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On a Class of Physically Admissible Variational Solutions to the Navier-Stokes-Fourier System

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Abstract. The main objective of the present paper is to introduce a class of admissible variational solutions to the Navier-Stokes-Fourier system of equations based on the second law of thermodynamics. We also show that the solutions exist globally in time regardless the size of the initial data. Finally, the question of the long-time behaviour of these solutions is being addressed.

Keywords: *Navier-Stokes-Fourier system, variational solutions, existence, long-time behaviour*

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1. Introduction

The term "variational solutions" in the title refers to a class of integrable functions satisfying a given system of partial differential equations in the sense of distributions. The classical example of hyperbolic conservation laws shows that these solutions may not be uniquely determined by the data; in other words, the corresponding mathematical problem is not, or at least not known to be, well-posed in the class of variational solutions. A common point of view is to interpret the distributional solutions as quantities arising as limits of sequences of (regular) functions solving a family of suitable, physically grounded, approximation problems. In practice, however, it is extremely difficult or rather impossible to determine the approximation procedure a given variational solution results from. Instead one hopes to identify the "ghost effect" of the approximation

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process expressed in terms of one or several intrinsic *admissibility criteria* to be imposed in order to choose the physically relevant solution. The main objective of the present paper is to introduce a class of admissible variational solutions to the Navier-Stokes-Fourier system based on the second law of thermodynamics, more specifically, on the "principal" of maximal mechanical energy dissipation.

A class of weak (variational) solutions to the Navier-Stokes-Fourier system has been constructed in the monograph [9]. However, it is not known, whether within this class, there are solutions satisfying energy identity (21) rather than inequality (11), and it is not immediately clear, whether they obey to the entropy production inequality of type (22). It appears, that both lastly mentioned properties are very important when one investigates large time behavior of these solutions. Indeed, in such a case, and provided the external force is potential, weak solutions converge to a rest state which is uniquely determined solely by the initial mass and initial total energy, see [11].

The main goal of the present paper is to show that such admissible variational solutions do exist. We shall reach this goal by using a special construction which takes advantage of regularization effects due to radiation observed in [4].

1.1. The Navier-Stokes-Fourier system. In accordance with the basic principles of classical continuum mechanics, the state of a fluid at each instant of time is fully characterized through the value of three macroscopic quantities: the *mass density* $\varrho = \varrho(t,x)$, the *velocity* $\mathbf{u} = \mathbf{u}(t,x)$, and the *absolute temperature* $\vartheta = \vartheta(t,x)$.

The time evolution of these quantities is governed by a system of conservation laws that are assumed to be obeyed by all fluids. Specifically, adopting the Eulerian reference system we have

$$\partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0 \tag{1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}\ \mathbb{S} + \varrho \mathbf{f} \tag{2}$$

$$\partial_t(\varrho e) + \mathrm{div}(\varrho e \mathbf{u}) + \mathrm{div}\ \mathbf{q} = \Phi - p\ \mathrm{div} \mathbf{u}, \tag{3}$$

where the *pressure* p, the *viscous stress tensor* $\mathbb{S}$, the *internal energy* e, the *heat flux* $\mathbf{q}$ as well as the *dissipation function* Φ are to be determined in terms of the state variables through a system of *constitutive relations*.

Here we have used *Stokes' law*

$$\mathbb{T} = \mathbb{S} - p\ \mathbb{I}$$

for the *Cauchy stress* $\mathbb{T}$ characterizing *fluids* among all materials in nature. Moreover, the heat flux $\mathbf{q}$ will by given by *Fourier's law*

$$\mathbf{q} = -\kappa \nabla_x \vartheta,$$

with a positive *heat conductivity coefficient* κ which may depend on other state variables as the case may be.

The constitutive relations for the pressure $p = p(\varrho, \vartheta)$ and the internal energy $e = e(\varrho, \vartheta)$ are interrelated through *Maxwell's equation*

$$\frac{\partial e}{\partial \varrho} = \frac{1}{\varrho^2}\Big(p - \vartheta \frac{\partial p}{\partial \vartheta}\Big)$$

yielding the possibility to write the pressure p as a sum

$$p = p_e + p_{th}, \quad p_e = \varrho^2 \frac{\partial e}{\partial \varrho}, \quad p_{th} = \vartheta \frac{\partial p}{\partial \vartheta},$$

with the *elastic pressure* p_e and the *thermal pressure* p_{th}.

Throughout the whole text, we shall assume the pressure p to be an affine function of the temperature ϑ. Accordingly, we have

$$p_e = p_e(\varrho), \quad p_{th} = \vartheta p_\vartheta(\varrho)$$

while the internal energy e reads

$$e = \varrho P_e(\varrho) + Q(\vartheta), \quad \text{with } P_e = \int_1^\varrho \frac{p_e(z)}{z^2} \, \mathrm{d}z.$$

Finally, we suppose

$$Q'(\vartheta) = c_v > 0$$

- the *specific heat at constant volume* - to be a positive constant. Accordingly, equation (3) can be rewritten as

$$\partial_t(c_v \varrho\vartheta) + \operatorname{div}(c_v \varrho\vartheta) - \operatorname{div}(\kappa \nabla_x \vartheta) = \Phi - \vartheta p_\vartheta \operatorname{div}\mathbf{u}, \tag{4}$$

where we have used the "renormalized equation"

$$\partial_t(\varrho P_e(\varrho)) + \operatorname{div}(\varrho P_e(\varrho)\mathbf{u}) + p_e(\varrho) \operatorname{div}\mathbf{u} = 0$$

which can be obtained via a straightforward manipulation of (1).

The system of equations (1), (2), and (4) defined on a spatial domain $\Omega \subset R^N$, $N = 2, 3$, will be supplemented with *conservative boundary conditions*

$$\mathbf{u}|_{\partial\Omega} = 0, \ \nabla_x \vartheta \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{5}$$

where the symbol $\mathbf{n}$ stands for the outer normal vector.

1.2. Mechanical energy dissipation. If the motion is smooth, one can multiply equation (2) on $\mathbf{u}$ to deduce the kinetic energy balance

$$\partial_t\Big(\frac{1}{2}\varrho|\mathbf{u}|^2\Big) + \mathrm{div}\Big(\frac{1}{2}\varrho|\mathbf{u}|^2\mathbf{u}\Big) + \mathrm{div}(p\mathbf{u}) = \mathrm{div}(\mathbb{S}\ \mathbf{u}) + p\ \mathrm{div}\mathbf{u} - \mathbb{S} : \nabla_x\mathbf{u} + \varrho\mathbf{f}\cdot\mathbf{u}.$$

On the other hand, as the *total energy*

$$E = \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho P_e(\varrho) + c_v\varrho\vartheta$$

is a conserved quantity, the dissipation function Φ introduced in (3) satisfies

$$\Phi = \mathbb{S} : \nabla_x\mathbf{u},$$

or, as the viscous stress is symmetric,

$$\Phi = \mathbb{S} : \mathbb{D}_x\mathbf{u},$$

with the symmteric velocity gradient

$$\mathbb{D}_x\mathbf{u} = \frac{1}{2}\Big(\nabla_x\mathbf{u} + {}^t\nabla_x\mathbf{u}\Big).$$

Now, the "standard" approach is to postulate a constitutive equation

$$\mathbb{S} = \mathcal{S}(\varrho, \vartheta, \mathbb{D}_x\mathbf{u}),$$

where the second law of thermodynamics is used to impose the structural condition

$$\mathcal{S}(\varrho, \vartheta, \mathbb{D}) : \mathbb{D} \geq 0$$

to be satisfied for any choice of the arguments ϱ, ϑ, and $\mathbb{D} \in R^{N\times N}_{sym}$. Under the additional hypothesis that $\mathbb{S}$ is a *linear* function of $\mathbb{D}_x\mathbf{u}$, this leads to the well-known constitutive equations characterizing the *Newtonian fluids*:

$$\mathbb{S} = \mu\Big(\nabla_x\mathbf{u} + {}^t\nabla_x\mathbf{u} - \frac{2}{N}\mathrm{div}\mathbf{u}\ \mathbb{I}\Big) + \xi\ \mathrm{div}\mathbf{u}\ \mathbb{I} \tag{6}$$

with the *shear viscosity coefficient* $\mu = \mu(\varrho, \vartheta) \geq 0$ and the *bulk viscosity coefficient* $\xi = \xi(\varrho, \vartheta) \geq 0$.

