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Uniform Convergence in Topological Groups

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Abstract. We improve the Basic Matrix Theorem of Antosik-Swartz in the framework of topological groups. We also obtain an equivalent form of this generalization, which improves the Uniform Convergence Principle of Qu Wenbo and Wu Junde in Proc. Amer. Math. Soc. 130 (2002), 3283 – 3285.

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1. Introduction

In [7] Qu Wenbo and Wu Junde pointed out that the Uniform Convergence Principle is an equivalent form of the Basic Matrix Theorem ([4]). The Uniform Convergence Principle is just the main result of [6] and also is studied by Swartz ([9]). Qu Wenbo and Wu Junde ([7]) consider a sequence $(f_j)_j$ of G-valued sequentially continuous functions defined on Ω, where G denotes an abelian topological group and Ω is a sequentially compact topological space. One of the basic ideas which allows the equivalence between these two results to be proved is the following:

> Let $(x_{ij})_{i,j}$ be a matrix in G such that $\lim_i x_{ij} = 0$ for each $j \in \mathbb{N}$. A sequence $(f_j)_j$ similar to the previous one can be constructed as follows: Let $\Omega = \{\frac{1}{i}, 0\}_{i=1}^{\infty}$ and let $f_j : \Omega \to G$ satisfy that $f_j(\frac{1}{i}) = x_{ij}$ and $f_j(0) = 0$ for each $i, j \in \mathbb{N}$.

Let $\mathcal{F}$ be a natural family (i.e., a subfamily of $P(\mathbb{N})$ which contains the finite subsets) with the property SC, which stands for subsequentially completed ([5], [1], see Definition 1). In the literature a family with property SC is also called a permeating family ([8]). Let $(x_{ij})_{i,j}$ be a matrix in a normed group E such that:

(a) $\lim_i x_{ij} = x_j$ exists for each $j \in \mathbb{N}$

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(b) for each sequence $(m_j)_j$ there exits a subsequence $(n_j)_j$ of $(m_j)_j$ such that $\left(\sum_{j=1}^{\infty} x_{in_j}\right)_i$ is a Cauchy sequence.

In this setting, the Basic Matrix Theorem asserts that $(x_{ij})_i$ are uniformly convergent on $j \in \mathbb{N}$. An equivalent form of the Basic matrix theorem is the following:

Let $\mathcal{F}$ be a natural family with property SC *and let $(x_{ij})_{i,j}$ be a matrix in a normed group E such that $\left(\sum_{j \in B} x_{ij}\right)_i$ is a Cauchy sequence for each $B \in \mathcal{F}$, then $(x_{ij})_i$ are Cauchy sequences uniformly on $j \in \mathbb{N}$.*

Many applications of the Basic Matrix Theorem in measure theory and Banach space theory have been found since its appearance (see, e.g., [4], [9] and [10]), such as generalizations of the uniform boundedness principle, the Banach-Steinhaus theorem and the classical Schur and Phillips lemmas.

In [2] we improved the Basic Matrix Theorem in the framework of a normed space by using a separation property (called the P_{c_0} property) of natural families. Here we extend this result to topological groups and, as a corollary, we also generalize Corollary 1 in [7], which allows the equivalence between the Basic Matrix Theorem and the Uniform Convergence Principle to be obtained. We prove that both generalizations are equivalent.

These results and our separation property P_{c_0} allow us to obtain, as a corollary, the Uniform Convergence Principle under weaker hypothesis than those that appear in [7].

2. Basic matrix theorem

In this section we first introduce the notation and definitions we need to extend our improvement of the Basic Matrix Theorem ([2]) to topological groups. We also prove its equivalent form (which is called Corollary 1). These results lead us to complete the third section with the mentioned relation with the Uniform Convergence Principle.

Definition 1. We say that $\mathcal{F}$ is a *natural family* if $\phi_0(\mathbb{N}) \subseteq \mathcal{F} \subseteq P(\mathbb{N})$, where $\phi_0(\mathbb{N})$ denotes the family of finite subsets of $\mathbb{N}$.

