Averaging of Perturbed One Sided Lipschitz Differential Inclusions

T. Donchev, M. Kamenskii and M. Quincampoix

Abstract. We consider one sided Lipschitz differential inclusions perturbed with multimap, satisfying compactness type conditions. The state space is Banach with uniformly convex dual. Averaging result on a finite interval is proved. The averaging of functional differential inclusions is also studied.

1. Introduction

In the paper we consider upper semicontinuous (USC) differential inclusions in Banach spaces with uniformly convex dual. The right-hand side is a sum of one sided Lipschitz (OSL) and satisfying compactness assumption multifunctions. We study an averaging method for such a differential inclusion. In the last section we describe briefly some extensions to the case of differential inclusions with time lags. The averaging method is very well presented in the works of Plotnikov et al. [12, 13, 14] in case of $E \equiv \mathbb{R}^n$. The averaging of differential inclusions (in Banach spaces) with OSL right-hand side is considered in [6] and under compactness type assumptions in [2]. Similar result is also obtained in [15] in R^n.

 Given a Banach space E with uniformly convex dual E^*. Denote by $J(x) = \{l \in E^* : |l| = |x|, \langle l, x \rangle = |x|^2\}$ the duality map. The Hausdorff distance is $D_H(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} |a - b|, \sup_{b \in B} \inf_{a \in A} |a - b|\}$, the support function is $\sigma(l, A) = \sup_{a \in A} \langle l, a \rangle$. Let X, Y be metric spaces with distances ρ_X

T. Donchev: Department of Mathematics, University of Architecture and Civil Engineering, 1 "Hr. Smirnenski" street, 1046 Sofia, Bulgaria; tdd51us@yahoo.com

M. Kamenskii: Faculty of Mathematics, University of Voronezh, Universitetskaya pl. 1, 394693 Voronezh, Russia; Mikhail@kam.vsu.ru

M. Quincampoix, Laboratoire de Mathématiques, Université de Bretagne Occidentale; 6 av. Victor Le Gorgeu B.P. 809; 29285 Brest Cedex, France; Marc.Quincampoix@univ-brest.fr

and ρ_Y. For a set $D \subset X$ denote by $D^\varepsilon := \{x \in X; \ \text{dist}(x, D) < \varepsilon\}$, where $\text{dist}(x, D) = \inf_{b \in D} \rho_X(x, b)$. The multimap $\Gamma : X \rightrightarrows Y$ is said to be upper semicontinuous (USC) when for all $x \in X$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $\Gamma(\{x\}^\delta) \subset \Gamma(x)^\varepsilon$. For $\Gamma : [0, T] \times X \rightrightarrows Y$ we say that $\Gamma(\cdot, \cdot)$ is almost USC when to $\varepsilon > 0$ there exists a compact $I_\varepsilon \subset [0, T]$ with Lebesgue measure $\text{meas}(I_\varepsilon) > T - \varepsilon$ such that Γ is USC on $I_\varepsilon \times X$.

For all conditions and notations used here, but not given explicitly we refer to [3] or [7].

Definition 1. The multifunction R from E into E is said to be *One Sided Lipschitz* (OSL) when there exists a constant L such that

$$\sigma(J(x - y), R(x)) - \sigma(J(x - y), R(y)) \leq L|x - y|^2 \quad \forall x, y \in E \ .$$

The paper contains two sections with introduction. In the next section we present our results. In the last section we present some extensions of our results and formulate some propositions, where our results are applicable.

2. The results

Consider the differential inclusion

$$\dot{x}(t) \in F(t, x) + G(\frac{t}{\varepsilon}, x), \ x(0) = x_0, \ t \in I = [0, a], \ x \in E \qquad (1)$$

under the following conditions:

A1. $F(\cdot, \cdot)$ and $G(\cdot, \cdot)$ have non-empty convex and compact values.

A2. $F(t, \cdot)$ is OSL with a constant L (not depending on t), it is almost USC and maps bounded sets into bounded.

A3. $G(\cdot, x)$ admits a (strongly) measurable selection, while $G(t, \cdot)$ is USC. Moreover,
$$\chi_E(G([0, a] \times \Omega)) \leq k\chi_E(\Omega)$$
for every bounded set $\Omega \subset E$.