Motivated by the idea of Rajagopal and Srinivasa [20], we adopt an alternative approach consisting in a judicious choice of Φ as a prescribed function of both $\mathbb{S}$ and $\mathbb{D}_x\mathbf{u}$ as well as of the remaining state variables as the case may be so that Φ is always non-negative. The constitutive equation relating $\mathbb{S}$ to $\mathbb{D}_x\mathbf{u}$ will be determined *implicitly* through the satisfaction of

$$\Phi(\mathbb{S}, \mathbb{D}_x\mathbf{u}) = \mathbb{S} : \mathbb{D}_x\mathbf{u}.$$

In order to be more specific assume, for simplicity, that Φ depends solely on $\mathbb{S}$ and $\mathbb{D}_x\mathbf{u}$. We shall say that

$$\Phi = \Phi(\mathbb{S}, \mathbb{D}) : R^{N\times N}_{sym} \times R^{N\times N}_{sym} \to [0, \infty]$$

is a *dissipation function* if

- Φ is lower semi-continuous on $R^{N\times N}_{sym} \times R^{N\times N}_{sym}$
- Φ is non-negative, $\Phi(0,0) = 0$
- for all $\mathbb{S}, \mathbb{D} \in R^{N\times N}_{sym}$:
$$\Phi(\mathbb{S}, \mathbb{D}) \geq \mathbb{S} : \mathbb{D} \tag{7}$$
- for any $\mathbb{D} \in R^{N\times N}_{sym}$ there exists a unique $\mathbb{S} = \mathcal{S}(\mathbb{D}) \in R^{N\times N}_{sym}$ solving
$$\Phi\Big(\mathcal{S}(\mathbb{D}), \mathbb{D}\Big) = \mathcal{S}(\mathbb{D}) : \mathbb{D}. \tag{8}$$

Consider now quantities ϱ, $\mathbf{u}$, ϑ, p, and $\mathbb{S}$ satisfying (1), (2), and (4) for a given dissipation function $\Phi(\mathbb{S}, \mathbb{D}_x\mathbf{u})$. Furthermore, let $\mathbf{u}$, ϑ comply with the conservative boundary conditions (5). Finally, suppose that the total energy of the system is a conserved quantity (if $\mathbf{f} = 0$), more specifically, we require

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega E(t)\ \mathrm{d}x = \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u}\ \mathrm{d}x. \tag{9}$$

A brief examination of the total energy balance yields

$$\int_\Omega \Phi(\mathbb{S}, \mathbb{D}_x\mathbf{u}) - \mathbb{S} : \mathbb{D}_x\mathbf{u}\ \mathrm{d}x = 0;$$

whence, in accordance with (7),

$$\Phi(\mathbb{S}, \mathbb{D}_x\mathbf{u}) = \mathbb{S} : \mathbb{D}_x\mathbf{u},$$

and, by virtue of (8), we infer that

$$\mathbb{S} = \mathcal{S}(\mathbb{D}_x\mathbf{u}).$$

Note that the same conclusion remains valid if equation (4) is replaced by an inequality

$$\partial_t(c_v\varrho\vartheta) + \mathrm{div}(c_v\varrho\vartheta) - \mathrm{div}(\kappa\nabla_x\vartheta) \geq \Phi - \vartheta p_\vartheta\ \mathrm{div}\mathbf{u}, \tag{10}$$

and if we postulate

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega E(t)\ \mathrm{d}x \leq \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u}\ \mathrm{d}x \tag{11}$$

instead of (9). One of the possible choices of Φ reads

$$\Phi(\mathbb{S},\mathbb{D}) = \Lambda(\mathbb{D})\mathcal{H}\Big(\mathbb{D}\Big) + \Lambda(\mathbb{D})\mathcal{H}^*\Big(\frac{\mathbb{S}}{\Lambda(\mathbb{D})}\Big), \tag{12}$$

where $\Lambda > 0$, $\mathcal{H}$ is a convex continuously differentiable function attaining its minimum at zero, and $\mathcal{H}^*$ its conjugate. Formula (12) gives rise to a constitutive relation

$$\mathbb{S} = \mathcal{S}(\mathbb{D}_x\mathbf{u}) = \Lambda(\mathbb{D}_x\mathbf{u})\partial\mathcal{H}(\mathbb{D}_x\mathbf{u}).$$

In particular, taking

$$\Lambda = 1,\ \mathcal{H}(\mathbb{D}) = \frac{\mu}{2}\ \left|P\ \mathbb{D}\right|^2 + N\xi\ \left|(I-P)\ \mathbb{D}\right|^2,$$

where P denotes the projection on the space of traceless tensors, we arrive at the constitutive equation (6) of linearly viscous fluids with constant viscosity coefficients. In accordance with Stokes' hypothesis it is customary to take the bulk viscosity $\xi = 0$, which necessarily implies

$$\mathcal{H}^*(\mathbb{S}) = \infty \text{ whenever } (I-P)\ \mathbb{S} \neq 0.$$

1.3. Entropy production rate. Equation (4) divided by ϑ gives rise to the entropy balance

$$\partial_t(\varrho s) + \operatorname{div}(\varrho s\mathbf{u}) - \operatorname{div}\Big(\frac{\kappa}{\vartheta}\nabla_x\vartheta\Big) = \frac{\Phi}{\vartheta} + \frac{\kappa}{\vartheta^2}|\nabla_x\vartheta|^2, \tag{13}$$

with the *specific entropy*

$$s = s(\varrho,\vartheta) = c_v\log(\vartheta) - P_\vartheta(\varrho), \quad P_\vartheta(\varrho) = \int_1^\varrho \frac{p_\vartheta(z)}{z^2}\,\mathrm{d}z.$$

Furthermore, corresponding to experimental results (see, for example, [24]), we shall assume $\kappa = \kappa(\vartheta)$ to be a function of the absolute temperature ϑ. In accordance with our previous discussion, we introduce the *entropy production rate*

$$\zeta = \zeta(\vartheta,\nabla_x\vartheta,\mathbb{S},\mathbb{D}_x\mathbf{u}) = \frac{\Phi(\mathbb{S},\mathbb{D}_x\mathbf{u})}{\vartheta} + \frac{\kappa(\vartheta)}{\vartheta^2}|\nabla_x\vartheta|^2;$$

more specifically, we set

$$\zeta = \zeta(\vartheta,\mathbf{d},\mathbb{S},\mathbb{D}) = \begin{cases} \frac{1}{\vartheta}\Phi(\mathbb{S},\mathbb{D}) + \frac{\kappa(\vartheta)}{\vartheta^2}|\mathbf{d}|^2 & \text{if } \vartheta > 0 \\ 0 & \text{if } \vartheta = \Phi(\mathbb{S},\mathbb{D}) = |\mathbf{d}| = 0 \\ \infty & \text{otherwise.} \end{cases} \tag{14}$$

Note that, in accordance with the second law of thermodynamics, ζ is a non-negative function attaining its global minimum at zero.

There is a principal difference between the entropy production rate ζ and its counterpart $\tilde{\zeta}$ introduced by Rajagopal and Srinivasa [20]; namely, ζ depends on both the "flux" $\mathbb{S}$ and the "affinities" $\nabla_x \vartheta$, $\mathbb{D}_x \mathbf{u}$ while $\tilde{\zeta}$ is a function of the "fluxes" $\mathbb{S}$, $\mathbf{q}$ only. With the notation introduced in (12) we would have

$$\tilde{\zeta} = \frac{\mathcal{H}^*(\mathbb{S})}{\vartheta} + \frac{|\mathbf{q}|^2}{\kappa(\vartheta)\vartheta^2}$$

which is "equivalent" to (14), where Φ is given by (12) with a suitably chosen Λ (cf. Section 6 in [20]).

1.4. Non-smooth motions. The solutions of (1), (2), (4) may develop singularities in a finite time provided no effective dissipation mechanism is present. The only way to continue these processes beyond the critical time is to introduce the concept of variational (weak) solutions satisfying the equations only in the sense of distributions. It comes as a striking fact that some of these generalized solutions *do* dissipate mechanical energy having reached the critical time even though there is no explicit dissipation term present in the equations.

From the physical viewpoint, the weak solutions are to be understood as suitable limits of the dissipative processes, where the dissipation due to the presence of viscosity or other mechanism is small or even negligible in the equations of motion (2) but it imposes a "ghost effect" on the thermal energy balance (4). Mathematically speaking, we have to abandon the equality sign in (4) to obtain, similarly to (10),

$$\partial_t(c_v \varrho \vartheta) + \operatorname{div}(c_v \varrho \vartheta \mathbf{u}) - \operatorname{div}(\kappa \nabla_x \vartheta) \geq \Phi - \vartheta p_\vartheta \operatorname{div}\mathbf{u} \tag{15}$$

provided the motion is not (known to be) smooth.

On the other hand, any physically admissible solution, smooth or not, should satisfy the total energy balance (9). It is easy to check that the system (1), (2), (15) *together with* (9) is, in fact, *equivalent* to (1), (2), (4) for smooth solutions satisfying the boundary conditions (5).

The truly dissipative processes are not expected to develop singularities in a finite time. The smoothing effect induced by viscosity in Newtonian fluids is strong enough to prevent the formation of discontinuities at least in the flows which are "close" to steady states (see the pioneering papers by Matsumura and Nishida [15], [16]). If, in addition, the fluid is *incompressible* and $N = 2$, the same result holds for any flow even in a regime close to turbulence. On the other hand, the existence of global in time regular solutions of the Navier-Stokes system describing the motion of a linearly viscous incompressible fluid in $\Omega \subset R^3$ represents one of the most challenging open problems of the theory of partial differential equations.

Singularities are more likely to occur for the hyperbolic-parabolic system (1), (2), (4) even though no explicit examples are known to the best of our knowledge (the solutions constructed by Xin [23] and Vaigant [22] correspond to irregular initial data containing a "vacuum" region and an unbounded driving force $\mathbf{f}$, respectively). Moreover, some of the possible singularities, once they appear, will propagate in time as indicated by the results of Hoff [12], [13]. Very roughly indeed, the principal difficulties when dealing with a compressible fluid may be characterized as possible concentrations or vanishing of the density corresponding to the "gravitational" collapse and the appearance of vacuum, respectively. In both cases, the velocity gradient becomes singular with possible side effects influencing the thermal energy balance (15).