Let $\mathcal{F}$ be a natural family and let $\sum_{i \geq 1} x_i$ be a series in an abelian topological group G. We say that the series $\sum_i x_i$ is *$\mathcal{F}$-convergent* (resp. *$\mathcal{F}$-Cauchy*, *$\mathcal{F}$-weakly convergent*, *$\mathcal{F}$-weakly Cauchy*) if the series $\sum_{i \in A} x_i$ is convergent (resp. Cauchy, weakly convergent, weakly Cauchy), for each $A \in \mathcal{F}$.

It is said that a natural family $\mathcal{F}$ has the *property* SC if for every infinite set $M \subseteq \mathbb{N}$ there exists an infinite set $P \subseteq M$ such that $P \in \mathcal{F}$.

Definition 2. [2] We will say that a natural family $\mathcal{F}$ has the *property* P_{c_0} if there exists a map $f : \mathbb{N} \to \mathbb{N}$ such that for every pair of sequences $(j_r)_r$ and

$(m_r)_r$ in $\mathbb{N}$ with $j_1 < m_1 < j_2 < m_2 < \cdots$ there exists an infinite set $M \subseteq \mathbb{N}$ and a set $B \in \mathcal{F}$ that verify

(a) $(m_{r-1}, m_r) \cap B = \{j_r\}$, for each $r \in M$.

(b) $card([m_{r-1}, m_r] \cap B) \le f(r)$, for each $r \in \mathbb{N} \setminus M$.

It can be checked that each natural family with property SC has the property P_{c_0}. However, it will be shown that there exist natural families which have the property P_{c_0} and lack property SC (see remark at the end of this section).

Definition 3. Let $(x_{ij})_{i,j}$ be a matrix in an abelian Hausdorff topological group G. We will say that $(x_{ij})_{i,j}$ is P_{c_0}-*Cauchy* if there exist a map $f : \mathbb{N} \to \mathbb{N}$ such that if $(j_r)_r$ and $(m_r)_r$ are sequences of natural numbers with $j_1 < m_1 < j_2 < m_2 < \dots$, then there exist a set $B \subseteq \mathbb{N}$ and an infinite set $M \subseteq \mathbb{N}$ with the properties

(a) for $r \in M$, $r > 1$: $(m_{r-1}, m_r) \cap B = \{j_r\}$

(b) for $r \in \mathbb{N} \setminus M$: $\text{card}([m_{r-1}, m_r] \cap B) \le f(r)$

(c) $\left(\sum_{j \in B} x_{ij}\right)_i$ is a Cauchy sequence.

We will say that $(x_{ij})_{i,j}$ is P_{c_0}-*convergent to* $x \in G$ if $(x_{ij})_{i,j}$ verifies the obove conditions (a), (b) and the condition

(c') $\left(\sum_{j \in B} x_{ij}\right)_i$ converges to x,

which replaces condition (c).

The following result is a generalization of the Basic Matrix Theorem ([4]) in terms of property P_{c_0}.

Theorem 1. *Let G be an abelian Hausdorff topological group and let $(x_{ij})_{i,j}$ be a matrix in G. Let us suppose that*

1. *for $j \in \mathbb{N}$: $(x_{ij})_i$ is a Cauchy sequence*

2. *$(x_{ij})_{i,j}$ is a P_{c_0}-Cauchy matrix.*

Then, $(x_{ij})_i$ are Cauchy sequences uniformly on $j \in \mathbb{N}$.

Proof. We first prove that $(x_{ij})_j$ converges to zero, for $i \in \mathbb{N}$. In the contrary, let $i_0 \in \mathbb{N}$ be such that $(x_{i_0 j})_j$ does not converge to zero. Let U be a neighbourhood of zero and let $(j_r)_r$ and $(m_r)_r$ be two sequences in $\mathbb{N}$ such that $j_1 < m_1 < j_2 < m_2 < \dots$ and $x_{i_0 j_r} \notin U$ for $r \in \mathbb{N}$. Let $B, M \subseteq \mathbb{N}$ be two infinite sets that verify the conditions (a), (b) and (c) of Definition 3. Then

$$\sum_{\substack{j \in B \\ j \in [1, m_r]}} x_{i_0 j} - \sum_{\substack{j \in B \\ j \in [1, m_{r-1}]}} x_{i_0 j} = x_{i_0 j_r} \notin U$$

for $r \in M$, $r > 1$. This contradicts the fact that $\sum_{j \in B} x_{i_0 j}$ is a convergent series.