Here, $\chi_E(\cdot)$ denotes Hausdorff measure of noncompactness defined by

$$\chi(A) = \inf\{r > 0 : A \text{ can be covered by finitely many balls with radius } \leq r\}.$$

We refer to [9] for the theory of measures of noncompactness and the condensing operators.

For $v(\cdot) \in C([0, a], E)$ define the operator

$$\Phi(v) = \{Sol(g) : \ g(t) \in G(t, v(t))\}.$$

Here $Sol(g)$ is the solution set of

$$\dot{x}(t) \in F(t, x(t)) + g(t), \quad x(0) = x_0, \tag{2}$$

on the interval $[0, a]$. By $Sol(g(t))$ we denote the solution set of the differential inclusion (2) on $[0, t]$.

Lemma 1. *Under the conditions* A1 *and* A2 *the following statements are true:*

1) $D_H\left(Sol(g_1(t)), Sol(g_2(t))\right) \le \int_0^t e^{L(t-s)}|g_1(s) - g_2(s)|\, ds$

2) *If* $\mathcal{L} = \{g: g(\cdot) \text{ measurable, } g(t) \in K \text{ for a.a. } t \in I\}$, *where* $K \subset E$ *is compact, then* $Sol(\mathcal{L})$ *is a* $C([0, a], E)$ *compact set.*

3) *The set* $Sol(g)$ *is a compact* R_δ *set.*

Proof. From Theorem 1 of [4] we know that the solution set of (2) is non-empty and $C(I, E)$ compact. Let $x(\cdot)$ be a solution of (2), when $g(\cdot)$ is replaced by $g_1(\cdot)$. We consider the multivalued mapping

$$H(t, u) = \left\{ v \in F(t, u) + g_2(t) \,\middle|\, \begin{array}{c} \langle J(x(t) - u), \dot{x}(t) - v\rangle \\ \le (L|x(t) - u| + |g_1(t) - g_2(t)|)|x(t) - u| \end{array} \right\}.$$

Using standard arguments one can prove that $H(\cdot, \cdot)$ is almost USC with non-empty convex and compact values. Since $H(t, y) \subset F(t, y)$ one has that the differential inclusion

$$\dot{y}(t) \in H(t, y), \quad y(0) = x_0$$

has a solution $y(\cdot)$ as it is shown in the Step 3 of the proof of Theorem 1 of [4]. Really, let $H(\cdot, x)$ be strongly measurable and let $\sigma(l, H(t, \cdot))$ be USC as a real valued function for every $l \in E^*$. Furthermore let $H(t, x) \subset F(t, x)$ (where $F(\cdot, \cdot)$ is almost USC with non-empty convex compact values, bounded on the bounded sets and $F(t, \cdot)$ is OSL) be non-empty convex and compact valued. We claim that the solution set R_1 of

$$\dot{x}(t) \in H(t, x), \quad x(0) = x_0$$

is non-empty compact valued and $\lim_{\varepsilon \to 0} D_H(R_1, R_\varepsilon) = 0$. Here R_ε is the solution set of

$$\dot{x}(t) \in \overline{co}\, H(t, x + U_\varepsilon) + U_\varepsilon, \quad x(0) = x_0 \ .$$

As it is shown in [4] the solution set R_{RP} of

$$\dot{x}(t) \in F(t, x), \quad x(0) = x_0 \tag{3}$$

is non-empty compact. Furthermore, if one has a subdivision $0 = \tau_0 < \tau_1 < \ldots < \tau_n = a$ of I and R_{DIRP} is the solution set of

$$\dot{x}(t) \in F(t, x(\tau_i)), \quad x(0) = x_0, \quad x(\tau_i) = \lim_{t \to \tau_i - 0} x(t) \ ,$$

then it is non-empty $C(I, E)$ compact and $\lim_{h \to 0} D_H(R_{DIRP}, R_{RP}) = 0$. Hence if $x_h(\cdot) \in R_{DIRP}$ then there exists a converging $x_{h_i}(\cdot)$ to some $x(\cdot) \in R_{RP}$. Consider

$$\dot{x}(t) \in H(t, x(\tau_i)), \quad x(0) = x_0, \quad x(\tau_i) = \lim_{t \to \tau_i - 0} x(t)$$

