On Some Local Geometric Properties
in Musielak-Orlicz Function Spaces

Henryk Hudzik and Wojciech Kowalewski

Abstract. Criteria for compactly locally uniformly rotund points in Musielak-Orlicz spaces equipped with the Luxemburg and the Orlicz-Amemiya norms are given. Next, criteria for compact local uniform rotundity and local uniform rotundity of the spaces for both norms are deduced. These properties are important because, for any Banach space X, both of them imply the Kadec-Klee property and this property, together with reflexivity, is equivalent to approximative compactness of X (see [9]). Approximative compactness of X gives that any nonempty convex and closed set in X is proximinal in X and the projection $P_A(\cdot)$ from X to A is a continuous operator (see [9], [12]).

Keywords: *Musielak-Orlicz space, Orlicz-Amemiya norm, Luxemburg norm, SU-point, compactly locally uniformly rotund point, compact local uniform rotundity, local uniform rotundity*

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1. Introduction

Denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of natural and real numbers, respectively. Let $(X, \|\ \|)$ be a real Banach space and X^* be its dual space. Let $S(X)$ and $B(X)$ denote the unit sphere and the unit ball of X, respectively. We say that $x^* \in S(X^*)$ is a support functional at $x \in X \setminus \{0\}$ if $\|x^*\| = 1$ and $x^*(x) = \|x\|$. The set of all support functionals at $x \in X \setminus \{0\}$ is denoted by $Grad(x)$. A point $x \in S(X)$ is said to be an *exposed point* (of $B(X)$) if there exists $x^* \in Grad(x)$ such that $x^* \notin Grad(y)$ whenever $y \in S(X)$ and $y \neq x$.

A point $x \in S(X)$ is said to be a point of *compact local uniform rotundity* (*local uniform rotundity*) (*CLUR*-point, (*LUR*-point) for short) (of $B(X)$) if for any sequence $(x_n)_{n=1}^{\infty}$ in $S(X)$ such that $\|x_n + x\| \to 2$, we have that (x_n) is a relatively compact set in S(X) (resp. $\|x_n - x\| \to 0$). If every $x \in S(X)$ is a *CLUR*-point (*LUR*-point), then we say that X is a *compactly locally uniformly rotund* (*locally uniformly rotund*) space – $X \in (CLUR)$ ($X \in (LUR)$) for short.

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Let (T, Σ, μ) be a nonatomic, complete and σ-finite measure space and L^o be the space of all σ-measurable real functions defined on T. A Banach space X is called a *Köthe space*, if it is a subspace of L^o such that

1° if $x \in L^o$, $y \in X$ and $|x(t)| \leq |y(t)|$ for μ-a.e. $t \in T$, then $x \in X$ and $\|x\| \leq \|y\|$

2° there exists $x \in X$ such that $supp(x) = T$, where $supp(x) := \{t \in T : x(t) \neq 0\}$.

This paper concerns Musielak-Orlicz spaces, which are Köthe spaces. Let for any Köthe space X, X_+ denote the positive cone in X.

A Köthe space X is said to be *monotonically complete* if for any sequence (x_n) in X_+ and any $x \in X$ the assumption $x_n \uparrow x$ implies $\|x_n\| \uparrow \|x\|$. The notation $x \uparrow x$ means that $x_n(t) \leq x_{n+1}(t) \leq \cdots \leq x(t)$ and $x_n(t) \to x(t)$ as $n \to \infty$ for μ-a.e $t \in T$. A Köthe space X is said to have the *Fatou property* if for any (x_n) in X with $0 \leq x_n \uparrow x$ and $\sup_n \|x_n\| < \infty$, $\sup_n x_n$ exists in X, $\sup_n x_n = x$ and $\|x_n\| \uparrow \|x\|$.

A function $\Phi : T \times \mathbb{R} \to [0, \infty]$ is said to be a *Musielak-Orlicz function* if Φ has the following properties:

(1) $\Phi(\cdot, u) \in L^o$ for any $u \in \mathbb{R}$;

(2) $\Phi(t, \cdot)$ is even, convex and left continuous on $[0, \infty)$;

(3) $\Phi(t, 0) = 0$, $\Phi(t, u) \to \infty$ as $u \to \infty$ and for μ-a.e. $t \in T$ there exists $u_t > 0$ satisfying $\Phi(t, u_t) < \infty$.

We write $\Phi > 0$ if the Orlicz function $\Phi(t, \cdot)$ vanishes only at zero for μ-a.e. $t \in T$. A function Ψ is called the *complementary function* of Φ in the sense of Young, if

$$\Psi(t, v) = \sup_{u \geq 0} \{u|v| - \Phi(t, u)\} \quad (t \in T, \ v \in \mathbb{R}).$$

Here and in the following "$t \in T$" means that we consider μ-almost all t from T. It is easy to see that Ψ is also a Musielak-Orlicz function. Let $p_-(t, u)$, $p_+(t, u)$ and $q_-(t, u)$, $q_+(t, u)$ denote the left and right derivatives of $\Phi(t, u)$ and $\Psi(t, u)$ at $u \in \mathbb{R}$, respectively.

We have the Young inequality

$$uv \leq \Phi(t, u) + \Psi(t, v) \ (t \in T, \ u, v \geq 0)$$

and

$$uv = \Phi(t, u) + \Psi(t, v) \iff \begin{cases} p_-(t, u) \leq v \leq p_+(t, u) & \text{for fixed } u \text{ or} \\ q_-(t, v) \leq u \leq q_+(t, v) & \text{for fixed } v. \end{cases}$$

Let $I_\Phi : L^o \to [0, \infty]$ be the modular defined by

$$I_\Phi(x) = \int_T \Phi(t, |x(t)|)d\mu \ .$$

The linear space

$$\left\{x \in L^o : I_\Phi(\lambda x) < \infty \text{ for some } \lambda > 0\right\}$$

equipped with the *Luxemburg norm*

$$\|x\|_\Phi = \inf\left\{\lambda > 0 : I_\Phi\left(\tfrac{x}{\lambda}\right) \le 1\right\}$$

or with the *Orlicz-Amemiya norm*

$$\|x\|_\Phi^o = \inf_{k>0}\left\{\frac{1}{k}\left(1 + I_\Phi(kx)\right)\right\}$$

is a Köthe space, denoted by L_Φ or L_Φ^o, respectively. We call them *Musielak-Orlicz function spaces*. These two norms are equivalent, namely

$$\|x\|_\Phi \le \|x\|_\Phi^o \le 2\|x\|_\Phi$$

for any $x \in L_\Phi$. Moreover, the linear subspace

$$\{x \in L^o : I_\Phi(\lambda x) < \infty \text{ for all } \lambda > 0\}$$

equipped with the Luxemburg norm or with the Orlicz norm induced from L_Φ, is a Köthe space and we denote it by E_Φ or E_Φ^o (according to the norm that is considered). It is well known that E_Φ is the subspace of all order continuous elements of L_Φ.

Let T_0 be any infinite subset of T. We say that Φ *satisfies condition* $\Delta_2(T_0)$ ($\Phi \in \Delta_2(T_0)$ for short), if for any $h > 1$ there exists $k > 1$ and a nonnegative function $f \in L^o$ with $\int_{T_0} f(t)d\mu < \infty$ such that

$$\Phi(t, hu) \le k\Phi(t, u) + f(t) \quad (t \in T_0, u \in \mathbb{R}).$$

If $\Phi \in \Delta_2(T)$, we write simply $\Phi_2 \in \Delta_2$. For any $x \in L_\Phi^o \setminus \{0\}$, we define

$$k^*(x) = \inf\left\{k \ge 0 : I_\Psi\left(p_+(k|x|)\right) \ge 1\right\}$$
$$k^{**}(x) = \sup\left\{k \ge 0 : I_\Psi\left(p_+(k|x|)\right) \le 1\right\}$$

and

$$K(x) = \begin{cases} \emptyset & \text{if } k^*(x) = +\infty \\ [k^*(x), k^{**}(x)] & \text{if } k^{**}(x) < +\infty \\ [k^*(x), \infty) & \text{if } k^*(x) < +\infty \text{ and } k^{**}(x) = +\infty . \end{cases}$$

It is clear that $k^*(x) \le k^{**}(x)$ for any $x \in L_\Phi^o \setminus \{0\}$. We will write k^*, k^{**} instead of $k^*(x), k^{**}(x)$, if it is clear which x is considered. It is known that for any $x \in L_\Phi \setminus \{0\}$, we have $\|x\|_\Phi^o = \frac{1}{k}\left(1 + I_\Phi(kx)\right)$ if and only if $k \in K(x)$ (see [17]).

We define that Φ is *upper (lower) affine* at $z \in \mathbb{R}_+ \setminus \{0\}$ if there is $w \in \mathbb{R}_+$ such that $\Phi(w) > \Phi(z)$ ($\Phi(w) < \Phi(z)$) and Φ is affine on the interval $[z, w]$ ($[w, z]$). For $z \in (-\infty, 0)$, the upper (lower) affinity of Φ at z is defined similarly. We introduce the following notations:

$$A_u(t) = \{z \in \mathbb{R} \,:\, \Phi(t, \cdot) \text{ is upper affine at z}\}$$
$$A_l(t) = \{z \in \mathbb{R} \,:\, \Phi(t, \cdot) \text{ is lower affine at z}\}$$
$$AS_u(t) = \{z \in A_u(t) : p^-(t, z) = p^+(t, z)\}$$
$$AS_l(t) = \{z \in A_l(t) : p^-(t, z) = p^+(t, z)\}$$

$$A_u(x) = \{t \in T \,:\, x(t) \in A_u(t)\} \qquad ANS_u(t) = A_u(t) \setminus AS_u(t)$$
$$A_l(x) = \{t \in T \,:\, x(t) \in A_l(t)\} \qquad ANS_l(t) = A_l(t) \setminus AS_l(t)$$

$$A_u^s(x) = \{t \in A_u(x) : x(t) \in AS_u(t)\} \qquad A_u^{ns}(x) = A_u(x) \setminus A_u^s(x)$$
$$A_l^s(x) = \{t \in A_l(x) : x(t) \in AS_l(t)\} \qquad A_l^{ns}(x) = A_l(x) \setminus A_l^s(x).$$

Lemma 1.1 ([8]). *Assume that Φ is a Musielak-Orlicz function such that $\Phi > 0$, $\frac{1}{u}\Phi(t, u) \to 0$ as $u \to 0$ for μ-a.e. $t \in T$ and $\Phi \in \Delta_2$. Then the following conditions are equivalent:*

(i) $\Psi \in \Delta_2$.
(ii) *For any $\varepsilon > 0$ there exist $\xi \in (0, 1)$ and a function $f : T \to \mathbb{R}_+$ such that $I_\Phi(f) < \varepsilon$ and $\Phi(t, \frac{u}{2}) \leq \frac{1-\xi}{2}\Phi(t, u)$ for μ-a.e. $t \in T$ and any $u \geq f(t)$.*

Remark 1.2 ([1]). Under the assumptions from Lemma 1.1, condition (ii) in Lemma 1.1 can be reformulated equivalently in the form:

(ii') There exist $\xi \in (0, 1)$ and a function $f : T \to \mathbb{R}_+$, $I_\Phi(f) < \infty$ such that $\Phi(t, \frac{u}{2}) \leq \frac{1-\xi}{2}\Phi(t, u)$ for μ-a.e. $t \in T$ and any $u \geq f(t)$.

Lemma 1.3 ([14]). *Let Φ be a real-valued Musielak-Orlicz function. Then there exists a sequence $(T_n)_{n=1}^\infty$ of pairwise disjoint, measurable sets of a positive and finite measure such that $T = \bigcup_{n=1}^\infty T_n$ and $\sup\{\Phi(t, u) \,:\, t \in T_n\} < \infty$ for any $n \in \mathbb{N}$ and $u \in \mathbb{R}_+$.*

It follows from Lemma 1.3 that $\chi_{T_n} \in E_\Phi$ for any $n \in \mathbb{N}$. Let us say that a sequence $(x_n)_{n=1}^\infty \in L^o$ converges to $x \in L^o$ *locally in measure* whenever $x_n\chi_A \to x\chi_A$ in measure for any $A \in \Sigma$ with $\mu(A) < \infty$.

Lemma 1.4 ([9]). *Let $x \in L_\Phi$, $(x_n) \subset L_\Phi$ and $\Phi \in \Delta_2$. If $x_n \to x$ locally in measure and $I_\Phi(x_n) \to I_\Phi(x) < \infty$, then $\|x_n - x\|_\Phi \to 0$.*

Lemma 1.5 ([3]). *Assume that Φ is a Musielak-Orlicz function such that $\Phi \in \Delta_2$ and $\Phi > 0$. Then for any $L > 0$ and $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|I_\Phi(x) - I_\Phi(y)| < \varepsilon$ for any $x, y \in L_\Phi$ satisfying $I_\Phi(x) \leq L$, $I_\Phi(y) \leq L$ and $I_\Phi(x - y) < \delta$.*

Lemma 1.6 ([2]). *Let Σ_L denote the σ-algebra of Lebesgue measurable sets in $\mathbb{R}$, $E \in \Sigma_L$, E be closed and bounded. Then there exist two sequences (E_n) and (F_n) in Σ_L such that $E_n, F_n \in \Sigma_L$, $F_n \cap E_n = \emptyset$, $E = F_n \cup E_n$, $\mu(E_n) = \mu(F_n) = \frac{1}{2}\mu(E)$ for any $n \in \mathbb{N}$ and*

$$\lim_{n\to\infty} \int_E v(t)(\chi_{E_n}(t) - \chi_{F_n}(t))dt = 0$$

for any integrable function v.