Let us prove that the sequences $(x_{ij})_i$ are Cauchy sequences uniformly on $j \in \mathbb{N}$. In the contrary, there exists a symmetric neighbourhood of zero U such that for each $k \in \mathbb{N}$ there exist $i > k$ and $j \in \mathbb{N}$ satisfying $x_{ij} - x_{kj} \notin U$. Let V be a closed symmetric neighbourhood of zero such that $V + V + V \subseteq U$. Let f be the map which appears in Definition 3. We now proceed by induction.

`Step 1.` The following argument is very similar to the remaining inductive steps:

(i) There exist $i_1 > k_1 > 1$ and $j_1 \in \mathbb{N}$ such that $x_{i_1 j_1} - x_{k_1 j_1} \notin U$.

(ii) Let V_{10} and V_{11} be symmetric neighbourhoods of zero such that $V_{10} + V_{10} \subseteq V$ and $V_{11} + V_{11} + \overset{3f(2)}{\cdots\cdots} + V_{11} \subseteq V_{10}$ (which means V_{11} must be added $3f(2)$ times).

(iii) Since $\lim_j x_{ij} = 0$, for each $i \in \mathbb{N}$, there is an integer $m_1 > j_1$ such that $x_{ij} \in V_{11}$, for each $j > m_1$ and each $i \in \{1, 2, \ldots i_1\}$.

(iv) Since $(x_{ij})_i$ is Cauchy, for each $j \in \mathbb{N}$, there exists $k_0 > i_1$ such that $\sum_{j \in C}(x_{ij} - x_{kj}) \in V$ for all $i, k > k_0$ and all $C \subseteq \{1, 2, \ldots, m_1\}$. Then there exist $i_2 > k_2 > k_0 > i_1$ and $j_2 > m_1 > j_1$ such that $x_{i_2 j_2} - x_{k_2 j_2} \notin U$.

`Step 2.` We apply this argument again with the following two differences:

(a) k_2 has been already defined.

(b) The neigbourhoods V_{20} and V_{21} must verify $V_{20} + V_{20} \subseteq V_{10}$ and $V_{21} + V_{21} + \overset{3f(3)}{\cdots\cdots} + V_{21} \subseteq V_{20}$.

We continue in this fashion to complete this inductive argument and so to obtain

- four sequences $(k_r)_r$, $(i_r)_r$, $(j_r)_r$ and $(m_r)_r$ of natural numbers such that $k_1 < i_1 < k_2 < i_2 < \ldots$ and $j_1 < m_1 < j_2 < m_2 < \ldots$.
- two sequences $(V_{r0})_r$ $(V_{r1})_r$ of symmetric neighbourhoods of zero such that $V_{r0} + V_{r0} \subseteq V_{(r-1)0}$ for $r \geq 1$ (where $V_{00} = V$) and $V_{r1} + V_{r1} + \overset{3f(r+1)}{\cdots\cdots} + V_{r1} \subseteq V_{r0}$.

These six sequences verify the following three properties:

a. $x_{i_r j_r} - x_{k_r j_r} \notin U$

b. $\sum_{j \in C}(x_{i_r j} - x_{k_r j}) \in V$ for $C \subseteq \{1, 2, \ldots m_{r-1}\}$

c. $x_{ij} \in V_{r1}$ for $i \in \{1, 2, \ldots i_r\}$ and $j \geq m_r$.