with a solution set R_{DI}^S. Obviously $\lim_{h \to 0} \rho(R_{DI}^S, R_{RP}) = 0$. Since R_{RP} is non-empty compact, one has that the net $\{R_{DI}^S\}_{h>0}$ is pre-compact in $C(I, E)$, i.e. every net $\{x_h(\cdot)\}_{h>0}$ has a converging to some $x(\cdot)$ subnet. Obviously that $x(\cdot) \in R_1$. Therefore one can conclude that $\lim_{h \to 0} \rho(R_{DI}^S, R_1) = 0$, i.e. R_1 is non-empty compact. The fact that $R_1 = \lim_{\varepsilon \to 0} R_\varepsilon$ is straightforward (see for instance [4]).

Due to the definition of H we have

$$\langle J(x(t) - y(t)), \dot{x}(t) - \dot{y}(t) \rangle \leq \big(L|x(t) - u| + |g_1(t) - g_2(t)| \big) |x(t) - y(t)|.$$

The last inequality implies

$$\frac{d}{dt} |x(t) - y(t)| \leq L|x(t) - y(t)| + |g_1(t) - g_2(t)|.$$

This fact proves 1).

Consider the multifunction $R(t, x) = F(t, x) + \overline{\mathrm{co}}\, K$. Obviously $R(\cdot, \cdot)$ is almost USC with non-empty convex and compact values and it is OSL. Therefore under Theorem 1 of [4], the solution set of $\dot{x}(t) \in R(t, x)$, $x(0) = x_0$ is non-empty $C(I, E)$ compact. Obviously $F(t, x) + g(t)$ is OSL and almost USC with non-empty convex compact values. From Theorem 1 of [4] we know that the solution set $Sol(g)$ of (2) is non-empty compact R_δ set. ∎

We will use the following lemma proved in [2].

Lemma 2. *Let Ω be a set of measurable on $[0, T]$ functions and*

$$\| \Omega(t) \| \leq \alpha(t), \quad \text{and} \quad \chi_E(\Omega(t)) \leq \beta(t) \text{ a.e.,}$$

where $\alpha, \beta \in L^1[0, T]$. Then for every $\delta > 0$ there exist a compact set $K_\delta \subset E$ and a measurable $I_\delta \subset [0, T]$ with $\mathrm{meas}(I_\delta) < \delta$ (recall that meas is the Lebesgue measure) such that for every $x \in \Omega$ there exists $g_\delta \in L^1([0, T], K_\delta)$ for which $|x(t) - g_\delta(t)| \leq \beta(t) + \delta$ for all $t \in [0, T] \setminus I_\delta$.

Recall that an operator Φ is condensing with respect to the measure of non-compactness φ (see [9]) if the inequality

$$\varphi(\Phi(\Omega)) \geq \varphi(\Omega)$$

implies that Ω is relatively compact.

Corollary 1. *Under the conditions* A1 *and* A2 *the operator* $\Phi(\cdot)$ *is condensing with respect to the following measure of noncompactness:*

$$\varphi(\Omega) = \{\sup_{t \in [0,a]} e^{-\alpha t} \chi_E(\Omega(t)), \mathrm{mod}_C\Omega)\},$$

where

$$\mathrm{mod}_C\Omega = \lim_{\delta \to 0} \sup_{x \in \Omega} \max_{|t_1 - t_2| \le \delta} |x(t_1) - x(t_2)|.$$

Proof. We have to prove that if $\varphi(\Phi(\Omega)) \ge \varphi(\Omega)$, then Ω is relatively compact. Let $x(\cdot) \in \Omega$ and let $y(\cdot) \in \Phi(x(\cdot))$. Then there exists $g(t) \in G(t, x)$ such that $y \in Sol(g)$. Due to Lemma 1 it holds

$$\mathrm{dist}(y(\cdot), Sol(g_\delta)) \le \int_0^t e^{L(t-s)} |g(s) - g_\delta(s)| \, ds.$$