Lemma 1.7 ([13]). *Let $x^* = \xi_v + \phi$ denote a linear, continuous functional on L_Φ^o such that $\|x^*\| = 1$, where ξ_v denotes the regular functional on L_Φ^o, generating by $v \in L_\Psi$ such that $\xi_v(y) = \int_T v(t)y(t)\,dt$ ($y \in L_\Phi^o$) and ϕ is a singular functional. Then x^* attains its norm at $x \in S(L_\Phi^o)$, that is, $x^* \in Grad(x)$ if and only if for some $k \in K(x)$ the following conditions hold:*

1. $I_\Phi(v) + \|\phi\| = 1$
2. $\|\phi\| = \phi(kx)$
3. $\langle kx, v \rangle = I_\Phi(kx) + I_\Psi(v)$, *i.e.* $v(t) \in \partial\Phi(t, kx(t))$ *for μ-a.e. $t \in T$.*

2. Results

We start with a lemma that will be important to prove our main results.

Lemma 2.1. *Let X denote the space L_Φ or L_Φ^o. Assume that $\Phi > 0$, $\frac{1}{u}\Phi(t, u) \to 0$ as $u \to 0$ for μ-a.e. $t \in T$, $\Phi \in \Delta_2$ and $\Psi \in \Delta_2$. Moreover, assume that $(x_n), (y_n) \subset S(X)$ and $\|x_n + y_n\| \to 2$. Then for any $\varepsilon > 0$ there exist numbers $\delta = \delta(\varepsilon) > 0$ and $n' = n'(\varepsilon) \in \mathbb{N}$ such that, for any $n > n'$ and $E \in \Sigma$, the condition $\|y_n\chi_E\| < \delta$ implies that $\|x_n\chi_E\| < \varepsilon$.*

Proof. Let us fix $\varepsilon > 0$. Since $\Phi \in \Delta_2$, so there exists $\sigma(\varepsilon) > 0$ such that $I_\Phi(x) < \sigma(\varepsilon)$ implies $\|x\|_\Phi < \varepsilon$ (see [3, Lemma 1.4]). Let f and ξ be the function and the constant in Lemma 1.1 for $\frac{\sigma(\varepsilon)}{3}$ instead of ε. Define

$$A_n = \{t \in E : |x_n(t)| \leq f(t)\}.$$

Then $I_\Phi(x_n\chi_{A_n}) \leq I_\Phi(f\chi_{A_n}) < \frac{\sigma(\varepsilon)}{3}$. Let $\delta \in (0, 1)$ be such that the conditions $I_\Phi(x) \leq 1$ and $I_\Phi(y) < \delta$ imply that

$$\left|I_\Phi(x + y) - I_\Phi(x)\right| < \frac{\xi\sigma(\varepsilon)}{6}.$$

Assume that $\|y_n\chi_E\|_\Phi < \delta$. Then $I_\Phi(y_n\chi_E) < \delta$. Hence

$$I_\Phi\left(\tfrac{x_n+y_n}{2}\chi_{E\setminus A_n}\right) \leq I_\Phi\left(\tfrac{x_n}{2}\chi_{E\setminus A_n}\right) + \frac{\xi\sigma(\varepsilon)}{6} \leq \frac{1-\xi}{2}I_\Phi(x_n\chi_{E\setminus A_n}) + \frac{\xi\sigma(\varepsilon)}{6}.$$

Let $n' \in \mathbb{N}$ be such that $I_\Phi\left(\frac{x_n+y_n}{2}\right) > 1 - \frac{\xi\sigma(\varepsilon)}{6}$ for any $n > n'$. Such a number $n' \in \mathbb{N}$ exists because $\|x_n + y_n\|_\Phi \to 1$ as $n \to \infty$ whence, by $\Phi \in \Delta_2$, we have $I_\Phi\left(\frac{x_n+y_n}{2}\right) \to 1$ as $n \to \infty$. By convexity of the function $\Phi(t, \cdot)$ on $[0, \infty]$, we have

$$\frac{\Phi\big(t, x_n(t)\big) + \Phi\big(t, y_n(t)\big)}{2} - \Phi\big(t, \tfrac{x_n(t)+y_n(t)}{2}\big) \geq 0.$$

for μ-a.e. $t \in T$. Therefore, for $n > n'$, we get

$$\frac{\xi\sigma(\varepsilon)}{6} > 1 - I_\Phi\left(\tfrac{x_n+y_n}{2}\right)$$

$$= \int_T \left[\frac{\Phi\big(t, x_n(t)\big) + \Phi\big(t, y_n(t)\big)}{2} - \Phi\big(t, \tfrac{x_n(t)+y_n(t)}{2}\big)\right] dt$$

$$\geq \frac{I_\Phi\big(x_n\chi_{E\setminus A_n}\big) + I_\Phi\big(y_n\chi_{E\setminus A_n}\big)}{2} - I_\Phi\big(\tfrac{x_n+y_n}{2}\chi_{E\setminus A_n}\big)$$

$$\geq \frac{I_\Phi\big(x_n\chi_{E\setminus A_n}\big) + I_\Phi\big(y_n\chi_{E\setminus A_n}\big)}{2} - \frac{1-\xi}{2}I_\Phi\big(x_n\chi_{E\setminus A_n}\big) - \frac{\xi\sigma(\varepsilon)}{6}$$

$$\geq \frac{\xi}{2}I_\Phi\big(x_n\chi_{E\setminus A_n}\big) - \frac{\xi\sigma(\varepsilon)}{6}.$$

Hence $\frac{\xi}{2}I_\Phi\big(x_n\chi_{E\setminus A_n}\big) \leq \frac{\xi\sigma(\varepsilon)}{3}$ which implies $I_\Phi\big(x_n\chi_{E\setminus A_n}\big) \leq \frac{2\sigma(\varepsilon)}{3}$. We get $I_\Phi\big(x_n\chi_E\big) < \sigma(\varepsilon)$ for $n > n'$ since $I_\Phi\big(x_n\chi_{A_n}\big) < \frac{\sigma(\varepsilon)}{3}$. Therefore, $\|x_n\chi_E\|_\Phi < \varepsilon$ for $n > n'$. ∎

Remark 2.2. Let us fix $y_n = x$ for any $n \in \mathbb{N}$ in Lemma 2.1. Then any sequence (x_n), satisfying the assumptions of that Lemma, is norm equi-continuous.

Proof. Let us fix $\varepsilon > 0$ and define $S_n = \bigcup_{i=1}^n T_i$, where $(T_i)_1^\infty$ is the sequence of sets from Lemma 1.3. Then, by the Beppo-Levi Theorem, we get $I_\Phi\big(x\chi_{S_n}\big) \to I_\Phi(x)$, whence $I_\Phi\big(x\chi_{T\setminus S_n}\big) \to 0$ and, by $\Phi \in \Delta_2$, $\|x\chi_{T\setminus S_n}\|_\Phi \to 0$. Then there exists $m \in \mathbb{N}$ such that $\|x\chi_{T\setminus S_m}\|_\Phi < \delta$, where δ is from Lemma 2.1. Therefore, by that Lemma, $\|x_n\chi_{T\setminus S_m}\|_\Phi < \varepsilon$ for any $n > n'$. Let $A = S_m$. Since, by $\Phi \in \Delta_2$, L_Φ is order continuous, so the element $x\chi_A$ is order continuous. Moreover $\mu(A) < \infty$. Let $B \subset A$ and $B \in \Sigma$. Then there exists $\sigma = \sigma(\varepsilon)$ such that, if $\mu(B) < \sigma$, then $\|x\chi_B\|_\Phi < \delta$, and it follows from Lemma 2.1 that $\|x_n\chi_B\|_\Phi < \varepsilon$ for any $n > n'$, which finishes the proof. ∎

Lemma 2.3. *Let us fix $y_n = x$ for any $n \in \mathbb{N}$ in Lemma 2.1. Let (x_n) be the sequence from that Lemma such that, in addition, $x_n \to x$ locally in measure. Then $\|x_n - x\| \to 0$ as $n \to \infty$.*

Proof. It is enough to show this Lemma for the Luxemburg norm. Let us fix $\varepsilon > 0$. We define S_n as in the proof of Remark 2.2. By Lemma 2.1 we get

that there exist $\delta < \frac{\varepsilon}{5}$ and $n' \in \mathbb{N}$ such that for any $E \in \Sigma$, if $\|x\chi_E\| < \delta$, then $\|x_n\chi_E\| < \frac{\varepsilon}{5}$ for any $n > n'$. By the assumption that $\Phi \in \Delta_2$ there exists $m \in \mathbb{N}$ such that $\|x\chi_{T\setminus S_m}\| < \delta$. So we have $\|x_n\chi_{T\setminus S_m}\| < \frac{\varepsilon}{5}$ for any $n' > n$. Since $\mu(S_m) < \infty$, so $x_n \xrightarrow{\mu} x$ in the set S_m. Without loss of generality we can assume that $x_n(t) \to x(t)$ for μ-a.e. $t \in T$. Since (x_n) is norm equi-continuous and $\mu(S_m) < \infty$, so there exist $n(\varepsilon)$ and $a(\varepsilon) > 0$ such that for any $E \in \Sigma$, if $\mu(E) < a(\varepsilon)$, then $\|x_n\chi_E\|_\Phi < \frac{\varepsilon}{5}$ for any $n > n(\varepsilon)$. Since L_Φ is order continuous, so there exists $b(\varepsilon) > 0$ such that for any $E \in \Sigma$, $E \subset S_m$, if $\mu(E) < b(\varepsilon)$, then $\|x\chi_E\|_\Phi < \frac{\varepsilon}{5}$. It follows from the Yegoroff theorem that there exists $A \in \Sigma$ such that $A \subset S_m$, $\mu(A) < min(a(\varepsilon), b(\varepsilon))$ and $x_n - x \to 0$ uniformly in $S_m \setminus A$, i.e. there exists $n_1(\varepsilon) \in \mathbb{N}$ such that $|x_n(t) - x(t)| \le 1$ for any $t \in S_m \setminus A$ and any $n > n_1(\varepsilon)$. Hence $\Phi\big(t, x_n(t) - x(t)\big) \le \Phi\big(t, 1\big)$ for any $t \in S_m \setminus A$. By the Lebesgue dominated convergence theorem, we get $I_\Phi\big((x_n - x)\chi_{S_m\setminus A}\big) \to 0$ as $n \to \infty$. Since $\Phi \in \Delta_2$, so we have $\big\|(x_n - x)\chi_{S_m\setminus A}\big\|_\Phi \to 0$, which means that there exists $n_2(\varepsilon) \ge \max(n(\varepsilon), n_1(\varepsilon))$ such that $\big\|(x_n - x)\chi_{S_m\setminus A}\big\|_\Phi < \frac{\varepsilon}{5}$ for any $n > n_2(\varepsilon)$. Finally,

$$\|x_n - x\|_\Phi \le \big\|(x_n - x)\chi_{T\setminus S_m}\big\|_\Phi + \big\|(x_n - x)\chi_{S_m\setminus A}\big\|_\Phi + \big\|(x_n - x)\chi_A\big\|_\Phi$$
$$< \big\|x_n\chi_{T\setminus S_m}\big\|_\Phi + \big\|x\chi_{T\setminus S_m}\big\|_\Phi + \frac{\varepsilon}{5} + \big\|x_n\chi_A\big\|_\Phi + \big\|x\chi_A\big\|_\Phi < \varepsilon$$

for any $n > n_2(\varepsilon)$. ∎

Lemma 2.4. *Let Φ be a Musielak-Orlicz function and $x \in S(L_\Phi)$. If $\Phi \in \Delta_2$, $x(t) \in Ext(\Phi\big(t, \cdot\big))$ for μ-a.e. $t \in T$ and $\mu(A_l(x))\mu(A_u(x)) > 0$, then x is not a CLUR-point.*

Proof. Let $\|x\|_\Phi = 1$. By $\Phi \in \Delta_2$, we get $I_\Phi\big(x\big) = 1$. Suppose that $\mu(A_l(x))\mu(A_u(x)) > 0$. For any $n \in \mathbb{N}$ we define the sets

$$A_u^n(x) = \left\{ t \in T : \begin{array}{l} x(t) \text{ and } x(t) + \frac{1}{n}sgn\big(x(t)\big) \\ \text{are in the same affinity interval of } \Phi\big(t, \cdot\big) \end{array} \right\}$$

$$A_l^n(x) = \left\{ t \in T : \begin{array}{l} x(t) \text{ and } x(t) - \frac{1}{n}sgn\big(x(t)\big) \\ \text{are in the same affinity interval of } \Phi\big(t, \cdot\big) \end{array} \right\}$$

Then $\bigcup_{n=1}^\infty A_u^n(x) = A_u(x)$, $\bigcup_{n=1}^\infty A_l^n(x) = A_l(x)$ and $A_u^n(x) \uparrow$, $A_l^n(x) \uparrow$. Hence there exists $m \in \mathbb{N}$ such that $\mu(A_u^m(x)) > 0$ and $\mu(A_l^m(x)) > 0$. Without loss of generality we can assume that $\mu(A_u^m(x) \cap A_l^m(x)) = 0$ (because of the fact that $x(t) \in Ext(\Phi\big(t, \cdot\big))$ for μ-a.e. $t \in T$). Since μ is nonatomic, so we can find

measurable sets B_u^m, B_l^m such that $B_u^m \subset A_u^m(x)$, $B_l^m \subset A_l^m(x)$ and

$$
\int_{B_u^m} \left[\Phi\big(t, x(t) + \frac{1}{m} sgn(x(t))\big) - \Phi\big(t, x(t)\big) \right] d\mu
$$
$$
= \int_{B_l^m} \left[-\Phi\big(t, x(t) - \frac{1}{m} sgn(x(t))\big) + \Phi\big(t, x(t)\big) \right] d\mu. \tag{1}
$$

Denote $A = B_u^m$, $B = B_l^m$. Then, from (1), we have

$$
I_\Phi\big((x + \tfrac{1}{m} sgn(x))\chi_A\big) + I_\Phi\big((x - \tfrac{1}{m} sgn(x))\chi_B\big) = I_\Phi\big(x\chi_{A \cup B}\big). \tag{2}
$$