The sequences $(j_r)_r$ and $(m_r)_r$ allow us to consider the sets B and M as those ones in Definition 3. We will prove that $\left(\sum_{j \in B} x_{ij}\right)_i$ is not a Cauchy sequence, which leads us to a contradiction. For $r \in M$, $r > 1$, we have that

$$\sum_{j \in B}(x_{i_r j} - x_{k_r j}) = \sum_{\substack{j \in B \\ j \leq m_{r-1}}} (x_{i_r j} - x_{k_r j}) + (x_{i_r j_r} - x_{k_r j_r}) + \sum_{\substack{j \in B \\ j \geq mr}} (x_{i_r j} - x_{k_r j}) \ . \quad (1)$$

From b. (see the items just above) it follows that

$$\sum_{\substack{j\in B\\ j\le m_{r-1}}} (x_{i_r j} - x_{k_r j}) \in V \ .$$

Let us check this property for $j \in B, j \geq mr$. Since V is closed, it is sufficient to prove that

$$\sum_{\substack{j\in B\\ j\in[m_r, m_{r+k}]}} (x_{i_r j} - x_{k_r j}) \in V \quad \text{for } k \in \mathbb{N} \ .$$

For $k = 1$ we consider the following two cases:

`Case 1.` If $(r+1) \notin M$, we have that $\operatorname{card}([m_r, m_{r+1}] \cap B) \leq f(r+1)$. Hence

$$\sum_{\substack{j\in B\\ j\in[m_r, m_{r+1}]}} (x_{i_r j} - x_{k_r j}) \in V_{r1} + V_{r1} + \ldots^{f(r+1)} + V_{r1} \subseteq V_{r0} \ .$$

`Case 2.` If $(r+1) \in M$ then $\operatorname{card}([m_r, m_{r+1}] \cap B) \leq 3$. Hence

$$\sum_{\substack{j\in B\\ j\in[m_r, m_{r+1}]}} (x_{i_r j} - x_{k_r j}) \in V_{r1} + V_{r1} + V_{r1} \subseteq V_{r0} \ .$$

It is easy to check, as before, that

$$\sum_{\substack{j\in B\\ j\in[m_{r+k-1}, m_{r+k}]}} (x_{i_r j} - x_{k_r j}) \in V_{(r+k-1)0}$$

for $k \in \mathbb{N}$. Since $V_{h0} + V_{h0} \subseteq V_{(h-1)0}$ for each $h \geq 1$, we have that

$$\begin{aligned}\sum_{\substack{j\in B\\ j\in[m_r, m_{r+k}]}} (x_{i_r j} - x_{k_r j}) &= \sum_{l=0}^{k-1} \sum_{\substack{j\in B\\ j\in[m_{r+l}, m_{r+l+1}]}} (x_{i_r j} - x_{k_r j}) \\ &\in V_{r0} + V_{(r+1)0} + \cdots + V_{(r+k-3)0} + V_{(r+k-2)0} + V_{(r+k-1)0}\end{aligned}$$

with

$$\begin{aligned}V_{r0} + V_{(r+1)0} + \cdots &+ V_{(r+k-3)0} + V_{(r+k-2)0} + V_{(r+k-1)0} \\ &\subseteq V_{r0} + V_{(r+1)0} + \cdots + V_{(r+k-3)0} + (V_{(r+k-2)0} + V_{(r+k-2)0}) \\ &\subseteq V_{r0} + V_{(r+1)0} + \cdots + (V_{(r+k-3)0} + V_{(r+k-3)0}) \subseteq \ldots \\ &\subseteq V_{r0} + V_{(r+1)0} + V_{(r+1)0} \\ &\subseteq V_{r0} + V_{r0} \\ &\subseteq V_{(r-1)0} \\ &\subseteq V.\end{aligned}$$

From equality (1) (see at the end of the last page) it follows that $\sum_{j\in B}(x_{i_r j} - x_{k_r j}) \notin V$ for $r \in M$, $r > 1$. This contradiction proves the theorem. ∎

If $\mathcal{F}$ is a natural family with property P_{c_0}, then the result above remains valid if the hypotheses 1. and 2. are replaced by the condition that $\left(\sum_{j\in B} x_{ij}\right)_i$ is a Cauchy sequence for each $B \in \mathcal{F}$.

