Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 24 (2005), No. 1, 29–74

Delta Waves for a Strongly Singular Initial-Boundary Hyperbolic Problem with Integral Boundary Condition

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Abstract. We investigate the existence and the singular structure of delta wave solutions to a semilinear hyperbolic equation with strongly singular initial and boundary conditions. The boundary conditions are given in nonlocal form with a linear integral operator involved. We construct a delta wave solution as a distributional limit of solutions to the regularized system. This determines the macroscopic behavior of the corresponding generalized solution in the Colombeau algebra $\mathcal{G}$ of generalized functions. We represent our delta wave as a sum of a purely singular part satisfying a linear system and a regular part satisfying a nonlinear system.

Keywords: *hyperbolic equation, integral condition, strongly singular data, delta wave, population dynamics*

MSC 2000: 46F30

1. Introduction

In the domain

$$\Pi = \{(x,t) \in \mathbb{R}^2 \,|\, 0 < x < L, t > 0\}$$

we study the following initial-boundary value problem for the first-order semilinear hyperbolic equation:

$$(\partial_t + \lambda(x,t)\partial_x)u = p(x,t)u + f(x,t,u), \qquad (x,t) \in \Pi \tag{1}$$

$$u(x,0) = a(x), \qquad x \in (0,L) \tag{2}$$

$$u(0,t) = \int_0^L h(x,t)u(x,t)\,dx, \qquad t \in (0,\infty)\,. \tag{3}$$

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This work was done while visiting the Institut für Mathematik, Universität Wien, supported by an ÖAD grant.

ISSN 0232-2064 / $ 2.50

Mathematical models of this kind stem from mathematical biology and serve for describing the age-dependent population dynamics (see [2, 3, 13, 23, 24]). In particular, the linear case of the problem, when $f(x,t,u)$ does not depend on u, arises in demography, where $u(x,t)$ is the population density of age x at time t, $a(x)$ is the initial density, $h(x,t)$ is the birth rate, $-p(x,t)$ is the death rate, and $f(x,t)$ is the migrant density. Nonlinear models of age structured populations are studied in [2, 3]. To model point-concentration of the initial density and the birth rate, we consider the data $a(x)$ and $h(x,t)$ to be strongly singular, of the Dirac delta type.

As well known, solutions to the classical initial-boundary semilinear hyperbolic problems in a single space variable are at least as singular as the initial and the boundary data. We therefore can expect for the nonclassical problem (1)–(3) that the multiplication of distributions appears in the right-hand sides of (1) and (3). Such multiplication in general cannot be performed within the distributional theory and, by this reason, is usually defined in differential algebras of generalized functions. In [14] we used the Colombeau algebra of generalized functions $\mathcal{G}(\overline{\Pi})$ [1, 5, 16] to prove a global existence-uniqueness result for (1)–(3) with $p(x,t) \equiv 0$ and smooth λ and f (this is a special case of the problem studied here). Nevertheless, the macroscopic behavior of the Colombeau solution remained unclear.

We here show that the system (1)–(3) has a delta wave solution (shortly, a delta wave) in the sense of [22], i.e., the sequence of approximate (or sequential) solutions obtained by regularizing all singular data has a weak limit which does not depend on a particular regularization. This determines the macroscopic behavior (the singular structure) of the Colombeau solution to the problem (1)–(3) with $p(x,t) \equiv 0$ and smooth λ and f. In the course of construction of the delta wave solution we show interaction and propagation of singularities.

Note that a delta wave solution in general does not satisfy the system in a differential-algebraic sense. Our paper brings one more example into the collection of delta wave solutions which are not distributional solutions.

The advantage of using delta wave solutions lies in the fact that, due to the procedure of their obtaining, they are stable with respect to regularizations. In contrast with this, if we use a priori defined intrinsic multiplication of distributions for obtaining distributional solutions, the result may be nonstable and noncorrect [6]. The concept of a delta wave solution has also other advantages. It serves us a solution concept for nonlinear systems and for linear systems with nonsmooth coefficients, for which the distributional theory is not well adapted. For the delta wave solutions of semilinear hyperbolic problems we refer the reader to the sources [6, 7, 10] and [14 – 22].

We split a delta wave into the sum of a regular part satisfying a nonlinear equation and a singular part satisfying a linear equation. The idea of nonlinear splitting goes back to [7, 18, 19, 22]. An important feature of the nonlinear splitting suggested here is a quite strong interdependence of the systems describing the singular and the regular parts. A similar phenomenon is discovered in [14] for a nonlocal problem with nonseparable boundary conditions, where the singular part of the nonlinear splitting depends on the regular part.

Delta wave solutions for initial-boundary semilinear hyperbolic problems were considered in [20, 14]. The paper [20] investigates the existence and structure of delta waves in a nonlinear boundary value problem for a second order hyperbolic equation where the boundary condition is nonlinear and the nonlinearity is given by a bounded smooth function. Both in [20] and [14] the right hand side of the differential equations is bounded, and in [20] it can be also sublinear with respect to u.

The paper is organized as follows. In Section 2 we describe in detail our splitting of a delta wave solution and state our main result. The proof is given in Sections 3 – 8. In particular, in Section 5 we show that our splitting procedure is correct. In Section 6 we are concerned with the regular part. Using the Cauchy criterion of the uniform convergency, we prove that the family of approximate solutions to the regular part uniformly converges on any compact subset of $\overline{\Pi}$. In Section 7 we deal with the singular part and prove that the sequence of approximate solutions to the singular part converges in $\mathcal{D}'(\overline{\Pi})$ to a function v. We then show that v actually represents the purely singular part of the initial problem and that it is the sum of measures concentrated on characteristic curves (see Sections 7 and 8).

2. Interaction and propagation of strong singularities and construction of a delta wave solution

We first list assumptions that will be made for the problem (1)–(3).

`Assumption 1.` $a(x) = a_s(x) + a_r(x)$, $\quad h(x,t) = b(x) \otimes c(t) = (b_s(x) + b_r(x)) \otimes (c_s(t) + c_r(t))$, where $a_s(x)$, $b_s(x)$, $c_s(t)$ and $a_r(x)$, $b_r(x)$, $c_r(t)$ are, respectively, singular and regular parts of the functions $a(x)$, $b(x)$, and $c(t)$.

`Assumption 2.` $a_s(x), b_s(x)$, and $c_s(t)$ are the finite sums of the Dirac measures at points, whose supports are as follows:

$$\begin{aligned}
\operatorname{supp} a_s(x) &= \{x_1^*, x_2^*, \dots, x_m^*\}, \text{where} 0 < x_1^* < \cdots < x_m^* < L. \\
\operatorname{supp} b_s(x) &= \{x_1, x_2, \dots, x_k\}, \text{where} 0 < x_1 < \cdots < x_k < L. \\
\operatorname{supp} c_s(t) &= \{t_1, t_2, \dots, t_l\}, \quad \text{where} 0 < t_1 < \cdots < t_l.
\end{aligned}$$

`Assumption 3.` $b_r(0) = 0$ and $b_r(L) = 0$.

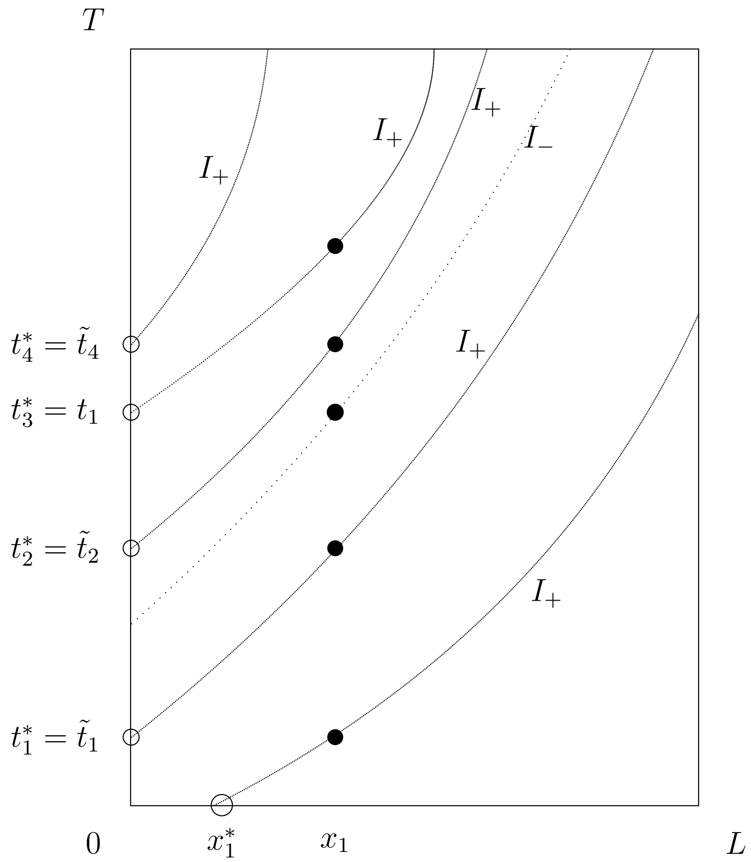
`Assumption 4.` $a_r(0) = 0$ and $c_r(0) = 0$.

`Assumption 5.` p, f, a_r, b_r, c_r are continuous and λ is continuously differentiable with respect to all their arguments, f is continuously differentiable with respect to u.

`Assumption 6.` $\lambda(x,t) > 0$ for all $(x,t) \in \overline{\Pi}$.

`Assumption 7.` f and $\nabla_u f$ are globally bounded with respect to (x,t) varying in compact subsets of $\overline{\Pi}$.

Assumption 4 serves to ensure the 0-order compatibility between (2) and (3). Assumption 6 is not restrictive from the mathematical point of view. Indeed, if $\lambda < 0$, we can replace the boundary condition (3) for the line $x = 0$ by this condition for the line $x = L$. If $\lambda = 0$, the boundary $\{(x,t) \,|\, x = 0 \text{ or } x = L\}$ of Π is characteristic. The condition of global boundedness imposed on $\nabla_u f$ means that f has at most linear growth with respect to u as $|u| \to \infty$. Note that the assumptions are not restrictive from the viewpoint of applications.



Recall that all characteristics of the differential equation (1) are solutions to the following initial problem for ordinary differential equation:

$$\frac{d\xi}{d\tau} = \lambda(\xi(\tau), \tau), \quad \xi(t) = x,$$

where $(x,t) \in \overline{\Pi}$. It is well known that, under Assumptions 5 and 6, for every $(x,t) \in \overline{\Pi}$ this problem has a unique C^1-solution which can be expressed in any of two forms $\xi = \omega(\tau; x, t)$ or $\tau = \tilde{\omega}(\xi; x, t)$.

Choose $\varepsilon_0 > 0$ so small that $x_1^* - \varepsilon_0 > 0$ and $t_1 - \varepsilon_0 > 0$. Some additional conditions on ε_0 will be put below. We will consider ε in the range $0 < \varepsilon \le \varepsilon_0$.

The following definitions are visualized in the figure.

Definition 1. Let I_- be the union of the characteristics $\omega(t; x_i, t_j)$ passing through the common points of the lines $x = x_i$ and $t = t_j$, where $i \le k$ and $j \le l$. Let I_-^ε be the union of the tubular neighborhoods $\{(\omega(\tau; x_i, t), \tau) \,|\, \tilde{\omega}(x_i; x_i - \varepsilon, t_j + \varepsilon) < t < \tilde{\omega}(x_i; x_i + \varepsilon, t_j - \varepsilon)\}$ around the characteristics contributing into I_-.

Definition 2. Let $I_+ = \bigcup_{n \ge 0} I_+[n]$ and, for $\varepsilon < \varepsilon_0$, $I_+^\varepsilon = \bigcup_{n \ge 0} I_+^\varepsilon[n]$, where $I_+[n]$ and $I_+^\varepsilon[n]$ are subsets of $\overline{\Pi}$ defined by induction as follows.

- $I_+[0]$ includes the characteristics $\omega(t; x_i^*, 0)$ and $\omega(t; 0, t_j)$ for all $i \le m$ and $j \le l$ (i.e. $I_+[0]$ is the union of the characteristics issuing from the singular points on $\partial\Pi$ caused by the initial data $a_s(x)$ and the boundary data $c_s(t)$).

  $I_+^\varepsilon[0]$ includes the tubular neighborhoods $\{(\omega(\tau; x, 0), \tau) \,|\, x_i^* - \varepsilon < x < x_i^* + \varepsilon\}$ and $\{(\omega(\tau; 0, t), \tau) \,|\, t_j - \varepsilon < t < t_j + \varepsilon\}$ around the characteristics contributing into $I_+[0]$.
- Let $n \ge 1$. If $I_+[n-1]$ includes the characteristic $\omega(t; x_i, \tilde{t})$, then $I_+[n]$ includes the characteristic $\omega(t; 0, \tilde{t})$.

  If $I_+^\varepsilon[n-1]$ includes the tubular neighborhood $\{(\omega(\tau; x_i, t), \tau) \,|\, \tilde{t} - \varepsilon^- < t < \tilde{t} + \varepsilon^+\}$ around the characteristic $\omega(t; x_i, \tilde{t})$, then $I_+^\varepsilon[n]$ includes the tubular neighborhood $\{(\omega(\tau; 0, t), \tau) \,|\, \tilde{\omega}(x_i - \varepsilon; x_i, \tilde{t} - \varepsilon^-) < t < \tilde{\omega}(x_i + \varepsilon; x_i, \tilde{t} + \varepsilon^+)\}$ around the characteristic $\omega(t; 0, \tilde{t})$.

Note that the parameters k, l, m, and l, on which the construction of I_-, I_-^ε, I_+, and I_+^ε is based, are predetermined by the problem due to Assumption 2. The set I_+ captures the propagation of all singularities. For characteristics contributing into I_+ (respectively, $I_+ \setminus I_+[0]$), denote their intersection points with the axis $x = 0$ by $t_1^*, t_2^*, \dots$ (respectively, $\tilde{t}_{i_1}, \tilde{t}_{i_2}, \dots$). We assume that $t_j^* < t_{j+1}^*$ for $j \ge 1$ and $i_n = p$ for $\tilde{t}_{i_n} = t_p^*$. Obviously, $\{t_1^*, t_2^*, \dots\} = \{t_1, \dots, t_l\} \cup \{\tilde{t}_{i_1}, \tilde{t}_{i_2}, \dots\}$. Let $\varepsilon_i^-(\varepsilon)$ and $\varepsilon_i^+(\varepsilon)$ be such that

$$I_+^\varepsilon \cap \{(x,t) \in \overline{\Pi} \,|\, x = 0\} = \bigcup_i \{(0,t) \,|\, t_i^* - \varepsilon_i^-(\varepsilon) < t < t_i^* + \varepsilon_i^+(\varepsilon)\}.$$

If $t_i^* = t_j$ for some $j \le l$, then $\varepsilon_i^-(\varepsilon) = \varepsilon_i^+(\varepsilon) = \varepsilon$. Observe that $\lim_{\varepsilon \to 0} \varepsilon_i^-(\varepsilon) = 0$ and $\lim_{\varepsilon \to 0} \varepsilon_i^+(\varepsilon) = 0$.

Assumption 8. $\tilde{\omega}(0; x_i, t_j) \neq t_s^*$, $\omega(0; x_i, t_j) \neq x_q^*$ for all $i \leq k$, $j \leq l$, $q \leq m$ and $t_s^* < t_l$.

This assumption means that no three different singularities caused by the initial and the boundary data hit at the same point. In other words, neither points $(x_q^*, 0)$ and (x_i, t_j) nor points $(0, t_s^*)$ and (x_i, t_j) are connected by any of characteristic curves. As a consequence, there exists ε_0 such that $I_-^\varepsilon \cap I_+^\varepsilon = \emptyset$ for each $\varepsilon \leq \varepsilon_0$. Assume that $(0,0) \notin I_-$. We choose ε_0 so small that $I_-^{\varepsilon_0}$ and $I_+^{\varepsilon_0}$ do not contain the point $(0,0)$. Clearly, $\bigcap_{\varepsilon>0} I_+^\varepsilon = I_+$ and $\bigcap_{\varepsilon>0} I_-^\varepsilon = I_-$.

Our aim is to show that the generalized solution to the problem (1)–(3), whose existence is shown in [14], admits an *associated distribution* or a *delta wave*. The latter means that the family $(u^\varepsilon)_{\varepsilon>0}$ of solutions to the system with regularized initial and boundary data

$$(\partial_t + \lambda(x,t)\partial_x)u^\varepsilon = p(x,t)u^\varepsilon + f(x,t,u^\varepsilon) \tag{4}$$

$$u^\varepsilon(x,0) = a_s^\varepsilon + a_r \tag{5}$$

$$u^\varepsilon(0,t) = (c_s^\varepsilon + c_r) \int_0^L (b_s^\varepsilon + b_r) u^\varepsilon \, dx \tag{6}$$

has a weak limit. Here

$$a_s^\varepsilon = a_s * \varphi_\varepsilon, \quad b_s^\varepsilon = b_s * \varphi_\varepsilon, \quad c_s^\varepsilon = c_s * \varphi_\varepsilon,$$

where mollifiers φ_ε are model delta nets, that is,

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\Big(\frac{x}{\varepsilon}\Big)$$

for an arbitrary fixed $\varphi \in \mathcal{D}(\mathbb{R})$ with $\int \varphi(x)\, dx = 1$. Note that

$$a_s^\varepsilon = O\Big(\frac{1}{\varepsilon}\Big), \quad b_s^\varepsilon = O\Big(\frac{1}{\varepsilon}\Big), \quad c_s^\varepsilon = O\Big(\frac{1}{\varepsilon}\Big) \tag{7}$$

and

$$\int_0^L |a_s^\varepsilon(x)|\, dx \leq C, \quad \int_0^L |b_s^\varepsilon(x)|\, dx \leq C, \quad \int_0^\infty |c_s^\varepsilon(t)|\, dt \leq C, \tag{8}$$

where C does not depend on ε. We will consider mollifiers φ with

$$\operatorname{supp}\varphi \subset [-1,1]. \tag{9}$$

This restriction makes no loss of generality, because if (9) is not true, then $\operatorname{supp}\varphi \subset [-d,d]$ for some $d > 0$. Therefore $\operatorname{supp}\varphi_\varepsilon \subset [-d\varepsilon, d\varepsilon]$ and it is enough to replace I_+^ε by $I_+^{d\varepsilon}$ to keep all arguments valid, with the result not depending on d. It follows from (9) that for all $\varepsilon > 0$

$$\begin{aligned} \int_{x_i^*-\varepsilon}^{x_i^*+\varepsilon} a_s^\varepsilon(x)\,dx = 1, &\quad 1 \le i \le m \\ \int_{x_j-\varepsilon}^{x_j+\varepsilon} b_s^\varepsilon(x)\,dx = 1, &\quad 1 \le j \le k \\ \int_{t_p-\varepsilon}^{t_p+\varepsilon} c_s^\varepsilon(t)\,dt = 1, &\quad 1 \le p \le l. \end{aligned} \tag{10}$$

Let T be an arbitrary positive real, $\Pi^T = \{(x,t) \in \Pi \,|\, t < T\}$. We will show that a delta wave splits up into the sum $w + v$ of the following kind. The function w corresponds to the regular part of the problem. More specifically, for every $T > 0$, the restriction of w to Π^T is the limit of w^ε in $\mathrm{C}(\Pi^T)$ as $\varepsilon \to 0$, where w^ε for every fixed $\varepsilon > 0$ is a continuous solution (the concept of a continuous solution will be defined below) to the nonlinear problem

$$(\partial_t + \lambda(x,t)\partial_x)w^\varepsilon = p(x,t)w^\varepsilon + f(x,t,w^\varepsilon) \tag{11}$$

$$w^\varepsilon(x,0) = a_r \tag{12}$$

$$w^\varepsilon(0,t) = c_r \int_0^L [(b_s^\varepsilon + b_r)w^\varepsilon + b_r v^\varepsilon]\,dx. \tag{13}$$

The function v corresponds to the singular part of the problem and is the limit of v^ε in $\mathcal{D}'(\Pi)$ as $\varepsilon \to 0$, where v^ε for every fixed $\varepsilon > 0$ is a continuous solution to the linear problem

$$(\partial_t + \lambda(x,t)\partial_x)v^\varepsilon = p(x,t)v^\varepsilon \tag{14}$$

$$v^\varepsilon(x,0) = a_s^\varepsilon \tag{15}$$

$$v^\varepsilon(0,t) = c_s^\varepsilon \int_0^L [(b_s^\varepsilon + b_r)w^\varepsilon + b_r v^\varepsilon]\,dx + c_r \int_0^L b_s^\varepsilon v^\varepsilon\,dx. \tag{16}$$

Fix $\varepsilon > 0$. Note that the problem (4)–(6) for a function $u^\varepsilon(x,t) \in \mathrm{C}^1(\Pi) \cap \mathrm{C}(\overline{\Pi})$ can be transformed in an equivalent system of integral-operator equations (see [14, p. 641]). We say that the problem (4)–(6) has a continuous solution u^ε if u^ε is continuous in $\overline{\Pi}$ and satisfies the corresponding system of integral-operator equations. Similarly we define the concept of a continuous solution to the problems (11)–(13) and (14)–(16).