Now denote $A_1^o = A$, $B_1^o = B$. Since the functions

$$
f(t) = \quad \Phi\big(t, x(t) + \frac{1}{m} sgn(x(t))\big) - \Phi\big(t, x(t)\big)
$$
$$
g(t) = -\Phi\big(t, x(t) - \frac{1}{m} sgn(x(t))\big) + \Phi\big(t, x(t)\big)
$$

are nonnegative, measurable and integrable, so they generate on $\Sigma \cap A$ and on $\Sigma \cap B$, respectively, the nonatomic measures $\nu = \nu_f$ and $\kappa = \kappa_g$:

$$
\nu(D) = \int_D \left[\Phi\big(t, x(t) + \tfrac{1}{m} sgn(x(t))\big) - \Phi\big(t, x(t)\big) \right] d\mu \qquad (\forall D \in \Sigma \cap A)
$$
$$
\kappa(D) = \int_D \left[-\Phi\big(t, x(t) - \tfrac{1}{m} sgn(x(t))\big) + \Phi\big(t, x(t)\big) \right] d\mu \quad (\forall D \in \Sigma \cap B).
$$

Hence there exist sets $A_1^1, A_2^1 \in \Sigma \cap A$ and $B_1^1, B_2^1 \in \Sigma \cap B$ such that

$$
\nu(A_1^1) = \nu(A_2^1), \ \kappa(B_1^1) = \kappa(B_2^1), \ A_0^1 = A_1^1 \cup A_2^1, \ B_0^1 = B_1^1 \cup B_2^1.
$$

Then we get

$$
I_\Phi\big((x + \tfrac{1}{m} sgn(x))\chi_{A_1^1}\big) + I_\Phi\big(x\chi_{A_2^1}\big)
$$
$$
= I_\Phi\big((x + \tfrac{1}{m} sgn(x))\chi_{A_2^1}\big) + I_\Phi\big(x\chi_{A_1^1}\big) \tag{3}
$$

$$
I_\Phi\big((x - \tfrac{1}{m} sgn(x))\chi_{B_2^1}\big) + I_\Phi\big(x\chi_{B_1^1}\big)
$$
$$
= I_\Phi\big((x - \tfrac{1}{m} sgn(x))\chi_{B_1^1}\big) + I_\Phi\big(x\chi_{B_2^1}\big) \tag{4}
$$

Let $F := B_2^1 \cup B_1^1 \cup A_2^1 \cup A_1^1 = A \cup B$ and define

$$
x_1 = x\chi_{T \setminus F} + x\chi_{B_2^1} + \left(x - \frac{1}{m} sgn(x) \right)\chi_{B_1^1} + x\chi_{A_1^1} + \left(x + \frac{1}{m} sgn(x) \right)\chi_{A_2^1}.
$$

By equalities (2) - (4) and the convexity of the function Φ, we have

$$I_\Phi(x_1) = I_\Phi\left(\tfrac{x_1+x}{2}\right) = I_\Phi(x) = 1.$$

In the same way we decompose the sets $A_i^n, B_i^n,\ n \geq 1,\ i = 1,\ldots,2^n$, into subsets $A_{2i-1}^{n+1},\ A_{2i}^{n+1},\ B_{2i-1}^{n+1},\ B_{2i}^{n+1}$ such that

$$A = \bigcup_{i=1}^{2^n} A_i^n, \qquad A_{2i-1}^{n+1} \cup A_{2i}^{n+1} = A_i^n$$

$$B = \bigcup_{i=1}^{2^n} B_i^n \qquad B_{2i-1}^{n+1} \cup B_{2i}^{n+1} = B_i^n$$

$$\nu(A_{2i-1}^{n+1}) = \nu(A_{2i}^{n+1}), \quad \nu(A_{2i-1}^{n+1} \cap A_{2i}^{n+1}) = 0$$
$$\kappa(B_{2i-1}^{n+1}) = \kappa(B_{2i}^{n+1}), \quad \kappa(B_{2i-1}^{n+1} \cap B_{2i}^{n+1}) = 0 \ .$$

Defining the sets

$$C_1^n = \bigcup_{k=1}^{2^{n-1}} A_{2k-1}^n, \quad C_2^n = \bigcup_{k=1}^{2^{n-1}} A_{2k}^n, \quad D_1^n = \bigcup_{k=1}^{2^{n-1}} B_{2k-1}^n, \quad D_2^n = \bigcup_{k=1}^{2^{n-1}} D_{2k}^n,$$

we get $\nu(C_1^n) = \nu(C_2^n) = \tfrac{1}{2}\nu(A)$ and $\kappa(D_1^n) = \kappa(D_2^n) = \tfrac{1}{2}\kappa(B)$. Hence

$$\begin{aligned}
I_\Phi\left((x + \tfrac{1}{m}sgn(x))\chi_{C_2^n}\right) &+ I_\Phi\left(x\chi_{C_1^n}\right) \\
&= I_\Phi\left((x + \tfrac{1}{m}sgn(x))\chi_{C_1^n}\right) + I_\Phi\left(x\chi_{C_2^n}\right)
\end{aligned} \tag{5}$$

$$\begin{aligned}
I_\Phi\left((x - \tfrac{1}{m}sgn(x))\chi_{D_2^n}\right) &+ I_\Phi\left(x\chi_{D_1^n}\right) \\
&= I_\Phi\left((x - \tfrac{1}{m}sgn(x))\chi_{D_1^n}\right) + I_\Phi\left(x\chi_{D_2^n}\right)
\end{aligned} \tag{6}$$

Define

$$x_n = x\chi_{T\setminus F} + x\chi_{D_1^n} + (x - \frac{1}{m}sgn(x))\chi_{D_2^1} + x\chi_{C_1^n} + (x + \frac{1}{m}sgn(x))\chi_{C_2^n}.$$

By equalities (5) and (6), we have

$$I_\Phi(x_n) = I_\Phi\left(\tfrac{x_n+x}{2}\right) = 1, \quad \text{whence} \quad \|x_n\|_\Phi = \left\|\tfrac{x_n+x}{2}\right\|_\Phi = 1.$$

Let $n < p$. Then

$$D_1^n = D_1^n \cap B = D_1^n \cap (D_1^p \cup D_2^p) = (D_1^n \cap D_1^p) \cup (D_1^n \cap D_2^p).$$

Hence $D_1^n \setminus D_1^p = D_1^n \setminus (D_1^n \cap D_1^p) = D_1^n \cap D_2^p$. In the same way we can prove that $D_2^n \setminus D_2^p = D_2^n \cap D_1^p$. Moreover, $(D_1^n \setminus D_1^p) \cap (D_2^n \setminus D_2^p) = \emptyset$. Therefore, by symmetry of the decomposition, we have

$$\frac{1}{4}\kappa(B) = \frac{1}{2}\kappa(D_1^n) = \kappa(D_1^n \cap D_1^p) = \kappa(D_1^n \cap D_2^p) = \kappa(D_1^n \setminus D_1^p),$$

whence $\kappa(D_1^n \setminus D_1^p) = \frac{1}{4}\kappa(B)$. Similarly we can show that $\kappa(D_2^n \setminus D_2^p) = \frac{1}{4}\kappa(B)$, whence, by the definition of the measure κ, we get

$$
\begin{aligned}
\frac{1}{4}\kappa(B) &= \kappa(D_1^n \setminus D_1^p) \\
&= I_\Phi\big(x\chi_{D_1^n \setminus D_1^p}\big) - I_\Phi\big((x - \tfrac{1}{m}sgn(x))\chi_{D_1^n \setminus D_1^p}\big) \\
&< I_\Phi\big(x\chi_{D_1^n \setminus D_1^p}\big) \\
&< I_\Phi\big(x\chi_{D_1^n \setminus D_1^p}\big) + I_\Phi\big((x - \tfrac{1}{m}sgn(x))\chi_{D_2^n \setminus D_2^p}\big).
\end{aligned}
\tag{7}
$$

Similarly

$$
\frac{1}{4}\nu(A) < I_\Phi\big(x\chi_{C_1^n \setminus C_1^p}\big) + I_\Phi\big((x + \tfrac{1}{m}sgn(x))\chi_{C_2^n \setminus C_2^p}\big).
\tag{8}
$$

Moreover,

$$
x_n - x_p = x\chi_{D_1^n \setminus D_1^p} + (x - \frac{1}{m}sgn(x))\chi_{D_2^n \setminus D_2^p} + x\chi_{C_1^n \setminus C_1^p} + (x + \frac{1}{m}sgn(x))\chi_{C_2^n \setminus C_2^p}
$$

and

$$
(D_1^n \setminus D_1^p) \cap (D_2^n \setminus D_2^p) \cap (C_1^n \setminus C_1^p) \cap (C_2^n \setminus C_2^p) = \emptyset.
$$

Therefore, by (7) and (8), we get for $n \neq p$

$$
\begin{aligned}
I_\Phi\big(x_n - x_p\big) &= I_\Phi\big(x\chi_{D_1^n \setminus D_1^p}\big) + I_\Phi\big((x - \tfrac{1}{m}sgn(x))\chi_{D_2^n \setminus D_2^p}\big) \\
&\quad + I_\Phi\big(x\chi_{C_1^n \setminus C_1^p}\big) + I_\Phi\big((x + \tfrac{1}{m}sgn(x))\chi_{C_2^n \setminus C_2^p}\big) \\
&> \frac{1}{4}\big[\nu(A) + \kappa(B)\big].
\end{aligned}
$$

This shows that (x_n) has no Cauchy subsequence, which contradicts the assumption that x is a $CLUR$-point. ∎

Theorem 2.5. *Let $\mu(T) < \infty$ and Φ be a Musielak-Orlicz function such that $\Phi > 0$, $\Phi < \infty$, $\frac{1}{u}\Phi(t,u) \to 0$ as $u \to 0$ for μ-a.e. $t \in T$. Then $x \in S(L_\Phi)$ is a $CLUR$-point if and only if the following conditions hold:*

   (i) $\Phi \in \Delta_2$

 (ii) (a) $\mu(A_l(x)) = 0$ *or*

       (b) $\Psi \in \delta_2$ *and* $\mu(A_u(x)) = 0$

(iii) $x(t) \in Ext(\Phi(t,\cdot))$ *for μ-a.e. $t \in T$.*

Proof. Necessity.

(i): The necessity of this condition follows by the fact that any $CLUR$-point is also an H-point and that $\Phi \in \Delta_2$ is necessary so that a point x would be an H-point (see [7]).

(iii): The necessity of this condition can be proved in a similar way as Lemma 2.4.

(ii): First suppose that $\mu(A_l(x))\mu(A_u(x)) > 0$. Then, by Lemma 2.4, we get that x is not a $CLUR$-point.

Now let $x \geq 0$, $\mu(A_l(x)) > 0$ and $\Psi \notin \Delta_2$. Let

$$\tilde{T} = \left\{ t \in T : \ \Phi\left(t, \tfrac{u}{2}\right) > \left(1 - \frac{1}{n}\right)\frac{1}{2}\Phi(t, u) \ \text{ for some } u > 0 \right\}.$$

Let N" be the smallest subset of $\mathbb{N}$ such that $(T \setminus \tilde{T}) \subset \bigcup_{n \in N^"} T_n =: \tilde{A}$, where (T_n) is the sequence from Lemma 1.3. Fixing $n \in \mathbb{N}$, $n \geq 2$, we define

$$f_n(t) = \sup\left\{ u > 0 : \ \Phi\left(t, \tfrac{u}{2}\right) > \left(1 - \frac{1}{n}\right)\frac{1}{2}\Phi(t, u) \right\},$$

where $\sup \emptyset := 0$, which means that $f_n(t) = 0$ for $t \in T \setminus \tilde{T}$. We will show that these functions are Σ-measurable. Let $Q_+ = (w_k)_{k=1}^\infty$ be the set of positive rational numbers. Since the function Φ is continuous, we have

$$f_n(t) = \sup\left\{ u_k \in Q_+ : \ \Phi\left(t, \tfrac{u_k}{2}\right) > \left(1 - \frac{1}{n}\right)\frac{1}{2}\Phi(t, u_k) \right\}.$$

For any fixed $k \in Q_+$ define the set

$$B_k = \left\{ t \in \tilde{T} : \ \Phi\left(t, \tfrac{u_k}{2}\right) > \left(1 - \frac{1}{n}\right)\frac{1}{2}\Phi(t, u_k) \right\}.$$

Next, define $g_{k,l}(t) = u_k \chi_{B_k \cap T_l}$, where (T_l) is the sequence of sets from Lemma 1.3. Then, we have

$$f_n(t) = \sup\{ g_{k,l}(t) : \ k, l \in \mathbb{N} \}.$$

Indeed, it is evident $f_n(t) = 0$ for any $t \in T \setminus \tilde{T}$ and that for any $t \in \tilde{T}$

$$\sup\{ g_{k,l}(t) : \ k, l \in \mathbb{N} \} \leq f_n(t).$$

On the other hand, taking $t \in \tilde{T}$ and $\varepsilon > 0$, one can find $u > 0$ such that $f_n(t) < u + \varepsilon$ and $\Phi\left(t, \tfrac{u}{2}\right) > \left(1 - \frac{1}{n}\right)\frac{1}{2}\Phi(t, u)$. Next, we can find $u_k \in Q_+$ such that $\Phi\left(t, \tfrac{u_k}{2}\right) > \left(1 - \frac{1}{n}\right)\frac{1}{2}\Phi(t, u_k)$ and $f_n(t) < u_k + \varepsilon$. Let $l_k \in \mathbb{N}$ be such that $t \in T_{l_k}$. Then we have $\Phi\left(t, \tfrac{g_{k,l_k}(t)}{2}\right) > \left(1 - \frac{1}{n}\right)\frac{1}{2}\Phi(t, g_{k,l_k}(t))$ and $f_n(t) < g_{k,l_k}(t) + \varepsilon$. Consequently, $\sup\{ g_{k,l}(t) : \ k, l \in \mathbb{N} \} \geq f_n(t)$. It is obvious that the functions $g_{k,l}$ are measurable, so f_n is measurable as well.