Let $g_\delta(\cdot)$ be chosen with accordance to Lemma 2. Thus

$$\mathrm{dist}(y(\cdot), Sol(g_\delta)) \le \int_0^t e^{L(t-s)}(\beta(s) + \delta) \, ds + C\delta,$$

where C is a constant depending only on L. Multiplying this inequality by $e^{-\alpha t}$ we obtain

$$e^{-\alpha t}\mathrm{dist}\big(y(\cdot), Sol(g_\delta)\big) \le \int_0^t e^{(L-\alpha)(t-s)}(\beta(s) + \delta) \, ds + e^{-\alpha t}C\delta$$

$$\le C\delta + \int_0^t e^{(L-\alpha)(t-s)}\varphi_1(\Omega) \, ds$$

$$\le C\delta + \frac{\varphi_1(\Omega)}{\alpha - L}.$$

Since $\delta > 0$ is arbitrary taking $\alpha \ge L + 2$ one obtains

$$\sup_t e^{-\alpha t}\mathrm{dist}\big(y(\cdot), Sol(g_\delta)\big) \le \frac{\varphi_1(\Omega)}{2},$$

which means $\varphi_1(\Omega) \le \varphi_1(\Phi(\Omega)) \le \frac{1}{2}\varphi_1(\Omega)$, i.e. $\varphi_1(\Omega) = 0$. Consequently $\chi_E(\Omega(t)) = 0$ for all t, and hence $\Omega(t)$ is relatively compact. Furthermore, $D_H(Sol(g(t), Sol(g_\delta(t))) \le C\delta$ and $\varphi(\Omega) = 0$ due to the Ascoli theorem. ∎

We will consider the case when $G(\cdot, x)$ is T periodic, i.e. there exists $T > 0$ such that $G(t + T, x) \equiv G(t, x)$. Define the averaged

$$G_0(x) = \left\{ \frac{1}{T} \int_0^T g(s) \, ds : \ g(s) \in G(s, x) \right\}.$$

The averaged system corresponding to (1) is

$$\dot{x}(t) \in F(t, x) + G_0(x), \quad x(0) = x_0. \tag{4}$$

Denote by X_ε the solution set of (1) and by X_0 the solution set of (4) on I. Analogously, $X_0(t)$ is the solution set on $|0, t|$. From Lemma 5.4.1 of [9] we know that $G_0(\cdot)$ is USC multimap.

The following theorem is the main result of the paper.

Theorem 1. *Under the conditions of Corollary 1, if X_0 is a bounded set and moreover, every local solution $x(\cdot) \in X_0$ can be extended on I, then for ε sufficiently close to 0^+ we have $X_\varepsilon \neq \emptyset$, i.e. there exists $x_\varepsilon(\cdot) \in X_\varepsilon$ defined on the whole interval I. Moreover, the map $\varepsilon \mapsto X_\varepsilon$ is USC at $\varepsilon = 0^+$.*

Proof. Evidently $x(\cdot) \in Sol_\varepsilon(g)$ iff $x \in \Phi_\varepsilon(x)$. Let $U(X_0, \delta) \subset C([0, a], E)$ be a δ – neighborhood of X_0. Suppose the contrary, i.e. there exist sequences $\varepsilon_n \mapsto 0$, $t_n \mapsto t^*$ and $\{x_n(\cdot)\}_{n=1}^\infty$, where $x_n(\cdot)$ is a (local) solution of (1) for ε replaced by ε_n. Furthermore, for $t \in [0, t_n)$ one has $x_n \in \Phi_{\varepsilon_n}(x_n)$, $\text{dist}(x_n(\cdot), X_0(t)) < \delta$ and $\text{dist}(x_n(\cdot), X_0(t_n)) = \delta$. Since $G(\cdot, \cdot)$ is bounded on the bounded sets one has that for every sufficiently small ε the solution set X_ε is non-empty. Moreover, there exists $0 < \gamma < t^*$ such that $t_n > t^* - \gamma$ for sufficiently large n and $\text{dist}(x_n(\cdot), X_0(t^* - \gamma)) > \frac{\delta}{2}$. Repeating the reasons of the Theorem 1 one can obtain $\varphi(\{x_n(t)\}) = 0$. This implies that passing to subsequences one obtains $x_n(\cdot) \mapsto x^*(\cdot)$ with respect to $C([0, t^* - \gamma], E)$ and (the corresponding to $x_n(\cdot)$) $g_n(\cdot) \rightharpoonup g^*(\cdot)$ with respect to $L_1([0, t^* - \gamma], E)$–weak. Due to Theorem 3 of [2] (see also [9, Lemma 5.4.2]), $g^*(t) \in G_0(x^*(t))$ for a.a. $t \in [0, t^* - \gamma]$. This implies that $x^*(\cdot)$ is a solution of (4) on $[0, t^* - \gamma]$. By the assumptions $x^*(\cdot)$ can be extended on the whole interval I. However

$$\text{dist}\big(x^*(\cdot), X_0(t^* - \gamma)\big) = \lim_{n \to \infty} \text{dist}\big(x_n(\cdot), X_0(t^* - \gamma)\big) \geq \frac{\delta}{2}$$

which is a contradiction.   ∎

3. Concluding remarks

In this section we will briefly discuss some extensions and applications of our results.