Proposition 1. *For every $\varepsilon \le \varepsilon_0$ there exist a unique continuous solution u^ε to the problem* (4)–(6)*, a unique continuous solution w^ε to the problem* (11)–(13)*, and a unique continuous solution v^ε to the problem* (14)–(16)*.*

Proof. From the proof of [14, Theorem 3] it follows that, if Assumptions 4 – 7 hold and ε is so small that $(0,0) \notin I_+^\varepsilon$, this problem has a unique continuous solution u^ε.

Fix $\varepsilon > 0$. The systems (11)–(13) and (14)–(16) are interdependent (v^ε appears in (13) and w^ε appears in (16)). That is, we look for the unknown pair $(w^\varepsilon, v^\varepsilon)$ solving the system (11)–(16). Note that we have zero-order compatibility of (12), (13) and of (15), (16), the former by Assumption 4 and the latter by Assumption 4 and the fact that $(0,0) \notin I_+^\varepsilon$. From the proof of [14, Theorem 3] it follows that under Assumptions 4 – 7 the problem has a unique continuous solution $(w^\varepsilon, v^\varepsilon)$. ∎

We are now prepared to state the main result of the paper.

Theorem 1. *Let Assumptions 1 – 8 hold. Let u^ε, for every $\varepsilon > 0$, be the continuous solution to the problem (4)–(6). Then*

$$u^\varepsilon \to w + v \quad \text{in } \mathcal{D}'(\Pi) \quad \text{as } \varepsilon \to 0,$$

where

- *for every $T > 0$, the restriction of w to Π^T is the limit of w^ε in $\mathrm{C}(\Pi^T)$ as $\varepsilon \to 0$ with w^ε being the continuous solution to the problem (11)–(13),*
- *$v = \lim_{\varepsilon\to 0} v^\varepsilon$ in $\mathcal{D}'(\Pi)$ with v^ε being the continuous solution to the problem (14)–(16). Furthermore, the restriction of v to $\Pi \setminus I_+$ is identically equal to 0.*

Given $u \in \mathcal{D}'(\Omega)$, we define c-sing supp u as follows. A point $(x,t) \in \Omega$ belongs to c-sing supp u if (x,t) does not have any neighborhoods on which u is continuous.

Corollary 1. $c\text{-sing supp}(w+v) = c\text{-sing supp}\, v = \operatorname{supp} v \subset I_+$.

This means that v actually represents the purely singular part of the initial problem. The proof of the corollary is straightforward. The proof of Theorem 1 consists of five lemmas whose proofs are given in Sections 4–8.

Lemma 1. *Let Assumptions 1, 2, 4 – 6, and 8 hold and v^ε be as in Theorem 1. Then*

$$v^\varepsilon \to 0 \quad \text{pointwise off } I_+ \quad \text{as } \varepsilon \to 0.$$

Lemma 2. *Let Assumptions 1 – 8 hold and $u^\varepsilon, v^\varepsilon$, and w^ε be as in Theorem 1. Then*

$$u^\varepsilon - v^\varepsilon - w^\varepsilon \to 0 \quad \text{in } \mathrm{L}^1_{loc}(\Pi) \quad \text{as } \varepsilon \to 0.$$

Lemma 3. *Let Assumptions 1 – 8 hold and w^ε be as in Theorem 1. Then*

$$w^\varepsilon \quad \text{converges in } \mathrm{C}(\overline{\Pi^T}) \quad \text{as } \varepsilon \to 0$$

for an arbitrary fixed $T > 0$.

Lemma 4. *Let Assumptions* 1 – 8 *hold and* v^ε *be as in Theorem* 1. *Then*

$$v^\varepsilon \quad \textit{converges in } \mathcal{D}'(\Pi) \quad \textit{as } \varepsilon \to 0.$$

Lemma 5. *Let Assumptions* 1 – 8 *hold,* v^ε *be as in Theorem* 1*, and* $v = \lim_{\varepsilon\to 0} v^\varepsilon$ *in* $\mathcal{D}'(\Pi)$*. Then* v *restricted to* $\Pi \setminus I_+$ *is identically equal to 0.*

Theorem 1 now follows from the embedding of $\mathrm{L}^1_{loc}(\Pi)$ into $\mathcal{D}'(\Pi)$.

3. Representation of the problems (11)–(13) and (14)–(16) in an integral-operator form

The problem (11)–(13) is equivalent to the integral-operator equation

$$w^\varepsilon(x,t) = (Rw^\varepsilon)(x,t) + \int_{\theta(x,t)}^t \Big[f(\xi,\tau,w^\varepsilon) + (pw^\varepsilon)(\xi,\tau)\Big]\Big|_{\xi=\omega(\tau;x,t)} d\tau \tag{17}$$

and to the corresponding linearized integral-operator equation

$$\begin{aligned} w^\varepsilon(x,t) = (Rw^\varepsilon)(x,t) + \int_{\theta(x,t)}^t \Big[& f(\xi,\tau,0) \\ & + w^\varepsilon\Big(\int_0^1 (\nabla_u f)(\xi,\tau,\sigma w^\varepsilon)\, d\sigma + p(\xi,\tau)\Big)\Big]\Big|_{\xi=\omega(\tau;x,t)} d\tau \end{aligned} \tag{18}$$

with boundary operator

$$(Rw^\varepsilon)(x,t) = \begin{cases} a_r(\omega(0;x,t)) & \text{if } \theta(x,t) = 0 \\ w^\varepsilon(0,\theta(x,t)) & \text{if } \theta(x,t) > 0 \end{cases} \tag{19}$$

Here

$$\theta(x,t) = \min_{(\omega(\tau;x,t),\tau)\in\partial\Pi} \tau.$$

The boundary function $w^\varepsilon(0,t)$ is given by (13).

The problem (14)–(16) is equivalent to the integral-operator equation

$$v^\varepsilon(x,t) = (B_\varepsilon v^\varepsilon)(x,t) + \int_{\theta(x,t)}^t (pv^\varepsilon)(\omega(\tau;x,t),\tau)\, d\tau \tag{20}$$

with boundary operator

$$(B_\varepsilon v^\varepsilon)(x,t) = \begin{cases} a_s^\varepsilon(\omega(0;x,t)) & \text{if } \theta(x,t) = 0 \\ v^\varepsilon(0,\theta(x,t)) & \text{if } \theta(x,t) > 0. \end{cases} \tag{21}$$

The function $v^\varepsilon(0,t)$ is given by the formula (16). Note that in (20) the integral operator is applied after the boundary operator. This allows us to rewrite (20) in the form

$$v^\varepsilon(x,t) = S(x,t)(B_\varepsilon v^\varepsilon)(x,t) \tag{22}$$

with continuous function

$$\begin{aligned} S(x,t) = 1 + \int_{\theta(x,t)}^{t} p(\omega(\tau;x,t),\tau)\,d\tau + \int_{\theta(x,t)}^{t} p(\omega(\tau;x,t),\tau)\,d\tau \\ \times \int_{\theta(x,t)}^{\tau} p(\omega(\tau_1;\omega(\tau;x,t),\tau),\tau_1)\,d\tau_1 + \dots . \end{aligned} \tag{23}$$

Equation (22) plays an important role in the proofs following.

4. Proof of Lemma 1

By (22) it suffices to show for every $(x,t) \in \Pi \backslash I_+$ that, if ε is small enough, then $(B_\varepsilon v^\varepsilon)(x,t) = 0$. If $\theta(x,t) = 0$, the latter is true by the equality $(B_\varepsilon v^\varepsilon)(x,t) = a_s^\varepsilon(\omega(0;x,t))$ and the fact that $(\omega(0;x,t),0) \notin I_+$. Consider the case that $\theta(x,t) > 0$. Since $\theta(x,t) \notin \left(\overline{I_+^\varepsilon} \cap \{(x,t)\,|\,x=0\}\right)$, the proof will be complete by showing that

$$\operatorname{supp} v^\varepsilon(0,t) \subset \left(\overline{I_+^\varepsilon} \cap \{(x,t)\,|\,x=0\}\right), \tag{24}$$

where $v^\varepsilon(0,t)$ is defined by (16). Observe that (24) is true for the first summand in (16). Indeed, by (9), Assumption 2, and the definition of I_+^ε,

$$\begin{aligned} \operatorname{supp}\left(c_s^\varepsilon \int_0^L [(b_s^\varepsilon + b_r)w^\varepsilon + b_r v^\varepsilon]\,dx\right) &\subset \bigcup_{i=1}^{l} [t_i - \varepsilon, t_i + \varepsilon] \\ &\subset \operatorname{supp} c_s^\varepsilon \\ &\subset \left(\overline{I_+^\varepsilon[0]} \cap \{(x,t)\,|\,x=0\}\right). \end{aligned} \tag{25}$$

To obtain (24), for the second summand in (16) we prove the inclusion

$$\operatorname{supp}\left(c_r \int_0^L b_s^\varepsilon v^\varepsilon\,dx\right) \subset \bigcup_{n\ge 1} [\tilde{t}_{i_n} - \varepsilon_{i_n}^-(\varepsilon), \tilde{t}_{i_n} + \varepsilon_{i_n}^+(\varepsilon)]. \tag{26}$$

Recall that $\tilde{t}_{i_n}$, $n \ge 1$, are intersection points of $I_+ \setminus I_+[0]$ with the axis $x=0$. Suppose (26) is false. Then there exists

$$\tau_1 \notin \bigcup_{n\ge 1} [\tilde{t}_{i_n} - \varepsilon_{i_n}^-(\varepsilon), \tilde{t}_{i_n} + \varepsilon_{i_n}^+(\varepsilon)] \tag{27}$$

such that

$$\int_0^L b_s^\varepsilon(x) v^\varepsilon(x,\tau_1)\,dx = \int_0^L b_s^\varepsilon(x)(B_\varepsilon v^\varepsilon)(x,\tau_1) S(x,\tau_1)\,dx \neq 0. \tag{28}$$

We fix such τ_1 and set

$$J_1 = \operatorname{supp} b_s^\varepsilon(x) \cap \operatorname{supp} v^\varepsilon(x,\tau_1). \tag{29}$$

By (28)

$$\mathrm{mes} J_1 \neq 0. \tag{30}$$

Assume that $\theta(x_0,\tau_1) = 0$ for some $x_0 \in J_1$. By (21) and (29),

$$(B_\varepsilon v^\varepsilon)(x_0,\tau_1) = a_s^\varepsilon(\omega(0;x_0,\tau_1)) \neq 0.$$

This means that $\omega(0;x_0,\tau_1) \in \operatorname{supp} a_s^\varepsilon \subset \overline{I_+^\varepsilon} \cap \{(x,t)\,|\,x = 0\}$. We conclude that $(x_0,\tau_1) \in \overline{I_+^\varepsilon[0]}$. Furthermore, from (29) we have $x_0 \in [x_i - \varepsilon, x_i + \varepsilon]$ for some $i \le k$. From the definition of I_+^ε it follows that, if $(x,t) \in \overline{I_+^\varepsilon[j]}$ and $x \in [x_i - \varepsilon, x_i + \varepsilon]$ for some $i \le k$, then $(0,t) \in \overline{I_+^\varepsilon[j+1]}$. Hence $(0,\tau_1) \in \overline{I_+^\varepsilon[1]}$. This contradicts (27).

Assume therefore that $\theta(x,\tau_1) > 0$ for all $x \in J_1$. Then in (28) we have $(B_\varepsilon v^\varepsilon)(x,\tau_1) = v^\varepsilon(0,\theta(x,\tau_1))$ and therefore

$$\int_0^L b_s^\varepsilon(x) v^\varepsilon(0,\theta(x,\tau_1)) S(x,\tau_1)\,dx \neq 0.$$

By (29) and (30) there exists $\tau_2 \in \theta(J_1,\tau_1)$ such that $v^\varepsilon(0,\tau_2) \neq 0$. It is clear that $\tau_2 < \tau_1$. Assume that $(0,\tau_2) \in \overline{I_+^\varepsilon}$. Let x_0 be such that $\theta(x_0,\tau_1) = \tau_2$. By the definition of I_+^ε, $(x_0,\tau_1) \in \overline{I_+^\varepsilon[j]}$ for some $j \ge 0$. Furthermore, $x_0 \in J_1$ and therefore $x_0 \in [x_i - \varepsilon, x_i + \varepsilon]$ for some $i \le k$. Hence $(0,\tau_1) \in \overline{I_+^\varepsilon[j+1]}$. This again contradicts (27).

Assume therefore that $(0,\tau_2) \notin \overline{I_+^\varepsilon}$. On the account of (16) and (25), we rewrite the condition $v^\varepsilon(0,\tau_2) \neq 0$ as

$$\int_0^L b_s^\varepsilon(x) v^\varepsilon(x,\tau_2)\,dx \neq 0.$$

Set

$$J_2 = \operatorname{supp} b_s^\varepsilon(x) \cap \operatorname{supp} v^\varepsilon(x,\tau_2).$$

Note that $\mathrm{mes} J_2 \neq 0$. Similarly to the above, if $\theta(x_0,\tau_2) = 0$ for some $x_0 \in J_2$, then $(0,\tau_2) \notin \overline{I_+^\varepsilon[1]}$, a contradiction with (27). We therefore assume that $\theta(\xi,\tau_2) > 0$ for all $x \in J_2$ and continue in this fashion, thereby constructing sequences $\tau_k \in \theta(I^{k-1}, \tau_{k-1})$ and $J_k = \operatorname{supp} b_s^\varepsilon(x) \cap \operatorname{supp} v^\varepsilon(x, \tau_k)$ for $k \geq 2$ such that $v^\varepsilon(0, \tau_k) \neq 0$ and $(0, \tau_k) \notin \overline{I_+^\varepsilon}$. By Assumptions 5 and 6, for some

$$k \leq \left\lceil \frac{T \max_{(x,t)\in\overline{\Pi^T}} |\lambda|}{x_1 - \varepsilon_0} \right\rceil$$

there exists $x_0 \in J_k$ such that $\theta(x_0, \tau_k) = 0$. This implies $(0, \tau_k) \in \overline{I_+^\varepsilon[1]}$, a contradiction with (27).

Thus (26) is true and the proof of Lemma 1 is complete. ∎

For the further reference observe that

$$\operatorname{supp} v^\varepsilon \subset \overline{I_+^\varepsilon}. \tag{31}$$

This fact is true by (21), (22), (24), and the definition of I_+^ε.

5. Proof of Lemma 2

Choose ε_0 so small that the number of connected components of $\Pi^T \cap I_+^{\varepsilon_0}$ and $\Pi^T \cap I_+$ coincide.

Definition 3. Given $T > 0$, let $\Pi_0^T = \{(x,t) \in \Pi^T \,|\, \omega(t; 0, 0) < x\}$ and $\Pi_1^T = \Pi^T \setminus \overline{\Pi_0^T}$. Let $n(T)$ and $\rho(T)$ be the number of connected components of $\Pi_1^T \setminus \overline{I_+^\varepsilon}$ (and $\Pi_1^T \setminus I_+$) and $\Pi_1^T \cap I_+^\varepsilon$ (and $\Pi_1^T \cap I_+$), respectively. We denote these components, respectively, by $\Pi^\varepsilon(1), \ldots, \Pi^\varepsilon(n(T))$ $(\Pi(1), \ldots, \Pi(n(T)))$ and $I_+^\varepsilon(1), \ldots, I_+^\varepsilon(\rho(T))$ $(I_+(1), \ldots, I_+(\rho(T)))$.

Clearly, $\Pi(i) = \bigcup_{\varepsilon>0} \Pi^\varepsilon(i)$ and $I_+(i) = \bigcap_{\varepsilon>0} I_+^\varepsilon(i)$. Observe that $\rho(T)$ does not depend on ε and either $n(T) = \rho(T)$ or $n(T) = \rho(T) + 1$. In the latter case, if $n(T) = \rho(T) + 1$, we define $I_+^\varepsilon(\rho(T) + 1) = \emptyset$. Given T, we choose ε_0 so small that for all $\varepsilon \leq \varepsilon_0$

$$\operatorname{supp} b_s^\varepsilon \subset (\omega(t_i^* + \varepsilon_i^+(\varepsilon); 0, t_i^* - \varepsilon_i^-(\varepsilon)), L]. \tag{32}$$

From (4)–(6), (11)–(13), and (14)–(16) it follows that the difference $u^\varepsilon - v^\varepsilon - w^\varepsilon$ satisfies the system

$$\begin{aligned}(\partial_t + \lambda(x,t)\partial_x)(u^\varepsilon - v^\varepsilon - w^\varepsilon) = {} & p(x,t)(u^\varepsilon - v^\varepsilon - w^\varepsilon) \\ & + F^\varepsilon(x,t)(u^\varepsilon - v^\varepsilon - w^\varepsilon) \\ & + f(x,t,u^\varepsilon) - f(x,t,u^\varepsilon - v^\varepsilon)\end{aligned} \tag{33}$$

$$(u^\varepsilon - v^\varepsilon - w^\varepsilon)|_{t=0} = 0 \tag{34}$$

$$\begin{aligned}(u^\varepsilon - v^\varepsilon - w^\varepsilon)|_{x=0} = c_s^\varepsilon \int_0^L b_s^\varepsilon(u^\varepsilon - w^\varepsilon)\,dx \\ + (c_s^\varepsilon + c_r)\int_0^L b_r(u^\varepsilon - w^\varepsilon - v^\varepsilon)\,dx \\ + c_r \int_0^L b_s^\varepsilon(u^\varepsilon - w^\varepsilon - v^\varepsilon)\,dx,\end{aligned} \tag{35}$$

where

$$F^\varepsilon(x,t) = \int_0^1 \Big(\nabla_u f\Big)(x,t,\sigma(u^\varepsilon - v^\varepsilon) + (1-\sigma)w^\varepsilon)\,d\sigma.$$

Claim 1. *$u^\varepsilon - v^\varepsilon - w^\varepsilon \to 0$ in $\mathrm{L}^1(\overline{\Pi_0^T})$ as $\varepsilon \to 0$.*

Proof. The problem (33)–(35) on Π_0^T reduces to the Cauchy problem (33)–(34). By Assumption 7, f is globally bounded and, by Lemma 1, $f(x,t,u^\varepsilon) - f(x,t,u^\varepsilon - v^\varepsilon) \to 0$ as $\varepsilon \to 0$ pointwise off I_+. By Lebesgue's dominated convergence theorem, $f(x,t,u^\varepsilon) - f(x,t,u^\varepsilon - v^\varepsilon) \to 0$ in $\mathrm{L}^1(\overline{\Pi_0^T})$ as $\varepsilon \to 0$. Applying Lebesgue's dominated convergence theorem to the functions $u^\varepsilon - v^\varepsilon - w^\varepsilon$ defined by (33)–(34), we obtain the claim. ∎

Claim 2. *$u^\varepsilon - v^\varepsilon - w^\varepsilon \to 0$ pointwise for $(x,t) \in \Pi(1)$ as $\varepsilon \to 0$.*

Proof. Taking into account Lemma 1, it is sufficient to prove that $u^\varepsilon - w^\varepsilon \to 0$ pointwise for $(x,t) \in \Pi(1)$ as $\varepsilon \to 0$. Fix an arbitrary $(x_0,t_0) \in \Pi(1)$. If ε is sufficiently small, the point (x_0,t_0) belongs to $\Pi^\varepsilon(1)$, where we have the following integral representation:

$$\begin{aligned}(u^\varepsilon - w^\varepsilon)(x,t) = c_r(\theta(x,t))\bigg[\int_0^L b_s^\varepsilon(\xi)(u^\varepsilon - w^\varepsilon)(\xi,\tau)\,d\xi \\ + \int_0^{z(\tau)} b_r(\xi)(u^\varepsilon - w^\varepsilon)(\xi,\tau)\,d\xi \\ + \int_{z(\tau)}^L b_r(\xi)(u^\varepsilon - w^\varepsilon - v^\varepsilon)(\xi,\tau)\,d\xi\bigg]\bigg|_{\tau=\theta(x,t)} \\ + \int_{\theta(x,t)}^t (u^\varepsilon - w^\varepsilon)(\xi,\tau)\Big[p(\xi,\tau) \\ + \int_0^1 (\nabla_u f)(\xi,\tau,\sigma u^\varepsilon + (1-\sigma)w^\varepsilon)\,d\sigma\Big]\Big|_{\xi=\omega(\tau;x,t)}\,d\tau.\end{aligned} \tag{36}$$