The fact that (15) may hold as a strict inequality or, equivalently, that there may be subtle dissipative mechanisms not related to the presence of the viscous stress $\mathbb{S}$ in (2), is not at odds with common physical intuition. Indeed such a scenario based on the phenomenon of *inertial energy dissipation* has been proposed even for the incompressible viscous fluids in three space dimensions (see [2], [8], and [18] among others). Note in this context that while the presence of singularities in a viscous and incompressible flow requires the velocity to be unbounded (the most recent result in this direction has been obtained by Escauriaza et al. [7]), the singularities in a flow of a compressible fluid may be perfectly "physical".

1.5. Summary of the main results. The arrangement of the paper is as follows. To begin with, we introduce the notion of a (physically) *admissible variational solution* of the system (1), (2), (4) compatible with the physical observations discussed above (see Section 2).

In the next step, we focus on linearly viscous fluids and propose a family of "physically relevant" approximate problems in order to construct the admissible solutions of the initial-boundary value problem for (1), (2), (4) (Section 3). The main idea is to exploit the regularizing effect of *radiation* already discussed in [4].

In Section 4, we show that the sequence of solutions of the approximate problems introduced in Section 3 tends to an admissible variational solution for a fairly general class of initial data. Thus we establish an *existence result* in the class of admissible variational solutions on an arbitrarily large time interval.

Finally, the asymptotic behaviour of solutions for a long time as well as related problems will be discussed in Section 5.

2. Physically admissible variational solutions

2.1. Equation of continuity. The physical principle of *mass conservation* is expressed through equation (1). Motivated by the previous discussion, we require ϱ to be a non-negative function satisfying the integral identity

$$\int_I \int_\Omega [\varrho H(\varrho)\varphi_t + \varrho H(\varrho)\mathbf{u} \cdot \nabla_x \varphi - h(\varrho) \operatorname{div}\mathbf{u}\ \varphi] \,\mathrm{d}x \,\mathrm{d}t = 0 \tag{16}$$

for any test function

$$\varphi \in \mathcal{D}(I \times \overline{\Omega}), \quad I \subset R \text{ an open interval,}$$

and for any

$$h \in BC[0,\infty), \quad h(0) = 0, \quad H(\varrho) = H(1) + \int_1^\varrho \frac{h(z)}{z^2} \,\mathrm{d}z. \tag{17}$$

In particular, ϱ, $\mathbf{u}$ solve the renormalized equation

$$\partial_t(\varrho H(\varrho)) + \operatorname{div}(\varrho H(\varrho)\mathbf{u}) + h(\varrho) \operatorname{div}\mathbf{u} = 0 \quad \text{in } \mathcal{D}'(I \times \Omega),$$

and the total mass

$$M = \int_\Omega \varrho(t) \,\mathrm{d}x$$

is a constant of motion.

Here, and always in what follows, we tacitly assume that all integrals make sense, that means, the quantities $\varrho H(\varrho)$, $\varrho H(\varrho)\mathbf{u}$, and $h(\varrho)$ div$\mathbf{u}$ are at least locally integrable on the set $I \times \overline{\Omega}$. The reader will have noticed that the no-slip boundary conditions imposed on the velocity $\mathbf{u}$ appear implicitly through the choice of the test functions in (16).

2.2. Momentum equation. If the motion is not smooth enough, one can expect the momentum equation (2) to be satisfied only in the sense of distributions, that means, the integral identity

$$\int_I \int_\Omega [\varrho\mathbf{u}\cdot\mathbf{w}_t + [\varrho\mathbf{u}\otimes\mathbf{u}] : \nabla_x\mathbf{w} + p \operatorname{div}\mathbf{w}] \,\mathrm{d}x \,\mathrm{d}t = \int_I \int_\Omega [\mathbb{S} : \nabla_x\mathbf{w} - \varrho\mathbf{f}\cdot\mathbf{w}] \,\mathrm{d}x \,\mathrm{d}t \tag{18}$$

holds for any vector valued test function

$$\mathbf{w} \in \mathcal{D}(I \times \Omega; R^N).$$

2.3. Mechanical energy dissipation. In accordance with the considerations in Section 1, we postulate the thermal energy balance in the form

$$\begin{aligned}\int_I \int_\Omega [c_v\varrho\vartheta\varphi_t + c_v\varrho\vartheta\mathbf{u} \cdot \nabla_x\varphi + \mathcal{K}(\vartheta)\Delta\varphi] \,\mathrm{d}x \,\mathrm{d}t \\ \le \int_I \int_\Omega [\vartheta p_\vartheta \operatorname{div}\mathbf{u}\varphi - \Phi(\mathbb{S}, \mathbb{D}_x\mathbf{u})\varphi] \,\mathrm{d}x \,\mathrm{d}t\end{aligned} \tag{19}$$

to be satisfied for any test function

$$\varphi \in \mathcal{D}(I \times \overline{\Omega}), \quad \varphi \geq 0, \quad \nabla_x \varphi \cdot \mathbf{n} = 0 \ \text{ on } \partial\Omega.$$

Here

$$\mathcal{K}(\vartheta) = \int_0^\vartheta \kappa(z) \, \mathrm{d}z,$$

and Φ is a given *dissipation function* introduced in Section 1.2.

In particular, we have

$$\partial_t (c_v \varrho \vartheta) + \mathrm{div}(c_v \varrho \vartheta \mathbf{u}) - \Delta \mathcal{K}(\vartheta) \geq \Phi(\mathbb{S}, \mathbb{D}_x \mathbf{u}) - \vartheta p_\vartheta \, \mathrm{div} \mathbf{u} \quad \text{in } \mathcal{D}'(I \times \Omega)$$

together with the total thermal energy balance

$$\int_\Omega c_v \varrho \vartheta(t_2) \, \mathrm{d}x - \int_\Omega c_v \varrho \vartheta(t_1) \, \mathrm{d}x \geq \int_{t_1}^{t_2} \int_\Omega [\Phi(\mathbb{S}, \mathbb{D}_x \mathbf{u}) - \vartheta p_\vartheta \, \mathrm{div} \mathbf{u}] \, \mathrm{d}x \, \mathrm{d}t$$

for a.a. $t_1 \leq t_2$, $t_1, t_2 \in I$.

At this stage, we need $\nabla_x \mathbf{u}$ to be at least locally integrable on $I \times \overline{\Omega}$ therefore it makes sense to postulate the no-slip boundary conditions for the velocity in the form

$$\mathbf{u} \in L^1_{loc}(I; W_0^{1,p}(\Omega; R^N)) \quad \text{with a certain } p \geq 1. \tag{20}$$

2.4. Total energy conservation. By virtue of the boundary conditions (5), the total energy E of the system should be conserved even in the class of variational solutions. More specifically, we require the integral identity

$$\int_\Omega E(t_2) \, \mathrm{d}x - \int_\Omega E(t_1) \, \mathrm{d}x = \int_{t_1}^{t_2} \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u} \, \mathrm{d}x \, \mathrm{d}t \tag{21}$$

to hold for a.a. $t_1 \leq t_2$, $t_1, t_2 \in I$.

2.5. Entropy production rate. A rigorous evaluation of the entropy production rate for a non-smooth motion is a difficult problem mainly because of possible appearance of vacuum zones at a finite time. Unlike its counterpart in the thermal energy balance – the dissipation function Φ – the rate of the "real" entropy production for non-smooth processes is not bounded below by the function ζ introduced in (14).

Instead one can assert only a *minimal* (or relaxed) entropy production rate expressed through the integral inequality

$$\begin{aligned} &\int_\Omega \varrho s(t_2) \, \mathrm{d}x - \int_\Omega \varrho s(t_1) \, \mathrm{d}x \\ &\qquad \geq \inf_{\substack{\Theta \in L^2(I; W^{1,2}(\Omega)) \\ \varrho\Theta = \varrho\vartheta}} \int_{t_1}^{t_2} \int_\Omega \zeta^{**}(\Theta, \nabla_x \Theta, \mathbb{S}, \mathbb{D}_x \mathbf{u}) \, \mathrm{d}x \, \mathrm{d}t \end{aligned} \tag{22}$$

for a.a. $t_1 \leq t_2$, $t_1, t_2 \in I$, where $\zeta = \zeta(\Theta, \mathbf{d}, \mathbb{S}, \mathbb{D})$ is the entropy production rate introduced in (14) and ζ^{**} denotes its bi-polar function with respect to all arguments $(\Theta, \mathbf{d}, \mathbb{S}, \mathbb{D}) \in R \times R^N \times R^{N\times N}_{sym} \times R^{N\times N}_{sym}$. We recall, invoking the standard definition, that ζ^{**} is a convex lower semicontinuous function, $0 \leq \zeta^{**} \leq \zeta$,

$$\zeta^{**}(\mathbf{Y}) = \sup\{L(\mathbf{Y}) \mid L \text{ affine},\ L \leq \zeta\}, \quad \mathbf{Y} \in R \times R^N \times R^{N\times N}_{sym} \times R^{N\times N}_{sym}.$$

Note that (22) implicitly includes

$$\vartheta(t,x) > 0 \quad \text{for a.a. } t \in I,\ x \in \Omega. \tag{23}$$

2.6. Admissible variational solutions. Motivated by the previous considerations, we introduce the concept of a physically admissible variational solution to the Navier-Stokes-Fourier system.