Let E be an arbitrary Banach space. Define

$$[x, y]_+ = \lim_{t \to 0^+} h^{-1}\{|x + hy| - |x|\}$$

(see [11, p.8]), $\mathbb{X} = C([-\tau, 0], E)$ and $\mathbb{X}_0 = \{\alpha \in \mathbb{X} : \|\alpha\|_\mathbb{X} = |\alpha(0)|_E\}$, $x_t(s) = x(t + s)$. Consider the functional differential inclusion

$$\dot{x}(t) \in F(t, x_t) + G(\frac{t}{\varepsilon}, x_t), \quad x_0 = \varphi, \ t \in I = [0, a], \quad x_t \in \mathbb{X}. \tag{5}$$

Definition 2. The multifunction $F : I \times \mathbb{X} \rightrightarrows E$ is said to be OSL (cf. [5]) when there exists a constant L such that for all $\alpha - \beta \in \mathbb{X}_0$ and $f_\alpha \in F(t, \alpha)$ there exists $f_\beta \in F(t, \beta)$ such that

$$[\alpha(0) - \beta(0), f_\alpha - f_\beta]_+ \leq L|\alpha(0) - \beta(0)|.$$

Since the given definition in the case where the duality map is single valued coincides with the above definition of OSL we keep this abbreviation for this case.

As in the previous section we assume:

F1. $F(\cdot, \cdot)$ and $G(\cdot, \cdot)$ have non-empty convex compact values.

F2. $F(\cdot, \cdot)$ is almost continuous and OSL with a constant not depending on t. It maps bounded sets into bounded ones.

F3. $G(\cdot, \alpha)$ admits a (strongly) measurable selection and it is T periodic for some $T > 0$. Furthermore $G(t, \cdot)$ is USC and

$$\chi_E(G([0, a] \times \Omega)) \leq k\chi_{\mathbb{X}}(\Omega)$$

for every bounded set $\Omega \subset X$, where $\chi_E(\cdot)$ and $\chi_{\mathbb{X}}(\cdot)$ denote the Hausdorff measure of non-compactness in the space E, respectively $\mathbb{X}$.

Let $g(t) \in G(t, v_t)$ be measurable, where $v : [-\tau, a] \to E$ is continuous. Denote by $Rsol(g)$ the solution set of

$$\dot{x}(t) \in F(t, x_t) + g(t), \quad x_0 = \varphi.$$

Using a similar fashion as in the previous section, as in Theorem 1 and Corollary 1 of [5] one can prove the following lemma.

Lemma 3. *Under the conditions* F1 *and* F2 *the following statements hold:*

1) $D_H\big(Rsol(g_1), Rsol(g_2)\big) \leq \displaystyle\int_0^a e^{L(a-s)}|g_1(s) - g_2(s)|\, ds.$

2) *If* $\mathcal{K} = \{g : g(\cdot) \text{ measurable } g(t) \in K \text{ for a.a. } t \in I\}$, *where* $K \subset E$ *is compact, then* $Rsol(\mathcal{L}) = \displaystyle\bigcup_{g \in \mathcal{K}} Rsol(g)$ *is* $C([0, a], E)$ *is a compact set.*

3) *The set* $Rsol(g)$ *is a non-empty compact* R_δ *set.*

Proof. From Theorem 1 of [5] we know that $Rsol(g)$ is non-empty $C(I, E)$ compact set. Let $x \in Rsol(g_1)$. Define the multifunction

$$G_\delta(t, \alpha) = \begin{cases} F(t, \alpha) & \text{for } x_t - \alpha \notin \mathbb{X}_0 \\[2mm] cl\left\{ u \in F(t, \alpha) \;\middle|\; \begin{array}{l} [x(t) - \alpha(0), \dot{x}(t) - u]_+ \\ \quad < L|x(t) - \alpha(0)| + \delta \end{array} \right\} & \text{for } x_t - \alpha \in \mathbb{X}_0 \backslash \{0\} \\[2mm] \dot{x}(t) & \text{for } x_t = \alpha. \end{cases}$$