Here

$$z(t) = \begin{cases} \omega(t;0,0) & \text{if } \theta(L,t) > 0, \\ L & \text{if } \theta(L,t) = 0. \end{cases} \tag{37}$$

Note that

$$\omega(t;0,\tau) \leq (t-\tau) \max_{(x,t)\in\overline{\Pi^T}} \lambda(x,t). \tag{38}$$

Since, for ε small enough,

$$|(u^\varepsilon - w^\varepsilon)(x_0,t_0)| \leq \max_{(x,t)\in\overline{\Pi^\varepsilon(1)}} |(u^\varepsilon - w^\varepsilon)(x,t)|,$$

it is sufficient to prove that

$$\max_{(x,t)\in\overline{\Pi^\varepsilon(1)}} |(u^\varepsilon - w^\varepsilon)(x,t)| = O(\varepsilon).$$

We start from the evaluation of the third integral in (36). On the account of (31), we represent it in the form

$$\begin{aligned}\int_{z(t)}^{L} b_r(u^\varepsilon - w^\varepsilon - v^\varepsilon)\,dx = & \int\limits_{[z(t),L]\times\{t\}\setminus(I_+^\varepsilon\cap\{(x,t)\,|\,x\in\mathbb{R}\})} b_r(u^\varepsilon - w^\varepsilon)\,dx \\ & + \int\limits_{[z(t),L]\times\{t\}\cap(I_+^\varepsilon\cap\{(x,t)\,|\,x\in\mathbb{R}\})} b_r(u^\varepsilon - w^\varepsilon - v^\varepsilon)\,dx.\end{aligned} \tag{39}$$

If $\theta(L,t) = 0$, this integral is equal to 0. Consider the case that $\theta(L,t) > 0$. To estimate the difference $u^\varepsilon - w^\varepsilon$ on $[z(t),L]\times\{t\}\setminus(I_+^\varepsilon\cap\{(x,t)\,|\,x\in\mathbb{R}\})$, we consider the corresponding problem

$$\begin{aligned}&(\partial_t + \lambda(x,t)\partial_x)(u^\varepsilon - w^\varepsilon) \\ &\qquad = \Big(p(x,t) + \int_0^1 (\nabla_u f)(x,t,\sigma u^\varepsilon + (1-\sigma)w^\varepsilon)\,d\sigma\Big)(u^\varepsilon - w^\varepsilon) \\ &(u^\varepsilon - w^\varepsilon)|_{t=0} = 0.\end{aligned} \tag{40}$$

By Assumption 7 this problem has only the trivial solution. Therefore the second integral in (39) is equal to 0. We now estimate the third integral. We have the integral equation

$$\begin{aligned}&(u^\varepsilon - v^\varepsilon - w^\varepsilon)(x,t) \\ &\qquad = \int_0^t (p+F)(\xi,\tau)(u^\varepsilon - w^\varepsilon - v^\varepsilon)(\xi,\tau)\big|_{\xi=\omega(\tau,x,t)}\,d\tau \\ &\qquad\quad + \int_0^t [f(\xi,\tau,u^\varepsilon) - f(\xi,\tau,u^\varepsilon - v^\varepsilon)]\big|_{\xi=\omega(\tau,x,t)}\,d\tau\end{aligned}$$

that corresponds to the Cauchy problem (33)–(34). Combining it with Assumption 7, we conclude that

$$\max_{(x,t)\in\overline{\Pi_0^T}\cap\overline{I_+^\varepsilon}} |u^\varepsilon - v^\varepsilon - w^\varepsilon| \leq C_1 \tag{41}$$

for a positive constant C_1 not depending on ε. Therefore the absolute value of the third integral in (39) is bounded from above by $C_1\varepsilon$. As a consequence,

$$\Big|\int_{z(t)}^{L} b_r(u^\varepsilon - w^\varepsilon - v^\varepsilon)\,dx\Big| \le C_2\varepsilon. \tag{42}$$

In the rest of the proof, C_i for $i \ge 1$ are positive constants that do not depend on ε. We will distinguish two cases.

`Case 1`: $\operatorname{supp} b_s^{\varepsilon_0} \subset [\omega(t_1^*;0,0), L]$. The first integral in (36) vanishes, since

$$\int_0^L b_s^\varepsilon(u^\varepsilon - w^\varepsilon)\,dx = \int\limits_{[z(t),L]\times\{t\}\setminus(I_+^\varepsilon\cap\{(x,t)\,|\,x\in\mathbb{R}\})} b_s^\varepsilon(u^\varepsilon - w^\varepsilon)\,dx \tag{43}$$

and $u^\varepsilon - w^\varepsilon$ on $[z(t), L]\times\{t\}\setminus(I_+^\varepsilon\cap\{(x,t)\,|\,x\in\mathbb{R}\})$ satisfies the problem (40) which has only the trivial solution. Taking into account (7), (36)–(38), and (42), similarly to [14, p. 644] we obtain the following estimate that holds on $\overline{\Pi^\varepsilon(1)}\cap\overline{\Pi^{\tau_0}}$:

$$|(u^\varepsilon - w^\varepsilon)(x,t)| \le \frac{C_2\varepsilon}{1 - q_0\tau_0},$$

where

$$\begin{aligned} q_0 = &\max_{(x,t,y)\in\overline{\Pi^T}\times\mathbb{R}} |(\nabla_u f)(x,t,y)| + \max_{(x,t)\in\overline{\Pi^T}} |p(x,t)| \\ &+ \max_{t\in[0,T]} |c_r(t)| \max_{x\in[0,L]} |b_r(x)| \max_{(x,t)\in\overline{\Pi^T}} |\lambda(x,t)|, \quad \tau_0 < q_0. \end{aligned}$$

Iterating this estimate at most $\lceil T/\tau_0\rceil$ times, each time using the final estimate for $|u^\varepsilon - w^\varepsilon|$ from a preceding iteration, we obtain the following bound that holds on $\overline{\Pi^\varepsilon(1)}$:

$$|(u^\varepsilon - w^\varepsilon)(x,t)| \le C_3\varepsilon. \tag{44}$$

This completes the proof in Case 1.

`Case 2`: $\operatorname{supp} b_s^{\varepsilon_0} \not\subset [\omega(t_1^*;0,0), L]$. We fix an arbitrary sequence $0 = t(0) < t(1) < t(2) < \cdots < t(M) = t_1^*$ such that $\operatorname{supp} b_s^{\varepsilon_0} \subset [\omega(t(j);0,t(j-1)), L]$. Since $\operatorname{supp} b_s^\varepsilon \subset \operatorname{supp} b_s^{\varepsilon_0}$ for $\varepsilon \le \varepsilon_0$, we can choose the same sequence for all $\varepsilon \le \varepsilon_0$. Given this sequence, we devide $\overline{\Pi^\varepsilon(1)}$ into a finite number of subsets

$$\Pi^\varepsilon(1,j) = \Big\{(x,t)\in\overline{\Pi^\varepsilon(1)}\,|\,\tilde\omega(x;0,t(j-1)) \le t \le \tilde\omega(x;0,t(j))\Big\}. \tag{45}$$

We prove (44) with an appropriate choice of C_3 separately for each of $\Pi^\varepsilon(1,j)$. Since $\operatorname{supp} b_s^\varepsilon \subset [\omega(t(1);0,0), L]$, the conditions of Case 1 are true for $\Pi^\varepsilon(1,1)$, and therefore the estimate (44) is true for this subset. The analog of (44) for $\Pi^\varepsilon(1,2)$ can be obtained in much the same way. We concentrate only on changes. For the first integral in (36) we use the representation (43) with one more summand in the right hand side

$$\int_{\omega(t;0,t(1))}^{z(t)} b_s^\varepsilon(u^\varepsilon - w^\varepsilon)\,dx.$$

The absolute value of this integral is bounded from above by $C_4\varepsilon$ due to (44) on $\Pi^\varepsilon(1,1)$.

To derive (42) on $\Pi^\varepsilon(1,2)$ with $\omega(t;0,t(1))$ in place of $z(t)$ and with new constant C_2, we observe that in the analog of (39) there appears the third summand

$$\int_{\omega(t;0,t(1))}^{z(t)} b_r(u^\varepsilon - w^\varepsilon)\,dx$$

that can be bounded from above by using (44) for $\Pi^\varepsilon(1,1)$. Apply similar arguments to all subsequent $\Pi^\varepsilon(1,j)$. Thus the estimate (44) is true for the whole $\Pi^\varepsilon(1)$. The proof of Claim 2 is complete. ∎

Claim 3. *The functions $u^\varepsilon - w^\varepsilon - v^\varepsilon$ are bounded on $\overline{I_+^\varepsilon(1)}$, uniformly in ε.*

Proof. Two cases are possible.

`Case 1:` $(0,t_1)\in I_+(1)$. We have

$$(u^\varepsilon - v^\varepsilon - w^\varepsilon)|_{x=0} = G^\varepsilon(t) + (c_s^\varepsilon + c_r)\int_0^{\omega(t;0,t_1-\varepsilon)} b_r(u^\varepsilon - w^\varepsilon - v^\varepsilon)\,dx \qquad (46)$$

for $t\in[t_1-\varepsilon,t_1+\varepsilon]$, where

$$\begin{aligned} G^\varepsilon(t) = (c_s^\varepsilon + c_r)&\int_{\omega(t;0,t_1-\varepsilon)}^{z(t)} (b_s^\varepsilon + b_r)(u^\varepsilon - w^\varepsilon)\,dx \\ &+ (c_s^\varepsilon + c_r)\int_{z(t)}^{L} b_s^\varepsilon(u^\varepsilon - w^\varepsilon)\,dx \\ &+ (c_s^\varepsilon + c_r)\int_{z(t)}^{L} b_r(u^\varepsilon - w^\varepsilon - v^\varepsilon)\,dx. \end{aligned} \qquad (47)$$

This representation follows from (32), (25), and (26). We now show that $G^\varepsilon(t)$ is bounded on $t\in[t_1-\varepsilon,t_1+\varepsilon]$. Since $[\omega(t;0,t_1-\varepsilon),z(t)]\times\{t\}\subset\overline{\Pi^\varepsilon(1)}$ for $t\in[t_1-\varepsilon,t_1+\varepsilon]$, the estimate (44) on $\overline{\Pi^\varepsilon(1)}$ applies for the difference $u^\varepsilon - w^\varepsilon$ under the first integral in (47). Using also (7) and (8), we conclude that the first summand in (47) is bounded uniformly in ε. Since $u^\varepsilon - w^\varepsilon\equiv 0$ on $\Pi_0^T\setminus I_+^\varepsilon$ (see (40)), the second integral is equal to 0. Applying (42) and (7) to the third summand, we see that $G^\varepsilon(t)$ are bounded on $[t_1-\varepsilon,t_1+\varepsilon]$, uniformly in ε.

Observe that

$$
\begin{aligned}
(u^\varepsilon - v^\varepsilon - w^\varepsilon)(x,t) = {} & (u^\varepsilon - v^\varepsilon - w^\varepsilon)(0,\theta(x,t)) \\
& + \int_{\theta(x,t)}^{t} p(\xi,\tau)(u^\varepsilon - v^\varepsilon - w^\varepsilon)(\xi,\tau)|_{\xi=\omega(\tau;x,t)}\, d\tau \\
& + \int_{\theta(x,t)}^{t} \Big[f(\xi,\tau,u^\varepsilon) - f(\xi,\tau,w^\varepsilon) \Big]\Big|_{\xi=\omega(\tau;x,t)} d\tau
\end{aligned}
\tag{48}
$$

for $(x,t) \in \overline{I_+^\varepsilon(1)}$, where the boundary function $(u^\varepsilon - w^\varepsilon - v^\varepsilon)(0,t)$ is given by (46). By Gronwall's argument applied to $|(u^\varepsilon - v^\varepsilon - w^\varepsilon)(x,t)|$, we easily obtain the estimate

$$
\begin{aligned}
& |(u^\varepsilon - v^\varepsilon - w^\varepsilon)(x,t)| \\
& \quad \le C_3 \Bigg[2T \max_{(x,t,y)\in\overline{\Pi^T}\times\mathbb{R}} |f(x,t,y)| + \max_{(x,t)\in\overline{I_+^\varepsilon(1)}} \Bigg| \Big[G^\varepsilon(\tau) + (c_s^\varepsilon + c_r)(\tau) \\
& \qquad \times \int_0^{\omega(\tau;0,t_1-\varepsilon)} b_r(\xi)(u^\varepsilon - v^\varepsilon - w^\varepsilon)(\xi,\tau)\, d\xi \Big]\Big|_{\tau=\theta(x,t)} \Bigg| \Bigg],
\end{aligned}
\tag{49}
$$

for $(x,t) \in \overline{I_+^\varepsilon(1)}$. By (38) we have

$$
\omega(t;0,t_1^* - \varepsilon_1^-(\varepsilon)) \le C_5 \varepsilon \tag{50}
$$

for $t \in [t_1^* - \varepsilon_1^-(\varepsilon), t_1^* + \varepsilon_1^+(\varepsilon)]$. Given a mollifier $\varphi(t)$, let

$$
q(\varepsilon) = C_3 C_5 \max_{x\in\left[0,\omega(t_1^*+\varepsilon_1^+(\varepsilon);0,t_1^*-\varepsilon_1^-(\varepsilon))\right]} |b_r(x)| \Big(\max_{t\in[0,T]} |\varphi(t)| + \varepsilon \max_{t\in[0,T]} |c_r(t)| \Big). \tag{51}
$$

By Assumptions 3 and 5,

$$
\lim_{\varepsilon\to 0} q(\varepsilon) = 0. \tag{52}
$$

We choose ε so small that

$$
q(\varepsilon) < 1. \tag{53}
$$

On the account of (49), (51), and (53), for sufficiently small ε we obtain

$$
\begin{aligned}
& \max_{(x,t)\in\overline{I_+^\varepsilon(1)}} |(u^\varepsilon - v^\varepsilon - w^\varepsilon)(x,t)| \\
& \qquad \le \tfrac{C_3}{1-q(\varepsilon)} \Big[2T \max_{(x,t,y)\in\overline{\Pi^T}\times\mathbb{R}} |f(x,t,y)| + \max_{t\in[t_1-\varepsilon,t_1+\varepsilon]} |G^\varepsilon(t)| \Big].
\end{aligned}
\tag{54}
$$

`Case 2:` $(0,t_1) \notin I_+(1)$. By (32), (25), and (26) we have the equality (46) with $t_1 - \varepsilon$ and $t_1 + \varepsilon$ replaced by $t_1^* - \varepsilon_1^-(\varepsilon)$ and $t_1^* + \varepsilon_1^+(\varepsilon)$, respectively, and with

$$
\begin{aligned}
G^\varepsilon(t) = c_r \int_{\omega(t;0,t_1^*-\varepsilon_1^-(\varepsilon))}^{z(t)} & (b_s^\varepsilon + b_r)(u^\varepsilon - w^\varepsilon)\, dx \\
& + c_r \int_{z(t)}^{L} b_s^\varepsilon (u^\varepsilon - w^\varepsilon)\, dx + c_r \int_{z(t)}^{L} b_r (u^\varepsilon - w^\varepsilon - v^\varepsilon)\, dx.
\end{aligned} \tag{55}
$$

To estimate the absolute value of the first integral in (55) we apply (44) on $\overline{\Pi^\varepsilon(1)}$ and (8). The second summand is equal to 0 (see (40)). For the third integral we use (42). It follows that $G^\varepsilon(t)$ is bounded on $[t_1^* - \varepsilon_1^-(\varepsilon), t_1^* + \varepsilon_1^+(\varepsilon)]$, uniformly in ε. The rest of the proof runs as in Case 1, the minor changes being in using (49) and (54) with $t_1^* - \varepsilon_1^-(\varepsilon)$ in place of $t_1 - \varepsilon$ and $t_1^* + \varepsilon_1^+(\varepsilon)$ in place of $t_1 + \varepsilon$. ∎

Claim 4. *For every $j \geq 1$:*

1. $u^\varepsilon - w^\varepsilon - v^\varepsilon \to 0$ *converges pointwise for* $(x,t) \in \Pi(j)$, *as* $\varepsilon \to 0$.
2. *the functions* $u^\varepsilon - w^\varepsilon - v^\varepsilon$ *are bounded on* $\overline{I_+^\varepsilon(j)}$, *uniformly in* ε.

Proof. Items 1 and 2 of the claim follow from the bounds

$$
\max_{(x,t)\in\overline{\Pi^\varepsilon(j)}} |(u^\varepsilon - w^\varepsilon)(x,t)| \leq A_j \varepsilon \tag{56}
$$

and

$$
\max_{(x,t)\in\overline{I_+^\varepsilon(j)}} |(u^\varepsilon - v^\varepsilon - w^\varepsilon)(x,t)| \leq A_j, \tag{57}
$$

respectively, where A_j are constants depending only on j. We prove (56) and (57) by induction on j. The base case of $j = 1$ is given by Claims 2 and 3. Assume that (56) and (57) are true for all $j < i$, $i \geq 2$, and prove these estimates for $j = i$.

To prove (56) for $j = i$, we follow the proof of Claim 2 with the following changes. We use the formula (36) with $\omega(\tau; 0, t_{i-1}^* + \varepsilon_{i-1}^+(\varepsilon))$ in place of $z(\tau)$. To estimate the third integral in the analog of (36), we represent it in the form

$$
\int_{\omega(t;0,t_{i-1}^*+\varepsilon_{i-1}^+(\varepsilon))}^{z(t)} b_r(u^\varepsilon - w^\varepsilon - v^\varepsilon)\, dx + \int_{z(t)}^{L} b_r(u^\varepsilon - w^\varepsilon - v^\varepsilon)\, dx,
$$

and apply the induction assumptions and (42). As a consequence, we obtain the estimate (42) with $z(t)$ replaced by $\omega(t; 0, t_{i-1}^* + \varepsilon_{i-1}^+(\varepsilon))$ and with a new constant C_2. Similarly to Claim 2, we distinguish two cases.

`Case 1:` $\operatorname{supp} b_s^{\varepsilon_0} \subset [\omega(t_i^*; 0, t_{i-1}^* - \varepsilon_{i-1}^-(\varepsilon_0)), L]$. On the account of (40), we can rewrite the first summand in the analog of (36) in the form

$$
\int_0^L b_s^\varepsilon (u^\varepsilon - w^\varepsilon)\, dx = \int\limits_{[\omega(t_i^*-\varepsilon_i^-(\varepsilon);0,t_{i-1}^*-\varepsilon_{i-1}^-(\varepsilon)),z(t)]\times\{t\}\setminus(I_+^\varepsilon\cap\{(x,t)\,|\,x\in\mathbb{R}\})} b_s^\varepsilon (u^\varepsilon - w^\varepsilon)\, dx.
$$

Applying (7) and (56) for $j < i$, we conclude that the absolute value of the integral is bounded from above by $C_7\varepsilon$. The rest of the proof for this case runs similarly to the proof of Claim 2 in Case 1.

`Case 2:` $\operatorname{supp} b_s^{\varepsilon_0} \not\subset [\omega(t_i^*; 0, t_{i-1}^* - \varepsilon_{i-1}^-(\varepsilon_0)), L]$. We fix an arbitrary sequence $t_{i-1}^* = t(0) < t(1) < t(2) < \cdots < t(M) = t_i^*$ such that $\operatorname{supp} b_s^{\varepsilon_0} \subset [\omega(t(j); 0, t(j-1)), L]$. Given this sequence, we devide $\overline{\Pi^\varepsilon(i)}$ into a finite number of subsets

$$\Pi^\varepsilon(i,j) = \Big\{(x,t) \in \overline{\Pi^\varepsilon(i)} \,|\, \tilde{\omega}(x; 0, t(j-1)) \le t \le \tilde{\omega}(x; 0, t(j))\Big\}. \tag{58}$$

Observe that the partition of $\overline{\Pi^\varepsilon(i)}$ is finite for every $\varepsilon > 0$ and the number of subsets does not depend on ε. We further apply arguments similar to those used in the proof of Claim 2 for Case 2.