Definition 2.1. *We shall say that ϱ, ϑ, $\mathbf{u}$ represent an admissible (variational) solution to the problem* (1), (2), (4), (5) *on an open time interval $I \subset R$ if*

- $\varrho, \vartheta \in L^1_{loc}(I; L^1(\Omega))$, $\mathbf{u} \in L^1_{loc}(I; W^{1,1}_0(\Omega; R^N))$
- $\varrho(t,x) \geq 0$, $\vartheta(t,x) > 0$ *for a.a.* $t \in I$, $x \in \Omega$
- *there exists* $\mathbb{S} \in L^1_{loc}(I; L^1(\Omega; R^{N\times N}_{sym}))$ *such that the integral relations* (16), (18), (19), (21), *and* (22) *hold for any possible choice of test functions in the appropriate class specified above.*

2.7. Remarks and comments.

2.7.1. As it has been made clear in Section 1, the admissible solution comply with the basic stipulation to be satisfied by any generalized solution:

(i) any classical (sufficiently smooth) solution of the problem represents an admissible solution in the sense of Definition 2.1

(ii) any admissible solution being smooth is a classical solution, in particular, the viscous stress tensor $\mathbb{S}$ is uniquely determined in terms of $\mathbb{D}_x\mathbf{u}$ through (8), and equation (4) holds with $\Phi = \mathbb{S} : \mathbb{D}_x\mathbf{u}$.

2.7.2. The so-called renormalized solutions satisfying (16) were introduced by DiPerna and P.-L. Lions [1]. The applications of the theory to problems in mathematical fluid dynamics are discussed at length in the monograph [14]. In particular, it can be shown that

$$\varrho \in C(J; L^1(\Omega)) \quad \text{for any compact subinterval } J \subset I. \tag{24}$$

Moreover, the validity of (24) can be extended to the whole interval I provided all quantities appearing in (16) are integrable on $I \times \Omega$. Thus the density ϱ possesses a well-defined *instantaneous value*

$$\varrho(t) \in L^1(\Omega) \quad \text{for any } t \in I.$$

2.7.3. The existence theory to be developed in Section 4 below requires certain coercivity properties of the elastic pressure p_e. Similarly to Chapter 7 in [9], we shall assume

$$\left\{ \begin{array}{l} p_e \in C[0,\infty) \cap C^1(0,\infty),\ p_e(0) = 0 \\ p_e'(\varrho) \geq c_1 \varrho^{\gamma-1} - c_2,\ c_1 > 0 \\ p_e(\varrho) \leq c_3 \varrho^{\gamma} + c_4, \end{array} \right\} \tag{25}$$

where

$$\gamma > \frac{N}{2}. \tag{26}$$

Taking the driving force $\mathbf{f}$ to be a bounded measurable function of t and x we easily deduce from the total energy balance (21) that

$$\varrho \mathbf{u} \in L^{\infty}_{loc}\big([t_0,\infty) \cap I; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^N)\big)$$

provided $t_0 \in I$ and $E(t_0)$ is finite. Thus one can use (18) in order to conclude

$$\varrho \mathbf{u} \in C\big([t_0,\infty) \cap I; L^{\frac{2\gamma}{\gamma+1}}_{weak}(\Omega; R^N)\big),$$

in particular, the momentum $\varrho\mathbf{u}$ has a well-defined value at each instant $t \geq t_0$ determined uniquely through the integral averages

$$\langle \varrho \mathbf{u}(t), \eta \rangle = \int_\Omega \varrho \mathbf{u}(t) \cdot \eta \ \mathrm{d}x, \quad \eta \in \mathcal{D}(\Omega).$$

2.7.4. The entropy production inequality (22) plays an important role in the asymptotic analysis of the solutions when time approaches infinity. The total entropy

$$\int_\Omega \varrho s(t) \ \mathrm{d}x$$

could be used as a "Lyapunov function" for the study of stability.

To be more specific, some preliminary observations are in order. First of all, assume that the dissipation function Φ can be written in the form

$$\Phi(\mathbb{S}, \mathbb{D}) = \Psi_1(\mathbb{S}) + \Psi_2(\mathbb{D}).$$

We claim that, without loss of generality, we can suppose that both Ψ_1 and Ψ_2 are convex lower semi-continuous functions, and

$$\Psi_1 = \Psi_2^*,$$

where the superscript $*$ denotes the conjugate function. Indeed, by virtue of hypothesis (7), we have

$$\Psi_2(\mathbb{D}) \geq \Psi_1^*(\mathbb{D}),$$

whence

$$\Psi_1(\mathbb{S}) + \Psi_2(\mathbb{D}) \geq \Psi_1(\mathbb{S}) + \Psi_1^*(\mathbb{D}) \geq \mathbb{S} : \mathbb{D}.$$

Now, as a direct consequence of the most right inequality,

$$\Psi_1(\mathbb{S}) \geq \Psi_1{}^{**}(\mathbb{S}),$$

therefore

$$\Psi_1(\mathbb{S}) + \Psi_2(\mathbb{D}) \geq \Psi_1^{**}(\mathbb{S}) + \Psi_1^*(\mathbb{D}) \geq \mathbb{S} : \mathbb{D}.$$

Thus

$$\Phi(\mathbb{S}, \mathbb{D}) = \mathbb{S} : \mathbb{D} \quad \text{implies} \quad \mathbb{S} \in \partial \Psi_1^*(\mathbb{D})$$

for any $\mathbb{D} \in R^{N\times N}_{sym}$ as claimed (see Proposition 5.1 in Chapter 1 in [6]).

If, moreover,

$$\Phi(\mathbb{S}, \mathbb{D}) = \mathcal{H}(\mathbb{D}) + \mathcal{H}^*(\mathbb{S}),$$

where

$$|M[\ \mathbb{D}\ ]|^2 \leq \mathcal{H}(\mathbb{D}) \leq \frac{\lambda}{2}|\mathbb{D}|^2,$$

with a linear mapping $M : R^{N\times N}_{sym} \to R^{N\times N}_{sym}$, then

$$\Phi(\mathbb{S}, \mathbb{D}) \geq |M[\ \mathbb{D}\ ]|^2 + \frac{1}{2\lambda}|\mathbb{S}|^2.$$

If, in addition

$$\kappa(\vartheta) \geq \kappa_0 \vartheta^2, \quad \text{with } \kappa_0 > 0, \tag{27}$$

then the (minimal) entropy production function ζ^{**} admits a lower bound

$$\zeta^{**}(\Theta, \nabla_x\Theta, \mathbb{S}, \mathbb{D}_x\mathbf{u}) \geq \frac{1}{\Theta}\Big(|M[\ \mathbb{D}_x\mathbf{u}\ ]|^2 + \frac{1}{2\lambda}|\mathbb{S}|^2\Big) + \kappa_0|\nabla_x\Theta|^2. \tag{28}$$

Thus under suitable "ellipticity" hypothesis imposed on M, we conclude that the admissible solutions for which the right expression in (28) vanishes correspond to the *static states* ϱ_s, ϑ_s, specifically,

$$\mathbf{u} = 0,\ \mathbb{S} = 0,\ \vartheta = \vartheta_s > 0\ \text{ – a constant}, \tag{29}$$

and ρ_s, where

$$\nabla_x p_e(\varrho_s) + \vartheta_s \nabla_x p_\vartheta(\varrho_s) = \varrho_s \mathbf{f}, \tag{30}$$

and where, necessarily, $\mathbf{f}$ must be a gradient of a scalar potential $F = F(x)$. This result can be considered as a rigorous confirmation of Onsager's *principle of minimum entropy production* (see Theorem 5.4 in [20]).