By Lemma 1.1 and Remark 1.2, we get that

$$I_\Phi(f_n) = \infty \quad \text{for any } n \geq 2. \tag{9}$$

Since the necessity of $\Phi \in \Delta_2$ has been already proved, we may assume that this condition is satisfied, whence it follows that Φ is a real-valued and consequently continuous function. Define

$$A_k = \left\{ t \in A_l(x) : \ \Phi(t, \cdot) \text{ is affine in } [x(t) - w_k, x(t)] \right\}.$$

Since $A_l(x) = \bigcup_k A_k$, so there exists $k_0 \in \mathbb{N}$ such that $\mu(A_{k_0}) > 0$. Let $a := w_{k_0}$, $A := A_{k_0} \cap \bigcup_{n \in N'} T_n$, where (T_n) is the sequence from Lemma 1.3 and N' is the biggest subset of $\mathbb{N}$ such that $A_{k_0} \supset \bigcup_{n \in N'} T_n$. Without loss of generality, we can assume that $1 \in N'$. Then $\Phi(t, \cdot)$ is affine on $[x(t) - a, x(t)]$ for any $t \in A$, i.e. $\Phi(t, u) = A(t)u + B(t)$ for any $u \in [x(t) - a, x(t)]$, where $A(t) = \frac{1}{a}(\Phi(t, x(t)) - \Phi(t, x(t) - a))$. Let $K := \int_A (\Phi(t, x(t)) - \Phi(t, x(t) - a))d\mu$.

Then

$$I_\Phi((x - a)\chi_A) = \int_A (A(t)(x(t) - a) + B(t))d\mu$$

$$= \int_A \Phi(t, x(t))d\mu - a \int_A A(t)d\mu \qquad (10)$$

$$= I_\Phi(x\chi_A) - K.$$

Without loss of generality (decreasing the set A if necessary) we can assume that

$$I_\Phi(f_n \chi_{T \setminus (A \cup \tilde{A})}) = \infty \quad \text{for any } n \geq 2. \qquad (11)$$

For any $n \in \mathbb{N}$, $n \geq 2$, define

$$C_n = \{t \in \tilde{T} : f_n(t) = +\infty\}, \quad N_f = \{n \geq 2 : \mu(C_n) > 0\}.$$

Let $\overline{N} = N_f \cup N' \cup N"$. We consider two cases.

Case I. If $N_0 = \mathbb{N} \setminus \overline{N}$ and $card(N_0) = \infty$, then, denoting $N_0 = (k_n)_{n=1}^\infty$, we have that

$$I_\Phi(f_{k_n} \chi_{T \setminus (A \cup \tilde{A})}) = \infty \quad \text{for any } n \in \mathbb{N} \qquad (12)$$

and

$$f_{k_n}(t) < \infty \quad \text{for } \mu - a.e \ t \in T \setminus (A \cup \tilde{A}) \ \text{and any } n \in \mathbb{N}. \qquad (13)$$

Define a sequence $(x_{k_n})_{n=1}^\infty$ in the following way. Let N_1 be the smallest subset of N_0 such that $I_\Phi(f_{k_1} \chi_{\bigcup_{n \in N_1} T_{k_n}}) > K$. Then there exists a set $D_1 \subset \bigcup_{n \in N_1} T_{k_n}$ such that $I_\Phi(f_{k_1} \chi_{D_1}) = K$. Let

$$x_1 = (x - a)\chi_A + x\chi_{\bigcup_{n \in N_f \cup N"} T_n} + f_{k_1}\chi_{D_1}.$$

We get $I_\Phi(x_1) = I_\Phi(x\chi_{A \cup \bigcup_{n \in N_f} T_n}) \leq 1$, whence $\|x_1\|_\Phi \leq 1$. By (12) and (13), we have that $I_\Phi(f_{k_n} \chi_{T \setminus (A \cup \bigcup_{n \in N_1} T_{k_n} \cup \bigcup_{n \in N_f} T_n)}) = \infty$ for any $n \in \mathbb{N}$. Let N_2 be the smallest subset of $N_0 \setminus N_1$ such that $I_\Phi(f_{k_2} \chi_{\bigcup_{n \in N_2} T_{k_n}}) > K$. Then there exists a set $D_2 \subset \bigcup_{n \in N_2} T_{k_n}$ such that $I_\Phi(f_{k_2} \chi_{D_2}) = K$. Let

$$x_2 = (x - a)\chi_A + x\chi_{\bigcup_{n \in N_f \cup N"} T_n \cup \bigcup_{n \in N_1} T_{k_n}} + f_{k_2}\chi_{D_2}.$$

We get $I_\Phi(x_2) = I_\Phi\big(x\chi_{A\cup\bigcup_{n\in N_f} T_n\cup\bigcup_{n\in N_1} T_{k_n}}\big) \leq 1$, whence $\|x_2\|_\Phi \leq 1$. In the same way we define for any $n \geq 3$

$$x_n = (x-a)\chi_A + x\chi_{\big(\bigcup_{m\in N_f\cup N"} T_n\cup \bigcup_{m\in\bigcup_{k=1}^{n-1} N_k} T_{km}\big)} + f_{k_n}\chi_{D_n},$$

obtaining that $I_\Phi(x_n) \uparrow I_\Phi(x)$, whence $\|x_n\|_\Phi \uparrow \|x\|_\Phi = 1$. Moreover, denoting $E_n = \bigcup_{m\in N_f} T_n \cup \bigcup_{m\in\bigcup_{k=1}^{n-1} N_k} T_{km}$, we have

$$\frac{x+x_n}{2} = (x-\frac{a}{2})\chi_A + x\chi_{E_n} + \frac{f_{k_n}+x}{2}\chi_{D_n} + \frac{x}{2}\chi_{T\setminus(E_n\cup D_n)}.$$

By the definition of f_{k_n}, (10) and $x \geq 0$, we get

$$
\begin{aligned}
1 \leftarrow \frac{I_\Phi(x) + I_\Phi(x_n)}{2} &\geq I_\Phi\big(\tfrac{x_n+x}{2}\big) \\
&\geq I_\Phi\big(x\chi_{A\cup E_n}\big) - \frac{K}{2} + I_\Phi\big(\tfrac{f_{k_n}}{2}\chi_{D_n}\big) \\
&\geq I_\Phi\big(x\chi_{A\cup E_n}\big) - \frac{K}{2} + \frac{1-\frac{1}{n+1}}{2}I_\Phi\big(f_{k_n}\chi_{D_n}\big) \\
&= I_\Phi\big(x\chi_{A\cup E_n}\big) - \frac{K}{2} + \frac{1-\frac{1}{n}}{2}K \;\rightarrow\; I_\Phi(x) = 1.
\end{aligned}
$$

Therefore $I_\Phi\big(\tfrac{x_n+x}{2}\big) \rightarrow 1$, whence $\big\|\tfrac{x_n+x}{2}\big\|_\Phi \rightarrow 1$. But

$$I_\Phi\big(x_n - x_m\big) \geq I_\Phi\big(f_{k_n}\chi_{D_n}\big) + I_\Phi\big(f_{k_m}\chi_{D_m}\big) = 2K > 0,$$

whence $\|x_n - x_m\|_\Phi \geq \min(2K, 1)$ for any $n \neq m$, which means that there exists no convergent subsequence of (x_n), and so x is not a $CLUR$-point.

    Case II. If $\mathbb{N} \setminus \overline{N} < \infty$ then, we may assume, without loss of generality (passing to a subsequence if necessary) that $\mu(C_n) >$ for any $n \in \mathbb{N} \setminus N'$. We also may assume without loss of generality (decreasing the subsets C_n if necessary) that $\mu(C_n \cap C_m) = 0$ for any natural $n \neq m$. Denote $N_1 := \mathbb{N} \setminus (N' \cup N") = (k_n)_{n=1}^\infty$ and $G_n = C_{k_n} \cap T_{m(k_n)}$, $n \in \mathbb{N}$, where for any $n \in \mathbb{N}$, a natural number $m(k_n)$ is chosen in such a way that $\mu(C_{k_n} \cap T_{m(k_n)}) > 0$. Then, by the definition of C_n, we get that

$$I_\Phi\big(f_{k_n}\chi_{G_n}\big) = \infty \quad \text{for any } n \in \mathbb{N}. \tag{14}$$

We define a sequence $(x_n)_{n=1}^\infty$ in the following way. Let

$$H_i = \Big\{t \in G_1 : \; \Phi\big(t, \tfrac{u_i}{2}\big) > \frac{1}{2}\Big(1 - \frac{1}{k_1}\Big)\Phi\big(t, u_i\big)\Big\} \quad \forall i \in \mathbb{N}.$$

Of course each H_i is a Σ-measurable set. Moreover $G_1 = \bigcup_{i \in \mathbb{N}} H_i$. Define

$$D_n = \bigcup_{i \leq n} H_i \quad \text{and} \quad p_n(t) = \max_{i \leq n} u_i \, \chi_{D_n}.$$

Then $D_n \uparrow G_1$ and $p_n(t) \to f_{k_1}(t)$ for any $t \in G_1$. By the Beppo-Levi Theorem, we get that $I_\Phi(p_n) \to I_\Phi(f_{k_1}\chi_{G_1})$. So we can find $m_1 \in \mathbb{N}$ such that $I_\Phi(p_{m_1}) > K$. Taking $J_1 \subset D_{m_1}$, $J_1 \in \Sigma$ such that $I_\Phi(p_{m_1}\chi_{J_1}) = K$, we define

$$x_1 = (x - a)\chi_A + x\chi_{\bigcup_{k \in N' \cup N''} T_k} + p_{m_1}\chi_{J_1}.$$

Then $I_\Phi(x_1) = I_\Phi(x\chi_{A \cup \bigcup_{k \in N' \cup N''} Tk}) \leq 1$, whence $\|x_1\|_\Phi \leq 1$. Using the sets G_n, $n \geq 2$ we define the successive elements x_n as

$$x_n = (x - a)\chi_A + x\chi_{\bigcup_{k \in N' \cup N'' \cup \{k_1,\ldots k_{n-1}\}} T_k} + p_{m_n}\chi_{J_n}.$$

Proceeding as in case I, we finish the proof of the necessity.

Sufficiency. First we will prove the sufficiency of conditions (i), (iii) and $\mu(A_l(x)) = 0$. Let $\|x\|_\Phi = \|x_n\|_\Phi = 1$ and $\left\|\frac{x_n + x}{2}\right\|_\Phi \to 1$. For any $n \in \mathbb{N}$ and $\sigma > 0$ we define the set

$$A_\sigma^n = \big\{t \in T : |x_n(t)| \leq |x(t)|, |x(t) - x_n(t)| \geq \sigma\big\}.$$

Using the same techniques as in [10] (proof of Theorem 5.3), we will show that

$$\lim_{n \to \infty} \mu(A_\sigma^n) = 0 \quad \text{for any } \sigma > 0. \tag{15}$$

We repeat this justification for clarity of the proof and because of the fact that we will need it in its next part. In the opposite case, passing to a subsequence of (x_n) if necessary, we may assume that there exist $\sigma_0 > 0, \varepsilon_0 > 0$ such that $\mu(A_{\sigma_0}^n) > \varepsilon_0$ for any $n \in \mathbb{N}$. Since $\Phi(t, u) \to \infty$ as $u \to \infty$ for any $t \in T$, so defining for any $m \in \mathbb{N}$ the sets $T_m = \{t \in T : \Phi(t, m) \geq \frac{10}{\varepsilon_0}\}$, we get

$$T_m \uparrow, \quad \bigcup_{m=1}^{\infty} T_m = T, \quad \mu(T \setminus T_m) \to 0 \text{ as } m \to \infty.$$

Hence there exists $m_0 \in \mathbb{N}$ such that $\mu(T \setminus T_{m_0}) < \frac{\varepsilon_0}{10}$. Moreover, $\Phi(t, m_0) < \frac{10}{\varepsilon_0}$ for any $t \in T \setminus T_{m_0}$. Denote $D_n = A_{\sigma_0}^n \cap (T \setminus T_{m_0})$ and define the sets

$$A_n = \{t \in A_{\sigma_0}^n : |x_n(t)| > m_0\}, \quad A_n' = \{t \in A_{\sigma_0}^n : |x(t)| > m_0\}.$$

Then

$$1 = I_\Phi(x_n) \geq I_\Phi(x_n\chi_{A_n \setminus D_n}) \geq I_\Phi(m_0\chi_{A_n \setminus D_n}) \geq \frac{10}{\varepsilon_0}\mu(A_n \setminus D_n),$$

whence $\mu(A_n) \leq \frac{\varepsilon_0}{10} + \mu(D_n) < \frac{\varepsilon_0}{5}$. Similarly $\mu(A'_n) < \frac{\varepsilon_0}{5}$. Since $\mu(A_l(x)) = 0$, so by the definition of $A^n_{\sigma_0}$, we get that there exits a measurable function $\delta : T \to (0,1)$ such that

$$\Phi\left(t, \tfrac{x_n(t)+x(t)}{2}\right) \leq \frac{1-\delta(t)}{2} \left[\Phi\left(t, x_n(t)\right) + \Phi\left(t, x(t)\right)\right] \tag{16}$$

for any $n \in \mathbb{N}$ and any $t \in A^n_{\sigma_0} \setminus (A_n \cup A'_n)$. Let $\delta_0 > 0$ be such that $\mu(E) < \frac{\varepsilon_0}{5}$, where $E = \{t \in T : \delta(t) < \delta_0\}$. Since $\Phi\left(t, \frac{\sigma_0}{2}\right) > 0$ for any $t \in A^n_{\sigma_0}$, so there exists $a_0 > 0$ satisfying $\mu(F) < \frac{\varepsilon_0}{5}$, where $F = \{t \in T : \Phi\left(t, \frac{\sigma_0}{2}\right) < a_0\}$. Let

$$G_n = A^n_{\sigma_0} \setminus (A_n \cup A'_n \cup E \cup F).$$

Then $\mu(G_n) > \frac{\varepsilon_0}{5}$ and for any $t \in G_n$, we have $\delta(t) \geq \delta_0$, $|x_n(t) - x(t)| \geq \sigma_0$, $\Phi\left(t, \frac{\sigma_0}{2}\right) \geq a_0$ and $|x_n(t)| \leq m_0$, $|x(t)| \leq m_0$. Therefore

$$
\begin{aligned}
0 \leftarrow \frac{I_\Phi(x_n) + I_\Phi(x)}{2} - I_\Phi\left(\tfrac{x_n+x}{2}\right) &\geq \int_{G_n} \delta(t) \frac{\Phi\left(t, x_n(t)\right) + \Phi\left(t, x(t)\right)}{2} dt \\
&\geq \delta_0 \int_{G_n} \Phi\left(t, \tfrac{x_n(t)+x(t)}{2}\right) dt \\
&\geq \delta_0 \int_{G_n} \Phi\left(t, \tfrac{x_n(t)-x(t)}{2}\right) dt \\
&\geq \delta_0 \int_{G_n} \Phi\left(t, \tfrac{\sigma_0}{2}\right) dt = \delta_0 a_0 \frac{\varepsilon_0}{5} > 0.
\end{aligned}
$$

