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Regularity of the Adjoint State for the Instationary Navier-Stokes Equations

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Abstract. In this article, we are considering imbeddings of abstract functions in spaces of functions being continuous in time. A family of functions depending on certain parameters is discussed in detail. In particular, this example shows that such functions do not belong to the space $C([0,T],H)$. In the second part, we investigate an optimal control problem for the instationary Navier-Stokes equation. We will answer the question, in what sense the initial value problem for the adjoint equation can be solved.

Keywords: *Vector-valued functions, imbeddings, Navier-Stokes equations, optimal control, regularity of adjoints*

MSC 2000: Primary 46E40, secondary 49N60

1. Introduction

In this paper, we will study the regularity of abstract functions. The discussed properties are heavily connected to the optimal control of instationary Navier-Stokes equations. Here, the gradient of a given objective functional is evaluated by means of an adjoint state. The adjoint state is itself the solution of an evolution equation. The discussion of abstract functions in the first part of the paper will reflect important properties of the adjoint state.

The aim of the present article is twofolded. At first, we want to shed light on imbeddings of abstract functions in spaces of continuous functions. We refer to the mostly classical results due to Lions [4]. Given a Gelfand triple $V \hookrightarrow H \hookrightarrow V'$, the space

$$W(0,T) = \left\{ y \in L^2(0,T;V) : \frac{\mathrm{d}}{\mathrm{d}t} y \in L^2(0,T;V') \right\}$$

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is continuously imbedded in $C([0,T],H)$. In a recent research paper of Amann [2], the question of compact imbeddings is considered. However, to the knowledge of the authors there are no further results in the literature generalizing the result of Lions substantially except the following one in the book of Dautray and Lions [4, XVIII.3.5, p. 521]. They wrote, that it suffices to require $\frac{\mathrm{d}}{\mathrm{d}t}y \in L^1(0,T;V')$ to get the continuity result $y \in C([0,T],H)$.

Since the adjoint state of the Navier-Stokes equation does not belong to $W(0,T)$ in general, the above mentioned result in [4] is used in different papers concerning the optimal control of the instationary Navier-Stokes equation, see e.g. [8, 10, 11, 14].

In the present paper, we will show that the more general imbedding result in [4] cannot be true. For this, we will discuss in detail a family of functions depending on certain parameters. Nevertheless, we want to point out that the incorrectness of the imbedding result in [4] does not influence the main results in the mentioned papers [8, 10, 11, 14]. The counterexamples show also that the result of Amann [2] is really sharp.

The second part of the article deals with the adjoint state of an optimal control problem for the instationary Navier-Stokes equations. Assuming certain regularity of the data, it belongs to the space

$$W(2,4/3;V,V') := \left\{ y \in L^2(0,T;V) : \ \frac{\mathrm{d}}{\mathrm{d}t}y \in L^{4/3}(0,T;V') \right\}.$$

As already mentioned, one cannot expect that this space is continuously imbedded in $C([0,T],H)$. Naturally, there arises the following question: is there an imbedding of $W(2,4/3;V,V')$ in $C([0,T],X)$, where X is a space of weaker topology than H?

The article is organized as follows: In Section 2, we construct families of functions and study regularity properties. In the second part, Section 3, we give a brief overview of the theory of optimal control for instationary Navier-Stokes equations. Finally, we present a regularity result of the adjoint state. A last example shows that this regularity cannot be improved by imbedding arguments.

2. Counterexamples

Here, we will deal with imbeddings of abstract functions in spaces of continuous functions. At first, we state the most classical result in this field. Let $V \hookrightarrow H \hookrightarrow V'$ be a Gelfand triple.

Theorem 2.1. *The space*

$$W(p,q;V,V') := \left\{ y \in L^p(0,T;V) : \ \frac{\mathrm{d}}{\mathrm{d}t}y \in L^q(0,T;V') \right\}$$

is continuously imbedded in $C([0,T],H)$ *if* $\frac{1}{p}+\frac{1}{q}\le 1$.

For the proof in the case $p=q=2$ we refer to [4, Theorem XVIII.1.2.1]. It can be easily adapted to the case $\frac{1}{p}+\frac{1}{q}=1$, cf. [5, Theorem IV.1.17].