As it is shown in [5] the map $G_\delta(\cdot, \cdot)$ is almost LSC and the differential inclusion

$$\dot{y}(t) \in G_\delta(t, y_t) + g_2(t), \quad y_0 = \varphi$$

has a solution $y_\delta(\cdot)$ such that

$$[x(t) - y(t), \dot{x}(t) - \dot{y}(t)]_+ \leq L|x(t) - y(t)| + |g_1(t) - g_2(t)| + \delta$$

when $x_t - y_t \in X_0$. Consequently,

$$|x(t) - y_\delta(t)| \leq \int_0^t e^{L(t-s)} \big(|g_1(s) - g_2(s)| + \delta \big) \, ds \ .$$

Since $Rsol(g_2)$ is compact one can find $y(\cdot) \in Rsol(g_2)$ such that

$$|x(t) - y(t)| \leq \int_0^t e^{L(t-s)} \big(|g_1(s) - g_2(s)| \big) \, ds \ .$$

The lemma is proved.   ∎

Define

$$G_0(\alpha) = \frac{1}{T} \overline{\int_0^T G(\tau, \alpha) \, d\tau} \ .$$

The averaged system is

$$\dot{x}(t) \in F(t, x_t) + G_0(x_t), \quad x_0 = \varphi. \tag{6}$$

The following theorem extends the main result of [8].

Theorem 2. *Under* F1, F2, F3, *if the solution set of* (6) *is non-empty bounded and every local solution is extendable on* I, *then the solution set* $Sol(\varepsilon)$ *of* (5) *is non-empty for sufficiently small* $\varepsilon > 0$ *and moreover the map* $\varepsilon \to Sol(\varepsilon)$ *is USC at* $\varepsilon = 0^+$.

The proof of Theorem 1 is valid (with obvious modifications) also in the case of Theorem 2 and is omitted.

Definition 3. ([1]) The closed set K is said to be a *viability domain* (for the differential inclusion (1) or for (4)) if for every $x_0 \in K$ there exists a solution remaining in K for every t. The set K is said to be *invariant* (for (1) or (4)) if for every $x_0 \in K$ each solution starting from x_0 remains in K for all t. Given a closed set K, the *viability kernel* $\text{Viab}_\varepsilon(K)$ is the largest closed subset (possibly empty) of K which is a viability domain. The largest invariant subset of K is called *invariant kernel*.

Let

$$R_\varepsilon(t) := \{y \in E : \text{ there exists a solution } x(\cdot) \text{ of (1) such that } y = x(t)\}$$

be the reachable set of (1) at the moment t. Analogously, by $R_0(t)$ we denote the reachable set of (4). Let $R_0(\cdot)$ be defined on $[0, \infty)$.

Definition 4. The reachable set $R_0(\cdot)$ is said to be *uniformly asymptotically semi stable* if for all $\eta > 0$ there exists $\delta(\eta) > 0$ such that for all $t_0 \in [0, \infty)$ it holds: If $ex(S(t_0), R_0(t_0)) < \delta(\eta, t_0)$ then the reachable set $S(t)$ of (4) with $x(t_0) \in S(t_0)$ exists and is bounded for every $t > t_0$, $ex(S(t), R_0(t)) < \eta$ and $\lim_{t \to \infty} ex(S(t), R_0(t)) = 0$. Here $ex(A, B) = \max_{a \in A} \min_{b \in B} |a - b|$.

The following theorem is an extension of Theorem 15.4 of [12].

Theorem 3. *Let all the conditions of Theorem* 1 *hold. If the reachable set* $R_0(\cdot)$ *is uniformly asymptotically stable and bounded on* $[0, \infty)$*, then the conclusion of Theorem* 1 *holds on* $[0, \infty)$.