We have obtained a contradiction. Therefore, for any $\sigma > 0$ and $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that $\mu(A^n_\sigma) < \varepsilon$ for any $n > N(\varepsilon)$, and moreover, for any $t \in T \setminus A^n_\sigma$, we have $|x_n(t)| > |x(t)|$ or $|x_n(t) - x(t)| < \sigma$. Denote $H = T \setminus A_u(x)$, $J = A_u(x)$. Proceeding similarly as in the proof of condition (15), we get that $x_n \to x$ in measure in the set H. Without loss of generality, we can assume that $x_n(t) \to x(t)$ μ-a.e. on H. Hence, by the Fatou Lemma and convexity of the function Φ, we have

$$\liminf_{n \to \infty} I_\Phi\left(x_n \chi_H\right) \geq I_\Phi\left(x \chi_H\right), \tag{17}$$

whence

$$\limsup_{n \to \infty} I_\Phi\left(x_n \chi_J\right) \leq I_\Phi\left(x \chi_J\right). \tag{18}$$

Suppose that $(x_n - x)\chi_J \xrightarrow{\mu} 0$. Then there exist $J_1 \subset J$ and $\varepsilon_0, \sigma_0 > 0$ such that $\mu(J_1) > 0$ and (without loss of generality) $\mu(E_n) > 0$ for any $n \in \mathbb{N}$, where $E_n = \{t \in J_1 : |x_n(t) - x(t)| > \sigma_0\}$. Let us fix $\varepsilon > 0$ such that $\varepsilon_0 - \varepsilon > 0$.

Equation (15) implies that there exists $N(\varepsilon)$ such that $\mu(A^n_{\sigma_0}) < \varepsilon$ for any $n > N(\varepsilon)$. Moreover, $|x_n(t)| > |x(t)|$ for any $t \in J_1 \setminus A^n_{\sigma_0}$ and $n > N(\varepsilon)$. If there exist a set $J_2 \subset J_1$ and an infinite sequence $(n_k) \subset \mathbb{N}$ such that, for any $k \in \mathbb{N}$ and any $t \in J_2$, the point x_{n_k} is outside of the affinity interval of the function $\Phi(t, \cdot)$, which has the "bottom endpoint" $x(t)$, then for any $k \in \mathbb{N}$ and any $t \in ((E_{n_k} \setminus (A_n \cup A'_n)) \cap J_2) \setminus A^{n_k}_{\sigma_0}$, we have

$$\Phi\left(t, \tfrac{x_{n_k}(t) + x(t)}{2}\right) \le \frac{1 - \delta(t)}{2}\left[\Phi\left(t, x_{n_k}(t)\right) + \Phi\left(t, x(t)\right)\right], \qquad (19)$$

where A_n and A'_n are defined as in the previous part of the proof (page 696). Now we proceed in the same way as in the proof of condition (15), obtaining that $(x_n - x)\chi_{J_2} \xrightarrow{\mu} 0$. Hence, without loss of generality, we may assume that for any $t \in J_1 \setminus A^n_{\sigma_0}$ all points $x_n(t)$ are in the affinity interval of the function $\Phi(t, \cdot)$, with "bottom endpoint" $x(t)$. Within this assumption, if there exist a set $J_2 \subset J_1$ and an infinite sequence $(n_k) \subset \mathbb{N}$ such that $x_{n_k}(t)x(t) < 0$ for any $k \in \mathbb{N}$ and $t \in J_2$, then for any $k \in \mathbb{N}$ and $t \in ((E_{n_k} \setminus (A_n \cup A'_n)) \cap J_2) \setminus A^{n_k}_{\sigma_0}$, the point $x_{n_k}(t)$ belongs to the different affinity interval of function $\Phi(t, \cdot)$ than the point $x(t)$, whence these points satisfy inequality (19), and we proceed again as in the previous part of the proof, obtaining that $(x_n - x)\chi_{J_2} \xrightarrow{\mu} 0$. Finally, we can assume that for any $t \in J_1 \setminus A^n_{\sigma_0}$ all points $x_n(t)$ are in the affinity interval of the function $\Phi(t, \cdot)$, which has "bottom endpoint" $x(t)$. Let

$$A_t = \frac{\Phi\left(t, x(t) + sgn(x(t))\tfrac{\sigma_0}{2}\right) - \Phi\left(t, x(t)\right)}{\tfrac{\sigma_0}{2}}.$$

Then there exist a set $C \subset J_1$ and $\delta > 0$ such that $\mu(C) \le \frac{\varepsilon_0 - \varepsilon}{2}$, where $C = \{t \in J_1 : A_t < \delta\}$. Hence $A_t\tfrac{\sigma_0}{2} + \Phi\left(t, x(t)\right) = \Phi\left(t, x(t) + sgn(x(t))\tfrac{\sigma_0}{2}\right)$ and $A_t \ge \delta$ for any $t \in E_n \setminus (C \cup A^n_{\sigma_0})$, and $\mu(E_n \setminus (C \cup A^n_{\sigma_0})) > \frac{\varepsilon_0 - \varepsilon}{2}$, because $\mu(E_n \setminus A^n_{\sigma_0}) > (\varepsilon_0 - \varepsilon)$. Therefore, for any $n > N(\varepsilon)$, we have

$$\frac{1}{2}I_\Phi\left(x_n\chi_{E_n\setminus(C\cup A^n_{\sigma_0})}\right) + \frac{1}{2}I_\Phi\left(x\chi_{E_n\setminus(C\cup A^n_{\sigma_0})}\right)$$

$$\ge I_\Phi\left(\tfrac{x_n+x}{2}\chi_{E_n\setminus(C\cup A^n_{\sigma_0})}\right)$$

$$\ge \int_{E_n\setminus(C\cup A^n_{\sigma_0})} \Phi\left(t, x(t) + sgn(x(t))\tfrac{\sigma_0}{2}\right)dt$$

$$\ge \delta\frac{\sigma_0}{2}\mu\left(E_n \setminus (C \cup A^n_{\sigma_0})\right) + I_\Phi\left(x\chi_{E_n\setminus(C\cup A^n_{\sigma_0})}\right)$$

$$\ge \delta\frac{\sigma_0}{2}\frac{(\varepsilon_0 - \varepsilon)}{2} + I_\Phi\left(x\chi_{E_n\setminus(C\cup A^n_{\sigma_0})}\right)$$

by using

$$I_\Phi\left(\tfrac{x_n+x}{2}\chi_{E_n\setminus(C\cup A^n_{\sigma_0})}\right) \;=\; \int\limits_{E_n\setminus(C\cup A^n_{\sigma_0})} \Phi\left(t,\tfrac{x_n(t)+x(t)}{2}\right)dt$$

$$\int\limits_{E_n\setminus(C\cup A^n_{\sigma_0})} \Phi\left(t,x(t)+sgn(x(t))\tfrac{\sigma_0}{2}\right)dt \;=\; \int\limits_{E_n\setminus(C\cup A^n_{\sigma_0})} \left[\Phi\left(t,x(t)\right)+A_t\frac{\sigma_0}{2}\right]dt \;.$$

Hence

$$I_\Phi\left(x_n\chi_{E_n\setminus(C\cup A^n_{\sigma_0})}\right) \geq \delta\sigma_0\frac{(\varepsilon_0-\varepsilon)}{2} + I_\Phi\left(x\chi_{E_n\setminus(C\cup A^n_{\sigma_0})}\right). \tag{20}$$

Consequently

$$\limsup_{n\to\infty}\left[I_\Phi\left(x_n\chi_{E_n\setminus(C\cup A^n_{\sigma_0})}\right) - I_\Phi\left(x\chi_{E_n\setminus(C\cup A^n_{\sigma_0})}\right)\right] \geq \delta\sigma_0\frac{(\varepsilon_0-\varepsilon)}{2}. \tag{21}$$

Then

$$0 \geq \limsup_{n\to\infty}\left[I_\Phi\left(x_n\chi_{J_1}\right) - I_\Phi\left(x\chi_{J_1}\right)\right]$$
$$\geq \limsup_{n\to\infty}\left[I_\Phi\left(x_n\chi_{E_n\setminus(C\cup A^n_{\sigma_0})}\right) - I_\Phi\left(x\chi_{E_n\setminus(C\cup A^n_{\sigma_0})}\right)\right]$$
$$\geq \delta\sigma_0\frac{(\varepsilon_0-\varepsilon)}{2}.$$

This contradiction implies that $x_n \to x$ in measure on the set J. Using Lemma 1.4 we finish the proof of sufficiency of conditions (i), (ii)a and (iii).

Now assume that conditions (i), (ii)b and (iii) hold. As in the previous part of the proof we show that

$$\lim_{n\to\infty}\mu(A^n_\sigma) = 0 \quad\text{for any } \sigma > 0, \tag{22}$$

where, in this case,

$$A^n_\sigma = \{t \in T : |x_n(t)| \geq |x(t)|, |x(t) - x_n(t)| \geq \sigma\}.$$

Denoting $J = A_l(x)$, $H = T \setminus J$ and again proceeding similarly as in the proof of condition (15), we get that $(x_n - x)\chi_H \xrightarrow{\mu} 0$. By Lemma 2.3 we have $\|(x_n - x)\chi_H\|_\Phi \to 0$, whence $I_\Phi\left((x_n - x)\chi_H\right) \to 0$. Then, by Lemma 1.5, we obtain

$$\lim_{n\to\infty}I_\Phi\left(x_n\chi_H\right) = I_\Phi\left(x\chi_H\right). \tag{23}$$

Suppose that $(x_n - x)\chi_J \xnrightarrow{\mu} 0$. Then there exist $J_1 \subset J$, $\varepsilon_0, \sigma_0 > 0$ such that $\mu(J_1) > 0$ and (without loss of generality) $\mu(E_n) > 0$ for any $n \in \mathbb{N}$, where

$E_n = \{t \in J_1 : |x_n(t) - x(t)| > \sigma_0\}$. Because of the analogous reasons as in the previous part of the proof we can assume that all points $x_n(t)$ are in the affinity interval of the function $\Phi(t, \cdot)$, which has the "top endpoint" $x(t)$, for any $t \in J_1 \setminus A_{\sigma_0}^n$. By (22) we get that there exists $n_1 \in \mathbb{N}$ such that $|x_n(t)| \le |x(t)|$ for any $n > n_1$ and μ-a.e. $t \in J_1 \setminus A_{\sigma_0}^n$. Defining $C \in \Sigma$ and elements A_t in the same way as previously (page 698), we get

$$0 \leftarrow I_\Phi(x\chi_J) - I_\Phi(x_n\chi_J) \geq I_\Phi(x\chi_{E_n \setminus (C \cup A_{\sigma_0}^n)}) - I_\Phi(x_n\chi_{E_n \setminus (C \cup A_{\sigma_0}^n)})$$

$$\geq \delta\sigma_0 \frac{(\varepsilon_0 - \varepsilon)}{2}.$$

This contradiction implies that $x_n \to x$ in measure on the set J. Using Lemma 1.4 we finish the proof of sufficiency of conditions (i), (ii)a) and (iii). ∎

Remark 2.6. Proposition 5.3 in [2] gives that $\Phi \in \Delta_2$ implies $\Phi < \infty$. Therefore, by condition (i) in Theorem 2.5, we get that $\Phi < \infty$ is necessary for the existence of a $CLUR$-point in $S(L_\Phi)$.

Remark 2.7. Let $T = \mathbb{R}$ and (T, Σ_L, μ) denote the Lebesgue measure space. Then the condition $a(\Phi) = 0$ is necessary for $L_\Phi \in (CLUR)$.