Remark 1. An application of Theorem 1 in measure theory, which can easily be proved, is the following one: Let $\mathcal{F}$ be a natural family with the property P_{c_0} and G an abelian topological group. If $\mu_n : \mathcal{F} \to G$ is a σ-additive measure for each $n \in \mathbb{N}$, and $(\mu_n(A))_n$ is Cauchy at each $A \in \mathcal{F}$, then for every pairwise disjoint $(A_j)_j \subseteq \mathcal{F}$, the sequence $(\mu_n(A_j))_n$ is uniformly Cauchy with respect to $j \in \mathbb{N}$ (let us observe that $B = \sup\{A \subseteq B\,;\ A \in \phi_0(\mathbb{N})\}$). Clearly, $(\mu_n)_n$ is uniformly strongly additive.

As a corollary of the previous theorem, we obtain a generalization of Corollary 1 in [7] which is an equivalent form of the Basic Matrix Theorem, and also it can be obtained as a consequence of the Uniform Convergence Principle of Qu Wenbo and Wu Junde. We also prove that our generalization is an equivalent form of Theorem 1.

Corollary 1. *Let G be an abelian Hausdorff topological group and let $(x_{ij})_{i,j}$ be a matrix in G. Let us suppose that*

1. *for $j \in \mathbb{N}$: $(x_{ij})_i$ converges to zero*

2. *the matrix $(x_{ij})_{i,j}$ is P_{c_0}-convergent to zero.*

Then the sequences $(x_{ij})_i$ converge to zero uniformly on $j \in \mathbb{N}$. In particular, $\lim_i x_{ii} = 0$.

As in Theorem 1, the result above remains valid if the hypotheses 1. and 2. are replaced by the condition that $\left(\sum_{j\in B} x_{ij}\right)_i$ is convergent to zero for each $B \in \mathcal{F}$, where $\mathcal{F}$ denotes natural family with property P_{c_0}. Let us prove that Theorem 1 and Corollary 1 are equivalent results.

Proof of equivalence. It is obvious that Theorem 1 implies Corollary 1. Conversely, let $(x_{ij})_{i,j}$ be a matrix, in an abelian Hausdorff topological group G that verifies the properties 1. and 2. in Theorem 1. If $(x_{ij})_i$ are not Cauchy sequences uniformly in $j \in \mathbb{N}$, then there exists a symmetric neigbourhood U of zero and three strictly increasing sequences $(k_r)_r$, $(i_r)_r$ and $(j_r)_r$ of natural numbers with $k_1 < i_1 < k_2 < i_2 < \ldots$ and $x_{i_r j_r} - x_{k_r j_r} \notin U$ for $r \in \mathbb{N}$. Let us consider the matrix $(x_{i_r j} - x_{k_r j})_{r,j}$. It is easy to check that $(x_{i_r j} - x_{k_r j})_{r,j}$ verifies the hypothesis of Corollary 1, but the sequences $(x_{i_r j} - x_{k_r j})_i$ don't converge uniformly on $j \in \mathbb{N}$. This contradiction proves the equivalence between Theorem 1 and Corollary 1. ∎

Remark 2. The following examples show that there exist natural families with property P_{c_0} that lack property SC. Let $\mathcal{B}_1$ be the family of sets $B \subseteq \mathbb{N}$ with the following properties:

(a) B and B^c have infinite even numbers and infinite odd numbers.

(b) The set $\{n \in \mathbb{N} : \{4n-1, 4n\} \subseteq B\}$ is finite.

Let $\mathcal{F}_1 = \mathcal{B}_1 \cup \phi_0(\mathbb{N})$. It can be seen (see [2]) that $\mathcal{F}_1$ has the property P_{c_0}. However $\mathcal{F}_1$ does not have the property SC (it is sufficient to observe that there is no infinite subset of the set of even numbers that belongs to $\mathcal{F}_1$).