Proof. Let $\eta > 0$ be given and let $\delta = \delta(\eta) > 0$. Fix $b > 0$ and apply Theorem 1 on the interval $[0, b]$. As we know there exists $\varepsilon(\delta) > 0$ such that for $\varepsilon < \varepsilon(\delta)$ the reachable set $R(t, \varepsilon)$ of (1) is contained in a $\frac{\delta}{2}$-neighborhood of the reachable set of (4). Let $R(b, \varepsilon) = A_\varepsilon$ be initial set of the (4). We will denote its reachable set by $R(t, A_\varepsilon)$. Then there exists $\Theta > 0$ such that $R(t, A_\varepsilon)$ is contained in $\frac{\delta}{2}$-neighborhood of $R(t, \varepsilon)$ for $t > \Theta$. We apply Theorem 1 on the interval $[b, \Theta]$ etc. One can prove by obvious application of the Zorn's lemma that the conclusion of Theorem 1 holds on $[0, \infty)$. ∎

Corollary 2. *Under the conditions of Theorem* 3 *the following statements hold:*

1) *Let the differential inclusion* (1) *admit a bounded invariant set* $K_\varepsilon \neq \emptyset$. *If* (4) *has a non-empty invariant set* $K_0 \neq \emptyset$*, then* $\lim ex(K_\varepsilon, K_0) = 0$.
2) *Let* K_ε *be a viability domain of* (1)*. If* $K_0 \neq \emptyset$ *is a viability domain of* (4)*, then* $K_0 \supset \limsup K_\varepsilon$.
3) *Let* K *be closed bounded set. If the viability kernel* $\text{Viab}_\varepsilon(K) \neq \emptyset$ *for all sufficiently small* $\varepsilon > 0$ *and* $\text{Viab}_0(K) \neq \emptyset$ *is a viability kernel of* (4)*, then* $\text{Viab}_0(K) \supset \limsup \text{Viab}_\varepsilon(K)$.
4) *The statement* 3) *holds true, when "viable" is replaced by "invariant".*

References

[1] Aubin, J.: *Viability Theory.* Boston: Birghäuser 1991.

[2] Couchouron, J. and M. Kamenski: *An abstract topological point of view and a general averaging principle in the theory of differential inclusions.* Nonlinear Analysis 42 (2000), 625 – 651.

[3] Deimling, K.: *Multivalued Differential Equations.* Berlin: De Gruyter 1992.

[4] Donchev, T.: *Semicontinuous differential inclusions.* Rend. Sem. Mat. Univ. di Padova 101 (1999), 147 – 160.

[5] Donchev, T.: *Functional differential inclusions involving dissipative and compact multifunctions.* Glasnik Matematički 33 (1998)(53), 51 – 60.

[6] Donchev, T.: *Averaging of differential inclusions in Banach spaces.* Ann. UACG XL (1998/99)II, 53 – 63.

[7] Hu, S. and N. Papageorgiou: *Handbook of Multivalued Analysis.* Vol. I: *Theory.* Dordrecht: Kluwer 1997. Vol. II: *Applications.* Dordrecht: Kluwer 2000.

[8] Janiak, T. and E. Luczak-Kumorek: *The theorem of midding for functional differential equations of neutral type.* Disc. Math. Diff. Incl. 11 (1991), 63 – 73.

[9] Kamenski, M., Obukhovskii, V. and P. Zecca: *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces.* Berlin: De Gruyter 2001.

[10] Kisielewicz, M.: *Method of averaging for differential equations with compact convex valued solutions.* Rend. Math. 9 (1976), 397 – 408.

[11] Lakshmikantham, V. and S. Leela: *Nonlinear Differential Equations in Abstract Spaces.* Oxford: Pergamon Press 1981.

[12] Plotnikov, V.: *Averaging Method in Control Problems.* Kiev–Odessa: Libid 1992 (in Russian).

[13] Plotnikov, V., Plotnikov, A. and A. Vityuk: *Differential Equations with Multivalued Right–Hand Side. Asymptotic Methods.* Odessa : Astro Print 1999 (in Russian).

[14] Plotnikov, V. and V. Savchenko: *About averaging of differential inclusions.* Ukr. Math. Journal 48 (1996), 1572 – 1575 (in Russian).

[15] Zverkova, T.: *Ground of the averaging method when the Lipschitz condition is violated.* Ann. Higher Schools – Sofia 23 (1987), 41 – 48 (in Russian).