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# Weighted Integrals of Holomorphic Functions on the Polydisc II

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**Abstract.** Let $\mathcal{L}^p_\alpha(U^n)$ denote the class of all measurable functions defined on the unit polydisc $U^n = \{z \in \mathbf{C}^n \mid |z_i| < 1,\ i = 1, ..., n\}$ such that

$$\|f\|^p_{\mathcal{L}_\alpha(U^n)} = \int_{U^n} |f(z)|^p \prod_{j=1}^n (1 - |z_j|^2)^{\alpha_j} dm(z_j) < \infty,$$

where $\alpha_j > -1, j = 1, ..., n$, and $dm(z_j)$ is the normalized area measure on the unit disk $U$, $H(U^n)$ the class of all holomorphic functions on $U^n$, and let $\mathcal{A}^p_\alpha(U^n) = \mathcal{L}^p_\alpha(U^n) \cap H(U^n)$ (the weighted Bergman space). In this paper we prove that for $p \in (0, \infty)$, $f \in \mathcal{A}^p_\alpha(U^n)$ if and only if the functions

$$\prod_{j \in S} (1 - |z_j|^2) \frac{\partial^{|S|} f}{\prod_{j \in S} \partial z_j} \big(\chi_S(1) z_1, \chi_S(2) z_2, ..., \chi_S(n) z_n\big)$$

belong to the space $\mathcal{L}^p_\alpha(U^n)$ for every $S \subseteq \{1, 2, ..., n\}$, where $\chi_S(\cdot)$ is the characteristic function of $S$, $|S|$ is the cardinal number of $S$, and $\prod_{j \in S} \partial z_j = \partial z_{j_1} \cdots \partial z_{j_{|S|}}$, where $j_k \in S$, $k = 1, ..., |S|$. This result extends Theorem 22 of Kehe Zhu in Trans. Amer. Math. Soc. 309 (1988) (1), 253 – 268, when $p \in (0, 1)$. Also in the case $p \in [1, \infty)$, we present a new proof.

**Keywords:** *Holomorphic function, weighted Bergman space, polydisc*

**MSC 2000:** Primary 32A10, secondary 32A36

## 1. Introduction

Let $z = (z_1, ..., z_n)$ and $w = (w_1, ..., w_n)$ be points in complex vector space $\mathbf{C}^n$, $U$ the unit disc in the complex plane $\mathbf{C}$, $U^n$ the open unit polydisc in $\mathbf{C}^n$ and let $H(U^n)$ be the class of all holomorphic functions $f$ defined on $U^n$.

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Let $\gamma = (\gamma_1, ..., \gamma_n)$ be a multi-index, $\gamma_k$ being nonnegative integers, we write

$$|\gamma| = \gamma_1 + \cdots + \gamma_n, \qquad \gamma! = \gamma_1! \cdots \gamma_n!, \qquad z^\gamma = z_1^{\gamma_1} \cdots z_n^{\gamma_n}.$$

For a holomorphic function $f$ we denote

$$D^\gamma f = \frac{\partial^{|\gamma|} f}{\partial z_1^{\gamma_1} \cdots \partial z_n^{\gamma_n}}.$$

Let $\alpha = (\alpha_1, ..., \alpha_n)$, with $\alpha_j > -1$ for $j = 1, ..., n$, and $p \in (0, \infty)$. The space $\mathcal{L}^p_\alpha(U^n) = \mathcal{L}^p_\alpha$ denotes the class of all measurable functions defined on the polydisc $U^n$ such that

$$\|f\|^p_{\mathcal{L}_\alpha} = \int_{U^n} |f(z)|^p \prod_{j=1}^n (1 - |z_j|^2)^{\alpha_j} dm(z_j) < \infty,$$

where $dm(z_j) = \frac{1}{\pi} r_j dr_j d\theta_j$ is the normalized area measure on the unit disk $U$. When $\alpha = \vec{0}$ we denote the space by $\mathcal{L}^p(U^n)$. The weighted Bergman space $\mathcal{A}^p_\alpha(U^n) = \mathcal{A}^p_\alpha$ is the intersection of $\mathcal{L}^p_\alpha$ and $H(U^n)$.