Proof. By Theorem 2.5 and Remark 2.6 we can assume that $\Phi < \infty$ and $\Phi \in \Delta_2$. Let $\mu(supp(a(\Phi))) > 0$. Define the sets $A_n = \{t \in T : a(\Phi(t, \cdot)) \le n\}$ and $B_n = A_n \cap T_n$, where (T_n) is the sequence from Lemma 1.3. Then $\bigcup B_n = T$, whence there exists $m \in \mathbb{N}$ such that $\mu(B_m) > 0$ and $\mu(B_m \cap supp(a(\Phi))) > 0$. Let us take a bounded, closed set A such that $A \subset B_m$, $\mu(A) > 0$ and $\mu(A \cap supp(a(\Phi))) > 0$. Let $B \in \Sigma_L$ be such that $B \ne \emptyset$, $A \cap B = \emptyset$ and $A \cup B = T$. Define

$$x = \frac{a(\Phi)}{2}\chi_A + c\chi_B,$$

where $c \in \mathbb{R}_+$ is such that $I_\Phi(c\chi_B) = I_\Phi(x) = 1$. We use Lemma 1.6 for $E := A$ and define

$$x_n = x + \frac{a(\Phi)}{2}[\chi_{E_n} - \chi_{F_n}],$$

where (E_n) and (F_n) are the sequences from Lemma 1.6. Then $A = E_n \cup F_n$, $F_n \cap E_n = \emptyset$,

$$x_n = \frac{a(\Phi)}{2}\chi_{E_n \cup F_n} + c\chi_B + \frac{a(\Phi)}{2}\chi_{E_n} - \frac{a(\Phi)}{2}\chi_{F_n} = a(\Phi)\chi_{E_n} + c\chi_B,$$

$x - x_n = \frac{a(\Phi)}{2}[\chi_{F_n} - \chi_{E_n}]$ and $supp(x - x_n) = A$. Moreover $I_\Phi(x) = I_\Phi(x_n) = 1$, whence $\|x\|_\Phi = \|x_n\|_\Phi = 1$. Since $\Phi < \infty$ and $a(\Phi)\chi_A \in E_\Phi$, so there exists $\lambda > 2$ such that

$$1 > d := I_\Phi(\lambda(x - x_n)) = I_\Phi(\lambda \tfrac{a(\Phi)}{2}[\chi_{E_n} - \chi_{F_n}]) > 0.$$

Then $\|x - x_n\|_\Phi > \frac{d}{\lambda} > 0$. Since $\Phi \in \Delta_2$, so for any $\lambda > 1$,

$$I_\Phi\big(\lambda(x_n - x)\big) = \int_A \Phi\big(t, \lambda(x_n - x)(t)\big)d\mu$$

$$\leq \; KI_\Phi\big(x_n - x\big) + \int_A f(t)d\mu$$

$$= \int_A f(t)d\mu \; < \; \infty.$$

Hence $x - x_n \in E_\Phi$, which implies that $\phi(x - x_n) = 0$ for any singular functional ϕ over L_Φ. Let $y \in S(L_\Psi^o)$. Then there exists $\lambda > 0$ such that $I_\Psi(\lambda y) < \infty$. By the Young inequality, we have

$$\int_T \frac{a(\Phi)}{2} y d\mu \leq \frac{1}{\lambda}\left[I_\Psi(\lambda y) + I_\Phi\big(\tfrac{a(\Phi)}{2}\big)\right] < \infty.$$

Therefore, the function $\frac{a(\Phi)}{2}y$ is integrable. Then by Lemma 1.6,

$$\langle x_n - x, y\rangle = \int_{supp(x_n - x)} y\frac{a(\Phi)}{2}[\chi_{E_n} - \chi_{F_n}]d\mu \to 0.$$

This means that $x_n \xrightarrow{w} x$, which shows that the condition $a(\Phi) = 0$ is necessary for property (H) of L_Φ (see. [5]). The observation that $(CLUR) \Rightarrow (H)$ finishes the proof. ∎

Remark 2.8. Let $c : T \to \mathbb{R}_+$ be a function such that $\Phi(t, \cdot)$ is affine on the interval $[0, c(t)]$ and $\Phi > 0$. Then the following statements hold:

1. If $[\mu(A_u(x)) > 0$ or $\Psi \notin \Delta_2]$ and $I_\Phi(c) \geq 1$, then there are no $CLUR$-points on the unit sphere $S(L_\Phi)$.
2. If $\mu(A_u(x)) = 0$ and $\Psi \in \Delta_2$, the following statements hold:

   (a) If $[\mu(A_l(x)) = 0$ and $I_\Phi(c) \geq 1]$ or $[\mu(A_l(x)) > 0$ and $I_\Phi(c) > 1]$, then there are no $CLUR$-points on the unit sphere $S(L_\Phi)$.
   (b) If $\mu(A_l(x)) > 0$ and $I_\Phi(c) = 1$, then the element c is the only $CLUR$-point on the unit sphere $S(L_\Phi)$.

3. If $I_\Phi(c) > 0$, then $L_\Phi \notin (CLUR)$.

Proof. Case 1. By the assumptions and Theorem 2.5 we have that, if $x \in S(L_\Phi)$ is a $CLUR$-point, then $I_\Phi(x) = 1$ and $A_l(x) = 0$, whence $|x(t)| > c(t)$ for t belonging to some set A, where $A \subset T$, $A \in \Sigma$ and $\mu(A) > 0$. Hence $I_\Phi(x) > 1$, a contradiction.

Case 2. In this case the proof is similar as in Case 1.

Case 3. Let us fix sets $A, B \in \Sigma$, a function $b : T \to \mathbb{R}_+$ and a number $\alpha \in \mathbb{R}_+$ such that $A \subset supp(c)$, $B \subset T \setminus A$, $\mu(A) > 0$, $\mu(B) > 0$, $0 < b(t) < c(t)$ for any $t \in A$, and let x be defined by

$$x(t) = b(t)\chi_A + \alpha\chi_B.$$

Then $I_\Phi(x) = 1$, whence $x \in S(L_\Phi)$ and, by condition (iii) from Theorem 2.5, we get that x is not a $CLUR$-point. ∎

Lemma 2.9. *If $\Psi \in \Delta_2$ and $\frac{1}{u}\Phi(t, u) \to 0$ as $u \to 0$ for μ-a.e. $t \in T$, then*

$$\sup_{\|x\|_\Phi^o = 1} \left\{ k > 0 : \ \|x\|_\Phi^o = \frac{1}{k}\left(1 + I_\Phi(kx)\right) \right\} < \infty.$$

Proof. This Lemma can be proved similarly as Lemma 1.6 in [10]. Although in that result the assumption of strict convexity of the function Φ instead of the assumption that $\frac{1}{u}\Phi(t, u) \to 0$ as $u \to 0$ for μ-a.e. $t \in T$ has been used, actually in the proof the important point is that for μ-a.e. $t \in T$, $\Phi(t, \cdot)$ is not affine in any neighbourhood of zero, but this property follows from the assumption that $\frac{1}{u}\Phi(t, u) \to 0$ as $u \to 0$ for μ-a.e. $t \in T$. ∎

Theorem 2.10. *Let Φ be a Musielak-Orlicz function such that $\Phi < \infty$, $\Phi > 0$, $\frac{1}{u}\Phi(t, u) \to 0$ as $u \to 0$ and $\frac{1}{u}\Phi(t, u) \to \infty$ as $u \to \infty$ for μ-a.e. $t \in T$. Moreover, let $x \in S(L_\Phi^o)$ and $k \in K(x)$. Then x is a $CLUR$-point if and only if*

  (i) $\Phi \in \Delta_2$,
  (ii) $\Psi \in \Delta_2$,
  (iii) $x(t) \in Ext(\Phi(t, \cdot))$ *for μ-a.e. $t \in T$,*
  (iv) $\mu(A_l^s(kx)) = 0$ *and* $\mu(A_u^s(kx)) = 0$,
  (v) *if $\mu(A_u^{ns}(kx)) > 0$, then $I_\Psi(p_-(k|x|)) = 1$,*
  (vi) *if $\mu(A_l^{ns}(kx)) > 0$, then $I_\Psi(p_+(k|x|)) = 1$.*

Proof. Necessity.

(i) Let $y \in S(L_\Phi^o)$ be a $CLUR$-point. Suppose that $\Phi \notin \Delta_2$. Then there exists $0 \neq x \in S(L_\Phi)$ such that $I_\Phi(x) < \infty$, $I_\Phi(\lambda x) = \infty$ for any $\lambda > 1$, and $sgn(x(t)) = sgn(y(t))$ for any $t \in supp(y)$. Let $x_0 = x/\|x\|_\Phi^o$. Then $I_\Phi(\lambda x_0) = \infty$ for any $\lambda > 2\|x\|_\Phi^o$. Define the sets

$$A_n = \left\{ t \in T : \ |x_0(t)| \leq n, \ |y_n(t)| \leq n \right\}$$

and $B_n = A_n \cap S_n$, with $S_n = \bigcup_{i=1}^n T_i$, where $(T_i)_1^\infty$ is the sequence from Lemma 1.3. By the Fatou property of L_Φ^o, we have $\|x_0\chi_{B_n}\|_\Phi^o \to \|x_0\|_\Phi^o$. Since

$I_\Phi\big(\lambda x_0\chi_{B_n}\big) < \infty$ for any $n \in \mathbb{N}$ and $\lambda \geq 2\|x\|_\Phi^o$, so $I_\Phi\big(\lambda x_0\chi_{T\setminus B_n}\big) = \infty$ for any $n \in \mathbb{N}$ and $\lambda \geq 2\|x\|_\Phi^o$. Hence $\big\|x_0\chi_{T\setminus B_1}\big\|_\Phi^o \geq 1/(2\|x_0\|_\Phi^o)$. Let $n_1 = 1$. By the Fatou property, we get $\big\|x_0\chi_{(T\setminus B_{n_1})\cap B_n}\big\|_\Phi^o \to \big\|x_0\chi_{T\setminus B_{n_1}}\big\|_\Phi^o$. Hence, there exists $n_2 > n_1$ such that

$$\big\|x_0\chi_{(T\setminus B_{n_1})\cap B_{n_2}}\big\|_\Phi^o \geq \frac{1}{4\|x_0\|_\Phi^o}.$$

Moreover, $I_\Phi\big(\lambda x_0\chi_{T\setminus B_{n_2}}\big) = \infty$ for any $\lambda \geq 2\|x\|_\Phi^o$. Hence $\big\|x_0\chi_{T\setminus B_{n_2}}\big\|_\Phi^o \geq 1/(2\|x_0\|_\Phi^o)$. Using again the Fatou property , we get $\big\|x_0\chi_{(T\setminus B_{n_2})\cap B_n}\big\|_\Phi^o \to \big\|x_0\chi_{T\setminus B_{n_2}}\big\|_\Phi^o$. Hence, there exists $n_3 > n_2$ such that

$$\big\|x_0\chi_{(T\setminus B_{n_2})\cap B_{n_3}}\big\|_\Phi^o \geq \frac{1}{4\|x_0\|_\Phi^o}.$$

Proceeding (by induction) analogously as above, we get a sequence $(n_k) \subset \mathbb{N}$ such that

$$\big\|x_0\chi_{(T\setminus B_{n_k})\cap B_{n_{k+1}}}\big\|_\Phi^o \geq \frac{1}{4\|x_0\|_\Phi^o}.$$

Denote $P_k = (T \setminus B_{n_k}) \cap B_{n_{k+1}}$. Since $P_k \cap P_l = \emptyset$ for $k \neq l$, so

$$\Big|\sum_{k=1}^\infty x_0\chi_{P_k}\Big| = \sum_{k=1}^\infty |x_0\chi_{P_k}| \leq |x_0| .$$

Moreover, $\phi(x_0\chi_{P_k}) = 0$ for any singular functional ϕ and any $k \in \mathbb{N}$. Let $v \in L_\Psi$. By the Hölder inequality, we have $|\langle |x_0|, |v|\rangle| \leq \|x_0\|_\Phi^o\|v\|_\Psi < \infty$. Therefore, $\sum_{k=1}^\infty\langle |x_0\chi_{P_k}|, |v|\rangle < \infty$, whence $\langle |x_0\chi_{P_k}|, |v|\rangle \to 0$ as $k \to \infty$. Hence also $\langle x_0\chi_{P_k}, v\rangle \to 0$. Thus we have that

$$x_0\chi_{P_k} \xrightarrow{w} 0 \quad \text{as } k \to \infty. \tag{24}$$

Similarly we show that $y\chi_{P_k} \xrightarrow{w} 0$ as $k \to \infty$, whence

$$y\chi_{T\setminus P_k} \xrightarrow{w} y \quad \text{as } k \to \infty. \tag{25}$$

Moreover, since $I_\Phi(x) < \infty$ and $\|x\|_\Phi^o > \|x\|_\Phi = 1$, so $\infty > I_\Phi(x_0) = \sum_{k=1}^\infty I_\Phi(x_0\chi_{P_k})$. Hence

$$I_\Phi\big(x_0\chi_{P_k}\big) \to 0 \quad \text{as } k \to \infty. \tag{26}$$

Let us take $t \in T$ such that $|y(t)| < \infty$. Then $t \in P_l$ for some $l \in \mathbb{N}$. Since $P_k \cap P_l = \emptyset$ for $k \neq l$, so $t \notin P_k$ for any $k > l$, whence $y\chi_{P_k} = 0$ for any $k > l$. This means that $y\chi_{P_k}(t) \downarrow 0$ for μ-a.e. $t \in T$. Therefore

$$\alpha y\chi_{T\setminus P_k}(t) \uparrow \alpha y(t) \quad \text{for } \mu\text{-a.e. } t \in T \quad \text{and any } \alpha > 0.$$

Thus, by the Beppo-Levi Theorem, we have

$$I_\Phi\big(\alpha y \chi_{T\setminus P_k}\big) \to I_\Phi(\alpha y) \quad \text{for any } \alpha > 0 \quad \text{as } k \to \infty. \tag{27}$$

Let $m > 0$ be such that $\|y\|_\Phi^o = \frac{1}{m}\big(1 + I_\Phi(my)\big)$. For any $k \in \mathbb{N}$ we define

$$y_k = y\chi_{T\setminus P_k} + \frac{1}{m}x_0\chi_{P_k}.$$

Then, by (26) and (27), we get

$$\begin{aligned}
\|y_k\|_\Phi^o &\leq \frac{1}{m}\big(1 + I_\Phi(my_k)\big) \\
&= \frac{1}{m}\big(1 + I_\Phi(my\chi_{T\setminus P_k}) + I_\Phi(x_0\chi_{P_k})\big) \to \frac{1}{m}\big(1 + I_\Phi(my)\big) = 1
\end{aligned}$$

for any $k \in \mathbb{N}$. Moreover, by the monotone completeness of the Orlicz-Amemiya norm and by the fact that $\big\|y\chi_{T\setminus P_k}\big\|_\Phi^o \leq \|y\|_\Phi^o$, we have $\lim_{k\to\infty}\|y_k\|_\Phi^o \geq 1$. Finally $\|y_k\|_\Phi^o \to 1 = \|y\|_\Phi^o$. By (24) and (25), we get

$$y_k \xrightarrow{w} y \quad \text{as } k \to \infty.$$

Since $sgn(x(t)) = sgn(y(t))$ for any $t \in supp(y)$, so

$$\|y_k - y\|_\Phi^o = \big\|y\chi_{P_k} + \tfrac{1}{m}x_0\chi_{P_k}\big\|_\Phi^o \geq \frac{1}{m}\|x_0\chi_{P_k}\|_\Phi^o > \frac{1}{4m\|x\|_\Phi^o}.$$

For this reason, y is not an H-point, whence it is not a $CLUR$-point as well.

