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On Representation, Boundedness and Convergence of Hankel-$K\{M_p\}'$ Generalized Functions

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Abstract. Under opportune assumptions on the defining sequence $\{M_p\}_{p=0}^{\infty}$, Hankel-$K\{M_p\}'$ generalized functions can be represented as

$$f = x^{-\mu-\frac{1}{2}}(Dx^{-1})^k F(x),$$

where $k \in \mathbb{N}$ and F is a continuous function on $I = (0, \infty)$ such that $M_r^{-1} F \in L^q(I)$ $(1 \leq q \leq \infty)$ for some $r \in \mathbb{N}$. A corresponding characterization of boundedness and convergence of Hankel-$K\{M_p\}'$ generalized functions is given.

Keywords: *Hankel transformation, Gelfand-Shilov spaces, generalized functions*

MSC 2000: Primary 46F05, secondary 46F12

1. Introduction

Let $\mu \geq -\frac{1}{2}$, and let $\{M_p\}_{p=0}^{\infty}$ be a sequence of continuous functions defined on $I = (0, \infty)$, such that

$$1 = M_0(x) \leq M_1(x) \leq M_2(x) \leq \dots \ . \tag{1}$$

We say that $K_\mu\{M_p\}$ is a *Hankel-$K\{M_p\}$ space (of order μ)* if it consists of all those functions $\varphi \in C^\infty(I)$ such that

$$\|\varphi\|_{\mu,p} = \max_{0\leq k\leq p} \sup_{x\in I} \left| M_p(x)(x^{-1}D)^k x^{-\mu-\frac{1}{2}} \varphi(x) \right| < \infty \quad (p \in \mathbb{N}).$$

The linear space $K_\mu\{M_p\}$ is endowed with the locally convex topology generated by the sequence of norms $\{\|\cdot\|_{\mu,p}\}_{p=0}^{\infty}$. We denote by $K_\mu\{M_p\}'$ the dual of $K_\mu\{M_p\}$.

The space $K_\mu\{M_p\}$ was introduced and studied by the author in [5], [3], [4] as a Hankel transformation related version of the Gelfand–Shilov $K\{M_p\}$

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spaces [1]. It encompasses various classes of test functions arising in the theory of the generalized Hankel transformation, such as the Zemanian spaces $\mathcal{B}_{\mu,a}$ ($a > 0$) [6] and $\mathcal{H}_\mu$ [7, Chapter 5], which are obtained for special choices of the defining sequence $\{M_p\}_{p=0}^\infty$. Additional examples are given in [5]. Thus the consideration of a general sequence $\{M_p\}_{p=0}^\infty$ of weights allows one to unify the treatment given to such a variety of spaces.

Let us consider the following conditions on $\{M_p\}_{p=0}^\infty$:

(A) To any $r, p \in \mathbb{N}$ there correspond $s \in \mathbb{N}$ and $b_{rp} > 0$ such that

$$M_r(x)M_p(x) \le b_{rp} M_s(x) \quad (x \in I).$$

(M) Each M_p ($p \in \mathbb{N}$) is *quasi-monotonic*: there exists $C_p > 0$ such that

$$M_p(x) \le C_p M_p(y) \quad (x, y \in I,\ x \le y).$$

(N) To every $p \in \mathbb{N}$ there corresponds $r \in \mathbb{N}$, $r > p$, such that the function

$$m_{pr}(x) = \frac{M_p(x)}{M_r(x)} \quad (x \in I)$$

lies in $L^1(I)$ and satisfies $\lim_{x\to\infty} m_{pr}(x) = 0$.

Under the assumptions (A), (M) and (N), the Fréchet space $K_\mu\{M_p\}$ is Montel and hence reflexive [5]. Therefore the weak and weak* topologies of $K_\mu\{M_p\}'$ coincide, and the weak* and strong topologies of $K_\mu\{M_p\}'$ share the same class of bounded sets as well as the same class of convergent sequences.

The following characterization of membership, boundedness and convergence in $K_\mu\{M_p\}'$ was obtained in [5]. From now on, $\|\cdot\|_q$ will stand for the usual $L^q(I)$-norm ($1 \le q \le \infty$).

Theorem 1.1. *Assume that* $\{M_p\}_{p=0}^\infty$ *satisfies the conditions* (A), (M) *and* (N). *Then:*

1. *A functional T belongs to $K_\mu\{M_p\}'$ if and only if, to every q, $1 < q \le \infty$, there corresponds $p \in \mathbb{N}$ such that*

$$T = x^{-\mu-\frac{1}{2}} \sum_{k=0}^{p} (Dx^{-1})^k \left[M_p(x) g_k(x)\right],$$

with $g_k \in L^q(I)$ ($k \in \mathbb{N}$, $0 \le k \le p$).