Weighted Bergman spaces of analytic functions of one variable have been studied, for example, in [3 – 6] and [11, 12, 16, 19] while the weighted Bergman spaces of analytic functions on the unit ball $B \subset \mathbf{C}^n$ have been studied, for example, in [1], [7 – 10] and [15, 17, 18] (see, also the references therein).

Throughout the rest of the paper $S$ denotes a subset of $\{1, ..., n\}$, $\chi_S(\cdot)$ is the characteristic function of $S$, $|S|$ denotes the cardinal number of $S$, and $\prod_{j \in S} \partial z_j = \partial z_{j_1} \cdots \partial z_{j_{|S|}}$, where $j_k \in S$, $k = 1, ..., |S|$.

In [1, p. 33] and [15] the authors proved the following theorem.

**Theorem A.** *Let $p \in (0, \infty)$, $\alpha = (\alpha_1, ..., \alpha_n)$, with $\alpha_j > -1$ for $j = 1, ..., n$, let $m$ be a fixed positive integer and $\mathbf{k} = (k_1, ..., k_n) \in (\mathbf{Z}_+)^n$. Let $f \in H(U^n)$, then $f \in \mathcal{A}^p_\alpha$ if and only if*

$$\left[\prod_{j=1}^n (1 - |z_j|^2)^{k_j}\right] \frac{\partial^{|\mathbf{k}|} f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}(z) \in \mathcal{L}^p_\alpha \qquad \forall \mathbf{k},\ |\mathbf{k}| = m.$$

*Moreover,*

$$\|f\|_{\mathcal{A}^p_\alpha} \asymp \sum_{|\mathbf{k}|=0}^{m-1} \left|\frac{\partial^{|\mathbf{k}|} f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}(0)\right| + \sum_{|\mathbf{k}|=m} \left\|\left[\prod_{j=1}^n (1 - |z_j|^2)^{k_j}\right] \frac{\partial^m f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}\right\|_{\mathcal{L}^p_\alpha}. \quad (1)$$

The above statement means that there are finite positive constants $C$ and $C'$ independent of $f$ such that the left and right hand sides $L(f)$ and $R(f)$ satisfy

$$CR(f) \le L(f) \le C'R(f)$$

for all holomorphic $f$.

In the proof of Theorem A, when $p \in [1, \infty)$, G. Benke and D. C. Chang used the weighted Bergman projection $\mathbf{B}_\alpha : \mathcal{L}^2_\alpha \to \mathcal{A}^2_\alpha$, which can be extended as a bounded operator from $\mathcal{L}^p_\alpha$ onto $\mathcal{A}^p_\alpha$. Case $p \in (0, 1]$ was considered by quite different method by author of this paper in [15].

Using also a Bergman type projection K. Zhu in [17] proved the following result.

**Theorem B.** *Suppose* $1 \le p < \infty$ *and* $f \in H(U^2)$, *then* $f \in \mathcal{A}^p(U^2)$ *if and only if the functions*

$$T_1 f(z_1, 0) = \big(1 - |z_1|^2\big)\frac{\partial f(z_1, 0)}{\partial z_1}$$
$$T_2 f(0, z_2) = \big(1 - |z_2|^2\big)\frac{\partial f(0, z_2)}{\partial z_2}$$
$$T_3 f(z_1, z_2) = \big(1 - |z_1|^2\big)\big(1 - |z_2|^2\big)\frac{\partial^2 f(z_1, z_2)}{\partial z_1 \partial z_2}$$

*are in* $\mathcal{L}^p(U^2)$. *Moreover,* $\|\cdot\|_{\mathcal{A}^p}$ *and*

$$\|f\|_* = |f(0,0)| + \|T_1 f\|_{\mathcal{L}^p} + \|T_2 f\|_{\mathcal{L}^p} + \|T_3 f\|_{\mathcal{L}^p}$$

*are equivalent norms on* $\mathcal{A}^p(U^2)$.

Closely related results on the unit disc and the unit ball in $\mathbf{C}^n$ or $\mathbf{R}^n$ can be found in [1 − 3] and [10 − 18]. The purpose of this note is to generalize Theorem B in the case $p \in (0, 1)$. We prove the following theorem.