To prove (57) for $j = i$, we follow the proof of Claim 3 with the following changes. Similarly to Claim 3, we distinguish two cases.

`Case 1:` $(0, t_k) \in I_+(i)$ for some $k \le l$. We use the formula (46) with t_k in place of t_1 and with

$$\begin{aligned} G^\varepsilon(t) = {} & (c_s^\varepsilon + c_r) \int\limits_{[\omega(t;0,t_i^*-\varepsilon_i^-(\varepsilon)),L]\times\{t\}\setminus(I_+^\varepsilon\cap\{(x,t)\,|\,x\in\mathbb{R}\})} (b_s^\varepsilon + b_r)(u^\varepsilon - w^\varepsilon)\,dx \\ & + (c_s^\varepsilon + c_r) \int\limits_{[\omega(t;0,t_i^*-\varepsilon_i^-(\varepsilon)),L]\times\{t\}\cap(I_+^\varepsilon\cap\{(x,t)\,|\,x\in\mathbb{R}\})} b_r(u^\varepsilon - w^\varepsilon - v^\varepsilon)\,dx. \end{aligned} \tag{59}$$

The estimation of the first summand is based on the inclusion

$$[\omega(t; 0, t_i^* - \varepsilon_i^-(\varepsilon)), L] \times \{t\} \setminus (I_+^\varepsilon \cap \{(x,t)\,|\,x \in \mathbb{R}\}) \subset \bigcup_{j=1}^{i} \overline{\Pi^\varepsilon(j)} \cup (\overline{\Pi_0^T} \setminus I_+^\varepsilon)$$

and on (56) which is given for $j < i$ by the induction assumptions, and for $j = i$ it is just proved. The estimation of the second summand is based on the inclusion

$$[\omega(t; 0, t_i^* - \varepsilon_i^-(\varepsilon)), L] \times \{t\} \cap (I_+^\varepsilon \cap \{(x,t)\,|\,x \in \mathbb{R}\}) \subset \bigcup_{j=1}^{i-1} \overline{I_+^\varepsilon(j)} \cup (\overline{\Pi_0^T} \cap \overline{I_+^\varepsilon}),$$

for $t \in [t_1 - \varepsilon, t_1 + \varepsilon]$, and on (57) for $j < i$.

`Case 2:` $(0, t_k) \notin I_+(i)$ for all $k \le l$. We use the formula (46) with $t_i^* - \varepsilon_i^-(\varepsilon)$ and $t_i^* + \varepsilon_i^+(\varepsilon)$ in place of $t_1 - \varepsilon$ and $t_1 - \varepsilon$, respectively, and with

$$\begin{aligned} G^\varepsilon(t) = {} & c_r \int\limits_{[\omega(t;0,t_i^*-\varepsilon_i^-(\varepsilon)),L]\times\{t\}\setminus(I_+^\varepsilon\cap\{(x,t)\,|\,x\in\mathbb{R}\})} (b_s^\varepsilon + b_r)(u^\varepsilon - w^\varepsilon)\,dx \\ & + c_r \int\limits_{[\omega(t;0,t_i^*-\varepsilon_i^-(\varepsilon)),L]\times\{t\}\cap(I_+^\varepsilon\cap\{(x,t)\,|\,x\in\mathbb{R}\})} (b_s^\varepsilon + b_r)(u^\varepsilon - w^\varepsilon - v^\varepsilon)\,dx. \end{aligned} \tag{60}$$

In order to prove the boundedness of $G^\varepsilon(t)$ we apply (56) for $j \le i$ and (57) for $j < i$. The rest of the proof for both cases runs similarly to the proof of Claim 4 in Case 1. ∎

From Claim 4, (56) for $j \le n(T)$, and (31) we conclude that the family $(u^\varepsilon - w^\varepsilon - v^\varepsilon)_{\varepsilon>0}$ is bounded on $\overline{\Pi_1^T}$ uniformly in ε and converges to 0 almost everywhere in $\overline{\Pi_1^T}$. By dominated convergence theorem this family converges to 0 in $\mathrm{L}^1(\Pi_1^T)$-norm. On the account of Claim 1, $(u^\varepsilon - w^\varepsilon - v^\varepsilon)_{\varepsilon>0}$ converges to 0 in $\mathrm{L}^1(\Pi^T)$-norm. Since T is arbitrary, this is precisely the assertion of Lemma 2.

6. Proof of Lemma 3

Given $T > 0$, we choose ε_0 so small that, for all $\varepsilon \le \varepsilon_0$, the conditions (32) and

$$q(\varepsilon)\exp\Big\{T \max_{(x,t)\in\overline{\Pi^T}} |p(x,t)|\Big\} < 1 \tag{61}$$

are fulfilled. Here $q(\varepsilon)$ is defined by (51). The condition (61) follows from (52).

Claim 1. *The family of functions w^ε converges in $\mathrm{C}(\overline{\Pi_0^T})$ as $\varepsilon \to 0$.*

Proof. For w^ε on $\overline{\Pi_0^T}$ we use the representation given by (18) and (19). Since $(Rw^\varepsilon)(x,t) = a_r(\omega(0;x,t))$ on $\overline{\Pi_0^T}$, the function w^ε for each $\varepsilon > 0$ satisfies the same Volterra integral equation of the second kind. This means that w^ε does not depend on ε, and for each $\varepsilon > 0$ it is equal to the same continuous function $w(x,t)$ that can be found from the integral equation (18) by the method of sequential approximation. The claim follows. ∎

Therewith we are done in $\overline{\Pi_0^T}$. Since $\overline{\Pi^T} = \overline{\Pi_0^T} \cup \overline{\Pi_1^T}$ and w^ε is continuous for each $\varepsilon > 0$ on $\overline{\Pi^T}$ (see Proposition 1 in Section 2), it remains to prove the convergence of w^ε in $\mathrm{C}(\overline{\Pi_1^T})$. We will check the Cauchy criterion of the uniform convergence of w^ε. Given $\delta > 0$, we have to show for some $\varepsilon_2 = \varepsilon_2(\delta)$ and every $\varepsilon_1 < \varepsilon_2$ that

$$\left|\Big(w^{\varepsilon_1} - w^{\varepsilon_2}\Big)(x,t)\right| \le \delta \tag{62}$$

for all $(x,t) \in \overline{\Pi_1^T}$.

Because of so strong interdependence of the problems (11)–(13) and (14)–(16), in the course of the proof of (62) we will need in parallel to prove some properties of v^ε. Let

$$M_k(\varepsilon) = \int_{t_k^*-\varepsilon_k^-(\varepsilon)}^{t_k^*+\varepsilon_k^+(\varepsilon)} |v^\varepsilon(0,t)|\,dt \tag{63}$$

$$K_k(\varepsilon_1,\varepsilon_2) = \int_{t_k^*-\varepsilon_k^-(\varepsilon_2)}^{t_k^*+\varepsilon_k^+(\varepsilon_2)} (v^{\varepsilon_1} - v^{\varepsilon_2})(0,t)\,dt. \tag{64}$$

We will prove by induction on j the following 5 assertions for $1 \le j \le n(T)$ ($n(T)$ as well as $\Pi(k)$ below are defined by Definition 3). Recall that $n(T)$ does not depend on ε_2. Throughout this section C is a large enough constant that does not depend on ε.

Assertion 1. For every $\delta > 0$, if ε_2 is small enough and $\varepsilon_1 < \varepsilon_2$, then (62) is true for all $\overline{\Pi^{\varepsilon_2}(j)} \cup \overline{I_+^{\varepsilon_2}(j)}$.

Assertion 2. The functions w^ε are bounded on $\overline{\Pi^{\varepsilon}(j)} \cup \overline{I_+^{\varepsilon}(j)}$, uniformly in $\varepsilon > 0$.

Assertion 3. The estimate $M_j(\varepsilon) \le C$ is true for all $\varepsilon > 0$.

Assertion 4. If ε_2 is small enough and $\varepsilon_1 < \varepsilon_2$, then $|K_j(\varepsilon_1, \varepsilon_2)| \le C\varepsilon_2$.

Assertion 5. $w^\varepsilon(x,t)$ converges in $\mathrm{C}\big(\bigcup_{k=1}^j \overline{\Pi(k)} \cup \overline{\Pi_0^T}\big)$ as $\varepsilon \to 0$.

Assertion 1 implies the Cauchy criterion of the uniform convergence of w^ε on $\overline{\Pi_1^T}$. Indeed, given $\delta > 0$, let ε_2 be so small that (62) is true for every $\varepsilon_1 < \varepsilon_2$ on each $\overline{\Pi^{\varepsilon_2}(j)} \cup \overline{I_+^{\varepsilon_2}(j)}$ for $j \le n(T)$. Recall that, for any $\varepsilon_2 > 0$,

$$\bigcup_{j=1}^{n(T)} \left(\overline{\Pi^{\varepsilon_2}(j)} \cup \overline{I_+^{\varepsilon_2}(j)}\right) = \overline{\Pi_1^T}.$$

It follows that (62) is true on $\overline{\Pi_1^T}$ for all $\varepsilon_1 < \varepsilon_2$. By the Cauchy criterion, w^ε uniformly converges on $\overline{\Pi_1^T}$.

The proof of Assertions 1 – 5 for $j = 1$ will be given by Claims 2 – 10. The induction step will be carried out by Claims 11–19.

To prove Assertion 1, we split $\overline{\Pi^{\varepsilon_2}(j)} \cup \overline{I_+^{\varepsilon_2}(j)}$ into four subsets:

$$\overline{\Pi^{\varepsilon_2}(j)} \cup \overline{I_+^{\varepsilon_2}(j)} = \overline{\Pi^{\varepsilon_2}(j)} \cup \left(\overline{\Pi^{\varepsilon_1}(j)} \cap \overline{I_+^{\varepsilon_2}(j)}\right) \cup \overline{I_+^{\varepsilon_1}(j)} \cup \left(\overline{I_+^{\varepsilon_2}(j)} \cap \overline{\Pi^{\varepsilon_1}(j+1)}\right),$$

where each two neighboring subsets have common border. We will prove Assertion 1 separately for each of the four subsets.

Claim 2. *The functions $w^\varepsilon(x,t)$ are bounded on $\overline{\Pi^\varepsilon(1)}$, uniformly in ε.*

Proof. We use the representation of w^ε by (17) and (19) restricted to $\overline{\Pi^\varepsilon(1)}$. In this representation, on the account of (13), (21), and (22), we have

$$\begin{aligned}\left(Rw^\varepsilon\right)(x,t) = {} & c_r(\theta(x,t)) \int_0^{\omega(\theta(x,t);0,0)} \left(b_s^\varepsilon + b_r\right)(\xi) w^\varepsilon(\xi, \theta(x,t))\, d\xi \\ & + c_r(\theta(x,t)) \int_{\omega(\theta(x,t);0,0)}^{L} \Big[\left(b_s^\varepsilon + b_r\right)(\xi) w(\xi,\tau) \\ & + S(\xi,\tau) a_s^\varepsilon(\omega(0;\xi,\tau))\Big]\Big|_{\tau=\theta(x,t)} d\xi,\end{aligned}$$

where $w(x,t) = w^\varepsilon(x,t)$ for all $(x,t) \in \overline{\Pi_0^T}$ and for all $\varepsilon > 0$ (see the proof of Claim 1). Taking into account (38) and the fact that $\theta(x,t) \le t$, similarly to [14, p. 646] we obtain the global estimate

$$\max_{(x,t)\in\overline{\Pi^\varepsilon(1)}} |w^\varepsilon(x,t)| \le \left(\frac{1}{1-q_1\tau_1}\right)^{\lceil \frac{T}{\tau_1}\rceil} P(E)\biggl(1 + \max_{x\in[0,L]} |a_r(x)| + \max_{(x,t)\in\overline{\Pi^T}} |S(x,t)| \max_\varepsilon \int_0^L |a_s^\varepsilon(x)|\,dx\biggr), \tag{65}$$

where

$$E = \max_{t\in[0,T]} |c_r(t)| \left(\max_{x\in[0,L]} |b_r(x)| + \max_\varepsilon \int_0^L |b_s^\varepsilon(x)|\,dx\right)$$

$$q_1 = (1+LE)\left(\max_{(x,t,y)\in\overline{\Pi^T}\times\mathbb{R}} |(\nabla_u f)(x,t,y)| + \max_{(x,t)\in\overline{\Pi^T}} |p(x,t)|\right) + E \max_{(x,t)\in\overline{\Pi^T}} |\lambda(x,t)|, \tag{66}$$

τ_1 is a real so small that

$$\tau_1 < q_1 \tag{67}$$

and

$$\operatorname{supp} b_s^\varepsilon \subset (\omega(k\tau_1;0,(k-1)\tau_1), L] \quad \text{for all} \quad 1 \le k \le \lceil T/\tau_1\rceil. \tag{68}$$

$P(E)$ is a polynomial of degree $\lceil T/\tau_1\rceil$ with positive coefficients depending on $f(x,t,0), L$, and T. The claim now follows by (8) and Assumptions 5 and 7. ∎

Claim 3. *Provided ε_2 is small enough, then*

1. *the estimate*
$$\left|\left(w^{\varepsilon_1'} - w^{\varepsilon_2'}\right)(x,t)\right| \le \delta \tag{69}$$
is true on $\overline{\Pi^{\varepsilon_2}(1)}$ for all $\varepsilon_2' \le \varepsilon_2$ and for all $\varepsilon_1' \le \varepsilon_2'$,
2. *the estimate* (62) *is true on $\overline{\Pi^{\varepsilon_2}(1)}$.*

Proof. Recall that $\overline{\Pi^{\varepsilon_2}(1)} \subset \overline{\Pi^{\varepsilon_1'}(1)}$ and $\overline{\Pi^{\varepsilon_2}(1)} \subset \overline{\Pi^{\varepsilon_2'}(1)}$. To represent $w^{\varepsilon_1'}$ and $w^{\varepsilon_2'}$ on $\overline{\Pi^{\varepsilon_2}(1)}$, we will use the system (11)–(13) restricted to $\overline{\Pi^{\varepsilon_2}(1)}$. For the difference $w^{\varepsilon_1'} - w^{\varepsilon_2'}$ we will employ the corresponding linearized integral-operator equation. We distinguish two cases.

`Case 1:` $\operatorname{supp} b_s^{\varepsilon_2} \subset [\omega(t_1^*;0,0), L]$. Using the fact that $w^\varepsilon(x,t) \equiv w(x,t)$ on $\overline{\Pi_0^T}$ for $\varepsilon > 0$, we obtain the integral equation

$$\left(w^{\varepsilon_1'} - w^{\varepsilon_2'}\right)(x,t) = c_r(\theta(x,t)) \int_0^{\omega(\theta(x,t);0,0)} b_r(\xi)(w^{\varepsilon_1'} - w^{\varepsilon_2'})(\xi,\theta(x,t))\,d\xi + S_1(x,t) + S_2(\theta(x,t)) + S_3(\theta(x,t)), \tag{70}$$

where

$$
\begin{aligned}
S_1(x,t) &= \int_{\theta(x,t)}^{t} \Big[\Big[p(\xi,\tau) + \int_0^1 (\nabla_u f)(\xi,\tau,\sigma w^{\varepsilon_1'} + (1-\sigma) w^{\varepsilon_2'})\, d\sigma \Big] \\
&\qquad \times (w^{\varepsilon_1'} - w^{\varepsilon_2'})(\xi,\tau) \Big] \Big|_{\xi=\omega(\tau;x,t)} d\tau \\
S_2(t) &= c_r(t) \int_{z(t)}^{L} (b_s^{\varepsilon_1'} - b_s^{\varepsilon_2'})(x) w(x,t)\, dx \\
S_3(t) &= c_r(t) \int_{z(t)}^{L} b_r(x) \Big(v^{\varepsilon_1'} - v^{\varepsilon_2'} \Big)(x,t)\, dx,
\end{aligned} \tag{71}
$$

and $z(t)$ is defined by (37). We now estimate $|S_2(t)|$ and $|S_3(t)|$. Since the function w is uniformly continuous on $\overline{\Pi_0^T}$, the properties (10) and (8) hold, and $\operatorname{supp} b_s^{\varepsilon} \subset \bigcup_{j=1}^{k} [x_j - \varepsilon, x_j + \varepsilon]$, we have

$$
\begin{aligned}
|S_2(t)| &\le \max_{t\in[0,T]} |c_r(t)| \int_{z(t)}^{L} \Big(|b_s^{\varepsilon_1'}| + |b_s^{\varepsilon_2'}| \Big) \\
&\qquad \sum_{j=1}^{k} \Big| w(x,t) - \chi_{[x_j-\varepsilon_2', x_j+\varepsilon_2']}(x) w(x_j,t) \Big|\, dx \\
&\qquad + \max_{t\in[0,T]} |c_r(t)| \sum_{j=1}^{k} \max_{(x_j,t)\in\overline{\Pi_0^T}} |w(x_j,t)| \Big| \int\limits_{x_j-\varepsilon_2'}^{x_j+\varepsilon_2'} \Big(b_s^{\varepsilon_1'} - b_s^{\varepsilon_2'} \Big)(x)\, dx \Big| \\
&\qquad \le C\varepsilon_2.
\end{aligned} \tag{72}
$$

Here $\chi_\Omega(x,t)$ denotes the characteristic function of a set Ω.

Taking into account (22) and changing coordinates (x,t) to $(\omega(0;x,t),t)$, we estimate $|S_3(t)|$ in the following way:

$$
\begin{aligned}
|S_3(t)| &= \Big| c_r(t) \int_{z(t)}^{L} \Big(b_r S \Big)(x,t) \Big(a_s^{\varepsilon_1'} - a_s^{\varepsilon_2'} \Big)(\omega(0;x,t))\, dx \Big| \\
&= \Big| c_r(t) \int_0^{\omega(0;L,t)} \frac{(b_r S)(\xi,t)}{(\partial_x \omega)(0;\xi,t)} \Big|_{\xi=\omega(t;x,0)} \Big(a_s^{\varepsilon_1'} - a_s^{\varepsilon_2'} \Big)(x)\, dx \Big| \\
&\le M_1 + M_2 + M_3,
\end{aligned} \tag{73}
$$

where

$$
\begin{aligned}
M_1 &= \max_{t\in[0,T]} |c_r(t)| \int_0^{x^*_{m(t)}+\varepsilon_2'} \Big(|a_s^{\varepsilon_1'}| + |a_s^{\varepsilon_2'}| \Big) \\
&\times \sum_{j=1}^{m(t)} \Big| \frac{(b_r S)(\omega(t;x,0),t)}{(\partial_x \omega)(0;\omega(t;x,0),t)} - \chi_{[x_j^*-\varepsilon_2', x_j^*+\varepsilon_2']}(x) \frac{(b_r S)(\omega(t;x_j^*,0),t)}{(\partial_x \omega)(0;\omega(t;x_j^*,0),t)} \Big|\, dx
\end{aligned}
$$

$$M_2 = \max_{t\in[0,T]} |c_r(t)| \sum_{j=1}^{m(t)} \max_{(\omega(t;x_j^*,0),t)\in\overline{\Pi_0^T}} \left| \frac{(b_rS)(\omega(t;x_j^*,0),t)}{(\partial_x\omega)(0,\omega(t;x_j^*,0),t)} \right|$$
$$\times \left| \int_0^{x^*_{m(t)}+\varepsilon_2'} \left(a_s^{\varepsilon_1'} - a_s^{\varepsilon_2'} \right)(x)\, dx \right|$$
$$M_3 = \max_{t\in[0,T]} |c_r(t)| \int_{x^*_{m(t)}+\varepsilon_2'}^{\omega(0;L,t)} \left(|a_s^{\varepsilon_1'}| + |a_s^{\varepsilon_2'}| \right) \left| \frac{(b_rS)(\omega(t;x,0),t)}{(\partial_x\omega)(0;\omega(t;x,0),t)} \right| dx.$$

Here, $m(t)$ denotes the number of indices $j \le m$ such that $x_j^*+\varepsilon_2' \in [0,\omega(0;L,t)]$. Similarly to [14, p. 644] we obtain the estimate for $|(w^{\varepsilon_1'} - w^{\varepsilon_2'})(x,t)|$ on the subset $\overline{\Pi^{\varepsilon_2}(1)} \cap \overline{\Pi^{\tau_1}}$:

$$\left| \left(w^{\varepsilon_1'} - w^{\varepsilon_2'} \right)(x,t) \right| \le \frac{C\varepsilon_2}{1-q_1\tau_1}, \tag{74}$$

where q_1 and τ_1 are defined by (66) and (67). Indeed, the second summand in the right-hand side of (73) is equal to 0 by (10). To estimate the first summand, we use (8) and the uniform continuity property for b_r, S, and λ on $\overline{\Pi}^T$. To estimate the third summand, we observe that the integral is equal to 0 if $\omega(0;L,t) \le x^*_{m(t)}+\varepsilon_2'$ and is actually from $x^*_{m(t)+1}-\varepsilon_2'$ to $\omega(0;L,t)$. In the latter case $\omega(0;L,t)-x^*_{m(t)+1}+\varepsilon_2' \le C\varepsilon_2'$. Combining this bound with the continuity of λ and the condition $\omega(t;\omega(0;L,t),0) = L$, we obtain $L-\omega(t;x^*_{m(t)+1}-\varepsilon_2',0) \le C\varepsilon_2'$. Since $b_r(L) = 0$ by Assumption 3, we conclude that

$$\begin{aligned}
\max_{x\in[x^*_{m(t)+1}-\varepsilon_2',\omega(0;L,t)]} \left| \left(b_rS \right)(\omega(t;x,0),t) \right| &= \max_{x\in[\omega(t;x^*_{m(t)+1}-\varepsilon_2',0),L]} \left| \left(b_rS \right)(x,t) \right| \\
&\le \max_{(x,t)\in[L-C\varepsilon_2',L]\times[0,T]} \left| \left(b_rS \right)(x,t) \right| \\
&\le C\varepsilon_2.
\end{aligned}$$

It follows that

$$|S_3(t)| \le C\varepsilon_2. \tag{75}$$

Using (70), (72), and (75), we derive (74) by Gronwall's argument applied to $|w^{\varepsilon_1'} - w^{\varepsilon_2'}|$. Iterating this estimate at most $\lceil T/\tau_1 \rceil$ times, each time using the final estimate for $|(w^{\varepsilon_1'} - w^{\varepsilon_2'})(x,t)|$ from a preceding iteration, we obtain the bound

$$\left| \left(w^{\varepsilon_1'} - w^{\varepsilon_2'} \right)(x,t) \right| \le \left(\frac{1}{1-q_1\tau_1} \right)^{\lceil \frac{T}{\tau_1} \rceil} P(E_1)C\varepsilon_2, \tag{76}$$

where

$$E_1 = \max_{(x,t)\in\overline{\Pi^T}} |c_r(t)b_r(x)|$$

and $P(E_1)$ is a polynomial of degree $\lceil T/\tau_1 \rceil$ with positive coefficients depending on L and T.