(ii) Let $\|x\|_\Phi^o = 1$. Since $\Phi \in \Delta_2$, so $(L_\Phi^o)^* = L_\Psi$, whence there exists $y \in S(L_\Psi)$ such that $\langle x, y \rangle = \|x\|_\Phi^o = 1$. Suppose that $\Psi \notin \Delta_2$. Then there exists $x_0 \in S(L_\Psi)$ such that $sgn(x_0(t)) = sgn(x(t))$ for any $t \in supp(x)$, $I_\Psi(x_0) < \infty$ and $I_\Psi(\lambda x_0) = \infty$ for any $\lambda > 1$. Let (T_n) be the sequence from Lemma 1.3 used for the function Ψ instead of Φ. We fix $\varepsilon \in (0,1)$ and define the sets

$$B_n = \{t \in T : |x_0(t)| \leq n, |x(t)| \leq n, |y(t)| \leq n\}, \quad C_n = B_n \cap T_n.$$

Then $C_n \uparrow$, $\bigcup_n C_n = T$, and since $\chi_{T_n} \in E_\Psi$ for any $n \in \mathbb{N}$, so $|x_0|\chi_{C_n} \leq n\chi_{C_n} \leq n\chi_{T_n} \in E_\Psi$. By the Beppo-Levi Theorem,

$$I_\Psi(x_0\chi_{C_n}) \uparrow I_\Psi(x_0), \quad I_\Psi(y\chi_{C_n}) \uparrow I_\Psi(y), \quad I_\Psi(x\chi_{C_n}) \uparrow I_\Psi(x).$$

Hence there exists $m \in \mathbb{N}$ such that

$$I_\Psi(x_0\chi_{T\setminus C_m}) < \frac{\varepsilon}{2}, \quad I_\Psi(y\chi_{T\setminus C_m}) < \frac{\varepsilon}{2}, \quad I_\Psi(x\chi_{T\setminus C_m}) < \frac{\varepsilon}{2}. \tag{28}$$

Denoting $S_n = \bigcup_{i=1}^n T_i$, we have $\chi_{S_n} = \sum_{i=1}^n \chi_{T_i} \in E_\Psi$. Moreover,

$$I_\Psi(\lambda x_0\chi_{T\setminus S_n}) = I_\Psi(\lambda x_0) - I_\Psi(\lambda x_0\chi_{S_n}) = \infty \quad \text{for any } \lambda > 1 \text{ and } n \in \mathbb{N}. \tag{29}$$

In particular, $I_\Psi \left(\lambda x_0 \chi_{T \setminus S_m} \right) = \infty$ for any $\lambda > 1$. Denote $T_0 = T \setminus S_m$. Then $C_m \subset T \setminus T_0$. Let

$$n_1 := \min \left\{ n > m : \; I_\Psi \left((1 + \tfrac{1}{2}) x_0 \chi_{S_n \setminus S_m} \right) > 1 - \varepsilon \right\}.$$

Since the measure is nonatomic, there exists $A_1 \subset S_{n_1} \setminus S_m$ such that $I_\Psi \left(\left(1 + \tfrac{1}{2} \right) x_0 \chi_{A_1} \right) = 1 - \varepsilon$. Moreover, $A_1 \subset S_{n_1} \setminus S_m$, whence $\chi_{A_1} \le \chi_{S_{n_1} \setminus S_m} \in E_\Psi$, which means that $\chi_{A_1} \in E_\Psi$. Using (29) we can extend this construction, namely for any $k > 1$ we choose the number

$$n_k := \min \left\{ n > n_{k-1} : \; I_\Psi \left(\left(1 + \tfrac{1}{2^k} \right) x_0 \chi_{S_n \setminus S_{n_{k-1}}} \right) > 1 - \varepsilon \right\}$$

and a set $A_k \subset S_{n_k} \setminus S_{n_{k-1}}$ such that $I_\Psi \left(\left(1 + \tfrac{1}{2^k} \right) x_0 \chi_{A_k} \right) = 1 - \varepsilon$. Then $\chi_{A_k} \in E_\Psi$. Moreover, $A_k \subset T \setminus S_m \subset T \setminus C_m$ for any $k \in \mathbb{N}$. This implies, by (28), that $I_\Psi \left(x_0 \chi_{A_k} \right) < \tfrac{\varepsilon}{2}$ for any $k \in \mathbb{N}$. Then $I_\Psi \left(x_0 \chi_{A_n} \right) < \varepsilon$ for any $n \in \mathbb{N}$ and $z_n := x_0 \chi_{A_n} \in E_\Psi$, because $A_n \subset C_k$ for some $k \in \mathbb{N}$. Hence there exists $x_n \in S(L_\Phi^o)$ such that $\langle x_n, z_n \rangle = \| z_n \|_\Psi$. Define the sequence

$$y_n = \left(\frac{1 - \varepsilon}{1 + \tfrac{1}{2^n}} \right) \left(z_n + y \chi_{T \setminus A_n} \right).$$

Then

$$I_\Psi \left(y_n \right) \le \left(\frac{1 - \varepsilon}{1 + \tfrac{1}{2^n}} \right) \left(I_\Psi \left(z_n \right) + I_\Psi \left(y \right) \right) \le \left(\frac{1 - \varepsilon^2}{1 + \tfrac{1}{2^n}} \right) < 1 \qquad (30)$$

and

$$\langle x_n, y_n \rangle \ge \left(\frac{1 - \varepsilon}{1 + \tfrac{1}{2^n}} \right) \int_{A_n} x_n(t) y_n(t) dt = \left(\frac{1 - \varepsilon}{1 + \tfrac{1}{2^n}} \right) \langle x_n, z_n \rangle \ge \frac{(1 - \varepsilon)^2}{(1 + \tfrac{1}{2^n})^2}, \qquad (31)$$

because $\left\| \left(1 + \tfrac{1}{2^n} \right) z_n \right\|_\Psi \ge I_\Psi \left(\left(1 + \tfrac{1}{2^n} \right) z_n \right) = 1 - \varepsilon$. Moreover,

$$\langle x, y_n \rangle = \int_{A_n} x(t) y_n(t) dt$$

$$= \left(\frac{1 - \varepsilon}{1 + \tfrac{1}{2^n}} \right) \left[\int_{A_n} x(t) z_n(t) dt + \int_{T \setminus A_n} x(t) y(t) dt \right]$$

$$= \left(\frac{1 - \varepsilon}{1 + \tfrac{1}{2^n}} \right) \left[\int_T x(t) y(t) dt + \int_{A_n} x(t) z_n(t) dt - \int_{A_n} x(t) y(t) dt \right]$$

$$\ge \left(\frac{1 - \varepsilon}{1 + \tfrac{1}{2^n}} \right) \left[\langle x, y \rangle - \int_{A_n} x(t) y(t) dt \right],$$

because $sgn(x(t)) = sgn(z_n(t))$ for any $t \in supp(y)$ and any $n \in \mathbb{N}$. By the Young inequality and condition (29), we get

$$\int\limits_{A_n} x(t)y(t)dt \leq I_\Phi\big(x\chi_{A_n}\big) + I_\Psi\big(y\chi_{A_n}\big) < \varepsilon, \tag{32}$$

whence $\langle x, y_n \rangle \geq \frac{(1-\varepsilon)^2}{1+\frac{1}{2^n}}$. Since $I_\Psi(y_n) < 1$, so by the definition of the Orlicz-Amemiya norm, we get

$$2 \geq \|x_n + x\|_\Phi^o \geq \langle x + x_n, y_n \rangle \geq \frac{(1-\varepsilon)^2}{1+\frac{1}{2^n}} + \frac{(1-\varepsilon)^2}{(1+\frac{1}{2^n})^2}.$$

Therefore, $\|x_n + x\|_\Phi^o \to 2$. Moreover, $\|x_n - x_m\|_\Phi^o \geq \|x\|_\Phi^o = 1$ for any $m \neq n$, because supports of the elements x_n are disjoint. This means that there is no Cauchy subsequence of (x_n), so x is not a $CLUR$-point.

(iii) Proof of this part follows in the same way as the proof of Lemma 2.4, so it is omitted.

(iv) Suppose that $\mu(A_l^s(kx)) > 0$, $\mu(A_u^s(kx)) > 0$ and $\|x\|_\Phi^o = 1$. Let (w_n) denote the set of all rational, positive numbers. Define the sets

$$A_n = \left\{ t \in A_l^s(kx) \ : \ \Phi(t, \cdot) \text{ is affine on } \left\{ \begin{array}{l} [kx(t) - w_n, kx(t)] \text{ if } k(t) > 0 \\ [kx(t), kx(t) + w_n] \text{ if } k(t) < 0 \end{array} \right\} \right\}.$$

Since $A_l^s(kx) = \bigcup_n A_n$, there exists $l \in \mathbb{N}$ such that $\mu(A_l) > 0$. Denote $a := |w_n|$, $A := A_l$ and define the new nonatomic measure on $\Sigma \cap A$:

$$\nu(B) = \int\limits_B \big[\Phi\big(t, kx(t)\big) - \Phi\big(t, kx(t) - a\big)\big]d\mu.$$

Now we proceed in the way similar as in the proof of Lemma 2.4, defining the elements

$$x_n = x\chi_{T \setminus A} + x\chi_{B_1^n} + (x - b \cdot sgn(x))\chi_{B_2^n},$$

where $b = \frac{a}{k}$, and the sets B_1^n, B_2^n are constructed in the same way as the sets C_1^n, C_2^n in part (ii) of the proof of Theorem 2.5. Since $\nu(B_1^n) = \nu(B_2^n) = \frac{1}{2}\nu(A)$, so

$$\begin{aligned} I_\Phi\big(kx\chi_{B_1^n}\big) + I_\Phi\big((kx - a \cdot sgn(x))\chi_{B_2^n}\big) \\ = I_\Phi\big(kx\chi_{B_2^n}\big) + I_\Phi\big((kx - a \cdot sgn(x))\chi_{B_1^n}\big). \end{aligned} \tag{33}$$

Then

$$\begin{aligned} I_\Phi\big(kx_n\big) &= I_\Phi\big(kx\chi_{T \setminus A}\big) + I_\Phi\big(kx\chi_{B_1^n}\big) + I_\Phi\big((kx - a \cdot sgn(x))\chi_{B_2^n}\big) \\ &= I_\Phi\big(kx\chi_{T \setminus A}\big) + \frac{1}{2}(I_\Phi\big(kx\chi_A\big) + I_\Phi\big((kx - a \cdot sgn(x))\chi_A\big)). \end{aligned} \tag{34}$$

By the definition of x_n, we get for $t \in T$

$$p_-(t, kx_n(t)) = p_-(t, kx(t)), \quad p_+(t, kx_n(t)) = p_+(t, kx(t)). \qquad (35)$$

Therefore $I_\Psi(p_+(kx_n)) = I_\Psi(p_+(kx))$, whence $k \in K(x_n)$ for any $n \in \mathbb{N}$. Then (34) implies that all norms $\|x_n\|_\Phi^o$ are equal. Since $\Phi \in \Delta_2$, so $L_\Phi^o = E_\Phi^o$ and $(L_\Phi^o)^* = L_\Psi$. By (35) and Lemma 1.7, there exists $y \in L_\Psi$ such that $\langle x, y \rangle = 1$, $I_\Psi(y) = 1$ and

$$p_-(t, kx_n(t)) \leq y(t) \leq p_+(t, kx_n(t)).$$

Then $\langle x_n, y \rangle = \|x\|_\Phi^o$ for any $n \in \mathbb{N}$ and

$$2 \geq \left\| x + \frac{x_n}{\|x_n\|_\Phi^o} \right\|_\Phi^o \geq \int_T \left(x(t) + \frac{x_n(t)}{\|x_n\|_\Phi^o} \right) y(t) dt = 2.$$

Hence, the above norm is equal to 2. Since all norms $\|x_n\|_\Phi^o$ are equal, so denoting $L := \|x_n\|_\Phi^o$, we get for $m > n$

$$k \left(\frac{x_n}{\|x_n\|_\Phi^o} - \frac{x_m}{\|x_m\|_\Phi^o} \right) = \frac{1}{L} k(x_n - x_m)$$

$$= \frac{1}{L} \left(kx \chi_{B_1^n \setminus B_1^m} + (kx - a \cdot sgn(x)) \chi_{B_2^n \setminus B_2^m} \right).$$

Proceeding as in the proof of part (ii) in Lemma 2.4, we get

$$\left\| \frac{x_n}{\|x_n\|_\Phi^o} - \frac{x_m}{\|x_m\|_\Phi^o} \right\|_\Phi^o > \frac{1}{k} \left(1 + \frac{1}{4L} \nu(A) \right).$$

This means that x is not a CLUR-point.