**Theorem 1.** *Let* $p \in (0, \infty)$, $\alpha = (\alpha_1, ..., \alpha_n)$, *with* $\alpha_j > -1$ *for* $j = 1, ..., n$, *and* $f \in H(U^n)$. *Then* $f \in \mathcal{A}^p_\alpha(U^n)$ *if and only if the functions*

$$T_S f = \prod_{j \in S}(1 - |z_j|^2)\frac{\partial^{|S|} f}{\prod_{j \in S} \partial z_j}(\chi_S(1) z_1, \chi_S(2) z_2, ..., \chi_S(n) z_n) \tag{2}$$

*belong to the space* $\mathcal{L}^p_\alpha(U^n)$ *for every* $S \subseteq \{1, 2, ..., n\}$. *Moreover,* $\|\cdot\|_{\mathcal{A}^p}$ *and* $\|\cdot\|_*$ *are equivalent norms on* $\mathcal{A}^p(U^n)$, *where*

$$\|f\|_* = |f(\vec{0})| + \sum_{S \subseteq \{1,...,n\}, S \ne \emptyset} \|T_S f\|_{\mathcal{L}^p_\alpha}.$$

**Remark 1.** Since $\emptyset \subset \{1, ..., n\}$, we can consider that $T_\emptyset f = f(0, ..., 0)$, that is, $T_\emptyset f$ can be considered as one of the functions in (2). If we accept this notation, then $\|f\|_* = \sum_{S \subseteq \{1,...,n\}} \|T_S f\|_{\mathcal{L}^p_\alpha}$.

**Remark 2.** To be more suggestive we explain here what condition (2) exactly means when $n = 3$. In this case it means that the following eight functions are in $\mathcal{L}^p_\alpha(U^3)$:

$$\begin{array}{ll} f(0,0,0), & (1-|z_1|^2)(1-|z_2|^2)\dfrac{\partial^2 f(z_1,z_2,0)}{\partial z_1\partial z_2} \\ (1-|z_1|^2)\dfrac{\partial f(z_1,0,0)}{\partial z_1}, & (1-|z_1|^2)(1-|z_3|^2)\dfrac{\partial^2 f(z_1,0,z_3)}{\partial z_1\partial z_3} \\ (1-|z_2|^2)\dfrac{\partial f(0,z_2,0)}{\partial z_2}, & (1-|z_2|^2)(1-|z_3|^2)\dfrac{\partial^2 f(0,z_2,z_3)}{\partial z_2\partial z_3} \\ (1-|z_3|^2)\dfrac{\partial f(0,0,z_3)}{\partial z_2}, & (1-|z_1|^2)(1-|z_2|^2)(1-|z_3|^2)\dfrac{\partial^3 f(z_1,z_2,z_3)}{\partial z_1\partial z_2\partial z_3}. \end{array}$$

## 2. Proof of the main result

To prove the main result we need an auxiliary result which is incorporated in the following theorem.

**Theorem C.** *Suppose* $0 < p < \infty$ *and* $\alpha > -1$. *Then there is a constant* $C = C(p,\alpha)$ *such that*

$$\int_U |f(z)|^p(1-|z|^2)^\alpha dm(z) \le C\left(|f(0)|^p + \int_U |f'(z)|^p(1-|z|^2)^{p+\alpha}dm(z)\right)$$

*for all* $f \in H(U)$.

This theorem is a special case of Theorem 1 in [10]. Another proof of this theorem can be found in [14].

**Proof of Theorem 1.** `Necessity`. The proof of this part of the theorem is a special case of the proof of Theorem 1 in [15].

`Sufficiency`. First, assume that $n = 2$. Let $z_2$ be, for a moment, fixed. Then by Theorem C we have

$$\int_U |f(z_1,z_2)|^p(1-|z_1|^2)^{\alpha_1}dm(z_1) \\ \le C\left(|f(0,z_2)|^p + \int_U \left|\frac{\partial f}{\partial z_1}(z_1,z_2)\right|^p (1-|z_1|^2)^{p+\alpha_1}dm(z_1)\right).$$