2. *A set $B \subset K_\mu\{M_p\}'$ is (weakly, weakly*, strongly) bounded if and only if, given $1 < q \le \infty$, there exist $p \in \mathbb{N}$, $C > 0$ and, for every $T \in B$, functions $g_{k,T} \in L^q(I)$ ($k \in \mathbb{N}$, $0 \le k \le p$) such that*

$$T = x^{-\mu-\frac{1}{2}} \sum_{k=0}^{p} (Dx^{-1})^k \left[M_p(x) g_{k,T}(x)\right]$$

with $\sum_{k=0}^{p} \|g_{k,T}\|_q \le C$.

3. *A sequence* $\{T_j\}_{j=0}^{\infty}$ *converges (weakly, weakly*, strongly) to zero in the space* $K_\mu\{M_p\}'$ *if, and only if, to every* $1 < q \leq \infty$ *there correspond* $p \in \mathbb{N}$ *and functions* $g_{k,j} \in L^q(I)$ $(k \in \mathbb{N},\ 0 \leq k \leq p)$ *such that*

$$T_j = x^{-\mu-\frac{1}{2}} \sum_{k=0}^{p} (Dx^{-1})^k \left[M_p(x) g_{k,j}(x)\right] \quad (j \in \mathbb{N})$$

with $\lim_{j\to\infty} \sum_{k=0}^{p} \|g_{k,j}\|_q = 0$.

Our purpose here is to adapt to $K_\mu\{M_p\}'$-spaces the technique used by A. Kaminski [2] for the Gelfand-Shilov $K\{M_p\}'$ spaces in order to simplify and improve the previous result as stated in the next.

Theorem 1.2. *Assume that* $\{M_p\}_{p=0}^{\infty}$ *satisfies the conditions* (A), (M) *and* (N). *Then:*

1. *A functional* f *belongs to* $K_\mu\{M_p\}'$ *if and only if there exist* $k, p \in \mathbb{N}$ *and a continuous function* F *on* I *such that*

$$f = x^{-\mu-\frac{1}{2}} (Dx^{-1})^k F(x)$$

with $M_p^{-1} F \in L^q(I)$ $(1 \leq q \leq \infty)$.

2. *A set* $B \subset K_\mu\{M_p\}'$ *is (weakly, weakly*, strongly) bounded if and only if there exist* $k, p \in \mathbb{N}$, $C > 0$ *and, for every* $f \in B$, *a function* g_f *continuous on* I, *such that*

$$f = x^{-\mu-\frac{1}{2}} (Dx^{-1})^k g_f(x)$$

with $\left\|M_p^{-1} g_f\right\|_q \leq C \quad (1 \leq q \leq \infty)$.

3. *A sequence* $\{f_j\}_{j=0}^{\infty}$ *converges (weakly, weakly*, strongly) to zero in the space* $K_\mu\{M_p\}'$ *if and only if there exist* $k, p \in \mathbb{N}$ *and functions* g_j *continuous on* I *such that*

$$f_j = x^{-\mu-\frac{1}{2}} (Dx^{-1})^k g_j(x) \quad (j \in \mathbb{N})$$

with $\lim_{j\to\infty} \left\|M_p^{-1} g_j\right\|_q = 0 \quad (1 \leq q \leq \infty)$.

Theorem 1.2 summarizes our main results, to be proved in Section 2 (see Theorems 2.4, 2.5 and 2.6). An example is exhibited in Section 3. Throughout the paper we shall assume that the defining sequence $\{M_p\}_{p=0}^{\infty}$ fulfils conditions (A), (M) and (N). Moreover, we shall adopt the practice of denoting by the same letter, usually C, suitable constants whose values need not coincide at different occurrences.

2. Main results

Under the conditions (M) and (N), to every $p \in \mathbb{N}$ there correspond $s \in \mathbb{N}$, $s > p$, and $C > 0$, such that $x \leq CM_s(x)$ $(x \in I)$. Indeed, associate $s \in \mathbb{N}$ to a given $p \in \mathbb{N}$ as in (N). In view of (1) and (M), we may write

$$x = \int_0^x d\xi \leq \int_0^x M_p(\xi) d\xi = \int_0^x \frac{M_p(\xi)}{M_s(\xi)} M_s(\xi)\, d\xi$$

and hence

$$x \leq CM_s(x) \int_0^\infty \frac{M_p(\xi)}{M_s(\xi)}\, d\xi = CM_s(x) \quad (x \in I).$$

The preceding observation will be useful in the sequel. We begin by proving three auxiliary results.