Let Q_1 be an infinite subset of $\mathbb{N}$ whose complementary set is also infinite. Let Q_2 and Q_3 be two infinite subsets of $\mathbb{N}$ such that at least one of them has an infinite complement. It can also be shown ([2]) that the following families have property P_{c_0} and lack property SC:

- $\mathcal{F}_\alpha = \{B \subseteq \mathbb{N} : B \cap Q_1 \text{ is infinite}\} \cup \phi_0(\mathbb{N})$
- $\mathcal{F}_\beta = \{B \subseteq \mathbb{N} : B \cap Q_1 \text{ and } B^c \cap Q_1 \text{ are infinite}\} \cup \phi_0(\mathbb{N})$
- $\mathcal{F}_\sigma = \{B \subseteq \mathbb{N} : B \cap Q_2 \text{ and } B \cap Q_3 \text{ are infinite}\} \cup \phi_0(\mathbb{N})$
- $\mathcal{F}_\theta = \{B \subseteq \mathbb{N} : B \cap Q_2, B \cap Q_3, B^c \cap Q_2 \text{ and } B^c \cap Q_3 \text{ are infinite}\} \cup \phi_0(\mathbb{N})$.

3. Uniform convergence principle

In [7], there is an equivalent form of the Basic Matrix Theorem, called the Uniform Convergence Principle. By considering the results of Section 2, we obtain an equivalent form of Theorem 1.

Lemma 1. *Let G be an abelian Hausdorff topological group and let $(x_{ij})_{i,j}$ be a matrix in G such that $(x_{ij})_i$ and $(x_{ij})_j$ are Cauchy sequences, for $j \in \mathbb{N}$ and $i \in \mathbb{N}$ respectively. Then:*

1. *The following statements are equivalent:*

  a. *$(x_{ij})_i$ are Cauchy sequences uniformly on $j \in \mathbb{N}$.*

  b. *$(x_{ij})_j$ are Cauchy sequences uniformly on $i \in \mathbb{N}$.*

  c. *For each neighbourhood of zero U, there exist $i_0, j_0 \in \mathbb{N}$ such that $x_{ij} - x_{kl} \in U$ for $i, k \geq i_0$ and $j, l \geq j_0$.*

2. *If the sequence $(x_{ij})_i$ converges to some x_j, for $j \in \mathbb{N}$, and $\lim_j x_{ij} = x_i$ for $i \in \mathbb{N}$, then, under one of the hypothesis given in 1., it follows that:*

  a. *The sequences $(x_{ij})_i$ converge to x_j uniformly on $j \in \mathbb{N}$.*

  b. *The sequences $(x_{ij})_j$ converge to x_i uniformly on $i \in \mathbb{N}$.*

Proof. Let us prove assertion 1. We first show that a. $\Rightarrow$ b. If b. is false, let U be a symmetric neighbourhood of zero such that for $k \in \mathbb{N}$ there exist $j > k$ and $i \in \mathbb{N}$ such that $x_{ij} - x_{ik} \notin U$. Let V be a symmetric neighbourhood of zero such that $V + V + V \subseteq U$. Let us consider the following properties:

(i) We can inductively construct three strictly increasing sequences $(k_r)_r$, $(j_r)_r$ and $(i_r)_r$ such that $k_1 < j_1 < k_2 < j_2 < \ldots$ and $x_{i_r j_r} - x_{i_r k_r} \notin U$ for $r \in \mathbb{N}$.

(ii) By $a.$, let $i_0 \in \mathbb{N}$ be such that $x_{pj} - x_{qj} \in V$ for $j \in \mathbb{N}$ and $p, q \geq i_0$.

(iii) The sequence $(x_{i_0 j})_j$ is a Cauchy sequence, and so let $j_0 \in \mathbb{N}$ be such that $x_{i_0 l} - x_{i_0 h} \in V$ for $l, h \geq j_0$.

For $r \in \mathbb{N}$ with $i_r > i_0$ and $k_r > j_0$, we have that

$$(x_{i_r j_r} - x_{i_r k_r}) = (x_{i_r j_r} - x_{i_0 j_r}) + (x_{i_0 j_r} - x_{i_0 k_r}) + (x_{i_0 k_r} - x_{i_r k_r}) \ .$$

From (ii) and (iii) (see the items just above) it follows that $(x_{i_r j_r} - x_{i_r k_r}) \in U$, which contradicts (i).