Multiplying this inequality by $(1-|z_2|^2)^{\alpha_2}dm(z_2)$, integrating over $U$ and applying Fubini's theorem we obtain

$$\begin{aligned}\int_{U^2} &|f(z_1,z_2)|^p(1-|z_1|^2)^{\alpha_1}(1-|z_2|^2)^{\alpha_2}dm(z_1)dm(z_2)\\ &\le C\Big(\int_U |f(0,z_2)|^p(1-|z_2|^2)^{\alpha_2}dm(z_2) \qquad (3)\\ &\quad+\int_U\Big(\int_U\left|\frac{\partial f}{\partial z_1}(z_1,z_2)\right|^p(1-|z_2|^2)^{\alpha_2}dm(z_2)\Big)(1-|z_1|^2)^{p+\alpha_1}dm(z_1)\Big).\end{aligned}$$

On the other hand, for a fixed $z_1$ by Theorem C we have

$$\begin{aligned}\int_U &\left|\frac{\partial f}{\partial z_1}(z_1,z_2)\right|^p(1-|z_2|^2)^{\alpha_2}dm(z_2)\\ &\le C\left(\left|\frac{\partial f}{\partial z_1}(z_1,0)\right|^p+\int_U\left|\frac{\partial^2 f}{\partial z_1\partial z_2}(z_1,z_2)\right|^p(1-|z_2|^2)^{p+\alpha_2}dm(z_2)\right). \qquad (4)\end{aligned}$$

Let

$$I=\int_{U^2}|f(z_1,z_2)|^p(1-|z_1|^2)^{\alpha_1}(1-|z_2|^2)^{\alpha_2}dm(z_1)dm(z_2).$$

From (3) and (4) and Fubini's theorem, we obtain

$$\begin{aligned}I\le C\Big(&\int_U|f(0,z_2)|^p(1-|z_2|^2)^{\alpha_2}dm(z_2)\\ &+\int_U\left|\frac{\partial f}{\partial z_1}(z_1,0)\right|^p(1-|z_1|^2)^{p+\alpha_1}dm(z_1) \qquad (5)\\ &\times\int_{U^2}\left|\frac{\partial^2 f}{\partial z_1\partial z_2}(z_1,z_2)\right|^p(1-|z_1|^2)^{p+\alpha_1}(1-|z_2|^2)^{p+\alpha_2}dm(z_1)dm(z_2)\Big).\end{aligned}$$

Applying Theorem C to the function $f(0,z_2)$ we obtain

$$\begin{aligned}\int_U &|f(0,z_2)|^p(1-|z_2|^2)^{\alpha_2}dm(z_2)\\ &\le C\left(|f(0,0)|^p+\int_U\left|\frac{\partial f}{\partial z_2}(0,z_2)\right|^p(1-|z_2|^2)^{p+\alpha_2}dm(z_2)\right). \qquad (6)\end{aligned}$$

From (5) and (6) the result follows in this case.

For $n\ge 3$ we use induction. Assume that we have proved the theorem for $k\le n-1$, where $k$ is the number of variables. Let $z'=(z_1,...z_{n-1})$. Then by inductive hypothesis we have

$$\begin{aligned}
&\int_{U^n} |f(z)|^p \prod_{j=1}^{n} (1-|z_j|^2)^{\alpha_j} dm(z_j) \\
&\quad = \int_U \bigg( \int_{U^{n-1}} |f(z', z_n)|^p \prod_{j=1}^{n-1} (1-|z_j|^2)^{\alpha_j} dm(z_j) \bigg) (1-|z_n|^2)^{\alpha_n} dm(z_n) \\
&\quad \le \int_U \bigg( |f(0', z_n)|^p \\
&\qquad + \sum_{S' \subseteq \{1,\dots,n-1\}} \int_{U^{|S'|}} \bigg| \frac{\partial^{|S'|} f}{\prod_{j\in S'} \partial z_j} (\chi_{S'}(1) z_1, \dots, \chi_{S'}(n-1) z_{n-1}, z_n) \bigg|^p \\
&\qquad \times \prod_{j\in S'} (1-|z_j|^2)^{\alpha_j+p} dm(z_j) \bigg) (1-|z_n|^2)^{\alpha_n} dm(z_n).
\end{aligned} \tag{7}$$