Lemma 2.1. *Let F be a continuous function on I such that there exists $p \in \mathbb{N}$ for which $M_p^{-1}F \in L^q(I)$ $(1 \leq q \leq \infty)$. Then, to every $k \in \mathbb{N}$ there corresponds $p_k \in \mathbb{N}$ with $p_k \geq p$, and a continuous function F_k on I such that $(Dx^{-1})^k F_k(x) = F(x)$ $(x \in I)$ and $M_{p_k}^{-1} F_k \in L^q(I)$ $(1 \leq q \leq \infty)$.*

Proof. The result is obvious for $k = 0$. Arguing by induction, fix $k \in \mathbb{N}$, $k \geq 1$. Choose $p_k \in \mathbb{N}$, $p_k \geq p$, and a continuous function F_k on I such that $(Dx^{-1})^k F_k(x) = F(x)$ $(x \in I)$ and $M_{p_k}^{-1} F_k \in L^q(I)$ $(1 \leq q \leq \infty)$. Using (A), (M) and (N) we may find $n, r, s, t \in \mathbb{N}$, $n > r > s > t > p_k$, such that

$$\int_0^\infty \frac{M_{p_k}(x)}{M_t(x)}\, dx < \infty \tag{2}$$

$$x \leq CM_s(x) \quad (x \in I) \tag{3}$$

$$M_s(x) M_t(x) \leq CM_r(x) \quad (x \in I) \tag{4}$$

and

$$\int_0^\infty \frac{M_r(x)}{M_n(x)}\, dx < \infty. \tag{5}$$

Our induction hypotheses, jointly with (M) and (2), yields

$$\begin{aligned}
\frac{1}{M_t(x)} \left| \int_0^x F_k(\xi)\, d\xi \right| &\leq C \int_0^\infty \left| \frac{F_k(\xi)}{M_t(\xi)} \right| d\xi \\
&= C \int_0^\infty \left| \frac{F_k(\xi)}{M_{p_k}(\xi)} \right| \frac{M_{p_k}(\xi)}{M_t(\xi)}\, d\xi \\
&\leq C \sup_{\xi \in I} \left| \frac{F_k(\xi)}{M_{p_k}(\xi)} \right| \int_0^\infty \frac{M_{p_k}(\xi)}{M_t(\xi)}\, d\xi
\end{aligned}$$

and hence

$$\frac{1}{M_t(x)}\left|\int_0^x F_k(\xi)\,d\xi\right| \le C\sup_{\xi\in I}\left|\frac{F_k(\xi)}{M_{p_k}(\xi)}\right| \quad (x\in I). \tag{6}$$

The function $\tilde{F}_k(x) = x\displaystyle\int_0^x F_k(\xi)\,d\xi$ $(x\in I)$ is continuous with

$$(Dx^{-1})^{k+1}\tilde{F}_k(x) = (Dx^{-1})^k F_k(x) = F(x) \quad (x\in I).$$

By (3) and (6),

$$\frac{\left|\tilde{F}_k(x)\right|}{M_t(x)} = \frac{x}{M_t(x)}\left|\int_0^x F_k(\xi)\,d\xi\right| \le CM_s(x)\sup_{\xi\in I}\left|\frac{F_k(\xi)}{M_{p_k}(\xi)}\right| \quad (x\in I).$$

Using (4) and (1) we get

$$\begin{aligned}\left|\tilde{F}_k(x)\right| &\le CM_s(x)M_t(x)\sup_{\xi\in I}\left|\frac{F_k(\xi)}{M_{p_k}(\xi)}\right| \\ &\le CM_r(x)\sup_{\xi\in I}\left|\frac{F_k(\xi)}{M_{p_k}(\xi)}\right| \\ &\le CM_n(x)\sup_{\xi\in I}\left|\frac{F_k(\xi)}{M_{p_k}(\xi)}\right| \quad (x\in I).\end{aligned} \tag{7}$$

Thus we find

$$\sup_{x\in I}\left|\frac{\tilde{F}_k(x)}{M_n(x)}\right| \le C\sup_{\xi\in I}\left|\frac{F_k(\xi)}{M_{p_k}(\xi)}\right| < \infty.$$

Moreover, for every $1\le q<\infty$, with the aid of (7), (1) and (5) we obtain

$$\begin{aligned}\int_0^\infty \left|\frac{\tilde{F}_k(x)}{M_n(x)}\right|^q dx &= \int_0^\infty \left|\frac{\tilde{F}_k(x)}{M_r(x)}\right|^q \left(\frac{M_r(x)}{M_n(x)}\right)^q dx \\ &\le C\sup_{\xi\in I}\left|\frac{F_k(\xi)}{M_{p_k}(\xi)}\right|^q \int_0^\infty \frac{M_r(x)}{M_n(x)}\,dx \\ &= C\sup_{\xi\in I}\left|\frac{F_k(\xi)}{M_{p_k}(\xi)}\right|^q < \infty.\end{aligned}$$