In the sequel, we will construct several functions that are in $W(p,q;V,V')$, where p,q do not meet the assumptions of the previous theorem. We prove that in the case $\frac{1}{p}+\frac{1}{q}>1$ there is no imbedding $W(p,q;V,V')\hookrightarrow C([0,T],H)$. We are also looking for a positive result of the following kind: for given p,q the space $W(p,q;V,V')$ is continuously imbedded in $C([0,T],X)$, where X is a space with weaker topology than H.

Consider the following example: Let $\Omega=[0,1]$ and $T>0$. Set $V:=H_0^1(\Omega)$, $H=L^2(\Omega)$, and V' induced by the H-scalar product such that $V\hookrightarrow H\hookrightarrow V'$ forms a Gelfand-triple. Define a function $f_{\alpha,k}$ over $\Omega\times[0,T]$ by

$$f_{\alpha,k}(x,t)=\sum_{n=1}^{\infty} n^{-\frac{1}{2}}e^{(-n^{\alpha}t)}\,\sin n^k\pi x, \tag{1}$$

where k is a natural number.

Lemma 2.2. *The function* $f_{\alpha,k}$ *given by* (1) *has the following properties:*

- **(i)** $f_{\alpha,k}\notin C([0,T];H)$ *for* $\alpha>0$
- **(ii)** $f_{\alpha,k}\in L^p(0,T;V)$ *for* $p<\frac{\alpha}{k+\frac{1}{2}}$
- **(iii)** $\frac{\mathrm{d}}{\mathrm{d}t}f_{\alpha,k}\in L^q(0,T;V')$ *for* $q<\frac{\alpha}{\alpha+\frac{1}{2}-k}$.

Proof. Set $v_n(x):=\sin n^k\pi x$. At first, observe that the functions v_n are orthogonal with respect to the H- as well as to the V-scalar product. It holds

$$|v_n|_H=\frac{1}{\sqrt{2}}\qquad\text{and}\qquad |v_n|_V=\frac{1}{\sqrt{2}}n^k\pi.$$

Now, we want to derive the V'-norm of v_n. Let $\phi\in V$ be a test function. After partial integration, we find using the Cauchy-Schwarz inequality

$$\begin{aligned}\langle v_n,\,\phi\rangle_{V',V}&=\int_0^1\sin(n^k\pi x)\,\phi(x)\,\mathrm{d}x\\&=\frac{1}{n^k\pi}\int_0^1\cos(n^k\pi x)\,\phi'(x)\,\mathrm{d}x\\&\le\frac{1}{\sqrt{2}n^k\pi}|\phi|_V.\end{aligned}$$

This allows us to conclude

$$|v_n|_{V'}\le\frac{1}{\sqrt{2}n^k\pi}.$$

Setting $\phi(x) := v_n(x)$, we obtain

$$|v_n|_{V'} = \frac{1}{\sqrt{2}\pi} n^{-k}.$$

`Claim (i)`: Let $f^N_{\alpha,K}$ be the function defined by the finite series

$$f^N_{\alpha,k}(x,t) = \sum_{n=1}^{N} n^{-\frac{1}{2}} e^{(-n^\alpha t)} \sin n^k \pi x.$$

For $t = 0$, we obtain

$$f^N_{\alpha,k}(x,0) = \sum_{n=1}^{N} n^{-\frac{1}{2}} \sin n^k \pi x.$$

Consequently, we find for the L^2-norm

$$\|f^N_{\alpha,k}(\cdot,0)\|^2_H = \sum_{n=1}^{N} \frac{1}{n} \int_0^T \sin^2 n^k \pi x \, \mathrm{d}x = \sum_{n=1}^{N} \frac{1}{2n}.$$

This series grows unboundedly for $N \to \infty$. Therefore, $f_{\alpha,k}(0)$ cannot belong to H, which implies that $f_{\alpha,k}$ is not in $C([0,T];H)$.