3. Approximate problems

3.1. A model problem with radiation terms. Consider the system

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0 \tag{31}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p^a = \operatorname{div} \mathbb{S} + \varrho \mathbf{f} \tag{32}$$

$$\partial(\varrho Q^a) + \operatorname{div}(\varrho Q^a \mathbf{u}) - \operatorname{div}(\kappa^a \nabla_x \vartheta) = \mathbb{S} : \mathbb{D}_x \mathbf{u} - p^a_{th} \operatorname{div}\mathbf{u} \tag{33}$$

supplemented with the boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \nabla_x \vartheta \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{34}$$

where

$$\mathbb{S} = 2\mu\Big(\mathbb{D}_x \mathbf{u} - \frac{1}{N}\operatorname{div}\mathbf{u}\ \mathbb{I}\Big) + \xi \operatorname{div}\mathbf{u}\ \mathbb{I},$$

with constants $\mu > 0,\ \xi \geq 0$, and

$$p^a(\varrho, \vartheta) = p_e(\varrho) + \vartheta p_\vartheta(\varrho) + \Big[\frac{d}{3}\vartheta^4 + \delta \varrho^\beta\Big]$$

$$Q^a(\varrho, \vartheta) = c_v \vartheta + \Big[d\frac{\vartheta^4}{\varrho}\Big]$$

$$\kappa^a(\vartheta) = \kappa(\vartheta) + \Big[\sigma \vartheta^\omega\Big]$$

$$p^a_{th}(\varrho, \vartheta) = \vartheta p_\vartheta(\varrho) + \Big[\frac{d}{3}\vartheta^4\Big].$$

The quantities in square brackets represent *regularization terms* we are going to discuss now. The quantity $d\vartheta^4/3$ in the pressure term as well as the related contribution $d\vartheta^4/\varrho$ to the thermal energy Q^a represents the influence of *radiation* effective at very high temperatures. Note that $d > 0$ stands for the Stefan-Boltzmann constant which is extremely small (see, for example, [17], [3]). Similarly, the contribution $\sigma\vartheta^\omega$, with $\sigma > 0$, $\omega >> 1$ is responsible for a very fast transfer of heat at high temperatures due to radiation effects (see [24]). Finally, the elastic pressure is augmented through the component $\delta\varrho^\beta$, $\delta > 0$, $\beta >> 1$ whose regularizing impact is discussed in Section 7 in [9]. The original state of the system is characterized by the initial conditions:

$$\varrho(0) = \varrho_0 \geq 0, \quad \varrho\mathbf{u}(0) = \mathbf{m}_0, \quad \vartheta(0) = \vartheta_0 > 0. \tag{35}$$

At this stage, very mild structural restrictions on the non-linear constitutive equations are required. As for the pressure, in addition to (25), (26), we need

$$\left\{ \begin{array}{l} p_\vartheta \in C^1[0,\infty),\ p_\vartheta(0) = 0 \\ p'_\vartheta(\varrho) \geq 0,\ p_\vartheta \leq c(1 + \varrho^\Gamma) \text{ for all } \varrho \geq 0 \end{array} \right\} \tag{36}$$

(with Γ a convenient positive constant which will be specified later), while the heat conductivity coefficient will satisfy

$$\kappa \in C^1[0,\infty),\ 0 < \underline{\kappa}(1+\vartheta^\alpha) \le \kappa(\vartheta) \le \overline{\kappa}(1+\vartheta^\alpha) \quad \text{for all } \vartheta > 0, \tag{37}$$

where $\alpha > 0$ will be specified below.

3.2. Approximate solutions. Problem (31 - 35) introduced above is mathematically tractable. As a matter of fact, an existence theory, even for the case when the viscosity coefficients are allowed to depend on the temperature, was developed in [4].

In addition to (25), (26), (36), and (37), suppose that

$$d > 0, \quad \delta > 0, \quad \sigma > 0, \quad \beta > \max\left\{2, \gamma, \frac{4\Gamma}{3}\right\}, \quad \omega > \max\{\alpha, 8\}$$

together with

$$\mathbf{f} \in L^\infty((0,T)\times\Omega),$$

where $\Omega \subset R^N$, $N = 2,3$ is a *bounded* domain with smooth boundary.

Furthermore, let the initial data belong to the classes:

$$\left\{ \begin{array}{l} \varrho_0 \in L^\beta(\Omega),\ \varrho_0(x) \ge 0 \text{ for a.a. } x \in \Omega \\ \mathbf{m}_0 \in L^1(\Omega, R^N),\ \frac{|\mathbf{m}_0|^2}{\varrho_0} \in L^1(\Omega) \\ \vartheta_0 \in L^\infty(\Omega),\ \vartheta(x) \ge \underline{\vartheta} > 0 \text{ for a.a. } x \in \Omega. \end{array} \right\}$$

Under the hypotheses stated above, we claim, in accordance with Theorem 3.1 in [4], that the problem (31 - 35) possesses a variational (weak) solution ϱ, $\mathbf{u}$, ϑ defined on $(0,T)$ and enjoying the following properties:

3.2.1. We have

$$\varrho \in L^\infty(0,T;L^\beta), \quad \mathbf{u} \in L^2(0,T;W_0^{1,2}(\Omega;R^N)),$$

and the renormalized continuity equation

$$\int_0^T \int_\Omega [\varrho H(\varrho)\varphi_t + \varrho H(\varrho)\mathbf{u}\cdot\nabla_x\varphi - h(\varrho)\,\mathrm{div}\mathbf{u}\ \varphi]\ \mathrm{d}x\,\mathrm{d}t = 0 \tag{38}$$

holds for any h, H as in (17), and any test function $\varphi \in \mathcal{D}((0,T)\times\overline{\Omega})$. In other words, the renormalized continuity equation is satisfied in the sense of distributions on the whole set $(0,T)\times R^N$ provided ϱ and $\mathbf{u}$ were extended to be zero outside Ω. Moreover,

$$\varrho \in C([0,T];L^1(\Omega)) \quad \text{and} \quad \varrho(0) = \varrho_0.$$

3.2.2. The pressure $p^a = p^a(\varrho, \vartheta)$ belongs to the class $L^r((0,T) \times \Omega)$ for a certain $r > 1$, and the momentum equation (32) is satisfied in $\mathcal{D}'((0,T) \times \Omega)$, with

$$\mathbb{S} = 2\mu\Big(\mathbb{D}_x \mathbf{u} - \frac{1}{N} \mathrm{div} \mathbf{u}\ \mathbb{I}\Big) + \xi \mathrm{div} \mathbf{u}\ \mathbb{I} \in L^2\big((0,T); L^2(\Omega; R^{N\times N}_{sym})\big). \tag{39}$$

Furthermore,

$$\varrho \mathbf{u} \in C\big([0,T]; L^{\frac{2\beta}{\beta+1}}_{weak}(\Omega; R^N)\big) \quad \text{and} \quad \varrho \mathbf{u}(0) = \mathbf{m}_0.$$

3.2.3. The total energy

$$E^a = \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho P_e(\varrho) + c_v \varrho \vartheta + \Big[\frac{\delta}{\beta - 1}\varrho^\beta + d\vartheta^4\Big]$$

satisfies

$$\int_\Omega E^a(t_2)\ \mathrm{d}x - \int_\Omega E^a(t_1)\ \mathrm{d}x = \int_{t_1}^{t_2} \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u}\ \mathrm{d}x\, \mathrm{d}t \tag{40}$$

for a.a. $0 < t_1 \leq t_2 < T$, and

$$\operatorname*{ess\,lim}_{t \to 0+} \int_\Omega E^a(t)\ \mathrm{d}x = \int_\Omega E^a_0\ \mathrm{d}x,$$

with

$$E^a_0 = \frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + \varrho_0 P_e(\varrho_0) + c_v \varrho_0 \vartheta_0 + \Big[\frac{\delta}{\beta-1}\varrho_0^\beta + d\vartheta_0^4\Big].$$

3.2.4. The entropy production inequality

$$\begin{aligned} &\int_0^T \int_\Omega [\varrho s^a \partial_t \varphi + \varrho s^a \mathbf{u} \cdot \nabla_x \varphi + \mathcal{K}^a_1(\vartheta)\Delta\varphi]\ \mathrm{d}x\, \mathrm{d}t \\ &\qquad \leq - \int_0^T \int_\Omega \Big(\frac{\Phi(\mathbb{S}, \mathbb{D}_x \mathbf{u})}{\vartheta} + \frac{\kappa^a(\vartheta)}{\vartheta^2}|\nabla_x \vartheta|^2\Big)\varphi\ \mathrm{d}x\, \mathrm{d}t \end{aligned} \tag{41}$$

holds for any test function

$$\varphi \in \mathcal{D}((0,T) \times \overline{\Omega}), \quad \varphi \geq 0, \quad \nabla_x \varphi \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{42}$$

Here, we have set

$$\begin{aligned} s^a(\varrho, \vartheta) &= c_v \log(\vartheta) - P_\vartheta(\varrho) + \Big[\frac{4d\vartheta^3}{3\varrho}\Big] \\ (\mathcal{K}^a_1)'(\vartheta) &= \frac{\kappa^a(\vartheta)}{\vartheta}, \quad \vartheta > 0, \end{aligned}$$

and Φ is the standard dissipation function for linearly viscous fluids:

$$\Phi(\mathbb{S},\mathbb{D}) = \mathcal{H}(\mathbb{D}) + \mathcal{H}^*(\mathbb{S}), \quad \mathcal{H}(\mathbb{D}) = \frac{\mu}{2}| < \mathbb{D} > |^2 + M\xi| \quad \mathbb{D}- < \mathbb{D} > |^2,$$

where the symbol $< \cdot >$ denotes the projection onto the space of traceless tensors. In particular, we have

$$\log(\vartheta),\ \vartheta^{\frac{\omega}{2}} \in L^2\big(0,T;W^{1,2}(\Omega)\big). \tag{43}$$

Moreover, the entropy satisfies the "initial conditions"

$$\operatorname*{ess\,liminf}_{t\to 0+} \int_\Omega \varrho s^a(t)\eta \ \mathrm{d}x \geq \int_\Omega (\varrho s^a)_0 \eta \ \mathrm{d}x \quad \text{for any } \eta \in \mathcal{D}(\Omega),\ \eta \geq 0, \tag{44}$$

with

$$(\varrho s^a)_0 = c_v \varrho_0 \log(\vartheta_0) - \varrho_0 P_\vartheta(\varrho_0) + \Big[\frac{4d\vartheta_0^3}{3}\Big].$$

3.2.5. The reader will have noticed that we have deliberately avoided the thermal energy balance (33) which is, at least in the class of smooth solutions, equivalent to (41). The reason why this equation (or the corresponding inequality) does not appear in the conclusion of Theorem 3.1 in [4] is simple: the available *a priori* estimates are not strong enough to render the term p_{th} div$\mathbf{u}$, or, more precisely, its component ϑ^4 div$\mathbf{u}$, integrable.