The proof of $b. \Rightarrow a.$ is similar to the previous one. It is obvious that for $c. \Rightarrow a.$ Analysis similar to that in the proof of $a. \Rightarrow b.$ shows that $c.$ follows from $a.$ and $b.$

We now prove assertion 2.a. (the proof of statement b. is similar). Let us check that $(x_j)_j$ is a Cauchy sequence. Let U be a symmetric neighbourhood of zero, under our hypothesis we can consider:

(iv) $(x_{ij})_j$ are Cauchy sequences uniformly on $i \in \mathbb{N}$ and so there exists $j_0 \in \mathbb{N}$ such that $x_{ip} - x_{iq} \in U$ for $i \in \mathbb{N}$ and $p, q \geq j_0$.

(v) Let V be a symmetric neighbourhood of zero such that $V + V + V \subseteq U$. We fix $p, q \geq j_0$. Let $i_0 \in \mathbb{N}$ be such that $x_p - x_{ip} \in V$ and $x_q - x_{iq} \in V$ for $i \geq i_0$. From the previous assertion (called (iv)) we have that $(x_p - x_q) = (x_p - x_{i_0 p}) + (x_{i_0 p} - x_{i_0 q}) + (x_{i_0 q} - x_q) \in V + V + V \subseteq U$. This proves that $(x_j)_j$ is a Cauchy sequence.

Since $(x_j)_j$ is a Cauchy sequence, the proof is now an easy consequence of assertion 1.c. ∎

In order to extend the Uniform Convergence Principle to a natural family with the property P_{c_0} (let us observe that in the Uniform Convergence Principle appears the property SC), we first define the *pointwisely* P_{c_0}*-convergence* of a series $\sum_i f_i$.

Definition 4. Let G be an abelian Hausdorff topological group and let Ω be a topological space. Let $(f_i)_i$ be a sequence of sequentially continuous G-valued functions defined on Ω. We will say that $\sum_i f_i$ is *pointwisely* $\mathcal{F}$*-convergent*, where $\mathcal{F}$ denotes a natural family, if for each $B \in \mathcal{F}$

(a) the series $\sum_{j \in B} f_j(\omega)$ converges for each $\omega \in \Omega$

(b) $\sum_{j \in B} f_j : \Omega \to G$ is sequentially continuous.

Definition 5. Let G be an abelian Hausdorff topological group and let Ω be a topological space. Let $(f_i)_i$ be a sequence of sequentially continuous G-valued functions defined on Ω. We will say that $\sum_i f_i$ is *pointwisely* P_{c_0}*-convergent* if there exist a map $f : \mathbb{N} \to \mathbb{N}$ such that if $(j_r)_r$ and $(m_r)_r$ are sequences of natural numbers with $j_1 < m_1 < j_2 < m_2 < \ldots$, then there exist $B \subseteq \mathbb{N}$ and an infinite set $M \subseteq \mathbb{N}$ with the properties that

(a) for each $r \in M$, $r > 1$: $(m_{r-1}, m_r) \cap B = \{j_r\}$
(b) for $r \in \mathbb{N} \setminus M$: $\operatorname{card}([m_{r-1}, m_r] \cap B) \leq f(r)$
(c) $\sum_{j \in B} f_j(\omega)$ converges for each $\omega \in \Omega$ and $\sum_{j \in B} f_j : \Omega \to G$ is sequentially continuous.

We now prove our mentioned generalization of the Uniform Convergence Principle.

Theorem 2. *Let G be an abelian Hausdorff topological group and let Ω be a sequentially compact topological space. Let $(f_i)_i$ be a sequence of sequentially continuous G-valued functions defined on Ω. If $\sum_i f_i$ is pointwisely P_{c_0}-convergent, then $\lim_j f_j(\omega) = 0$ uniformly with respect to $\omega \in \Omega$.*

Proof. As in the proof of Theorem 1, it can be shown that $(f_j(\omega))_j$ converges to zero for $\omega \in \Omega$. If the convergence is not uniform on $\omega \in \Omega$, then there exist a symmetric neighbourhood of zero U, a strictly increasing sequence $(j_k)_k$ in $\mathbb{N}$ and a sequence $(\omega_k)_k$ in Ω such that $f_{j_k}(\omega_k) \notin U$. Let $(w_{n_k})_k$ be a subsequence that converges to some $\omega_0 \in \Omega$.