`Case 2:` $\operatorname{supp} b_s^{\varepsilon_2} \not\subset [\omega(t_1^*; 0, 0), L]$. We devide $\overline{\Pi^{\varepsilon_2}(1)}$ into a finite number of subsets $\Pi^{\varepsilon_2}(1, j)$, $j \le M$, defined by (45) with ε replaced by ε_2. Note that, if $j < M$, then $\Pi^{\varepsilon_2}(1, j)$ actually does not depend on ε_2. We prove an analog of (76) with an appropriate choice of τ_1, P, and C, separately for each of $\Pi^{\varepsilon_2}(1, j)$. Since $\operatorname{supp} b_s^{\varepsilon_2} \subset [\omega(t(1); 0, 0), L]$, the conditions of Case 1 are true for $\Pi^{\varepsilon_2}(1, 1)$, and therefore the estimate (76) is true for this subset. Thus, provided ε_2 is small enough, the estimate (69) holds on $\Pi^{\varepsilon_2}(1, 1)$.

The analog of (76) for $\Pi^{\varepsilon_2}(1, 2)$ can be obtained in much the same way. We concentrate only on changes. On $\Pi^{\varepsilon_2}(1, 2)$ we use the integral equation

$$
\begin{aligned}
&\left(w^{\varepsilon_1'} - w^{\varepsilon_2'}\right)(x, t) \\
&\quad = c_r(\theta(x, t)) \int_0^{\omega(\theta(x,t); 0, t(1))} b_r(\xi)(w^{\varepsilon_1'} - w^{\varepsilon_2'})(\xi, \theta(x, t))\, d\xi \\
&\qquad + S_1(x, t) + S_2(\theta(x, t)) + S_3(\theta(x, t)) + S_4(\theta(x, t)),
\end{aligned}
\tag{77}
$$

where

$$
S_4(t) = c_r(t) \int_{\omega(t; 0, t(1))}^{z(t)} \left[b_s^{\varepsilon_1'}(x) \left(w^{\varepsilon_1'} - w^{\varepsilon_2'}\right)(x, t) + \left(b_s^{\varepsilon_1'} - b_s^{\varepsilon_2'}\right)(x) w^{\varepsilon_2'}(x, t) \right] dx.
$$

We now bound $|S_2(t) + S_4(t)|$. Observe that $[\omega(t; 0, t(1)), z(t)] \times \{t\} \subset \Pi^{\varepsilon_2}(1, 1)$ if $t \in [t(1), t(2)]$. By Proposition 1, Claim 1, and the estimate (69) on $\Pi^{\varepsilon_2}(1, 1)$, we conclude that w^ε converges in $C(\Pi^{\varepsilon_2}(1, 1) \cup \overline{\Pi_0^T})$ to a continuous function $w(x, t)$. Using the equality $w^{\varepsilon_1'}(x, t) = w^{\varepsilon_2'}(x, t) = w(x, t)$ on $\overline{\Pi_0^T}$, similarly to (72) we derive the bound

$$
\begin{aligned}
&|S_2(t) + S_4(t)| \\
&\quad \le \max_{t \in [0, T]} |c_r(t)| \max_{(x,t) \in \Pi(1,1)} |w^{\varepsilon_1'} - w^{\varepsilon_2'}| \int_{\omega(t; 0, t(1))}^{z(t)} |b_s^{\varepsilon_1'}|\, dx \\
&\qquad + \max_{t \in [0, T]} |c_r(t)| \int_{\omega(t; 0, t(1))}^{L} \left(|b_s^{\varepsilon_1'}| + |b_s^{\varepsilon_2'}| \right) \\
&\qquad \times \left| w^{\varepsilon_2'}(x, t) - \sum_{j=1}^{k} \chi_{[x_j - \varepsilon_2', x_j + \varepsilon_2']}(x) w(x_j, t) \right| dx \\
&\qquad + \max_{t \in [0, T]} |c_r(t)| \sum_{j=1}^{k} \max_{(x_j, t) \in \overline{\Pi_0^T} \cup \Pi(1,1)} |w(x_j, t)| \left| \int_{x_j - \varepsilon_2'}^{x_j + \varepsilon_2'} \left(b_s^{\varepsilon_1'} - b_s^{\varepsilon_2'}\right)(x)\, dx \right| \\
&\quad \le C\varepsilon_2.
\end{aligned}
$$

Now, using (75), we conclude that (76) holds for $\Pi^{\varepsilon_2}(1,2)$ with new τ_1, P, and C.

Similar arguments apply to the subsets $\Pi^{\varepsilon_2}(1,j)$. Thus the estimate

$$\left|\left(w^{\varepsilon_1'} - w^{\varepsilon_2'}\right)(x,t)\right| \le C\varepsilon_2 \tag{78}$$

is true for the whole $\overline{\Pi^{\varepsilon_2}(1)}$ in both cases. The estimate (69) follows from (78), where ε_2 is chosen small enough. The proof of Item 1 is complete.

Item 2 is a straightforward consequence of Item 1. ∎

Claim 4. *Provided ε_2 is small enough,* (62) *is true on* $\overline{\Pi^{\varepsilon_1}(1)} \cap \overline{I_+^{\varepsilon_2}(1)}$.

Proof. The functions $w^{\varepsilon_1}(x,t)$ and $w^{\varepsilon_2}(x,t)$ on $\overline{\Pi^{\varepsilon_1}(1)} \cap \overline{I_+^{\varepsilon_2}(1)}$ are represented by (17) restricted to $\overline{\Pi^{\varepsilon_1}(1)}$ and $\overline{I_+^{\varepsilon_2}(1)}$, respectively.

Proposition 1 together with Claims 1 and 3 (Item 1) imply, for each fixed $\varepsilon_2 > 0$ as small as in Claim 3, the Cauchy criterion of the uniform convergece of w^ε on $\overline{\Pi_0^T} \cup \overline{\Pi^{\varepsilon_2}(1)}$. Therefore

$$w^\varepsilon(x,t) \text{ converges in } \mathrm{C}\left(\overline{\Pi_0^T} \cup \overline{\Pi^{\varepsilon_2}(1)}\right) \text{ as } \varepsilon \to 0. \tag{79}$$

As above, the limit function will be denoted by $w(x,t)$. Now, using (32), we have the representation

$$\begin{aligned}
&\left(w^{\varepsilon_1} - w^{\varepsilon_2}\right)(x,t) \\
&\quad = \left[c_r(\tau)\int_{\omega(\tau;0,t_1^*-\varepsilon_1^-(\varepsilon_2))}^{L} \Big((b_s^{\varepsilon_1} + b_r)(w^{\varepsilon_1} - w^{\varepsilon_2})\Big)(\xi,\tau)\,d\xi\right]\Bigg|_{\tau=\theta(x,t)} \\
&\qquad + \left[c_r(\tau)\int_{\omega(\tau;0,t_1^*-\varepsilon_1^-(\varepsilon_2))}^{L} \left(b_s^{\varepsilon_1} - b_s^{\varepsilon_2}\right)(\xi)\right. \\
&\qquad \times \left.\left(w^{\varepsilon_2}(x,t) - \sum_{j=1}^{k} \chi_{[x_j-\varepsilon_2,x_j+\varepsilon_2]}(x) w(x_j,t)\right) d\xi\right]\Bigg|_{\tau=\theta(x,t)} \\
&\qquad + \left[c_r(\tau)\sum_{j=1}^{k} w(x_j,t)\int_{\omega(\tau;0,t_1^*-\varepsilon_1^-(\varepsilon_2))}^{L} \left(b_s^{\varepsilon_1} - b_s^{\varepsilon_2}\right)(\xi)\,d\xi\right]\Bigg|_{\tau=\theta(x,t)} \\
&\qquad + S_3(\theta(x,t)) \\
&\qquad + \left[c_r(\tau)\int_{0}^{\omega(\tau;0,t_1^*-\varepsilon_1^-(\varepsilon_2))} \Big(b_r(w^{\varepsilon_1} - w^{\varepsilon_2})\Big)(\xi,\tau)\,d\xi\right]\Bigg|_{\tau=\theta(x,t)} \\
&\qquad + S_1(x,t) + S_5(\theta(x,t)),
\end{aligned} \tag{80}$$

where

$$S_5(t) = c_r(t) \int_0^{\omega(t;0,t_1^*-\varepsilon_1^-(\varepsilon_2))} \big(b_r v^{\varepsilon_2}\big)(x,t)\,dx.$$

For the absolute values of the first four summands in (80) we obtain the upper bound $C\varepsilon_2$ by the following argument. Note that $[\omega(t;0,t_1^*-\varepsilon_1^-(\varepsilon_2)),L]\times\{t\} \subset \overline{\Pi_0^T} \cup \overline{\Pi^{\varepsilon_2}(1)}$ for $t\in[t_1^*-\varepsilon_1^-(\varepsilon_2), t_1^*-\varepsilon_1^-(\varepsilon_1)]$. For the first summand the bound now follows from (76), (8), and Claims 1 and 3 (Item 2). For the second summand we apply (79) and (8). For the third one we use the properties (10). For $|S_3(t)|$ we use the estimate (75).

To prove the upper bound $C\varepsilon_2$ for $|S_5(t)|$ we need an estimate for v^ε on $\overline{I_+^\varepsilon(1)}$. To obtain it we consider two cases.

`Case 1:` $(0,t_1)\in I_+(1)$. Taking into account (32), we represent w^ε and v^ε on $\overline{I_+^\varepsilon(1)}$ in the form

$$\begin{aligned} w^\varepsilon(x,t) = {} & \Big[c_r(S_6^\varepsilon+S_7^\varepsilon)\Big](\theta(x,t)) \\ & + \int_{\theta(x,t)}^t f(\omega(\tau;x,t),\tau,w^\varepsilon)\,d\tau + \int_{\theta(x,t)}^t p(\omega(\tau;x,t),\tau)w^\varepsilon\,d\tau \end{aligned} \tag{81}$$

and

$$v^\varepsilon(x,t) = \Big[c_s(S_6^\varepsilon+S_7^\varepsilon)\Big](\theta(x,t)) + \int_{\theta(x,t)}^t p(\omega(\tau;x,t),\tau)v^\varepsilon\,d\tau, \tag{82}$$

where

$$\begin{aligned} S_6^\varepsilon(t) &= \int_{\omega(t;0,t_1^*-\varepsilon_1^-(\varepsilon))}^L \big(b_s^\varepsilon+b_r\big)(x)w^\varepsilon(x,t)\,dx + \int_{z(t)}^L \big(b_r v^\varepsilon\big)(x,t)\,dx \\ S_7^\varepsilon(t) &= \int_0^{\omega(t;0,t_1^*-\varepsilon_1^-(\varepsilon))} b_r(x)(w^\varepsilon+v^\varepsilon)(x,t)\,dx. \end{aligned} \tag{83}$$

Summing up, we have

$$\begin{aligned} w^\varepsilon(x,t)+v^\varepsilon(x,t) = {} & \Big[(c_r+c_s^\varepsilon)S_6^\varepsilon\Big](\theta(x,t)) + \Big[(c_r+c_s^\varepsilon)S_7^\varepsilon\Big](\theta(x,t)) \\ & + \int_{\theta(x,t)}^t f(\omega(\tau;x,t),\tau,w^\varepsilon)\,d\tau \\ & + \int_{\theta(x,t)}^t p(\omega(\tau;x,t),\tau)(w^\varepsilon+v^\varepsilon)\,d\tau. \end{aligned} \tag{84}$$

By (21), (22), (8), Proposition 1, and Claim 2, $S_6^\varepsilon(t)$ is a continuous function and satisfies the uniform in ε estimate

$$|S_6^\varepsilon(t)| \le C, \qquad t\in[t_1^*-\varepsilon_1^-(\varepsilon), t_1^*+\varepsilon_1^+(\varepsilon)]. \tag{85}$$

By Proposition 1, $S_7^\varepsilon(t)$ is continuous. We now derive an upper bound for $|S_7^\varepsilon(t)|$. Applying the method of sequential approximation to the function $w^\varepsilon + v^\varepsilon$ given by the formula (84), we obtain the estimate

$$\max_{(x,t)\in\overline{I_+^\varepsilon}(1)} |(w^\varepsilon + v^\varepsilon)(x,t)|$$
$$\le \Big[T \max_{(x,t,y)\in\overline{\Pi^T}\times\mathbb{R}} |f(x,t,y)| + \max_{t\in[t_1-\varepsilon,t_1+\varepsilon]} |c_r(t) + c_s^\varepsilon(t)|$$
$$\times \Big(C + \max_{t\in[t_1-\varepsilon,t_1+\varepsilon]} \Big|S_7^\varepsilon(t)|\Big)\Big] \exp\Big\{T \max_{(x,t)\in\overline{\Pi^T}} |p(x,t)|\Big\}.$$

By (61),

$$\max_{(x,t)\in\overline{I_+^\varepsilon(1)}} |w^\varepsilon + v^\varepsilon|$$
$$\le \frac{C}{1 - q(\varepsilon)C\exp\Big\{T \max\limits_{(x,t)\in\overline{\Pi^T}} |p|\Big\}} \tag{86}$$
$$\times \Big[\max_{(x,t,y)\in\overline{\Pi^T}\times\mathbb{R}} |f(x,t,y)| + \max_{t\in[t_1-\varepsilon,t_1+\varepsilon]} |c_r(t) + c_s^\varepsilon(t)|\Big].$$

From (50), (86), (7), and Assumption 3 we conclude that

$$|S_7^\varepsilon(t)| \le C \max_{x\in[0,\omega(t;0,t_1-\varepsilon)]} |b_r(x)| \le C \max_{x\in[0,C_5\varepsilon]} |b_r(x)| \le C\varepsilon \tag{87}$$

for $t \in [t_1^* - \varepsilon_1^-(\varepsilon), t_1^* + \varepsilon_1^+(\varepsilon)]$. Combining (20) and (22) with (82), we obtain

$$v^\varepsilon(x,t) = c_s^\varepsilon(\theta(x,t))\Big(S_6^\varepsilon + S_7^\varepsilon\Big)(\theta(x,t))S(x,t), \tag{88}$$

where the functions $|(S_6^\varepsilon + S_7^\varepsilon)(\theta(x,t))|$ on $\overline{I_+^\varepsilon(1)}$ are bounded uniformly in ε. The latter is true by (85) and (87). The formula (88) implies

$$v^\varepsilon = O\Big(\frac{1}{\varepsilon}\Big) \tag{89}$$

for $(x,t) \in \overline{I_+^\varepsilon(1)}$. Taking into account (88), (50), (7), and Assumption 3, we derive the bound

$$|S_5(t)| \le C \max_{x\in[0,\omega(t;0,t_1-\varepsilon_2)]} |b_r(x)| \le C \max_{x\in[0,C_5\varepsilon_2]} |b_r(x)| \le C\varepsilon_2. \tag{90}$$

`Case 2:` $(0,t_1) \notin I_+(1)$. The proof is much the same as for Case 1. The only difference is in evaluation of $|S_5(t)|$, where v^ε on $\overline{I_+^\varepsilon(1)}$ is now given by

$$v^\varepsilon(x,t) = \Big[c_r(\tau)\int_{\omega(\tau;0,0)}^L b_s^\varepsilon(\xi)S(\xi,\tau)a_s^\varepsilon(\omega(0;\xi,\tau))\,d\xi\Big]\Big|_{\tau=\theta(x,t)} S(x,t). \tag{91}$$

Hence (89) in this case is true by (7) and continuity of λ. Therefore for $|S_5(t)|$ the estimate (90) holds.

We now return to (80) and, taking into account (90), estimate $|w^{\varepsilon_1} - w^{\varepsilon_2}|$ following the proof of (76). As a result, the bound (62) is true for sufficiently small ε_2. ∎

Claim 5. *Provided* ε_2 *is small enough,* (62) *is true on* $\overline{I_+^{\varepsilon_1}(1)}$.

Proof. Note that w^{ε_1} and w^{ε_2} on $\overline{I_+^{\varepsilon_1}(1)}$ are defined by the same formula (81). Therefore

$$\begin{aligned}\big(w^{\varepsilon_1} - w^{\varepsilon_2}\big)(x,t)& \\ &= \Big[c_r\big(S_6^{\varepsilon_1} - S_6^{\varepsilon_2}\big)\Big](\theta(x,t)) + \Big[c_r\big(S_7^{\varepsilon_1} - S_7^{\varepsilon_2}\big)\Big](\theta(x,t)) + S_1(x,t).\end{aligned} \tag{92}$$

The upper bound $C\varepsilon_2$ for the absolute value of the second summand follows from (87). To estimate the first summand, we use the equality

$$\begin{aligned}\big(S_6^{\varepsilon_1} - S_6^{\varepsilon_2}\big)(t)& \\ &= \int_{\omega(t;0,t_1^*-\varepsilon_1^-(\varepsilon_2))}^{L} \Big[\big(b_s^{\varepsilon_1} + b_r\big)w^{\varepsilon_1} - \big(b_s^{\varepsilon_2} + b_r\big)w^{\varepsilon_2} + b_r\big(v^{\varepsilon_1} - v^{\varepsilon_2}\big)\Big](x,t)\,dx \\ &\quad + \int_{\omega(t;0,t_1^*-\varepsilon_1^-(\varepsilon_1))}^{\omega(t;0,t_1^*-\varepsilon_1^-(\varepsilon_2))} \big(b_r w^{\varepsilon_1}\big)(x,t)\,dx.\end{aligned}$$

The absolute value of the first summand is already estimated in the proof of Claim 4. For the second summand we can apply Claim 2, since

$$[\omega(t;0,t_1^*-\varepsilon_1^-(\varepsilon_1)), \omega(t;0,t_1^*-\varepsilon_1^-(\varepsilon_2))] \times \{t\} \subset \overline{\Pi^{\varepsilon_1}(1)}$$

for $t \in [t_1^* - \varepsilon_1^-(\varepsilon_2), t_1^* + \varepsilon_1^+(\varepsilon_2)]$. As a consequence,

$$\Big|\Big(S_6^{\varepsilon_1} - S_6^{\varepsilon_2}\Big)(t)\Big| \le C\varepsilon_2. \tag{93}$$

Applying the method of sequential approximation to $w^{\varepsilon_1} - w^{\varepsilon_2}$ given by (92), on the account of (93) and the upper bound $C\varepsilon_2$ for the second summand in (92), we derive the bound

$$\begin{aligned}\Big|\Big(w^{\varepsilon_1} - w^{\varepsilon_2}\Big)(x,t)\Big|& \\ &\le C\varepsilon_2 \exp\Big\{T\Big(\max_{(x,t,y)\in\overline{\Pi^T}\times\mathbb{R}} |(\nabla_u f)(x,t,y)| + \max_{(x,t)\in\overline{\Pi^T}} |p(x,t)|\Big)\Big\}.\end{aligned} \tag{94}$$

This implies (62), provided ε_2 is chosen small enough. ∎

Claim 6. *Provided ε_2 is small enough,* (62) *is true on* $\overline{I_+^{\varepsilon_2}(1)} \cap \overline{\Pi^{\varepsilon_1}(2)}$.