To complete the proof it suffices to take $p_{k+1} = n$ and $F_{k+1} = \tilde{F}_k$. ∎

Lemma 2.2. *Let $\mathcal{M}$ denote a family of continuous functions on I with the property that $\sup_{F\in\mathcal{M}}\left\|M_p^{-1}F\right\|_q \le A$ $(1\le q\le\infty)$ for some $p\in\mathbb{N}$ and $A>0$. Then, given $k\in\mathbb{N}$, there exist $p_k\in\mathbb{N}$, $p_k\ge p$, $C_k>0$, and for each $F\in\mathcal{M}$ a function $g_{k,F}$, continuous on I, such that $(Dx^{-1})^k g_{k,F}(x) = F(x)$ $(x\in I)$ and $\sup_{F\in\mathcal{M}}\left\|M_{p_k}^{-1}g_{k,F}\right\|_q \le C_k$ $(1\le q\le\infty)$.*

Proof. The result holds trivially for $k=0$. Proceeding by induction, fix $k \in \mathbb{N}$, $k \geq 1$. Let $p_k \in \mathbb{N}$, $p_k \geq p$, $C_k > 0$, and for each $F \in \mathcal{M}$ let $g_{k,F}$ be a continuous function on I such that $(Dx^{-1})^k g_{k,F}(x) = F(x)$ $(x \in I)$ with $\sup_{F\in\mathcal{M}} \left\|M_{p_k}^{-1} g_{k,F}\right\|_q \leq C_k$ $(1 \leq q \leq \infty)$. As in the proof of Lemma 2.1, for each $F \in \mathcal{M}$ we may construct a function $\tilde{g}_{k,F}$, continuous on I, satisfying

$$(Dx^{-1})^{k+1}\tilde{g}_{k,F}(x) = F(x) \quad (x \in I)$$
$$\sup_{x\in I}\left|\frac{\tilde{g}_{k,F}(x)}{M_n(x)}\right| \leq C \sup_{\xi\in I}\left|\frac{g_{k,F}(\xi)}{M_{p_k}(\xi)}\right|$$

and

$$\int_0^\infty \left|\frac{\tilde{g}_{k,F}(x)}{M_n(x)}\right|^q \, dx \leq C \sup_{\xi\in I}\left|\frac{g_{k,F}(\xi)}{M_{p_k}(\xi)}\right|^q \quad (1 \leq q < \infty)$$

for some $n \in \mathbb{N}$, $n > p_k$, where the positive constant C does not depend on F. To complete the proof it suffices to pick $p_{k+1} = n$ and $g_{k+1,F} = \tilde{g}_{k,F}$, and to take into account the induction hypotheses. ∎

The next result can be analogously established.

Lemma 2.3. *Let $\{F_j\}_{j=0}^\infty$ be a sequence of continuous functions on I such that there exists $p \in \mathbb{N}$ for which $\lim_{j\to\infty}\left\|M_p^{-1}F_j\right\|_q = 0$ $(1 \leq q \leq \infty)$. Then, to every $k \in \mathbb{N}$ there correspond $p_k \in \mathbb{N}$, $p_k \geq p$, and continuous functions $F_{k,j}$ on I $(j \in \mathbb{N})$ such that $(Dx^{-1})^k F_{k,j}(x) = F_j(x)$ $(j \in \mathbb{N},\ x \in I)$ and $\lim_{j\to\infty}\left\|M_{p_k}^{-1}F_{k,j}\right\|_q = 0$ $(1 \leq q \leq \infty)$.*

At this point we address to the characterization of those elements in the dual of the space $K_\mu\{M_p\}$.

Theorem 2.4. *The following statements are equivalent:*

1. *The functional f lies in $K_\mu\{M_p\}'$.*
2. *There exist $k, p \in \mathbb{N}$ and a continuous function F on I such that*
$$f = x^{-\mu-\frac{1}{2}}(Dx^{-1})^k F(x) \tag{8}$$
and
$$M_p^{-1}F \in L^q(I) \tag{9}$$
for any q, $1 \leq q \leq \infty$.
3. *There exist $k, p \in \mathbb{N}$ and a continuous function F on I satisfying* (8)*, such that* (9) *holds for some q, $1 \leq q \leq \infty$.*
4. *There exist $k, p \in \mathbb{N}$ and a continuous function F on I satisfying* (8)*, such that* (9) *holds for $q = \infty$.*