On the other hand we have

$$\begin{aligned}
&\int_U \bigg| \frac{\partial^{|S'|} f}{\prod_{j\in S'} \partial z_j} (\chi_{S'}(1) z_1, \dots, \chi_{S'}(n-1) z_{n-1}, z_n) \bigg|^p \prod_{j\in S'} (1-|z_n|^2)^{\alpha_n} dm(z_n) \\
&\qquad \le C \bigg( \bigg| \frac{\partial^{|S'|} f}{\prod_{j\in S'} \partial z_j} (\chi_{S'}(1) z_1, \dots, \chi_{S'}(n-1) z_{n-1}, 0) \bigg|^p \\
&\qquad + \int_U \bigg| \frac{\partial^{|S'|+1} f}{\prod_{j\in S'} \partial z_j \cdot \partial z_n} (\chi_{S'}(1) z_1, \dots, \chi_{S'}(n-1) z_{n-1}, z_n) \bigg|^p \\
&\qquad \times (1-|z_n|^2)^{\alpha_n+p} dm(z_n) \bigg).
\end{aligned} \tag{8}$$

Applying Fubini's theorem in (7), then substituting (8) in the obtained inequality, using the inequality

$$\int_U |f(0', z_n)|^p (1-|z_n|^2)^{\alpha_n} \le C \bigg( |f(\vec{0})|^p + \int_U \bigg| \frac{\partial f}{\partial z_n}(0', z_n) \bigg|^p (1-|z_n|^2)^{p+\alpha_n} dm(z_n) \bigg)$$

and the fact that every $S \subseteq \{1, \dots, n\}$ can be written as $S = S' \cup \{\emptyset\}$ or $S = S' \cup \{n\}$ where $S'$ is a subset of $\{1, \dots, n-1\}$, we obtain the result. ∎

**Remark 3.** Throughout the proof we use the letter $C$ for a constant which may vary from line to line.

**Corollary 1.** *Let $p \in (0, \infty)$ and $f \in H(U^n)$ be such that $f(z) = 0$ whenever $\prod_{j=1}^n z_j = 0$. Then, for some $C > 0$ independent of $f$ it holds*

$$\int_{U^n} |f(z)|^p \prod_{j=1}^{n} (1-|z_j|^2)^{\alpha_j} dm(z_j) \le C \int_{U^n} \bigg| \frac{\partial^n f}{\partial z_1 \cdots \partial z_n}(z) \bigg|^p \prod_{j=1}^{n} (1-|z_j|^2)^{\alpha_j+p} dm(z_j).$$

By Theorem A and Theorem 1 we see that the following result is true.

**Corollary 2.** *Let* $p \in (0,\infty)$, $\alpha = (\alpha_1, ..., \alpha_n)$, *with* $\alpha_j > -1$ *for* $j = 1, ..., n$, *let* $m$ *be a fixed positive integer and* $\mathbf{k} = (k_1, ..., k_n) \in (\mathbf{Z}_+)^n$. *Let* $f \in H(U^n)$, *then the following conditions are equivalent:*

**(a)** $f \in \mathcal{A}^p_\alpha$

**(b)** $\left[\prod_{j=1}^n (1-|z_j|^2)^{k_j}\right] \frac{\partial^{|\mathbf{k}|} f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}(z) \in \mathcal{L}^p_\alpha \qquad \forall \mathbf{k},\ |\mathbf{k}| = m;$

**(c)** *For every* $S \subseteq \{1, 2, ..., n\}$, *the functions*

$$T_S f = \prod_{j\in S}(1-|z_j|^2)\frac{\partial^{|S|} f}{\prod_{j\in S}\partial z_j}(\chi_S(1)z_1, \chi_S(2)z_2, ..., \chi_S(n)z_n),$$

*are in* $\mathcal{L}^p_\alpha(U^n)$.

*Moreover,*

$$\|f\|_{\mathcal{A}^p_\alpha} \asymp \sum_{|\mathbf{k}|=0}^{m-1}\left|\frac{\partial^{|\mathbf{k}|} f}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}(0)\right| + \sum_{|\mathbf{k}|=m}\left\|\left[\prod_{j=1}^n(1-|z_j|^2)^{k_j}\right]\frac{\partial^m f}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}\right\|_{\mathcal{L}^p_\alpha}$$
$$\asymp \|f\|_*.$$

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