Let us check that the matrix $(f_j(\omega_{n_i}))_{i,j}$ verifies the hypothesis of Theorem 1. It is obvious that $(f_j(\omega_{n_i}))_i$ converges to $f_j(\omega_0)$. If $(j_r)_r$ and $(m_r)_r$ are two sequences in $\mathbb{N}$ such that $j_1 < m_1 < j_2 < m_2 < \dots$, let $B, M \subseteq \mathbb{N}$ be the corresponding sets that appear in Definition 5. It is clear that $\left(\sum_{j \in B} f_j(\omega_{n_i})\right)_i$ converges to $\sum_{j \in B} f_j(\omega_0)$. By Theorem 1 and Lemma 1, it follows that $\left(f_j(\omega_{n_i})\right)_j$ converges to zero uniformly on $i \in \mathbb{N}$. This contradicts that $f_{i_{n_k}}(\omega_{n_k}) \notin U$ for $k \in \mathbb{N}$. ∎

If $\mathcal{F}$ is a natural family with property P_{c_0}, then the result above remains valid if $\sum_i f_i$ is pointwisely $\mathcal{F}$-convergent, instead of pointwisely P_{c_0}-convergent.

Remark 3 (Uniform Convergence Principle). Let G be an abelian Hausdorff topological group and let Ω be a sequentially compact topological space. Let $(f_i)_i$ be a sequence of sequentially continuous G-valued functions defined on Ω. If each strictly increasing sequence $\{m_j\}$ in $\mathbb{N}$ has a subsequence $\{n_j\}$ such that for each $\omega \in \Omega$, the series $\sum_j f_{n_j}(\omega)$ converges and $\sum_j f_{n_j} : \Omega \to G$ is sequentially continuous, then Qu Wenbo and Wu Junde proved in [7] that $\lim_j f_j(\omega) = 0$ uniformly with respect to $\omega \in \Omega$.

This result considers a sequence of functions $(f_i)_i$ such that $\sum_i f_i$ is pointwisely $\mathcal{F}$-convergent, where $\mathcal{F}$ denotes a natural family with property SC. Theorem 2 is similar to this one, but we consider the property P_{c_0} instead of the property SC.

Let us check that Theorem 1 and Theorem 2 are two equivalent results.

Proof of equivalence. It is obvious that Theorem 1 implies Theorem 2. By Remark 2, it follows that Corollary 1 is equvalent to Theorem 1. Hence, it is sufficient to prove that Theorem 2 implies Corollary 1.

Let $(x_{ij})_{i,j}$ be a matrix with the conditions that appear in Corollary 1. Let $\Omega = \{\frac{1}{i}, 0\}_{i=1}^{\infty}$. For $x, y \in \Omega$ we define $d(x,y) = |x - y|$. It is clear that (Ω, d) is a sequentially compact topological space. Let $f_j : \Omega \to G$ be the map defined by $f_j(\omega) = x_{ij}$, for $\omega = \frac{1}{i}$, $f_j(0) = 0$. It is easy to check that f_j is continuous.

Let $(j_r)_r$ and $(m_r)_r$ be two sequences in $\mathbb{N}$ with $j_1 < m_1 < j_2 < m_2 < \dots$ and let $B, M \subseteq \mathbb{N}$ the corresponding sets that appear in Definition 3. The sequence $(f_j)_j$ is pointwisely P_{c_0}-convergent and, by Theorem 2, $\lim_j f_j(\frac{1}{i}) = \lim_j x_{ij} = 0$ uniformly on $i \in \mathbb{N}$. Lemma 1 allow us to complete the proof. ∎

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