(v) By the definition of $K(x)$, we have $I_\Psi(p_-(kx)) \leq 1$. Suppose that $\mu(A_u^{ns}(kx)) > 0$ and $I_\Psi(p_-(kx)) < 1$. Similarly as in Case (iv), we get a set $A \subset A_u^{ns}(kx)$ and a number $a \in \mathbb{R}_+$ such that $\Phi(t, \cdot)$ is affine in the interval $[kx(t), kx(t) + a]$ for any $t \in A$. By the definition of $K(x)$, we have $I_\Psi(p_+(kx)) \geq 1$. Then there exists $B \subset A$, $B \in \Sigma$ such that

$$I_\Psi(p_+(kx)\chi_B) + I_\Psi(p_-(kx)\chi_{T \setminus B}) \leq 1. \qquad (36)$$

We decompose the set B into B_1^n, B_2^n (see the proof of Lemma 2.4) and define

$$x_n = x \chi_{T \setminus B} + x \chi_{B_1^n} + (x + a) \chi_{B_1^n}.$$

Constructing an adequate measure and proceeding as in the proof of neccesity of condition (iv), we get that the values of all modulars $I_\Phi(kx_n)$ are equal. Then for any $t \in T$,

$$p_-(t, kx_n(t)) \geq p_-(t, kx(t)), \quad p_+(t, kx_n(t)) = p_+(t, kx(t)). \qquad (37)$$

Therefore $I_\Psi\left(p_+(kx_n)\right) = I_\Psi\left(p_+(kx)\right) \geq 1$, whence $k \in K(x_n)$ for any $n \in \mathbb{N}$. Moreover, then we have that also all norms $\|x_n\|_\Phi^o$ are equal. Since it follows from (36) that $I_\Psi\left(p_-(kx_n)\right) \leq 1$, so there exists a function $y : T \to \mathbb{R}$ such that $p_-(t, kx_n(t)) \leq y(t) \leq p_+(t, kx_n(t))$ and $I_\Psi(y) = 1$. Moreover, by (37)

$$p_-(t, kx(t)) \leq p_-(t, kx_n(t)) \leq y(t) \leq p_+(t, kx(t)) = p_+(t, kx_n(t)),$$

whence, by Lemma 1.7, we get that all elements x_n and the element x have the same support functional. Now we proceed as in the proof of necessity of condition (iv).

(vi) The necessity of condtion (vi) can be proved in the same way as the necessity of condition (v).

Sufficiency. Let $\Phi \in \Delta_2$, $\Psi \in \Delta_2$, $\|x_n - x\|_\Phi^o \to 2$ and let k, k_n $(n = 1, 2, \dots)$ be positive numbers such that

$$1 = \|x_n\|_\Phi^o = \frac{1}{k_n}\left(1 + I_\Phi\left(k_n x_n\right)\right) = \|x\|_\Phi^o = \frac{1}{k}\left(1 + I_\Phi\left(kx\right)\right).$$

Fix $\varepsilon > 0$ and let $S_n = \bigcup_{i=1}^n T_i$, where $(T_i)_1^\infty$ is the sequence of sets from Lemma 1.3. Then, by the fact that $\Phi \in \Delta_2$, there exists $m \in \mathbb{N}$ such that $\left\|x\chi_{T \setminus S_m}\right\|_\Phi < \delta(\sigma)$, where $\delta(\sigma)$ is such that $\left\|x_n\chi_{T \setminus S_m}\right\|_\Phi < \sigma$ for any $n > n'$ (n' is from Lemma 2.1). We may choose δ, σ in such a way that $\max(\delta, \sigma) < \frac{\varepsilon}{2}$. Define

$$D = \left\{t \in S_m : \ kx(t) \in Ext(\Phi(t, \cdot))\right\}.$$

Then, in the same way as it was done in the proof of condition (15), we show that

$$(k_n x_n - kx)\chi_D \overset{\mu}{\to} 0. \tag{38}$$

Now assume that $\mu(A_u^{ns}(kx)) > 0$. Then it follows from conditions (v) and (vi) that $\mu(A_l^{ns}(kx)) = 0$. Let $A = A_u^{ns} \cap S_m$. So, $S_m \setminus D = A$. By (38) we have $(k_n x_n - kx)\chi_{S_m \setminus A} \overset{\mu}{\to} 0$ and, without loss of generality, we get $k_n x_n(t) \to kx(t) \to 0$ for μ-a.e. $t \in S_m \setminus A$. Moreover, $kx(t)$ are points of smoothness for the functions $\Phi(t, \cdot)$ for μ-a.e. $t \in S_m \setminus A$, whence $p_-(t, k_n x_n(t)) \to p_-(t, kx(t))$ for μ-a.e. $t \in S_m \setminus A$. By the Fatou Lemma, we have then

$$\liminf_{n \to \infty} I_\Psi\left(p_-(k_n x_n)\chi_{S_m \setminus A}\right) \geq I_\Psi\left(p_-(kx)\chi_{S_m \setminus A}\right). \tag{39}$$

It follows from the definition of $K(x)$ and from condition (v) that

$$I_\Psi\left(p_-(k_n x_n)\right) \leq 1 = I_\Psi\left(p_-(kx)\right). \tag{40}$$

Then, by (39) and (40), we get

$$\limsup_{n \to \infty} I_\Psi\left(p_-(k_n x_n)\chi_A\right) \leq I_\Psi\left(p_-(kx)\chi_A\right). \tag{41}$$

For any $n \in \mathbb{N}$ we define the set

$$A_\sigma^n = \big\{ t \in T : |k_n x_n(t)| \leq |kx(t)|, |kx(t) - k_n x_n(t)| \geq \sigma \big\}.$$

Since for any $n \in \mathbb{N}$ and $t \in A_\sigma^n$, $kx(t)$ and $k_n x_n(t)$ are points of strict convexity of the function $\Phi(t, \cdot)$, so proceeding as within (38), we get

$$\lim_{n \to \infty} \mu(A_\sigma^n) = 0 \quad \text{for any } \sigma > 0. \tag{42}$$

We will show that

$$(k_n x_n - kx)\chi_A \xrightarrow{\mu} 0. \tag{43}$$

Suppose that $(x_n - x)\chi_A \xrightarrow{\mu} 0$. Then there exist $A_1 \subset J$ and $\varepsilon_0, \sigma_0 > 0$ such that $\mu(A_1) > 0$ and (without loss of generality) $\mu(E_n) > 0$ for any $n \in \mathbb{N}$, where $E_n = \{ t \in A_1 : |x_n(t) - x(t)| > \sigma_0 \}$. Let us fix $\varepsilon > 0$ such that $\varepsilon_0 - \varepsilon > 0$. Equation (42) implies that there exists $N(\varepsilon)$ such that $\mu(A_{\sigma_0}^n) < \varepsilon$ for any $n \in \mathbb{N}$. Moreover, $|k_n x_n(t)| > |kx(t)|$ for any $t \in A_1 \setminus A_{\sigma_0}^n$ and $n > N(\varepsilon)$. Similarly as in the proof of the sufficiency part of Theorem 2.5 (page 698), we can assume that for any $t \in A_1 \setminus A_{\sigma_0}^n$, all points $|k_n x_n(t)|$ are in the affinity interval of the function $\Phi(t, \cdot)$, which has the "bottom endpoint" $|kx(t)|$. Then

$$p_-(t, k_n x_n(t)) = p_-(t, kx(t) + sgn(x(t))\sigma)$$

for any $n \in \mathbb{N}$ and $t \in A_1 \setminus A_{\sigma_0}^n$. Therefore

$$p_-(t, k_n x_n(t)) - p_-(t, kx(t)) = p_-(t, kx(t) + sgn(x(t))\sigma) - p_-(t, kx(t)) \tag{44}$$

for any $n \in \mathbb{N}$ and $t \in A_1 \setminus A_{\sigma_0}^n$. Hence there exists $a_0 > 0$ such that $\mu(C) < \frac{\varepsilon_0 - \varepsilon}{4}$, where

$$C = \big\{ t \in A_1 : p_-(t, kx_n(t) + sgn(x(t))\sigma) - p_-(t, kx(t)) < a_0 \big\}.$$

Then $\mu(E_n \setminus (A_{\sigma_0}^n \cup C)) > \frac{3}{4}(\varepsilon_0 - \varepsilon)$ for any $n \in \mathbb{N}$ and, by (44),

$$p_-(t, k_n x_n(t)) - p_-(t, kx(t)) \geq a_0 \tag{45}$$

for any $t \in E_n \setminus (A_{\sigma_0}^n \cup C)$ and $n \in \mathbb{N}$. Since $\Psi(t, a_0) > 0$ for any $t \in T$, so there exist $b_0 > 0$ and $D \in \Sigma$ defined by

$$D = \{ t \in S_m : \Psi(t, a_0) < b_0 \}$$

and such that $\mu(D) < \frac{\varepsilon_0 - \varepsilon}{4}$. Then $\mu(E_n \setminus (A_{\sigma_0}^n \cup C \cup D)) > \frac{\varepsilon_0 - \varepsilon}{2}$ for any $n \in \mathbb{N}$ and, by (45),

$$\Psi(t, p_-(t, k_n x_n(t)) - p_-(t, kx(t))) \geq b_0$$

for any $t \in E_n \setminus (A_{\sigma_0}^n \cup C \cup D)$ and $n \in \mathbb{N}$. By superadditivity of the function $\Psi(t, \cdot)$, we get

$$\Psi(t, p_-(t, k_n x_n(t))) - \Psi(t, p_-(t, kx(t))) \geq b_0 \qquad (46)$$

for any $t \in E_n \setminus (A_{\sigma_0}^n \cup C \cup D)$. Denote $F_n = E_n \setminus (A_{\sigma_0}^n \cup C \cup D)$. Then

$$I_\Psi (p_-(k_n x_n)\chi_{F_n}) - I_\Psi (p_-(t, kx_n)\chi_{F_n}) \geq b_0\mu(\chi_{F_n}) \geq \frac{b_0(\varepsilon_0 - \varepsilon)}{2}. \qquad (47)$$

This contradicts inequality (41) because

$$0 \geq \limsup_{n \to \infty} \left[I_\Psi (p_-(k_n x_n)\chi_A) - I_\Psi (p_-(kx)\chi_A) \right]$$
$$\geq \limsup_{n \to \infty} \left[I_\Psi (p_-(k_n x_n)\chi_{F_n}) - I_\Psi (p_-(kx)\chi_{F_n}) \right]$$
$$\geq \frac{b_0(\varepsilon_0 - \varepsilon)}{2}.$$

Therefore (43) is true, and together with (38), it gives

$$(k_n x_n - kx)\chi_{S_m} \xrightarrow{\mu} 0. \qquad (48)$$

By Remark 2.2, the sequence (x_n) is norm equi-continuous and, by Lemma 2.9, we have $\overline{k} = \max(\sup_n k_n, k) < \infty$. Since $\|k_n x_n \chi_E\|_\Phi^o \leq \overline{k}\|x_n \chi_E\|_\Phi^o$ for any $E \in \Sigma$, so norm equi-continuity of (x_n) implies norm equi-continuity of $(k_n x_n)$. Moreover, $\mu(S_m) < \infty$, so proceeding as in the proof of Lemma 2.3, we get $\|(k_n x_n - kx)\chi_{S_m}\|_\Phi^o \to 0$ as $n \to \infty$. Therefore

$$|k_n - k| = |k_n\|x_n\|_\Phi^o - k\|x\|_\Phi^o|$$
$$\leq \|k_n x_n - kx\|_\Phi^o$$
$$\leq \|(k_n x_n - kx)\chi_{S_m}\|_\Phi^o + \|(k_n x_n - kx)\chi_{T \setminus S_m}\|_\Phi^o$$
$$\leq \|(k_n x_n - kx)\chi_{S_m}\|_\Phi^o + \|k_n x_n\chi_{T \setminus S_m}\|_\Phi^o + \|kx\chi_{T \setminus S_m}\|_\Phi^o$$
$$< \|(k_n x_n - kx)\chi_{S_m}\|_\Phi^o + \overline{k}\varepsilon$$

for any $n > n'$. This means that $k_n \to k$, whence, by (48), we get $(x_n - x)\chi_{S_m} \xrightarrow{\mu} 0$. Then, by Lemma 2.3, we have $\|(x_n - x)\chi_{S_m}\|_\Phi^o \to 0$ as $n \to \infty$. Moreover, $\|(x_n - x)\chi_{T \setminus S_m}\|_\Phi^o < \varepsilon$, which gives that $\|(x_n - x)\chi_{T \setminus S_m}\|_\Phi^o \to 0$ as $n \to \infty$. ∎

From Theorems 2.5 and 2.10, we can easily get criteria for the CLUR-property and the LUR-property of L_Φ and L_Φ^o. We should note here that criteria for the LUR-property of L_Φ have been originally given by Kamińska in [14].

Theorem 2.11. *The following conditions are equivalent:*

    (i) $L_\Phi \in (LUR)$

   (ii) $L_\Phi \in (CLUR)$

  (iii)   **(a)** $\Phi \in \Delta_2$,

         (b) $\Phi(t, \cdot)$ *is strictly convex on* $\mathbb{R}$ *for* μ-*a.e.* $t \in T$.

Proof. (iii)$\Rightarrow$(i). Let $\|x\|_\Phi = \|x_n\|_\Phi = 1$ and $\left\|\frac{x_n + x}{2}\right\|_\Phi \to 1$. Assume that conditions (a) and (b) in (iii) hold. Similarly as in the proof of the sufficiency of Theorem 2.5 we show that $x_n \to x$ in measure. By (a) in condition (iii), we get $I_\Phi(x_n) = I_\Phi(x) = 1$. Using Lemma 1.4 we finish the proof of this part.

(i)$\Rightarrow$(ii). This is true by the definitions of the properties LUR and CLUR.

(ii)$\Rightarrow$(iii). Since the Kadec-Klee property of L_Φ implies conditions (a) and (b) in (iii) (see [5]), so using the fact that $CLUR$ implies the Kadec-Klee property, we get the necessity of these conditions for the property $CLUR$ of L_Φ. ∎

Theorem 2.12. *The following conditions are equivalent:*

    (i) $L_\Phi^o \in (LUR)$,

   (ii) $L_\Phi^o \in (CLUR)$,

  (iii)   **(a)** $\Phi \in \Delta_2$,

         (b) $\Psi \in \Delta_2$,

         (c) $\Phi(t, \cdot)$ *is strictly convex on* $\mathbb{R}$ *for* μ-*a.e.* $t \in T$.

Proof. (i)$\Rightarrow$(ii). This is true by the definitons of the properties LUR and CLUR.

(ii)$\Rightarrow$(iii). Since the Kadec-Klee property of L_Φ^o implies conditions (a) and (c) in (iii) (see [9], proof of Theorem 3.4) so, using the fact that $CLUR$ implies the Kadec-Klee property, we get the necessity of these conditions for property $CLUR$ of L_Φ. Condition (b) in (iii) is necessary by Theorem 2.10.

(iii)$\Rightarrow$(i). We can proceed as in the proof of the sufficiency of Theorem 2.10, which completes the proof. ∎

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