Proof. If $f \in K_\mu\{M_p\}'$ then Theorem 1.1 yields $p \in \mathbb{N}$ and $g_i \in L^\infty(I)$ $(i \in \mathbb{N},\ 0 \le i \le p)$ satisfying

$$f = x^{-\mu-\frac{1}{2}} \sum_{i=0}^{p} (Dx^{-1})^i [M_p(x) g_i(x)].$$

Hence $f = x^{-\mu-\frac{1}{2}} \sum_{i=0}^{p} (Dx^{-1})^i G_i(x)$, where $G_i(x) = M_p(x) g_i(x)$ $(i \in \mathbb{N},\ 0 \le i \le p;\ x \in I)$ are measurable functions such that $M_p^{-1} G_i$ $(i \in \mathbb{N},\ 0 \le i \le p)$ are bounded. Apply (A), (M) and (N) to choose $n, r, s, t \in \mathbb{N}$, $n > r > s > t > p$, in such a way that

$$\int_0^\infty \frac{M_p(x)}{M_t(x)}\, dx < \infty \tag{10}$$

$$x \le C M_s(x) \quad (x \in I) \tag{11}$$

$$M_s(x) M_t(x) \le C M_r(x) \quad (x \in I) \tag{12}$$

and

$$\int_0^\infty \frac{M_r(x)}{M_n(x)}\, dx < \infty. \tag{13}$$

Fix $i \in \mathbb{N}$, $0 \le i \le p$. By (M) and (10),

$$\begin{aligned}
\frac{1}{M_t(x)} \left| \int_0^x G_i(\xi)\, d\xi \right| &\le C \int_0^\infty \left| \frac{G_i(\xi)}{M_p(\xi)} \right| \frac{M_p(\xi)}{M_t(\xi)}\, d\xi \\
&\le C \sup_{\xi \in I} \left| \frac{G_i(\xi)}{M_p(\xi)} \right| \int_0^\infty \frac{M_p(\xi)}{M_t(\xi)}\, d\xi \qquad (14)\\
&= C \sup_{\xi \in I} \left| \frac{G_i(\xi)}{M_p(\xi)} \right| \quad (x \in I).
\end{aligned}$$

The function

$$\tilde{G}_i(x) = x \int_0^x G_i(\xi)\, d\xi \quad (x \in I)$$

is continuous and satisfies

$$(Dx^{-1}) \tilde{G}_i(x) = G_i(x) \quad (x \in I).$$

By (11), (14), (12) and (1),

$$\begin{aligned}
\left| \tilde{G}_i(x) \right| &\le C M_s(x) M_t(x) \sup_{\xi \in I} \left| \frac{G_i(\xi)}{M_p(\xi)} \right| \\
&\le C M_r(x) \sup_{\xi \in I} \left| \frac{G_i(\xi)}{M_p(\xi)} \right| \qquad (15)\\
&\le C M_n(x) \sup_{\xi \in I} \left| \frac{G_i(\xi)}{M_p(\xi)} \right| \quad (x \in I).
\end{aligned}$$

Consequently,

$$\sup_{x\in I}\left|\frac{\tilde{G}_i(x)}{M_n(x)}\right| \le C\sup_{\xi\in I}\left|\frac{G_i(\xi)}{M_p(\xi)}\right| < \infty.$$

Moreover, using (15), (1) and (13), we get

$$\int_0^\infty \left|\frac{\tilde{G}_i(x)}{M_n(x)}\right|^q dx \le C\sup_{\xi\in I}\left|\frac{G_i(\xi)}{M_p(\xi)}\right|^q \int_0^\infty \frac{M_r(x)}{M_n(x)}\,dx = C\sup_{\xi\in I}\left|\frac{G_i(\xi)}{M_p(\xi)}\right|^q < \infty$$

whenever $1 \le q < \infty$. With the aid of Lemma 2.1 we obtain a continuous function F_i on I and a nonnegative integer $s_i \ge n$ for which

$$(Dx^{-1})^{p-i}F_i(x) = \tilde{G}_i(x) \quad (x\in I)\ , \qquad M_{s_i}^{-1}F_i \in L^q(I) \quad (1\le q\le\infty).$$

Set

$$F = \sum_{i=0}^{p} F_i, \quad m = \max_{0\le i\le p} s_i.$$

Then F is continuous on I, $f = x^{-\mu-\frac{1}{2}}(Dx^{-1})^{p+1}F(x)$, and $M_m^{-1}F \in L^q(I)$ $(1 \le q \le \infty)$. Thus we have established that 1. implies 2.

It is apparent that assertion 2 implies assertion 3.