In the present setting, however, the necessary estimates will follow from (43) provided $\omega \gg 1$ has been chosen large enough. If this is the case, a direct inspection of the proof of Theorem 3.1 in [4] shows that the solutions constructed there will also satisfy a family of "rescaled" entropy inequalities:

$$\begin{aligned}
&\int_0^T \int_\Omega \Big[\Big(\frac{c_v}{1-\lambda}\varrho\vartheta^{1-\lambda} + \frac{4d}{4-\lambda}\vartheta^{4-\lambda}\Big)\varphi_t \ \mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^T \int_\Omega \Big(\Big[\frac{c_v}{1-\lambda}\varrho\vartheta^{1-\lambda} + \frac{4d}{4-\lambda}\vartheta^{4-\lambda}\Big]\mathbf{u}\Big) \cdot \nabla_x \varphi\Big] \ \mathrm{d}x\,\mathrm{d}t \\
&+ \int_0^T \int_\Omega \mathcal{K}^a_\lambda(\vartheta)\Delta\varphi \ \mathrm{d}x\,\mathrm{d}t \\
&\qquad \leq -\int_0^T \int_\Omega \Big(\frac{\Phi(\mathbb{S},\mathbb{D}_x\mathbf{u})}{\vartheta^\lambda} + \lambda\frac{\kappa^a(\vartheta)}{\vartheta^{\lambda+1}}|\nabla_x\vartheta|^2\Big)\varphi \ \mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_0^T \int_\Omega \Big[4\Big(\frac{d}{3} - \frac{d}{4-\lambda}\Big)\vartheta^{4-\lambda} + \vartheta^{1-\lambda}p_\vartheta\Big]\mathrm{div}\mathbf{u}\ \varphi \ \mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{45}$$

for $\lambda \in [0,1)$. Note that the case $\lambda = 0$ corresponds to (33) while inequality (41) may be viewed as the "limit" for $\lambda \to 1$. The test functions are the same as in (42), and we have denoted

$$(\mathcal{K}^a_\lambda)'(\vartheta) = \frac{\kappa^a(\vartheta)}{\vartheta^\lambda}.$$

Furthermore, one can use the total energy equality (40) together with the arguments employed in Section 4.3.4 of Chapter 4 in [9] in order to conclude that the thermal energy satisfies the "genuine" (in contrast with (44)) initial conditions

$$\operatorname*{ess\,lim}_{t\to 0+} \int_\Omega (\varrho Q^a)(t)\eta \,\mathrm{d}x = \int_\Omega (\varrho Q^a)_0 \eta \,\mathrm{d}x \quad \text{for any } \eta \in \mathcal{D}(\Omega), \tag{46}$$

where

$$(\varrho Q^a)_0 = c_v \varrho_0 \vartheta_0 + \left[d\vartheta_0^4 \right].$$

4. Existence

The focus of this paper is to develop an existence theory for the problem (1), (2), and (4) within the framework of the admissible solutions introduced in Section 2. To this end, we derive first suitable estimates and then pass to the limit for vanishing parameters d, δ, and σ in the family of approximate solutions constructed above.

4.1. Energy estimates. A direct application of Gronwall's lemma to (40) gives rise to the estimates

$$\int_\Omega E^a(t) \,\mathrm{d}x \le E_0^a \exp\Big(Tc\big(\|\mathbf{f}\|_{L^\infty((0,T)\times\Omega)}, M \big) \Big) \quad \text{for a.a. } t \in (0,T), \tag{47}$$

where $M = \int_\Omega \varrho \,\mathrm{d}x$ is the total mass independent of t. For the sake of simplicity, we keep the initial data ϱ_0, $\mathbf{m}_0$, ϑ_0 the same for any choice of the parameters d, δ, σ. Accordingly, the initial energy remains bounded, and we deduce the estimates

$$\sqrt{\varrho}|\mathbf{u}| \quad \text{bounded in } L^\infty(0,T;L^2(\Omega)) \tag{48}$$

$$\varrho \quad \text{bounded in } L^\infty(0,T;L^\gamma(\Omega)) \tag{49}$$

$$\varrho\vartheta \quad \text{bounded in } L^\infty(0,T;L^1(\Omega)), \tag{50}$$

and

$$d \operatorname*{ess\,sup}_{t\in(0,T)} \int_\Omega \vartheta^4(t) \,\mathrm{d}x \le c \tag{51}$$

independent of d, δ, and σ.

4.2. Entropy estimates. As a consequence of the entropy production inequality (41), we get

$$\int_\Omega \varrho s^a(t) \,\mathrm{d}x \ge \int_\Omega (\varrho s^a)_0 \,\mathrm{d}x + \int_0^t \int_\Omega \frac{\Phi(\mathbb{S}, \mathbb{D}_x \mathbf{u})}{\vartheta} + \frac{\kappa^a(\vartheta)}{\vartheta^2} |\nabla_x \vartheta|^2 \,\mathrm{d}x \,\mathrm{d}s \tag{52}$$

for a.a. $t \in (0,T)$. Thus, by virtue of (50), (51),

$$\varrho \log(\vartheta)\ ,\ \varrho P_\vartheta(\varrho) \quad \text{are bounded in } L^\infty\big(0,T;L^1(\Omega)\big) \tag{53}$$

$$\frac{\mathbb{S}}{\sqrt{\vartheta}}\ ,\ \frac{<\mathbb{D}_x \mathbf{u}>}{\sqrt{\vartheta}} \quad \text{are bounded in } L^2(0,T;L^2\big(\Omega;R^{N\times N}_{sym})\big) \tag{54}$$

uniformly with respect to d, δ, and σ. Moreover, assuming $\alpha \geq 2$ in (37) we obtain

$$\nabla_x \log(\vartheta) \quad \text{is bounded in } L^2(0,T;L^2(\Omega;R^N)) \tag{55}$$

$$\nabla_x \vartheta^{\frac{\alpha}{2}} \quad \text{is bounded in } L^2(0,T;L^2(\Omega;R^N)), \tag{56}$$

and, finally,

$$\sigma \int_0^T \int_\Omega |\nabla_x \vartheta^{\frac{\omega}{2}}|^2 \, \mathrm{d}x \, \mathrm{d}t \leq c. \tag{57}$$

4.3. Dissipation estimates. Relation (45) evaluated for $\lambda = 0$ and $\varphi = 1$ yields

$$\int_0^T \int_\Omega \big[\, |\mathbb{S}|^2 + |<\mathbb{D}_x \mathbf{u}>|^2 \,\big] \, \mathrm{d}x \, \mathrm{d}t$$
$$\leq c\Big(1 + \int_0^T \int_\Omega [d|\nabla_x \vartheta^4||\mathbf{u}| + \vartheta p_\vartheta(\varrho)|\mathrm{div}\mathbf{u}|] \, \mathrm{d}x \, \mathrm{d}t\Big).$$

At this stage, we recall the following version of Korn's inequality

$$\|\mathbf{v}\|_{W_0^{1,q}(\Omega;R^N)} \leq c(q)\, \|<\mathbb{D}_x \mathbf{v}>\|_{L^q(\Omega;R^{N\times N}_{sym})} \tag{58}$$

for any $\mathbf{v} \in W_0^{1,q}(\Omega;R^N), 1 < q < \infty$ (cf. Proposition 2.4, Chapter 3 in [21]). Under the hypothesis

$$\Gamma \begin{cases} < \frac{\gamma}{2} & \text{if} \quad N = 2 \\ = \frac{\gamma}{3} & \text{for } N = 3 \end{cases}$$

in (36) and $\alpha \geq 2$ in (37) we can use (49), (56), (57), and (58) in order to conclude

$$\mathbf{u} \ \text{ is bounded in } L^2\big(0,T;W_0^{1,2}(\Omega;R^N)\big) \tag{59}$$

$$\mathbb{S} \ \text{ is bounded in } L^2\big(0,T;L^2(\Omega;R^{N\times N}_{sym})\big) \tag{60}$$

provided $0 < d \leq \sigma < 1$. Taking $0 < \lambda < 1$ we deduce from (45) with $\varphi = 1$

$$\lambda \int_0^T \int_\Omega |\nabla_x \vartheta^{\frac{\alpha+1-\lambda}{2}}|^2 + \sigma|\nabla_x \vartheta^{\frac{\omega+1-\lambda}{2}}|^2 \, \mathrm{d}x \, \mathrm{d}t \leq c. \tag{61}$$

We shall need the following elementary observation (see Lemma 4.1 in [5]).