Proof. We follow the proof of Claim 4 with the following changes. On the account of (32), for $w^{\varepsilon_1} - w^{\varepsilon_2}$ on $\overline{I_+^{\varepsilon_2}(1)} \cap \overline{\Pi^{\varepsilon_1}(2)}$ we have the representation (80) with $t_1^* - \varepsilon_1^-(\varepsilon_2)$ replaced by $t_1^* + \varepsilon_1^+(\varepsilon_1)$ in the fifth summand, with

$$S_5(t) = c_r(t)\Bigg[\int_{\omega(t;0,t_1^*+\varepsilon_1^+(\varepsilon_1))}^{\omega(t;0,t_1^*-\varepsilon_1^-(\varepsilon_1))} \big(b_r v^{\varepsilon_1}\big)(x,t)\,dx - \int_0^{\omega(t;0,t_1^*-\varepsilon_1^-(\varepsilon_2))} \big(b_r v^{\varepsilon_2}\big)(x,t)\,dx\Bigg], \tag{95}$$

and with one more summand

$$\Bigg[c_r(\tau)\int_{\omega(\tau;0,t_1^*+\varepsilon_1^+(\varepsilon_1))}^{\omega(\tau;0,t_1^*-\varepsilon_1^-(\varepsilon_2))} \Big(b_r(w^{\varepsilon_1} - w^{\varepsilon_2})\Big)(\xi,\tau)\,d\xi\Bigg]\Bigg|_{\tau=\theta(x,t)}. \tag{96}$$

To estimate the absolute value of the latter expression, we use Claims 4 and 5. To estimate $|S_5(t)|$, to both integrals we apply the same argument that was used for evaluation of $|S_5(t)|$ in the proof of Claim 4. ∎

By Claims 3 – 6, Assertion 1 is true for $j = 1$.

Claim 7. *The functions w^ε are bounded on $\overline{I_+^\varepsilon(1)}$, uniformly in $\varepsilon > 0$.*

Proof. The claim follows from (81), (89), (50), and Assumptions 5 and 7. ∎

By Claims 2 and 7, Assertion 2 is true for $j = 1$.

Claim 8. *The family of functions w^ε converges in* $\mathrm{C}(\overline{\Pi_0^T} \cup \overline{\Pi(1)})$ *as* $\varepsilon \to 0$.

Proof. Since, by Proposition 1, each w^ε is continuous, it suffices to prove the convergence separately on $\overline{\Pi_0^T}$ and $\overline{\Pi(1)}$. On the former domain the convergence is given by Claim 2. The convergence on the latter domain follows by the Cauchy criterion which holds by Claims 3–5, and the fact that $\overline{\Pi(1)} \subset \overline{\Pi^{\varepsilon_2}(1)} \cup \overline{I_+^{\varepsilon_2}(1)}$ for every $\varepsilon_2 > 0$. ∎

Thus, Assertion 5 is true for $j = 1$. In the sequel the limit function will be denoted by $w(x,t)$.

Claim 9. *The estimate $M_1(\varepsilon) \le C$ is true for all $\varepsilon > 0$.*

Proof. `Case 1:` $(0, \tilde{t}_1) \in I_+(1)$. Then

$$M_1(\varepsilon) = \int_{t_1^*-\varepsilon_1^-(\varepsilon)}^{t_1^*+\varepsilon_1^+(\varepsilon)} \left| c_r(t)\int_{z(t)}^L b_s^\varepsilon(x) S(x,t) a_s^\varepsilon(\omega(0;x,t))\,dx\right| dt$$
$$\le \max_{(x,t)\in\overline{\Pi_0^T}} \left|\frac{(c_r S)(x,t)}{(\partial_t\omega)(0;x,t)}\right| \int_0^L |a_s^\varepsilon(x)|\,dx \int_0^L |b_s^\varepsilon(x)|\,dx \le C.$$

The estimate follows from (8) and Assumptions 5 and 6.

`Case 2:` $(0, t_1) \in I_+(1)$. Then

$$M_1(\varepsilon) = \int_{t_1-\varepsilon}^{t_1+\varepsilon} |c_s^\varepsilon(t)(S_6^\varepsilon + S_7^\varepsilon)(t)|\, dt \le C.$$

This estimate follows from (85), (87), and (8). ∎

Thus, Assertion 3 is true for $j = 1$.

Claim 10. *If ε_2 is small enough and $\varepsilon_1 < \varepsilon_2$, then*

$$|K_1(\varepsilon_1, \varepsilon_2)| \le C\varepsilon_2.$$

Proof. We will consider ε_2 as small as in Claims 3 – 6. We will use (16) restricted to $[t_1^* - \varepsilon_1^-(\varepsilon_2), t_1^* + \varepsilon_1^+(\varepsilon_2)]$ and (22).

`Case 1:` $(0, \tilde{t}_1) \in I_+(1)$. By (25) and (26) we have $c_s^\varepsilon = 0$, and therefore

$$K_1(\varepsilon_1, \varepsilon_2) = \int_{t_1^*-\varepsilon_1^-(\varepsilon_2)}^{t_1^*+\varepsilon_1^+(\varepsilon_2)} c_r(t) \int_{z(t)}^{L} \Big(b_s^{\varepsilon_1} v^{\varepsilon_1} - b_s^{\varepsilon_2} v^{\varepsilon_2}\Big)(x,t)\, dx\, dt.$$

Applying (22) restricted to $\overline{\Pi_0^T}$ and changing coordinates $(x,t) \to (x,\xi) = (x, \omega(0; x, t))$, we obtain

$$K_1(\varepsilon_1, \varepsilon_2) = \sum_{(q,d)\in E} \int_{x_d^*-\varepsilon_2}^{x_d^*+\varepsilon_2} \int_{x_q-\varepsilon_2}^{x_q+\varepsilon_2} T(x,\xi) \Big[b_s^{\varepsilon_1}(x) a_s^{\varepsilon_1}(\xi) - b_s^{\varepsilon_2}(x) a_s^{\varepsilon_2}(\xi)\Big]\, dx\, d\xi,$$

where

$$T(x,\xi) = \frac{(c_r S)(x,t)}{(\partial_t \omega)(0; x, t)}\Big|_{t=\tilde{\omega}(x;\xi,0)}$$

is a continuous function in x and ξ and E is the set of pairs of indices $q \le k$ and $d \le m$ such that $\omega(0; x_q, \tilde{t}_1) = x_d^*$. Evidently, if $(0, \tilde{t}_1) \in I_+(1)$, then there exists at least one pair (q,d) that satisfies the latter condition. Then

$$\begin{aligned}
|K_1(\varepsilon_1, \varepsilon_2)| \le \sum_{q,d} & \max_{(x,\xi)\in[x_q-\varepsilon_2, x_q+\varepsilon_2]\times[x_d^*-\varepsilon_2, x_d^*+\varepsilon_2]} |T(x,\xi) - T(x_q, x_d^*)| \\
& \times \left[\int_{x_q-\varepsilon_2}^{x_q+\varepsilon_2} |b_s^{\varepsilon_1}|\, dx \int_{x_d^*-\varepsilon_2}^{x_d^*+\varepsilon_2} |a_s^{\varepsilon_1}|\, dx + \int_{x_q-\varepsilon_2}^{x_q+\varepsilon_2} |b_s^{\varepsilon_2}|\, dx \int_{x_d^*-\varepsilon_2}^{x_d^*+\varepsilon_2} |a_s^{\varepsilon_2}|\, dx\right] \\
& + \sum_{q,d} \left\{ |T(x_q, x_d^*)| \left| \int_{x_q-\varepsilon_2}^{x_q+\varepsilon_2} b_s^{\varepsilon_1}\, dx \int_{x_d^*-\varepsilon_2}^{x_d^*+\varepsilon_2} (a_s^{\varepsilon_1} - a_s^{\varepsilon_2})\, dx \right.\right. \\
& \left.\left. + \int_{x_d^*-\varepsilon_2}^{x_d^*+\varepsilon_2} a_s^{\varepsilon_2}\, dx \int_{x_q-\varepsilon_2}^{x_q+\varepsilon_2} (b_s^{\varepsilon_1} - b_s^{\varepsilon_2})\, dx \right| \right\}.
\end{aligned}$$

The last two summands are equal to 0 by (10). The first summand is bounded from above by $C\varepsilon_2$ by (8) and the continuity property for $T(x,t)$. This completes the proof in Case 1.

`Case 2:` $(0,t_1) \in I_+(1)$. Then the second summand in (16) is equal to 0, and therefore

$$K_1(\varepsilon_1,\varepsilon_2) = \int_{t_1-\varepsilon_2}^{t_1+\varepsilon_2} \Big[c_s^{\varepsilon_1}(t)\Big(S_6^{\varepsilon_1} + S_7^{\varepsilon_1}\Big)(t) - c_s^{\varepsilon_2}(t)\Big(S_6^{\varepsilon_2} + S_7^{\varepsilon_2}\Big)(t)\Big]\,dt. \tag{97}$$

We hence have

$$\begin{aligned}
|K_1(\varepsilon_1,\varepsilon_2)| = \Bigg| &\int_{t_1-\varepsilon_2}^{t_1+\varepsilon_2} \Big(c_s^{\varepsilon_1} S_7^{\varepsilon_1} - c_s^{\varepsilon_2} S_7^{\varepsilon_2}\Big)(t)\,dt \\
&+ \int_{t_1-\varepsilon_2}^{t_1+\varepsilon_2} \Big(c_s^{\varepsilon_1} - c_s^{\varepsilon_2}\Big)(t)\Big(S_6^{\varepsilon_1}(t) - S_6^{\varepsilon_1}(t_1)\Big)\,dt \\
&+ S_6^{\varepsilon_1}(t_1)\int_{t_1-\varepsilon_2}^{t_1+\varepsilon_2} \Big(c_s^{\varepsilon_1} - c_s^{\varepsilon_2}\Big)(t)\,dt \\
&+ \int_{t_1-\varepsilon_2}^{t_1+\varepsilon_2} c_s^{\varepsilon_2}(t)\Big(S_6^{\varepsilon_1} - S_6^{\varepsilon_2}\Big)(t)\,dt\Bigg| \qquad (98) \\
\le &\int_{t_1-\varepsilon_2}^{t_1+\varepsilon_2} \Big(|c_s^{\varepsilon_1}||S_7^{\varepsilon_1}| + |c_s^{\varepsilon_2}||S_7^{\varepsilon_2}|\Big)\,dt \\
&+ \int_{t_1-\varepsilon_2}^{t_1+\varepsilon_2} \Big(|c_s^{\varepsilon_1}| + |c_s^{\varepsilon_2}|\Big)\Big|(S_6^{\varepsilon_1}(t) - S_6^{\varepsilon_1}(t_1))\Big|\,dt \\
&+ \int_{t_1-\varepsilon_2}^{t_1+\varepsilon_2} |c_s^{\varepsilon_2}|\Big|\Big(S_6^{\varepsilon_1} - S_6^{\varepsilon_2}\Big)(t)\Big|\,dt.
\end{aligned}$$

Combining (87) with (8), we obtain the upper bound $C\varepsilon_2$ for the first summand in the right-hand side of the inequality (98). The same bound for the last summand follows from (93) and (8). It remains to estimate the second summand. Without loss of generality we assume that $t \ge t_1$. By (32) we have the representation

$$\begin{aligned}
S_6^{\varepsilon_1}(t) - S_6^{\varepsilon_1}(t_1) = &\int_{\omega(t;0,t_1-\varepsilon_1)}^{L} \Big(b_s^{\varepsilon_1} + b_r\Big)(x)\Big(w^{\varepsilon_1}(x,t) - w^{\varepsilon_1}(x,t_1)\Big)\,dx \\
&+ \int_{\omega(t_1;0,t_1-\varepsilon_1)}^{\omega(t;0,t_1-\varepsilon_1)} b_r(x) w^{\varepsilon_1}(x,t_1)\,dx \\
&+ \int_{z(t)}^{L} (b_r S)(x,t) a_s^{\varepsilon_1}(\omega(0;x,t))\,dx \\
&- \int_{\omega(t_1;0,0)}^{\omega(t_1;L,t)} \Big(b_r S\Big)(x,t_1) a_s^{\varepsilon_1}(\omega(0;x,t_1))\,dx
\end{aligned} \tag{99}$$

$$- \int_{\omega(t_1;L,t)}^{L} \Big(b_r S\Big)(x,t_1) a_s^{\varepsilon_1}(\omega(0;x,t_1))\,dx.$$

Changing coordinates $x \to \omega(0;x,t)$ in the third integral and $x \to \omega(0;x,t_1)$ in the fourth integral, after simple computation we have

$$\begin{aligned}\Big|S_6^{\varepsilon_1}(t) - S_6^{\varepsilon_1}(t_1)\Big| &\le \left(\int_0^L |b_s^{\varepsilon}|\,dx + L \max_{x\in[0,L]} |b_r(x)|\right) \times M_1 \\ &\quad + M_2 + \Big[M_3 + M_4\Big] \int_0^L |a_s^{\varepsilon_1}|\,dx \\ &\le C\varepsilon_2. \end{aligned} \tag{100}$$

with

$$\begin{aligned} M_1 &= \max_{|t-t_1|\le\varepsilon_1,(x,t)\in\overline{\Pi_0^T}\cup\overline{\Pi(1)}} \Big[\Big|\Big(w^{\varepsilon_1} - w\Big)(x,t)\Big| \\ &\quad + \Big|w(x,t) - w(x,t_1)\Big| + \Big|w(x,t_1) - w^{\varepsilon_1}(x,t_1)\Big|\Big] \\ M_2 &= \max_{(x,t)\in\overline{\Pi^{\varepsilon_1}(1)}} |w^{\varepsilon_1}(x,t)| \max_{x\in[0,L]} |b_r(x)| \Big|\omega(t;0,t_1-\varepsilon_1) - \omega(t_1;0,t_1-\varepsilon_1)\Big| \\ M_3 &= \max_{(x,t)\in[0,L]\times[t_1-\varepsilon_1,t_1+\varepsilon_1]} \left|\frac{(b_rS)(\omega(t;x,0),t)}{(\partial_x\omega)(0;\omega(t;x,0),t)} - \frac{(b_rS)(\omega(t_1;x,0),t_1)}{(\partial_x\omega)(0;\omega(t_1;x,0),t_1)}\right| \\ M_4 &= \max_{x\in[\omega(t_1-\varepsilon_1;L,t_1+\varepsilon_1),L]} |b_r(x)| \max_{(x,t)\in\overline{\Pi^T}} |S(x,t)|. \end{aligned}$$

The latter estimate in (100) is true by Claims 2, 7, and 8, estimates (8), and Assumptions 3 and 5. Claim 9 follows. ∎

Therefore, Assertion 4 is true for $j = 1$.

Induction assumption. We assume that Assertions 1–5 are true for $j \le i-1$, $i \ge 2$.

Claim 11. *The functions $w^{\varepsilon}(x,t)$ are bounded on $\overline{\Pi^{\varepsilon}(i)}$, uniformly in $\varepsilon > 0$.*

Proof. The proof is similar to the proof of Claim 2. The function $w^{\varepsilon}(x,t)$ on $\overline{\Pi^{\varepsilon}(i)}$ is defined by the formula (17), where

$$\begin{aligned}(Rw^{\varepsilon})(x,t) = c_r(\theta(x,t)) \Big[&\int_0^{\omega(\tau;0,t^*_{i-1}+\varepsilon^+_{i-1}(\varepsilon))} \Big(b_s^{\varepsilon} + b_r\Big)(\xi) w^{\varepsilon}(\xi,\tau)\,d\xi \\ &+ \int_{\omega(\tau;0,t^*_{i-1}+\varepsilon^+_{i-1}(\varepsilon))}^{L} \Big(b_s^{\varepsilon} + b_r\Big)(\xi) w^{\varepsilon}(\xi,\tau)\,d\xi \\ &+ \int_{\omega(\tau;0,t^*_{i-1}+\varepsilon^+_{i-1}(\varepsilon))}^{z(\tau)} (b_rS)(\xi,\tau) v^{\varepsilon}(0,\theta(\xi,\tau))\,d\xi \end{aligned}$$

$$+ \int_{z(\tau)}^{L} (b_r S)(\xi,\tau) a_s^\varepsilon(\omega(0;\xi,\tau))\,d\xi \Big]\Big|_{\tau=\theta(x,t)}.$$

Taking into account (8) and Assertions 2 and 3 for $j < i$, we conclude that the last three summands are bounded uniformly in ε. Similarly to [14, p. 646], we obtain the global estimate

$$\max_{(x,t)\in\overline{\Pi_1^\varepsilon(i)}} |w^\varepsilon(x,t)| \le \left(\frac{1}{1-q_1\tau_1}\right)^{\lceil \frac{T}{\tau_1}\rceil} P(E) \times \left(1 + \max_{(x,t)\in\left(\bigcup_{j=1}^{i-1}\overline{\Pi(j)}\right)\cup\overline{I_+^\varepsilon(i-1)}\cup\overline{\Pi_0^T}} |w^\varepsilon(x,t)|\right), \tag{101}$$

where q_1, τ_1, and E are defined by (66), (67), and (68), and $P(E)$ is a polynomial of degree $\lceil T/\tau_1\rceil$ with positive coefficients depending on $f(x,t,0)$, L, and T. The claim now follows from Assertion 2 for $j \le i-1$. ∎

Claim 12. *Provided ε_2 is small enough, then*

1. *for all $\varepsilon_2' \le \varepsilon_2$ and for all $\varepsilon_1' \le \varepsilon_2'$ the estimate* (69) *is true on $\overline{\Pi(i)} \setminus I_+^{\varepsilon_2}(i)$,*
2. (62) *is true on $\overline{\Pi^{\varepsilon_2}(i)}$.*

Proof. We fix an arbitrary sequence $t_{i-1}^* = t(0) < p_1 < t(2) < \dots < t(M) = t_i^* - \varepsilon_i^-(\varepsilon_2)$ such that $M \ge 2$, $t(1) > t_{i-1}^* + \varepsilon_{i-1}^+(\varepsilon_2)$, and $\operatorname{supp} b_s^{\varepsilon_2} \subset [\omega(t(i);0,t(i-1)),L]$. We can do so due to (32). Given this sequence, we devide $\overline{\Pi(i)} \setminus I_+^{\varepsilon_2}(i)$ into a finite number of subsets

$$\Pi^{\varepsilon_2}(i,j) = \left\{(x,t) \in \overline{\Pi(i)} \setminus I_+^{\varepsilon_2}(i) \,|\, \tilde\omega(x;0,t(j-1)) \le t \le \tilde\omega(x;0,t(j))\right\}.$$

Note that, if $j < M$, then $\Pi^{\varepsilon_2}(i,j)$ actually does not depend on ε_2.