Let us prove that assertion 3 implies assertion 4. Suppose that there exist $k, p \in \mathbb{N}$ and a continuous function F on I satisfying $f = x^{-\mu-\frac{1}{2}}(Dx^{-1})^k F(x)$ and $M_p^{-1}F \in L^q(I)$ for some q, $1 \le q \le \infty$. Using (A), (M) and (N), choose $n > s > t > p$ such that

$$\int_0^\infty \frac{M_p(x)}{M_t(x)}\,dx < \infty \tag{16}$$

$$x \le CM_s(x) \quad (x\in I) \tag{17}$$

and

$$M_s(x)M_t(x) \le CM_n(x) \quad (x\in I). \tag{18}$$

Then the function

$$\tilde{F}(x) = x\int_0^x F(\xi)\,d(\xi) \quad (x\in I)$$

is continuous with $f = x^{-\mu-\frac{1}{2}}(Dx^{-1})^{k+1}\tilde{F}(x)$. A combination of (17), (18) and (M) yields

$$\begin{aligned}
\sup_{x\in I}\left|\frac{\tilde{F}(x)}{M_n(x)}\right| &\le C\sup_{x\in I}\left|\frac{M_s(x)}{M_n(x)}\int_0^x F(\xi)\,d\xi\right| \\
&= C\sup_{x\in I}\frac{M_s(x)M_t(x)}{M_n(x)M_t(x)}\left|\int_0^x F(\xi)\,d\xi\right| \\
&\le C\int_0^\infty \left|\frac{F(\xi)}{M_p(\xi)}\right|\frac{M_p(\xi)}{M_t(\xi)}\,d\xi. \qquad (19)
\end{aligned}$$

In case $q = 1$, it follows from (19) and (1) that

$$\sup_{x\in I}\left|\frac{\tilde{F}(x)}{M_n(x)}\right| \le C\int_0^\infty \left|\frac{F(\xi)}{M_p(\xi)}\right| d\xi < \infty.$$

If $q = \infty$, conditions (19) and (16) lead us to

$$\sup_{x\in I}\left|\frac{\tilde{F}(x)}{M_n(x)}\right| \le C\sup_{\xi\in I}\left|\frac{F(\xi)}{M_p(\xi)}\right|\int_0^\infty \frac{M_p(\xi)}{M_t(\xi)}\, d\xi < \infty.$$

Finally, if $1 < q < \infty$ then (19), the Hölder inequality, (1) and (16) give

$$\begin{aligned}\sup_{x\in I}\left|\frac{\tilde{F}(x)}{M_n(x)}\right| &\le C\left\{\int_0^\infty \left|\frac{F(\xi)}{M_p(\xi)}\right|^q d\xi\right\}^{\frac{1}{q}}\left\{\int_0^\infty \left(\frac{M_p(\xi)}{M_t(\xi)}\right)^{q'} d\xi\right\}^{\frac{1}{q'}}\\ &\le C\left\{\int_0^\infty \left|\frac{F(\xi)}{M_p(\xi)}\right|^q d\xi\right\}^{\frac{1}{q}}\left\{\int_0^\infty \frac{M_p(\xi)}{M_t(\xi)}\, d\xi\right\}^{\frac{1}{q'}} < \infty.\end{aligned}$$

Here q' denotes the exponent conjugate to q. Thus, (8) and (9) hold for $k+1$ instead of k, $\tilde{F}$ instead of F, n instead of p, and $q = \infty$. This establishes 4.

To complete the proof, assume there exist $k, p \in \mathbb{N}$ and a continuous function F on I such that $f = x^{-\mu-\frac{1}{2}}(Dx^{-1})^k F(x)$ and $M_p^{-1}F \in L^\infty(I)$. Such a representation of f ensures that $f \in K_\mu\{M_p\}'$. This follows from the estimate

$$\begin{aligned}|\langle f,\varphi\rangle| &= \left|(-1)^k\int_0^\infty F(x)(x^{-1}D)^k x^{-\mu-\frac{1}{2}}\varphi(x)\, dx\right|\\ &\le \sup_{x\in I}\left|\frac{F(x)}{M_p(x)}\right|\sup_{x\in I}\left|M_r(x)(x^{-1}D)^k x^{-\mu-\frac{1}{2}}\varphi(x)\right|\int_0^\infty \frac{M_p(x)}{M_r(x)}\, dx\end{aligned} \tag{20}$$

valid for all $\varphi \in K_\mu\{M_p\}$, where $r > p$ has been chosen according to (N). Thus assertion 4 implies assertion 1 and we are done. ∎

Next we characterize boundedness in $K_\mu\{M_p\}'$.

Theorem 2.5. *The following four statements are equivalent:*

1. *The set $B \subset K_\mu\{M_p\}'$ is (weakly, weakly*, strongly) bounded.*
2. *There exist $k, p \in \mathbb{N}$, $C > 0$, and for every $f \in B$ a function g_f continuous on I such that*

$$f = x^{-\mu-\frac{1}{2}}(Dx^{-1})^k g_f(x) \tag{21}$$

and

$$\left\|M_p^{-1}g_f\right\|_q \le C \tag{22}$$

for any q, $1 \le q \le \infty$.