Lemma 1. *Let $\Omega \subset R^N$, $N \geq 2$ be a bounded Lipschitz domain, and $r \geq 1$ a given constant. Let $\varrho \geq 0$ be a measurable function satisfying*

$$0 < M \leq \int_\Omega \varrho \, \mathrm{d}x, \quad \int_\Omega \varrho^\beta \, \mathrm{d}x \leq K$$

for $\beta > \frac{2N}{N+2}$. Then there exists a constant $c = c(M,K)$ such that

$$\|v\|_{L^2(\Omega)} \leq c(M,K)\Big(\|\nabla_x v\|_{L^2(\Omega;R^N)} + \Big[\int_\Omega \varrho |v|^{\frac{1}{r}} \, \mathrm{d}x\Big]^r\Big)$$

for any $v \in W^{1,2}(\Omega)$.

Now, by virtue of Lemma 1 and (61),

$$\lambda \|\vartheta^{\frac{\alpha+1-\lambda}{2}}\|^2_{L^2(0,T;W^{1,2}(\Omega))} + \lambda\sigma \|\vartheta^{\frac{\omega+1-\lambda}{2}}\|^2_{L^2(0,T;W^{1,2}(\Omega))} \leq c \tag{62}$$

for any $0 < \lambda \leq 1$. Moreover, by the same token, estimates (53), (55) imply that

$$\log(\vartheta) \text{ is bounded in } L^2(0,T;W^{1,2}(\Omega)) \tag{63}$$

uniformly with respect to δ, d, and σ.

4.4. Refined density estimates and convergence. Now, we test momentum equation (32) with

$$\mathbf{w} = \psi \mathcal{B}\Big(\rho^\nu \star \omega_\epsilon - \frac{1}{|\Omega|}\int_\Omega \rho^\nu \star \omega_\epsilon \, \mathrm{d}x\Big), \quad \nu > 0,$$

where ψ is a convenient function in $C_0^\infty(I)$, $\omega_\epsilon \to \delta(0)$ is a regularizing sequence in variable t and $\mathcal{B}$ is the Bogovskii operator satisfying

$$\mathrm{div}\mathcal{B}(v) = v, \quad \|\mathcal{B}(v)\|_{L^q(\Omega)} \leq c(q)\|\mathbf{g}\|_{L^q(\Omega)}, \quad \|\nabla \mathcal{B}(v)\|_{L^p(\Omega)} \leq c(p)\|v\|_{L^p(\Omega)},$$

provided $1 < q, p < \infty$, $\mathbf{g} \in L^q(\Omega; R^N)$, $v = \mathrm{div}\mathbf{g} \in L^p(\Omega)$ and $\mathbf{g} \cdot \mathbf{n}|_{\partial\Omega} = 0$ (cf. e.g. Lemma 3.17 in [19]).

We observe that estimate (45) together with hypothesis $\omega > 8$ yield

$$d\vartheta^4 \to 0 \text{ in } L^2(0,T;L^2(\Omega)) \quad \text{for } d \to 0 \tag{64}$$

provided $0 < d \leq \sigma \leq 1$. Consequently, the "radiative pressure" component in the momentum equation becomes negligible in the limit process for small values of δ, d, and σ. Therefore one can repeat step by step the arguments used in Section 7.9.5 in [19] in order to obtain

$$\int_0^T \int_\Omega [p_e(\varrho)\varrho^\nu + \delta \varrho^{\beta+\nu}] \, \mathrm{d}x \, \mathrm{d}t \leq c \quad \text{for a certain } \nu > 0. \tag{65}$$

Now, having established all the necessary estimates we are able to pass to the limit for δ, d, and σ approaching zero, at least in equations (31), (32). Denoting the approximate solutions constructed above as ϱ^a, $\mathbf{u}^a$, ϑ^a, $a \approx (\delta, d, \sigma)$, it is easy to show that

$$\varrho^a \to \varrho \qquad \text{in } \ C\big([0,T]; L^{\gamma}_{weak}(\Omega)\big) \tag{66}$$

$$\mathbf{u}^a \to \mathbf{u} \ \text{ weakly in } \ L^2\big(0,T; W_0^{1,2}(\Omega; R^N)\big) \tag{67}$$

$$\vartheta^a \to \vartheta \ \text{ weakly in } \ L^2\big(0,T; W^{1,2}(\Omega)\big) \tag{68}$$

and

$$\varrho^a \mathbf{u}^a \to \varrho \mathbf{u} \qquad \text{in } \ C\big([0,T]; L^{\frac{2\gamma}{\gamma+1}}_{weak}(\Omega; R^N)\big) \tag{69}$$

at least for a chosen subsequence.

In view of (64), we are in the situation considered in Section 7.5 of [9]. Accordingly, as in [9], we can prove the following, nowadays standard properties:

1) The effective pressure identity in the form

$$\overline{T_k(\varrho)p_e(\varrho)} + \vartheta \overline{T_k(\varrho)p_\vartheta(\varrho)} - \overline{T_k(\varrho)\mathrm{div}\mathbf{u}}$$
$$= \overline{T_k(\varrho)}\ \overline{p_e(\varrho)} + \vartheta \overline{T_k(\varrho)}\ \overline{p_\vartheta(\varrho)} - \overline{T_k(\varrho)}\mathrm{div}\mathbf{u},$$

where

$$T_k(s) = \begin{cases} s & \text{if } s \in [0,k) \\ k & \text{if } s \in [k,\infty) \end{cases}$$

and where the over-lined quantities denote corresponding weak limits in $L^1(I \times \Omega)$.

2) A consequence of this result is the boundedness of defect measure

$$\limsup_a \|T_k(\varrho^a) - T_k(\varrho)\|_{L^{\gamma+1}(I\times\Omega)}$$

uniformly with respect to δ, d, σ and k.

3) The lastly mentioned property implies that weak limits (66), (67) satisfy renormalized continuity equation (16).

Then it is not difficult to show that

$$\varrho^a \to \varrho \quad \text{(strongly) in } C([0,T]; L^1(\Omega)), \tag{70}$$

and that the the limits (66)–(68) represent a physically admissible solution of (1), (2) in the sense of Sections 2.2.1, 2.2.2. Obviously,

$$\varrho(0) = \varrho_0 \quad \text{and} \quad \varrho\mathbf{u}(0) = \mathbf{m}_0, \tag{71}$$

and the “limit” stress tensor $\mathbb{S}$ in (2) is given through

$$\mathbb{S} = 2\mu < \mathbb{D}_x \mathbf{u} > + \xi \mathrm{div} \mathbf{u} \ \mathbb{I}. \tag{72}$$

Moreover, by the same token, one can pass to the limit in the total energy balance (40) in order to obtain (21). Here, in addition,

$$\operatorname{ess} \lim_{t \to 0+} \int_\Omega E(t) \ \mathrm{d}x = E_0, \quad E_0 = \frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + \varrho_0 P_e(\varrho_0) + c_v \varrho_0 \vartheta_0. \tag{73}$$

4.5. Refined temperature estimates. From relations (50), (56) and (57) we find by interpolation that

$$\int_{\{\varrho \geq \epsilon\}} [\vartheta^{(\alpha+1)p} + \sigma^p \vartheta^{(\omega+1)p} \, \mathrm{d}x\mathrm{d}t \leq c(\epsilon), \quad \epsilon > 0$$

with some $p > 1$. Further, pursuing our overall strategy of Section 5.2 in [9], we use in equation (45) with $\lambda = 0$ test function $\varphi = \psi(\eta - \inf_{(t,x)} \eta(t,x))$, where ψ is a convenient function in $C_0^\infty(I)$ and η solves the Neumann problem

$$\Delta \eta = B(\varrho(t)) - \frac{1}{|\Omega|} \int_\Omega B(\varrho(t)), \quad \nabla \eta(t) \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \int_\Omega \eta(t) = 0$$

with B a smooth non-increasing function which is 0 on $(\infty, \frac{\epsilon}{2})$ and -1 on (ϵ, ∞). After a long calculation, we obtain the bound

$$\int_{\{\varrho < \epsilon\}} [\vartheta^{\alpha+1} + \sigma \vartheta^{\omega+1}] \, \mathrm{d}x\mathrm{d}t \leq c(\epsilon).$$

Summarizing, we assert the estimate

$$\int_0^T \int_\Omega [\vartheta^{\alpha+1} + \sigma \vartheta^{\omega+1}] \ \mathrm{d}x \, \mathrm{d}t \leq c \tag{74}$$

uniformly with any choice of (bounded) δ, d, and σ. Taking advantage of (45) we easily show that the sequence $\varrho^a \vartheta^a$ verifies assumptions of a generalized Aubin-Lions' lemma (see Lemma 6.3 in [9]), and we infer that $\varrho^a \vartheta^a \to \varrho\vartheta$ (strongly) in $L^2(0,T; W^{-1,2}(\Omega))$. Now, we employ (68) to get $\varrho^a (\vartheta^a)^2 \to \varrho\vartheta^2$ in $\mathcal{D}'(I \times \Omega)$ together with the pointwise convergence of the temperature field

$$\vartheta^a(t,x) \to \vartheta(t,x) \quad \text{for a.a. } (t,x) \in \{\varrho > 0\} \tag{75}$$

with $\{\varrho > 0\} = \{(t,x) \in (0,T) \times \Omega \mid \varrho(t,x) > 0\}$.