To obtain (69) for $\Pi^{\varepsilon_2}(i,1)$, we first derive (62) for $\Pi^{\varepsilon_2}(i,1)$. Doing this, we follow the proof of Claim 3 (Item 1) for Case 1 with the following changes. Throughout the proof ε_1' and ε_2' are replaced by ε_1 and ε_2, respectively. We use the formulas (70) and (71) with $\omega(t;0,t_{i-1}^* + \varepsilon_{i-1}^+(\varepsilon_2))$ in place of $z(t)$ and with

$$S_2(t) = c_r(t) \int_{\omega(t;0,t_{i-1}^*)}^{L} \Big[(b_s^{\varepsilon_1} - b_s^{\varepsilon_2}) w^{\varepsilon_2} + b_s^{\varepsilon_1}(w^{\varepsilon_1} - w^{\varepsilon_2})\Big](x,t)\,dx + c_r(t) \int_{\omega(t;0,t_{i-1}^*+\varepsilon_{i-1}^+(\varepsilon_2))}^{L} b_r(x)\Big(w^{\varepsilon_1} - w^{\varepsilon_2}\Big)(x,t)\,dx.$$

To estimate $|S_2|$, we use Assertions 1, 2, and 5 for $j < i$. To estimate the analog of $|S_3|$, we use Assertions 4 and 5 for $j < i$. As a result, (62) is true on $\Pi^{\varepsilon_2}(i,1)$.

Note that $\Pi^{\varepsilon_2}(i,1)$ does not depend on ε_1 and ε_2. By Assertion 1 for $j=i-1$ and (62) for $\Pi^{\varepsilon_2}(i,1)$, that we have just proved, we conclude that w^ε converges in $\mathrm{C}(\Pi^{\varepsilon_2}(i,1))$. The estimate (69) is hence true on $\Pi^{\varepsilon_2}(i,1)$.

The estimate (69) for $\Pi^{\varepsilon_2}(i,2)$ follows similarly to the proof of Claim 3 (Item 1) for Case 2 (see the proof of (69) for $\Pi^{\varepsilon_2}(1,2)$). The minor change is that we estimate $|S_2(t)+S_4(t)|$ using Assertion 5 for $j=i-1$ and (69) for $\Pi^{\varepsilon_2}(i,1)$, that we have just proved. Similar arguments apply to the subsets $\Pi^{\varepsilon_2}(i,j)$, $j\le M$. As a result, (69) is true for the whole $\overline{\Pi(i)}\setminus I_+^{\varepsilon_2}(i)$.

Item 1 follows. Item 2 is straightforward consequence of Item 1. ∎

Claim 13. *Provided* ε_2 *is small enough, then* (62) *is true on* $\overline{\Pi^{\varepsilon_1}(i)}\cap\overline{I_+^{\varepsilon_2}(i)}$.

Proof. The proof is similar to the proof of Claim 4. We concentrate only on the changes that here appear. We choose ε_2 so small that the condition (32) with $t_1^*+\varepsilon_1^+(\varepsilon_2)$ replaced by $t_i^*+\varepsilon_i^+(\varepsilon_2)$ and $t_1^*-\varepsilon_1^-(\varepsilon_2)$ replaced by $t_i^*-\varepsilon_i^-(\varepsilon_2)$ is true. As in Claim 4, we distinguish two cases.

`Case 1`: There exists $j\le l$ such that $(0,t_j)\in I_+(i)$. We use (80) with t_j^* in place of t_1^*, where $z(t)$ in $S_3(t)$ is replaced by $\omega(t;0,t_j-\varepsilon_2)$. For the first four summands in the analog of (80) we use Assertions 1 – 4 for $j<i$ and Claim 12. The upper bound for $|S_5(t)|$ follows from the bounds

$$|S_6^\varepsilon(t)|\le C,\qquad t\in[t_i^*-\varepsilon_i^-(\varepsilon),t_i^*+\varepsilon_i^+(\varepsilon)] \tag{102}$$

$$|S_7^\varepsilon(t)|\le C\varepsilon,\quad t\in[t_i^*-\varepsilon_i^-(\varepsilon),t_i^*+\varepsilon_i^+(\varepsilon)], \tag{103}$$

where $S_6^\varepsilon(t)$ and $S_7^\varepsilon(t)$ are defined by (83) with $t_i^*-\varepsilon_i^-(\varepsilon)$ in place of $t_1^*-\varepsilon_1^-(\varepsilon)$ and $\omega(t;0,t_{i-1}^*+\varepsilon_{i-1}^+(\varepsilon))$ in place of $z(t)$. The estimate (102) is true by the arguments used for obtaining (85), Assertions 2 and 3 for $j<i$, and Claim 11. The estimate (103) is obtained similarly to (87). The estimates (102) and (103) imply (89).

`Case 2`: $(0,t_j)\notin I_+(i)$ for $j\le l$. On the account of (32), (25), and (26), on $\overline{I_+^\varepsilon(i)}$ we have the equality

$$\begin{aligned} v^\varepsilon(x,t) &= \left[c_r(\tau)\int_{\omega(\tau;0,t_{i-1}^*+\varepsilon_{i-1}^+(\varepsilon_2))}^{z(\tau)} b_s^\varepsilon(\xi)S(\xi,\tau)v^\varepsilon(0,\theta(\xi,\tau))\,d\xi\right]\Bigg|_{\tau=\theta(x,t)} S(x,t)\\ &+\left[c_r(\tau)\int_{z(\tau)}^{L} b_s^\varepsilon(\xi)S(\xi,\tau)a_s^\varepsilon(\omega(0;\xi,\tau))\,d\xi\right]\Bigg|_{\tau=\theta(x,t)} S(x,t). \end{aligned} \tag{104}$$

For the second summand we apply the same arguments as in the proof of Claim 4 for Case 2. For the first summand we apply (7), Assertion 3 for $j<i$ and the

inclusion

$$\Big\{(0,\theta(x,t))\,\Big|\,t\in[t_i-\varepsilon_i^-(\varepsilon),t_i+\varepsilon_i^+(\varepsilon)],x\in[\omega(t;0,t_{i-1}^*+\varepsilon_{i-1}^+(\varepsilon_2)),z(t)]\Big\}$$
$$\subset\bigcup_{j=1}^{i-1}\Big(\overline{I_+^\varepsilon(j)}\cap\{(x,t)\,|\,x=0\}\Big).$$

Therefore (89) is true on $\overline{I_+^\varepsilon(i)}$. The claim follows. ∎

From what has already been proved we conclude that (89) holds on $\overline{I_+^\varepsilon(j)}$ for every $j\le n(T)$. Since $v^\varepsilon(x,t)=S(x,t)a_s^\varepsilon(\omega(0;x,t))$ for $(x,t)\in\overline{\Pi_0^T}$, (89) holds on $\overline{\Pi_0^T}$.

By (31), $\operatorname{supp}v^\varepsilon\subset\overline{I}_+^\varepsilon$. Therefore

$$v^\varepsilon=O\left(\frac{1}{\varepsilon}\right)\quad\text{on}\quad\overline{\Pi^T}. \tag{105}$$

We will need this property in the sequel.

Claim 14. *Provided ε_2 is small enough, then (62) is true on $\overline{I_+^{\varepsilon_1}(i)}$.*

Proof. We follow the proof of Claim 5 with the following changes. We use (83) and (92) with $t_1^*-\varepsilon_1^-(\varepsilon)$ and $z(t)$ to be replaced by $t_i^*-\varepsilon_i^-(\varepsilon)$. To estimate the absolute value of the first summand in the analog of (92), we use Assertions 1 – 3 for $j<i$ and Claims 11 – 13. As a consequence,

$$\Big|\Big(S_6^{\varepsilon_1}-S_6^{\varepsilon_2}\Big)(x,t)\Big|\le C\varepsilon_2,\qquad t\in[t_i^*-\varepsilon_i^-(\varepsilon),t_i^*+\varepsilon_i^+(\varepsilon)]. \tag{106}$$

To estimate the second summand, we use (103). ∎

Claim 15. *Provided ε_2 is small enough, then (62) is true on $\overline{I_+^{\varepsilon_2}(i)}\cap\overline{\Pi^{\varepsilon_1}(i+1)}$.*

Proof. We follow the proof of Claim 6 with the following changes. For $w^{\varepsilon_1}-w^{\varepsilon_2}$ on $\overline{I_+^{\varepsilon_2}(i)}\cap\overline{\Pi^{\varepsilon_1}(i+1)}$ we use (80) with $t_1^*-\varepsilon_1^-(\varepsilon_2)$ replaced by $t_i^*-\varepsilon_i^-(\varepsilon_2)$ in the fifth summand and with one more summand (96), where $t_1^*+\varepsilon_1^+(\varepsilon_1)$ and $t_1^*-\varepsilon_1^-(\varepsilon_2)$ are replaced by $t_i^*+\varepsilon_i^+(\varepsilon_1)$ and $t_i^*-\varepsilon_i^-(\varepsilon_2)$, respectively. In the representation of $S_3(t)$ we now have $\omega(t;0,t_{i-1}^*-\varepsilon_{i-1}^-(\varepsilon_2))$ in place of $z(t)$. For $S_5(t)$ we now use the formula (95) with $t_i^*-\varepsilon_i^-(\varepsilon_1)$, $t_i^*+\varepsilon_i^+(\varepsilon_1)$, and $t_i^*-\varepsilon_i^-(\varepsilon_2)$ in place of $t_1^*-\varepsilon_1^-(\varepsilon_1)$, $t_1^*+\varepsilon_1^+(\varepsilon_1)$, and $t_1^*-\varepsilon_1^-(\varepsilon_2)$, respectively.

To estimate the absolute value of the analog of (96) we use the Claims 13 and 14. ∎

By Claims 12 – 15, Assertion 1 is true for $j=i$.

Claim 16. *The functions $w^\varepsilon(x,t)$ are bounded on $\overline{I_+^\varepsilon(i)}$, uniformly in $\varepsilon>0$.*

Proof. Note that the function $w^\varepsilon(x,t)$ on $\overline{I_+^\varepsilon(i)}$ is defined by (81) and (83) with $t_i^* - \varepsilon_i^-(\varepsilon)$ in place of $t_1^* - \varepsilon_1^-(\varepsilon)$ and $\omega(t; 0, t_{i-1}^* + \varepsilon_{i-1}^+(\varepsilon))$ in place of $z(t)$. The claim follows from (102), (103), and Assumptions 5 and 7. ∎

Assertion 2 for $j = i$ follows from Claims 11 and 16.

Claim 17. *The family of functions w^ε converges in $\mathrm{C}(\bigcup_{j=1}^i \overline{\Pi(j)} \cup \overline{\Pi_0^T})$ as $\varepsilon \to 0$.*

Proof. Since, by Proposition 1, each w^ε is continuous, it suffices to prove the convergence separately on $\overline{\Pi_0^T}$ and $\bigcup_{j=1}^i \overline{\Pi(j)}$. On the former domain the convergence is ensured by Claim 2. The convergence on the latter domain follows by the Cauchy criterion which holds by Assertion 1 for $j \le i$, and the fact that $\bigcup_{j=1}^i \overline{\Pi(j)} \subset \bigcup_{j=1}^i (\overline{\Pi^{\varepsilon_2}(j)} \cup I_+^{\varepsilon_2}(j))$ for every $\varepsilon_2 > 0$. ∎

By Claim 17, Assertion 5 holds for $j = i$.

Claim 18. *The estimate $M_i(\varepsilon) \le C$ is true for all $\varepsilon > 0$.*

Proof. `Case 1:` $(0, \tilde{t}_i) \in I_+(i)$. We have

$$\begin{aligned}
M_i(\varepsilon) &= \int_{t_i^*-\varepsilon_i^-(\varepsilon)}^{t_i^*+\varepsilon_i^+(\varepsilon)} \Bigg| c_r(t) \int_0^{z(t)} b_s^\varepsilon(x) S(x,t) v^\varepsilon(0, \theta(x,t))\, dx \\
&\quad + c_r(t) \int_{z(t)}^L b_s^\varepsilon(x) S(x,t) a_s^\varepsilon(\omega(0; x,t))\, dx \Bigg|\, dt \\
&\le \max_{(x,t)\in\overline{\Pi_1^T}} \left| \frac{(c_r S)(x,t)}{(\partial_t \theta)(x,t)} \right| \sum_{j=1}^{i-1} \int_{t_j^*-\varepsilon_j^-(\varepsilon)}^{t_j^*+\varepsilon_j^+(\varepsilon)} |v^\varepsilon(0,t)|\, dt \int_0^L |b_s^\varepsilon(x)|\, dx \\
&\quad + \max_{(x,t)\in\overline{\Pi_0^T}} \left| \frac{(c_r S)(x,t)}{(\partial_t \omega)(0; x,t)} \right| \int_0^L |a_s^\varepsilon(x)|\, dx \int_0^L |b_s^\varepsilon(x)|\, dx \\
&\le C,
\end{aligned}$$

where $z(t)$ is defined by (37). This estimate is true by (8) and Assertion 3 for $j < i$.

`Case 2:` $(0, t_j) \in I_+(i)$ for some $j \le l$. We have

$$M_i(\varepsilon) = \int_{t_j-\varepsilon}^{t_j+\varepsilon} |c_s^\varepsilon(t)(S_6^\varepsilon + S_7^\varepsilon)(t)|\, dt \le C,$$

This estimate is true by (102), (103), and (8). ∎

Claim 18 implies Assertion 3 for $j = i$.

Claim 19. *The estimate $|K_i(\varepsilon_1, \varepsilon_2)| \le C\varepsilon_2$ is true for ε_2 so small that Assertion 1 holds for all $j \le i$.*

Proof. We follow the proof of Claim 10 with the changes listed below. Similarly to Claim 10, we distinguish two cases.

`Case 1:` $(0,\tilde{t}_i) \in I_+(i)$. We have $K_i(\varepsilon_1,\varepsilon_2) = K_i^1(\varepsilon_1,\varepsilon_2) + K_i^2(\varepsilon_1,\varepsilon_2)$, where

$$K_i^1(\varepsilon_1,\varepsilon_2) = \int_{t_i^*-\varepsilon_i^-(\varepsilon_2)}^{t_i^*+\varepsilon_i^+(\varepsilon_2)} c_r(t) \int_0^{z(t)} \Big(b_s^{\varepsilon_1} v^{\varepsilon_1} - b_s^{\varepsilon_2} v^{\varepsilon_2}\Big)(x,t)\,dx\,dt$$

$$K_i^2(\varepsilon_1,\varepsilon_2) = \int_{t_i^*-\varepsilon_i^-(\varepsilon_2)}^{t_i^*+\varepsilon_i^+(\varepsilon_2)} c_r(t) \int_{z(t)}^{L} \Big(b_s^{\varepsilon_1} v^{\varepsilon_1} - b_s^{\varepsilon_2} v^{\varepsilon_2}\Big)(x,t)\,dx\,dt.$$

If $z(t) < L$, then $[z(t),L]\times\{t\} \subset \overline{\Pi_0^T}$. Therefore $|K_i^2(\varepsilon_1,\varepsilon_2)|$ can be estimated in the same way as $|K_1(\varepsilon_1,\varepsilon_2)|$ was estimated in the proof of Claim 10 for Case 1. The minor change is that now E will denote the set of pairs of indices $q \le k$ and $d \le m$ such that $\omega(0;x_q,\tilde{t}_i) = x_d^*$. It remains to estimate $|K_i^1(\varepsilon_1,\varepsilon_2)|$. Applying (21) and (22) restricted to $\overline{\Pi_1^T}$ and changing coordinates $(x,t) \to (x,\xi) = (x,\theta(x,t))$,we obtain

$$K_i^1(\varepsilon_1,\varepsilon_2) = \sum_{(q,d)\in J} \int_{t_d^*-\varepsilon_d^-(\varepsilon_2)}^{x_d+\varepsilon_d^+(\varepsilon_2)} \int\limits_{x_q-\varepsilon_2}^{x_q+\varepsilon_2} Q_1(x,t)\Big[(b_s^{\varepsilon_1} - b_s^{\varepsilon_2})(x) v^{\varepsilon_1}(0,t)$$
$$- b_s^{\varepsilon_2}(x)(v^{\varepsilon_1} - v^{\varepsilon_2})(0,t)\Big]\,dx\,dt,$$

where

$$Q_1(x,\xi) = \frac{(c_r S)(x,\tau)}{(\partial_t\theta)(x,\tau)}\Big|_{\tau=\tilde{\omega}(x;0,t)}$$

and J is the set of pairs of indices $q \le k$ and $d \le i-1$ such that $\omega(x_q;0,t_d^*) = \tilde{t}_i$. Obviously, at least one of the sets E or J is nonempty. To estimate $|K_i^1(\varepsilon_1,\varepsilon_2)|$, in addition to the arguments used for estimation of $|K_2(\varepsilon_1,\varepsilon_2)|$ we apply Assertions 3 and 4 for $j < i$.

`Case 2:` $(0,t_j) \in I_+(i)$ for some $j \le l$. We use (97) and (98) with t_1 replaced by t_j. To estimate the first and the third summands in the analog of (98), we apply (8), (103), and (106). To estimate the second summand, we use the representation

$$S_6^{\varepsilon_1}(t) - S_6^{\varepsilon_1}(t_j) = \int_{\omega(t;0,t_j-\varepsilon_1)}^{L} \Big(b_s^{\varepsilon_1} + b_r\Big)(x)\Big(w^{\varepsilon_1}(x,t) - w^{\varepsilon_1}(x,t_j)\Big)\,dx$$
$$+ \int_{\omega(t_j;0,t_j-\varepsilon_1)}^{\omega(t;0,t_j-\varepsilon_1)} b_r(x) w^{\varepsilon_1}(x,t_j)\,dx$$

$$
\begin{aligned}
&+\left[\int_{z(t)}^{L}(b_rS)(x,t)a_s^{\varepsilon_1}(\omega(0;x,t))\,dx\right.\\
&\left.-\int_{z(t_j)}^{L}(b_rS)(x,t_j)a_s^{\varepsilon_1}(\omega(0;x,t_j))\,dx\right]\\
&+\left[\int_{\omega(t;0,t_j-\varepsilon_1)}^{z(t)}(b_rS)(x,t)v^{\varepsilon_1}(0,\theta(x,t))\,dx\right.\\
&\left.-\int_{\omega(t_j;0,t_j-\varepsilon_1)}^{z(t_j)}(b_rS)(x,t_j)v^{\varepsilon_1}(0,\theta(x,t_j))\,dx\right].
\end{aligned}
\tag{107}
$$

The absolute value of the first three summands are estimated similarly to estimation of $|S_6^{\varepsilon_1}(t)-S_6^{\varepsilon_1}(t_1)|$ in the proof of Claim 10 (see (100)). In addition to the arguments used for (100), we apply Assertion 2 for $j<i$ and Claims 11, 16, and 17. We now concentrate on the last summand. Let us rewrite it in the form

$$
\begin{aligned}
&\int_{\theta(z(t),t)}^{t_j-\varepsilon_1}\left[\frac{(b_rS)(x,t)}{(\partial_x\theta)(x,t)}\bigg|_{x=\omega(t;0,\tau)}-\frac{(b_rS)(x,t_j)}{(\partial_x\theta)(x,t_j)}\bigg|_{x=\omega(t_j;0,\tau)}\right]v^{\varepsilon_1}(0,\tau)\,d\tau\\
&+\int_{\theta(z(t_j),t_j)}^{\theta(z(t),t)}\frac{(b_rS)(x,t)}{(\partial_x\theta)(x,t)}\bigg|_{x=\omega(t;0,\tau)}v^{\varepsilon_1}(0,\tau)\,d\tau.
\end{aligned}
$$

The absolute value of this expression is less than or equal to

$$
\begin{aligned}
&\left[\max_{(x,t)\in[0,L]\times[t_j-\varepsilon_1,t_j+\varepsilon_1]}\left|\frac{(b_rS)(x,t)}{(\partial_x\theta)(x,t)}\bigg|_{x=\omega(t;0,\tau)}-\frac{(b_rS)(x,t_j)}{(\partial_x\theta)(x,t_j)}\bigg|_{x=\omega(t_j;0,\tau)}\right|\right.\\
&\left.+\max_{x\in[\omega(t_j-\varepsilon_1;z(t_j+\varepsilon_1),t_j+\varepsilon_1),z(t_j-\varepsilon_1)]}|b_r(x)|\max_{(x,t)\in\overline{\Pi^T}}|S(x,t)|\right]\\
&\times\sum_{r=1}^{i-1}\int_{t_r^*-\varepsilon_r^-(\varepsilon_1)}^{t_r^*+\varepsilon_r^+(\varepsilon_1)}|v^{\varepsilon_1}(0,t)|\,dt\\
&\le C\varepsilon_2.
\end{aligned}
$$

The latter bound is true by Assertion 3 for $j<i$ and Assumption 3. ∎

We conclude that Assertion 4 holds for $j=i$. Thus the induction step is done and the proof of Lemma 3 is complete.