3. *There exist $k, p \in \mathbb{N}$, $C > 0$, and for every $f \in B$ a function g_f continuous on I such that* (21) *holds and* (22) *is satisfied for some q, $1 \le q \le \infty$.*

4. *There exist $k, p \in \mathbb{N}$, $C > 0$, and for every $f \in B$ a function g_f continuous on I such that* (21) *holds and* (22) *is satisfied for $q = \infty$.*

Proof. If B is a bounded subset of $K_\mu\{M_p\}'$, then Theorem 1.1 yields $p \in \mathbb{N}$, $A > 0$, and for every $f \in B$ functions $g_{i,f} \in L^\infty(I)$ $(i \in \mathbb{N},\ 0 \le i \le p)$ such that

$$f = x^{-\mu-\frac{1}{2}} \sum_{i=0}^{p} (Dx^{-1})^i [M_p(x) g_{i,f}(x)]$$

with $\sum_{i=0}^{p} \|g_{i,f}\|_\infty \le A$. Hence $f = x^{-\mu-\frac{1}{2}} \sum_{i=0}^{p} (Dx^{-1})^i G_{i,f}(x)$, where $G_{i,f}(x) = M_p(x) g_{i,f}(x)$ $(i \in \mathbb{N},\ 0 \le i \le p;\ x \in I)$ are measurable functions satisfying

$$\sum_{i=0}^{p} \left\| M_p^{-1} G_{i,f} \right\|_\infty \le A.$$

Fix $i \in \mathbb{N}$, $0 \le i \le p$. Arguing as in the proof that 1. implies 2. in Theorem 2.4, we may find $n \in \mathbb{N}$, $n > p$, $\tilde{A} > 0$, and continuous functions $\tilde{G}_{i,f}$ on I such that $(Dx^{-1})\tilde{G}_{i,f}(x) = G_{i,f}(x)$ $(x \in I)$,

$$\sup_{x \in I} \left| \frac{\tilde{G}_{i,f}(x)}{M_n(x)} \right| \le C \sup_{\xi \in I} \left| \frac{G_{i,f}(\xi)}{M_p(\xi)} \right| \le \tilde{A},$$

and

$$\int_0^\infty \left| \frac{\tilde{G}_{i,f}(x)}{M_n(x)} \right|^q dx \le C \sup_{\xi \in I} \left| \frac{G_{i,f}(\xi)}{M_p(\xi)} \right|^q \le \tilde{A},$$

where $\tilde{A}$ does not depend upon $f \in B$. By virtue of Lemma 2.2, there exist $s_i \in \mathbb{N}$, $s_i \ge n$, $C_i > 0$, and for every $f \in B$ a continuous function $F_{i,f}$ on I such that

$$(Dx^{-1})^{p-i} F_{i,f}(x) = \tilde{G}_{i,f}(x) \quad (x \in I)$$

and

$$\left\| M_{s_i}^{-1} F_{i,f} \right\|_q \le C_i \quad (1 \le q \le \infty).$$

Setting

$$g_f = \sum_{i=0}^{p} F_{i,f}, \quad m = \max_{0 \le i \le p} s_i, \quad C = \sum_{i=0}^{p} C_i$$

we encounter that g_f is continuous on I, $f = x^{-\mu-\frac{1}{2}} (Dx^{-1})^{p+1} g_f(x)$ and that $\|M_m^{-1} g_f\|_q \le C$ $(1 \le q \le \infty)$ where neither $m \in \mathbb{N}$ nor $C > 0$ depend on $f \in B$. Thus we have established that 1. implies 2.

It is apparent that assertion 2 implies assertion 3.

To prove that assertion 3 implies assertion 4, assume there exist $k, p \in \mathbb{N}$, $A > 0$, and for each $f \in B$ a continuous function g_f on I satisfying $f = x^{-\mu-\frac{1}{2}}(Dx^{-1})^k g_f(x)$ with $\left\|M_p^{-1} g_f\right\|_q \leq A$ for some q, $1 \leq q \leq \infty$. The argument in the proof that 3. implies 4. in Theorem 2.4 allows one to find $n \in \mathbb{N}$, $n > p$, $\tilde{A} > 0$, and for any $f \in B$ a function $\tilde{g}_f$ continuous on I such that

$$f = x^{-\mu-\frac{1}{2}}(Dx^{-1})^{k+1}\tilde{g}_f(x)$$

with

$$\sup_{x\in I}\left|\frac{\tilde{g}_f(x)}{M_n(x)}\right| \leq C \sup_{\xi\in I}\left|\frac{g_f(\xi)}{M_p(\xi)}\right| \leq \tilde{A}$$

if $q = \infty$, or

$$\sup_{x\in I}\left|\frac{\tilde{g}_f(x)}{M_n(x)}\right| \leq C\left\{\int_0^\infty \left|\frac{g_f(\xi)}{M_p(\xi)}\right|^q d\xi\right\}^{\frac{1}{q}} \leq \tilde{A}$$

if $1 \leq q < \infty$. This establishes 4.

