

Conversely, let $x_n \notin [x_i]_{i\neq n}$ and let $(\{x_n\}, T)$ be not exact. Then there exists a bounded linear operator T_1 defined by $T_1(\{f(x_i)\}_{i\neq n}) = f$, $f \in E^*$ such that $(\{x_i\}_{i\neq n}, T_1)$ is a retro Banach frame for E^*. Therefore, by retro Banach frame inequality, $[x_i]_{i\neq n} = E$. This gives $x_n \in [x_i]_{i\neq n}$, a contradiction.

Proof of Theorem 3.5. Let $(\{x_n\}, T)$ $(\{x_n\} \subset E, T : (E^*)_d \to E^*)$ be a retro Banach frame for E^* with respect to $(E^*)_d$. Put $\phi_n = \pi(x_n)$, $n \in \mathbb{N}$. Then $\{\phi_n\} \subset E^{**}$ is total over E^*. Therefore, by Lemma 2.1 there is a bounded linear operator $U : \{\{f(x_n)\} : f \in E^*\} \to E^*$ given by $U(\{f(x_n)\}) = f$, $f \in E^*$ and $\|\{f(x_n)\}\| = \|f\|_{E^*}$. Thus $(\{x_n\}, U)$ is a normalized tight retro Banach frame for E^* with respect to $\{\{f(x_n)\} : f \in E^*\}$.

Further, we may assume, without loss of generality, that $\{\phi_n\}$ is finitely linearly independent. Then, by Lemma 2.2, for each $n \in \mathbb{N}$, there exists an $f_n \in E^*$ such that $\phi_i(f_n) = 0$, $i = 1, 2, \ldots, n-1$ and $\phi_n(f_n) = 1$. This gives $f_n(x_i) = 0$, $i = 1, 2, \ldots, n-1$ and $f_n(x_n) = 1$. Define $\{y_n\} \subset E$ by

$$y_1 = x_1, \qquad y_n = x_n - \sum_{i=1}^{n-1} f_i(x_n) y_i, \qquad n = 2, 3, \ldots\,.$$

Put $\psi_n = \pi(y_n)$, $n \in \mathbb{N}$. Then $\{\psi_n\}$ is total over E^*. Therefore, by Lemma 2.1 again, there exists a bounded linear operator $V : \{\{f(y_n)\} : f \in E^*\} \to E^*$ such that $(\{y_n\}, V)$ is a normalized tight retro Banach frame for E^* with respect to $\{\{f(y_n)\} : f \in E^*\}$. Further since $y_n \notin [y_i]_{i \neq n}$, by Lemma 3.1 $(\{y_n\}, V)$ is an exact retro Banach frame for E^*. ∎

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