Finally, due to (63), (75) on one hand and due to the weak lower semicontinuity of the (convex) function ξ^{**} on the other hand, it is easy to pass to the limit in the entropy inequality (41) in order to conclude

$$\int_\Omega \varrho s(t_2) \ \mathrm{d}x - \int_\Omega \varrho s(t_1) \ \mathrm{d}x \geq \int_{t_1}^{t_2} \int_\Omega \zeta^{**}(\vartheta, \nabla_x \vartheta, \mathbb{S}, \mathbb{D}_x \mathbf{u}) \ \mathrm{d}x \ \mathrm{d}t, \tag{76}$$

with the entropy production rate ζ^{**} introduced in (22). Moreover, in accordance with (44), we claim

$$\operatorname{ess}\liminf_{t\to 0+} \int_\Omega \varrho s(t)\eta \,\mathrm{d}x \ge \int_\Omega \varrho_0 s_0 \eta \,\mathrm{d}x \quad \text{for any } \eta \in \mathcal{D}(\Omega),\ \eta \ge 0, \tag{77}$$

with

$$s_0 = c_v \log(\vartheta_0) - P_\vartheta(\varrho_0).$$

Finally, it is easy to observe that (74) implies

$$\sigma \int_0^T \int_\Omega (\vartheta^a)^{\omega+1-\lambda} \,\mathrm{d}x\,\mathrm{d}t \to 0 \text{ for } \sigma \to 0 \quad \text{whenever } \lambda > 0. \tag{78}$$

Consequently, one can use the technique based on concept of *biting or renormalized limit* exactly as in Section 7.5.5 in [9] to let the parameters d and σ to go to zero in (45), first for fixed $\lambda > 0$, then for $\lambda \to 0$, in order to deduce that the limit quantities satisfy the thermal energy inequality (19). As pointed out in Section 7.5.5 in [9], such a procedure requires, as a final step, to modify the limit temperature on possible "vacuum regions" where ϱ vanishes, which converts (76) to the weaker statement (22). Finally, we have

$$\operatorname{ess}\lim_{t\to 0+} \int_\Omega \varrho\vartheta(t)\eta \,\mathrm{d}x = \int_\Omega \varrho_0\vartheta_0 \,\mathrm{d}x \quad \text{for any } \eta \in \mathcal{D}(\Omega). \tag{79}$$

4.6. Conclusion. We have proved the following existence result.

Theorem 1. *Let $\Omega \subset R^N$, $N = 2, 3$ be a bounded domain with smooth boundary. Assume the pressure p can be written in the form*

$$p(\varrho, \vartheta) = p_e(\varrho) + \vartheta p_\vartheta(\varrho),$$

where $p_e \in C[0,\infty) \cap C^1(0,\infty)$, $p_e(0) = 0$,

$$p_e'(\varrho) \ge c_1 \varrho^{\gamma-1} - c_2, \quad p_e(\varrho) \le c_3(1 + \varrho^\gamma) \quad \text{for } \varrho > 0, c_1 > 0,$$

$p_\vartheta \in C^1[0,\infty)$, $p_\vartheta(0) = 0$,

$$p_\vartheta'(\varrho) \ge 0, \quad p_\vartheta(\varrho) \le c_4(1 + \varrho^\Gamma) \quad \text{for } \varrho \ge 0,$$

with $\gamma > \frac{N}{2}$ and

$$\Gamma \begin{cases} < \frac{\gamma}{2} & \text{if } \ N = 2 \\ = \frac{\gamma}{3} & \text{for } N = 3. \end{cases}$$

Furthermore, let $\mathbb{S}$ be given through

$$\mathbb{S} = 2\mu < \mathbb{D}_x \mathbf{u} > + \xi \mathrm{div} \mathbf{u}\ \mathbb{I}, \quad \mu > 0,\ \xi \geq 0 \text{ constant},$$

with the corresponding dissipation function Φ. Let $\mathbf{f} = \mathbf{f}(t,x)$ be a bounded measurable function on $(0,T) \times \Omega$, and let $\kappa \in C^1[0,\infty)$ is a positive function such that

$$0 < \underline{\kappa}(1 + \vartheta^\alpha) \leq \kappa(\vartheta) \leq \overline{\kappa}(1 + \vartheta^\alpha) \quad \text{for any } \vartheta \geq 0, \text{ with } \alpha \geq 2.$$

Finally, suppose that the initial data satisfy

$$\varrho_0 \in L^\gamma(\Omega),\ \varrho_0 \geq 0, \quad \mathbf{m}_0 \in L^1(\Omega),\ \frac{|\mathbf{m}_0|^2}{\varrho_0} \in L^1(\Omega), \quad \vartheta \in L^\infty(\Omega),\ \vartheta \geq \underline{\vartheta} > 0.$$

Then the problem (1), (2), (4) *supplemented with the boundary conditions* (5) *possesses an admissible variational solution ϱ, $\mathbf{u}$, ϑ (in the sense specified in Section* 2*) defined on the time interval* $(0,T)$ *and satisfying the initial conditions*

$$\varrho(t) \to \varrho_0 \qquad \text{in } \ L^1(\Omega) \text{ for } t \to 0+$$

$$\varrho\mathbf{u}(t) \to \mathbf{m}_0 \ \text{ weakly in } \ L^{\frac{2\gamma}{\gamma+1}} \text{ for } t \to 0+$$

$$\operatorname*{ess\,lim}_{t\to 0+} \int_\Omega \varrho\vartheta\eta \ \mathrm{d}x = \int_\Omega \varrho_0\vartheta_0\eta \ \mathrm{d}x \quad \text{for any } \eta \in \mathcal{D}(\Omega).$$

5. The large time behaviour

The decisive piece of information in order to study the asymptotic behaviour of the admissible solutions for large times is the entropy production inequality (22).

We start with an elementary observation that the infimum in (22) is attained through a function $\overline{\vartheta}$. Indeed, in accordance with the hypotheses imposed in Theorem 1,

$$\zeta^{**}(\Theta, \mathbf{d}, \mathbb{S}, \mathbb{D}) \geq c|\nabla_x \Theta|^2, \quad c > 0;$$

whence any minimizing sequence on a time interval $[t_1, t_2]$ converges weakly in the space $L^2(t_1, t_2; W^{1,2}(\Omega))$ to a minimizer $\overline{\vartheta}_{t_1,t_2}$. On the other hand, it is easy to check, using the integral additivity property, that one can take

$$\overline{\vartheta}_{T_1,T_2} = \overline{\vartheta}_{t_1,t_2} \quad \text{whenever } [t_1,t_2] \subset [T_1,T_2].$$

Thus there exists a function

$$\overline{\vartheta} \in L^2(0,T; W^{1,2}(\Omega)),\ \overline{\vartheta} = \vartheta \text{ on the set } \{\varrho > 0\}$$

such that for a.a. $t_1, t_2 \in (0,T),\ t_1 \le t_2$,

$$\int_\Omega \varrho s(t_2)\ \mathrm{d}x - \int_\Omega \varrho s(t_1)\ \mathrm{d}x \\ \ge \int_{t_1}^{t_2} \int_\Omega \zeta^{**}(\overline{\vartheta}, \nabla_x \overline{\vartheta}, \mathbb{S}, \mathbb{D}_x \mathbf{u})\ \mathrm{d}x\,\mathrm{d}t\ .$$

If the time evolution of the state variables ϱ, $\mathbf{u}$, and ϑ is viewed as a (conservative) dynamical system, then the entropy production should vanish in the long run while the system itself should stabilize to a single "equilibrium" or static state. Satisfaction of such a scenario can be considered as a good criterion of the physical relevance of the chosen class of *weak* solutions. Here, we conclude with a result in this direction which can be viewed as a particular case covered by Theorem 4.1 in [11].

Theorem 2. *In addition to the hypotheses of Theorem* 1*, assume that the elastic pressure component* p_e *is a strictly increasing function of the density* ϱ*. Let* $\varrho(t)$*,* $\mathbf{u}$*, and* ϑ *be an admissible variational solution of problem* (1)*,* (2)*,* (4)*, and* (5) *in the sense of Definition* 2.1 *on a time interval* (T_0, ∞)*. Furthermore, let*

$$\mathbf{f} = \mathbf{f}(x), \quad \mathbf{f} \in C(\overline{\Omega}; R^N)$$

be a given function. Then
either:

(i) *there exists a scalar potential* $F \in C^1(\overline{\Omega})$ *such that*

$$\mathbf{f} = \nabla_x F \quad \text{in } \Omega,$$

and

$$\left\{ \begin{array}{l} \varrho(t) \to \varrho_s \text{ in } L^\gamma(\Omega),\ \text{ as } t \to \infty \\ \displaystyle \operatorname*{ess\,lim}_{t\to\infty} \int_\Omega \varrho |\mathbf{u}|^2(t)\ \mathrm{d}x = 0 \\ \displaystyle \operatorname*{ess\,lim}_{t\to\infty} \int_\Omega \varrho\vartheta(t)\ \mathrm{d}x = M_0 \vartheta_s \end{array} \right\}$$

$\vartheta_s > 0$ *is a constant, and* $\varrho_s = \varrho_s(x)$ *is a static equilibrium solving*

$$\nabla_x p_e(\varrho_s) + \vartheta_s \nabla_x p_\vartheta(\varrho_s) = \varrho_s \nabla_x F \quad \text{in } \Omega;$$

or:

(ii) $\displaystyle \operatorname*{ess\,lim}_{t\to\infty} \int_\Omega E(t)\ \mathrm{d}x = \infty.$

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