Finally, assertion 4 and (20) with g_f instead of F ($f \in B$) yield 1. ∎

Convergence in $K_\mu\{M_p\}'$ is described next.

Theorem 2.6. *The following statements are equivalent:*

1. *The sequence $\{f_j\}_{j=0}^\infty$ converges (weakly, weakly*, strongly) to zero in $K_\mu\{M_p\}'$.*
2. *There exist $k, p \in \mathbb{N}$ and continuous functions F_j on I ($j \in \mathbb{N}$) such that*

$$f_j = x^{-\mu-\frac{1}{2}}(Dx^{-1})^k F_j(x) \quad (j \in \mathbb{N}) \tag{23}$$

and

$$\lim_{j\to\infty}\left\|M_p^{-1}F_j\right\|_q = 0 \tag{24}$$

for any q, $1 \leq q \leq \infty$.

3. *There exist $k, p \in \mathbb{N}$ and continuous functions F_j on I ($j \in \mathbb{N}$) such that (23) holds and (24) is satisfied for some q, $1 \leq q \leq \infty$.*
4. *There exist $k, p \in \mathbb{N}$ and continuous functions F_j on I ($j \in \mathbb{N}$) such that (23) holds and (24) is satisfied for $q = \infty$.*
5. *There exist $k, p \in \mathbb{N}$, $C > 0$, and continuous functions F_j on I ($j \in \mathbb{N}$), such that (23) holds,*

$$\left\|M_p^{-1}F_j\right\|_\infty \leq C \quad (j \in \mathbb{N}),$$

and $\lim_{j\to\infty} F_j(x) = 0$ for almost all $x \in I$.

Proof. Parts 3. and 5. follow trivially from 2. and 4., respectively. It follows essentially by the arguments in the proof of the corresponding results in Theorem 2.5 with the aid of Lemma 2.3 in place of Lemma 2.2 that 1. implies 2. and 3. implies 4. (we omit the details). Finally, (N) and the Lebesgue dominated convergence theorem applied to the integrals

$$\langle f_j, \varphi \rangle = (-1)^k \int_0^\infty F_j(x)(x^{-1}D)^k x^{-\mu-\frac{1}{2}} \varphi(x)\, dx \quad (j \in \mathbb{N},\ \varphi \in K_\mu\{M_p\})$$

show that 5. implies 1. ∎

3. An example

Theorems 2.4, 2.5 and 2.6 characterize membership, boundedness and convergence in the dual of a wide range of spaces arising in connection with the generalized Hankel transformation (see [5]). Let us record the following special case of Theorem 1.2 for the Zemanian space $\mathcal{H}_\mu = K_\mu\{(1+x^2)^p\}$ (see [7, Chapter 5]).

Corollary 3.1. *Let $\mathcal{H}'_\mu$ denote the dual of $\mathcal{H}_\mu$. Then:*

1. *A functional f belongs to $\mathcal{H}'_\mu$ if, and only if, there exist $k, p \in \mathbb{N}$ and a continuous function F on I such that*

$$f = x^{-\mu-\frac{1}{2}}(Dx^{-1})^k F(x)$$

with $(1+x^2)^{-p}F(x) \in L^q(I)$ $(1 \le q \le \infty)$.

2. *A set $B \subset \mathcal{H}'_\mu$ is (weakly, weakly*, strongly) bounded if, and only if, there exist $k, p \in \mathbb{N}$, $C > 0$ and, for every $f \in B$, a function g_f continuous on I, such that*

$$f = x^{-\mu-\frac{1}{2}}(Dx^{-1})^k g_f(x)$$

with $\|(1+x^2)^{-p}g_f(x)\|_q \le C \quad (1 \le q \le \infty)$.

3. *A sequence $\{f_j\}_{j=0}^\infty$ converges (weakly, weakly*, strongly) to zero in $\mathcal{H}'_\mu$ if, and only if, there exist $k, p \in \mathbb{N}$ and functions g_j continuous on I such that*

$$f_j = x^{-\mu-\frac{1}{2}}(Dx^{-1})^k g_j(x) \quad (j \in \mathbb{N})$$

with $\lim_{j\to\infty} \|(1+x^2)^{-p}g_j(x)\|_q = 0 \quad (1 \le q \le \infty)$.

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