7. Proof of Lemma 4

Given an arbitrary test function $\varphi\in\mathcal{D}(\Pi)$, we have to show the convergence of $\langle v^\varepsilon(x,t),\varphi(x,t)\rangle$ as $\varepsilon\to 0$. Fix $T=T(\varphi)>0$ such that $\operatorname{supp}\varphi\subset\Pi^T$. Let

$$
v^\varepsilon(x,t)=\sum_{i=0}^{\infty}v_i^\varepsilon(x,t), \tag{108}
$$

where

$$\operatorname{supp} v_0^\varepsilon(x,t) = \operatorname{supp}\Big\{ v^\varepsilon(x,t)\Big|_{\overline{\Pi_0 \cap I_+^\varepsilon}} \Big\}, \quad \Pi_0 = \big\{(x,t) \in \Pi \,|\, x > \omega(t;0,0)\big\}$$

$$\operatorname{supp}\Big\{ v_i^\varepsilon(x,t)\Big|_{\overline{\Pi^T}} \Big\} = \operatorname{supp}\Big\{ v^\varepsilon(x,t)\Big|_{\overline{I_+^\varepsilon(i)}} \Big\}, \qquad i \le \rho(T),$$

and $\rho(T)$ is as in Definition 3. Clearly, $v_i^\varepsilon(x,t)$ for $i \ge 1$ is supported on one of the connected components of $(\overline{\Pi} \setminus \Pi_0) \cap \overline{I_+^\varepsilon}$. The representation (108) is true because $\operatorname{supp} v^\varepsilon \subset \overline{I_+^\varepsilon}$ by (31). Since

$$\langle v^\varepsilon(x,t), \varphi(x,t)\rangle = \Big\langle \sum_{i=0}^{\rho(T)} v_i^\varepsilon(x,t), \varphi(x,t) \Big\rangle,$$

it suffices to prove that $\langle v_i^\varepsilon(x,t), \varphi(x,t)\rangle$ converges as $\varepsilon \to 0$ separately for each $0 \le i \le \rho(T)$.

Claim 1. *$\langle v_0^\varepsilon(x,t), \varphi(x,t)\rangle$ converges as $\varepsilon \to 0$.*

Proof. By (21) and (22),

$$v_0^\varepsilon(x,t) = S(x,t) a_s^\varepsilon(\omega(0;x,t)). \tag{109}$$

Let us compute the action

$$\begin{aligned} \langle v_0^\varepsilon(x,t), \varphi(x,t)\rangle &= \int_{\Pi_0^T} S(x,t) a_s^\varepsilon(\omega(0;x,t))\varphi(x,t)\, d(x,t) \\ &= \int_{\tilde\Pi_0^T} \frac{S(\xi,t)\varphi(\xi,t)}{(\partial_x \omega)(0;\xi,t)}\Big|_{\xi=\omega(t;x,0)} a_s^\varepsilon(x)\, d(x,t), \end{aligned}$$

where $\tilde\Pi_0^T = \Big\{(x,t) \in \mathbb{R}^2 \,|\, (\omega(t;x,0),t) \in \Pi_0^T\Big\}$. It follows easily that

$$\langle v_0^\varepsilon(x,t), \varphi(x,t)\rangle \ \xrightarrow{\varepsilon \to 0}\ \sum_{i=1}^m \int_{\tilde\Pi_0^T \cap \{(x,t)\,|\, x = x_i^*\}} \frac{S(\xi,t)\varphi(\xi,t)}{\omega_\xi(0;\xi,t)}\Big|_{\xi=\omega(t;x_i^*,0)} dt,$$

Here we used a simple change of coordinates $F_0 : (x,t) \to (\omega(0;x,t),t)$, where F_0 maps Π_0^T to $\tilde\Pi_0^T$. Since $\tilde\Pi_0^T$ is a bounded domain, the claim is proved. ∎

Claim 2. *$\langle v_1^\varepsilon(x,t), \varphi(x,t)\rangle$ converges as $\varepsilon \to 0$.*

Proof. Two cases are possible.

`Case 1:` $(0,t_1) \in I_+(1)$. Let

$$S_8^\varepsilon(t) = \int_0^L [(b_s^\varepsilon + b_r) w^\varepsilon + b_r v^\varepsilon]\, dx.$$

From (13) we conclude that $c_r(t)S_8^\varepsilon(t) = w^\varepsilon(0,t)$. By Lemma 3, the family of functions $w^\varepsilon(0,t)$ is uniformly convergent on $[0,T]$. Since $c_r(t)$ is an arbitrary continuous function, the same assertion is true for the family of functions $S_8^\varepsilon(t)$. Hence there exists a continuous function $S_8^0(t)$ such that

$$\lim_{\varepsilon\to 0} S_8^\varepsilon(t) = S_8^0(t) \quad \text{in } \mathrm{C}[0,T].$$

Therefore, if $|t-t_1| \le C\varepsilon_2$, we have

$$\left|S_8^\varepsilon(t) - S_8^0(t_1)\right| \le \left|\left(S_8^\varepsilon - S_8^0\right)(t)\right| + \left|S_8^0(t) - S_8^0(t_1)\right| \le C\varepsilon_2.$$

Note that $S_8^\varepsilon(t) = S_6^\varepsilon(t) + S_7^\varepsilon(t)$ whenever $t \in [t_1-\varepsilon, t_1+\varepsilon]$. Using the representation (88), we conclude that

$$\begin{aligned}
\langle v_1^\varepsilon(x,t), \varphi(x,t)\rangle &= \int_{I_+^{\varepsilon_0}(1)} c_s^\varepsilon(\theta(x,t))S_8^\varepsilon(\theta(x,t))S(x,t)\varphi(x,t)\,d(x,t)\\
&= \int_{\tilde I_+^{\varepsilon_0}(1)} c_s^\varepsilon(t)S_8^\varepsilon(t)\left.\frac{S(x,\tau)\varphi(x,\tau)}{(\partial_t\tilde\omega)(0;x,\tau)}\right|_{\tau=\tilde\omega(x;0,t)} d(x,t)\\
&\overset{\varepsilon\to 0}{\longrightarrow}\; S_8^0(t_1)\int_{\tilde I_+^{\varepsilon_0}(1)\cap\{(x,t)\,|\,t=t_1\}} \left.\frac{S(x,\tau)\varphi(x,\tau)}{(\partial_t\tilde\omega)(0;x,\tau)}\right|_{\tau=\tilde\omega(x;0,t_1)} dx.
\end{aligned}$$

Here

$$\tilde I_+^{\varepsilon_0}(1) = \left\{(x,t)\in\mathbb{R}^2 \,|\, (x,\tilde\omega(x;0,t)) \in I_+^{\varepsilon_0}(1)\right\}$$

is a bounded domain. This implies that the latter integral is finite, and therefore the claim in this case is true.

`Case 2:` $(0,t_1) \notin I_+(1)$. Using the representation of v_1^ε given by (104), consider the action

$$\begin{aligned}
&\langle v_1^\varepsilon(x,t), \varphi(x,t)\rangle\\
&\quad= \Big\langle c_r(\theta(x,t)) \int_{\omega(\theta(x,t);0,0)}^L b_s^\varepsilon(\xi) a_s^\varepsilon\big(\omega\big(0;\xi,\theta(x,t)\big)\big)\\
&\qquad \times S(\xi,\theta(x,t))\,d\xi\, S(x,t), \varphi(x,t)\Big\rangle\\
&\quad= \int_{I_+^{\varepsilon_0}(1)} \int_{\omega(\theta(x,t);0,0)}^L c_r\big(\theta(x,t)\big) b_s^\varepsilon(\xi) a_s^\varepsilon\big(\omega\big(0;\xi,\theta(x,t)\big)\big)\\
&\qquad \times S(\xi,\theta(x,t))S(x,t)\varphi(x,t)\,d\xi\,d(x,t).
\end{aligned}$$

Changing coordinates $(\xi,x,t) \to (\xi,x,\omega(0;\xi,\theta(x,t)))$, we convert the latter expression into

$$\int_\Omega b_s^\varepsilon(\xi)a_s^\varepsilon(t)\Big[c_r(\theta(x,\tau))S(\xi,\theta(x,\tau))S(x,\tau)\varphi(x,\tau)$$
$$\times\Big[(\partial_t\omega)(0;\xi,\theta(x,\tau))(\partial_t\theta)(x,\tau)\Big]^{-1}\Big]\Big|_{\tau=\tilde\omega(x;0,\tilde\omega(\xi;t,0))} d(\xi,x,t),$$

where

$$\Omega=\Big\{(\xi,x,t)\,|\,t=\omega(0;\xi,\theta(x,\tau)),(x,\tau)\in I_+^{\varepsilon_0}(1),\omega(\theta(x,\tau);0,0)<\xi<L\Big\}$$

is a bounded domain. Let us denote by $J(i)$ the set of those pairs (j,r) that $j\le k$, $r\le m$, and $\omega(0;x_j,t_i^*)=x_r^*$. It is now clear that, as $\varepsilon\to 0$, $\langle v_1^\varepsilon(x,t),\varphi(x,t)\rangle$ converges to

$$c_r(t_1^*)\sum_{(j,r)\in J(1)}\frac{S(x_j,t_1^*)}{(\partial_t\omega)(0;x_j,t_1^*)}\int_{\Omega\cap\{(\xi,x,t)\,|\,\xi=x_j,t=x_r^*\}}\frac{S(x,t)\varphi(x,t)}{(\partial_t\theta)(x,t)}\Big|_{t=\tilde\omega(x;0,t_1^*)}dx.$$

Since the sum is finite, the claim follows. ∎

Claim 3. *$\lim_{\varepsilon\to 0}v_1^\varepsilon(0,t)=C_1\delta(t-t_1^*)$ in $\mathcal{D}'(\mathbb{R}_+)$, where C_1 is a real constant.*

Proof. Take a test function $\psi(t)\in\mathcal{D}(\mathbb{R}_+)$ and compute the action $\langle v_1^\varepsilon(0,t),\psi(t)\rangle$. Similarly to the proof of Claim 2, we obtain that

$$(0,t_1)\in I_+(1):\ v_1^\varepsilon(0,t)\xrightarrow{\varepsilon\to 0}S_8^0(t_1)S(0,t_1)\delta(t-t_1)$$
$$(0,t_1)\notin I_+(1):\ v_1^\varepsilon(0,t)\xrightarrow{\varepsilon\to 0}c_r(t_1^*)\sum_{\substack{\{j\le k:\\(j,r)\in J(1),r\le m\}}}\frac{S(x_j,t_1^*)}{(\partial_t\omega)(0;x_j,t_1^*)}S(0,t_1^*)\delta(t-t_1^*),$$

which proves the claim. ∎

Claim 4. *$\langle v_j^\varepsilon(x,t),\varphi(x,t)\rangle$ for $1\le j\le\rho(T)$ converges as $\varepsilon\to 0$.*

Proof. We prove the claim, using induction on j. The base case of $j=1$ is given by Claims 2 and 3. We make the following assumptions for $j\le i-1$.

`Assumption 1.` $\langle v_j^\varepsilon(x,t),\varphi(x,t)\rangle$ converges as $\varepsilon\to 0$.

`Assumption 2.` $\lim_{\varepsilon\to 0}v_j^\varepsilon(0,t)=C_j\delta(t-t_j^*)$ in $\mathcal{D}'(\mathbb{R}_+)$, where C_j is a real constant.

Prove the claim for $j=i$. Similarly to Claim 2, we distinguish two cases.

`Case 1:` $(0,t_j)\in I_+(i)$ for some $j\le l$. The claim follows similarly to the proof of Claim 2 for Case 1.

`Case 2:` $(0,t_j) \notin I_+(i)$, for $j \le l$. We use (104), (21), and (22), and represent v_i^ε in the form

$$v_i^\varepsilon(x,t) = \left[c_r(\tau) \int_{\omega(\tau;0,t_{i-1}^*+\varepsilon_{i-1}^+(\varepsilon_0))}^{z(\tau)} b_s^\varepsilon(\xi) v^\varepsilon(0,\theta(\xi,\tau)) S(\xi,\tau)\,d\xi \right]\Big|_{\tau=\theta(x,t)} S(x,t)$$
$$+ \left[c_r(\tau) \int_{z(\tau)}^{L} \left(b_s^\varepsilon v_0^\varepsilon \right)(\xi,\tau)\,d\xi \right]\Big|_{\tau=\theta(x,t)} S(x,t),$$

where $z(t)$ is defined by (37). The convergence of the second summand follows from Claim 1. We now prove the convergence of the first summand. Consider the action

$$\Big\langle c_r\big(\theta(x,t)\big) \int_{\omega(\theta(x,t);0,t_{i-1}^*+\varepsilon_{i-1}^+(\varepsilon))}^{z(\theta(x,t))} b_s^\varepsilon(\xi) v^\varepsilon\big(0,\theta(\xi,\theta(x,t))\big) S\big(\xi,\theta(x,t)\big)\,d\xi$$
$$\times S(x,t), \varphi(x,t) \Big\rangle$$
$$= \int_{I_+^{\varepsilon_0}(i)} \int_{\omega(\theta(x,t);0,t_{i-1}^*+\varepsilon_{i-1}^+(\varepsilon_0))}^{z(\theta(x,t))} b_s^\varepsilon(\xi) \Big[c_r(\tau) v^\varepsilon\big(0,\theta(\xi,\tau)\big) S(\xi,\tau) \Big]\Big|_{\tau=\theta(x,t)}$$
$$\times S(x,t)\varphi(x,t)\,d\xi\,d(x,t)$$
$$= \sum_{(j,q)\in N(i)} \int_{x_j^*-\varepsilon_0}^{x_j^*+\varepsilon_0} \int_{t_q^*-\varepsilon_q^-(\varepsilon_0)}^{t_q^*+\varepsilon_q^+(\varepsilon_0)} \int_0^{\zeta(\xi,t)} \Big[D(\xi,x,t)\varphi\big(x,\tilde\omega(x;0,\tilde\omega(\xi;0,t))\big)$$
$$- \chi_{[x_j-\varepsilon,x_j+\varepsilon]\times[t_q^*-\varepsilon_q^-(\varepsilon),t_q^*+\varepsilon_q^+(\varepsilon)]}(\xi,t) D(x_j,x,t_q^*)\varphi\big(x,\tilde\omega(x;0,t_i^*)\big) \Big]$$
$$\times b_s^\varepsilon(\xi) v_q^\varepsilon(0,t)\,dx\,dt\,d\xi$$
$$+ \sum_{(j,q)\in N(i)} \int_{x_j^*-\varepsilon_0}^{x_j^*+\varepsilon_0} b_s^\varepsilon(\xi)\,d\xi \int_{t_q^*-\varepsilon_q^-(\varepsilon_0)}^{t_q^*+\varepsilon_q^+(\varepsilon_0)} v_q^\varepsilon(0,t)\,dt$$
$$\times \int_0^{\zeta(\xi,t)} D(x_j,x,t_q^*)\varphi\big(x,\tilde\omega(x;0,t_i^*)\big)\,dx,$$

where $N(i)$ is the set of pairs (j,q) such that $\omega(x_j;0,t_q^*) = t_i^*$,

$$\zeta(\xi,t) = \begin{cases} L & \text{if } \theta(L,T) \ge \tilde\omega(\xi;0,t) \\ \omega(T;0,\tilde\omega(\xi;0,t)) & \text{if } \theta(L,T) < \tilde\omega(\xi;0,t) \end{cases} \tag{110}$$

and

$$D(\xi, x, t) = \Big[c_r(\theta(x,\tau)) S(\xi, \theta(x,\tau)) S(x,\tau) \\ \times \Big[(\partial_t\theta)(\xi, \theta(x,\tau))(\partial_t\theta)(x,\tau) \Big]^{-1} \Big] \Big|_{\tau=\tilde{\omega}(x;0,\tilde{\omega}(\xi;0,t))}.$$

It is immediate now that

$$\langle v_i^\varepsilon(x,t), \varphi(x,t)\rangle \stackrel{\varepsilon\to 0}{\longrightarrow} \sum_{(j,q)\in N(i)} C_q \int_0^{\zeta(x_j,t_i^*)} D(x_j, x, t_q^*) \varphi(x, \tilde{\omega}(x;0,t_i^*))\, dx.$$

This convergence is true due to the second induction assumption, the condition (10), the continuity of c_r, p, λ, and φ, and the uniform in ε boundedness of $\int_0^T |v^\varepsilon(0,t)|\, dt$, that is proved in Lemma 3.

Take a test function $\psi(t) \in \mathcal{D}(\mathbb{R}_+)$ and consider the action $\langle v_i^\varepsilon(0,t),\, \psi(t)\rangle$. Computation similar to the above shows that

$$v_i^\varepsilon(0,t) \stackrel{\varepsilon\to 0}{\longrightarrow} c_r(t_i^*) \sum_{(j,r)\in J(i)} \frac{S(x_j, t_i^*)}{(\partial_t\omega)(0;x_j,t_i^*)} S(0,t_i^*)\delta(t-t_i^*) \\ + c_r(t_i^*) \sum_{(j,q)\in N(i)} C_q \frac{S(x_j,t_i^*)}{(\partial_t\omega)(x_j;0,t_i^*)} S(0,t_i^*)\delta(t-t_i^*),$$

thereby proving the required convergence. ∎

Lemma 4 is proved.

8. Proof of Lemma 5

To prove the lemma, it suffices to show that every point of $\Pi \setminus I_+$ has a neighborhood on which $u = 0$. As soon as this is done, the lemma will follow from the standard argument of the elementary distribution theory (see [12, 2.2.1]).

Let us fix an arbitrary $(x,t) \in \Pi \setminus I_+$ and show that there exists a neighborhood $Y \subset \Pi \setminus I_+$ of (x,t) such that the restriction of v to Y is equal to 0. We choose Y such that $\partial Y \cap I_+ = \emptyset$. This is possible because $\Pi \setminus I_+$ is open. We now prove that $v = 0$ on Y. By the definition of convergence in $\mathcal{D}'$, if $v = \lim_{\varepsilon\to 0} v^\varepsilon$ in $\mathcal{D}'(\Pi)$, then $v = \lim_{\varepsilon\to 0} v^\varepsilon$ in $\mathcal{D}'(Y)$. On the account of Lemma 4, it suffices to prove the convergence of v^ε to 0 in $\mathcal{D}'(Y)$. For this purpose we use Lebesgue's dominated convergence theorem (see [9, 1.5.1]) which extends in an obvious way to the families of functions $(f_\varepsilon)_{\varepsilon>0} \in L^1_{loc}$.

The function v^ε is in $L^1_{loc}(Y)$ by Proposition 1. By Lemma 1, v^ε converges to 0 pointwise on Y as $\varepsilon \to 0$. By the conditions imposed on λ, each component of $\Pi \setminus I_+$ is bounded. Since Y is included in one of the components, Y is bounded. Clearly, there exists $T > 0$ such that $Y \subset \Pi^T$. Since $\bigcap_{\varepsilon>0} I_+^\varepsilon = I_+$, $\partial Y \cap I_+ = \emptyset$, and $\operatorname{supp} v^\varepsilon \subset \overline{I_+^\varepsilon}$ (see (31)), there exists $\tilde{\varepsilon}$ such that $v^\varepsilon = 0$ on Y for all $\varepsilon \geq \tilde{\varepsilon}$. By (105), that was proved in Section 6, it follows that $v^\varepsilon = O\big(\frac{1}{\varepsilon}\big)$ on Π^T. This implies the uniform in ε estimate

$$|v^\varepsilon| \leq \frac{C}{\tilde{\varepsilon}}$$

on Y. Premises of the Lebesgue's dominated convergence theorem hold for v^ε on Y. Therefore $v^\varepsilon \to 0$ in $\mathcal{D}'(Y)$ and $v = 0$ on Y.

Since $(x,t) \in \Pi \setminus I_+$ is arbitrary, the lemma is true.

Remark. From the proofs of Lemmas 4 and 5 it follows that v is indeed a measure concentrated on I_+.

From the construction of I_+ (see Definition 2) it follows that in general the density of singularity curves increases as time progresses.

Acknowledgement. I am thankful to the members of the DIANA group for their hospitality during my stay at the Vienna university.

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Received 14.08.2003; in revised form 28.11.2004