`Claim (ii)`: Again, we consider the finite series $f^N_{\alpha,k}$. We want to estimate the $L^p(0,T;V)$-norm of $f^N_{\alpha,k}$. Using Hölders inequality, we derive first

$$\begin{aligned}\|f^N_{\alpha,k}\|_{L^p(V)} &= \left(\int_0^T \left(\sum_{n=1}^{N} n^{-\frac{1}{2}} e^{(-n^\alpha t)} |v_n|_V \right)^p \mathrm{d}t \right)^{\frac{1}{p}} \\ &\le \sum_{n=1}^{N} \left(\int_0^T \left(n^{-\frac{1}{2}} e^{(-n^\alpha t)} |v_n|_V \right)^p \mathrm{d}t \right)^{\frac{1}{p}}.\end{aligned}$$

The integral on the right-hand side can be computed by

$$\begin{aligned}\int_0^T \left(n^{-\frac{1}{2}} e^{(-n^\alpha t)} |v_n|_V \right)^p \mathrm{d}t &= \int_0^T \left(\frac{\pi}{\sqrt{2}} \right)^p n^{p(k-\frac{1}{2})} e^{(-pn^\alpha t)} \, \mathrm{d}t \\ &\le \frac{1}{p} \left(\frac{\pi}{\sqrt{2}} \right)^p n^{p(k-\frac{1}{2})-\alpha}.\end{aligned}$$

Hence, we arrive at the estimate

$$\|f^N_{\alpha,k}\|_{L^p(V)} \le \frac{\pi}{\sqrt{2}\sqrt[p]{p}} \sum_{n=1}^{N} n^{k-\frac{1}{2}-\frac{\alpha}{p}}.$$

This series will be finite for $N \to \infty$, if

$$k - \frac{1}{2} - \frac{\alpha}{p} < -1$$

or equivalently

$$p < \frac{\alpha}{k + \frac{1}{2}}$$

holds.

`Claim (iii):` Similarly, the $L^q(0, T; V')$-estimate can be proven. We shall begin with

$$\begin{aligned} \left\| \frac{\mathrm{d}}{\mathrm{d}t} f_{\alpha,k}^N \right\|_{L^q(V')} &= \left(\int_0^T \left(\sum_{n=1}^N n^{\alpha - \frac{1}{2}} e^{(-n^\alpha t)} \left| v_n \right|_{V'} \right)^q \mathrm{d}t \right)^{\frac{1}{q}} \\ &\leq \sum_{n=1}^N \left(\int_0^T \left(n^{\alpha - \frac{1}{2}} e^{(-n^\alpha t)} \left| v_n \right|_{V'} \right)^q \mathrm{d}t \right)^{\frac{1}{q}}. \end{aligned}$$

We find for the time integral

$$\begin{aligned} \int_0^T \left(n^{\alpha - \frac{1}{2}} e^{(-n^\alpha t)} \left| v_n \right|_{V'} \right)^q \mathrm{d}t &= \left(\frac{1}{\sqrt{2}\pi} \right)^q n^{q(\alpha - k - \frac{1}{2})} \int_0^T e^{(-qn^\alpha t)} \, \mathrm{d}t \\ &\leq \frac{1}{q} \left(\frac{1}{\sqrt{2}\pi} \right)^q n^{q(\alpha - k - \frac{1}{2}) - \alpha}. \end{aligned}$$

This implies

$$\left\| \frac{\mathrm{d}}{\mathrm{d}t} f_{\alpha,k}^N \right\|_{L^q(V')} \leq \frac{1}{\sqrt{2}\sqrt[q]{q}\,\pi} \sum_{n=1}^N n^{\alpha - k - \frac{1}{2} - \frac{\alpha}{q}}.$$

The series on the right hand side is uniformly bounded for

$$\alpha - \frac{1}{2} - k - \frac{\alpha}{q} < -1,$$

which is equivalent to

$$q < \frac{\alpha}{\alpha + \frac{1}{2} - k},$$

and completes the proof. ∎

Remark 2.3. For $\alpha = p(k + \frac{1}{2}) + \varepsilon$ with some fixed $\varepsilon > 0$, we find that (i) and (ii) are automatically fulfilled. Moreover, we obtain from (iii)

$$q < \frac{pk + \frac{p}{2} + \varepsilon}{(p-1)k + \frac{p}{2} + \varepsilon + \frac{1}{2}}.$$

If k is sufficiently large, the value of q is arbitrary close to

$$p' = \frac{p}{p-1} = \frac{1}{1-\frac{1}{p}}.$$

This shows, that the proposition of Theorem 2.1 is sharp.

Further, we can conclude that there is no imbedding of $W(2,1;V,V')$ in $C([0,T],H)$ as stated in [4, p. 521].

In the following, we will denote by $H^s(\Omega)$ for $-1 \le s \le 0$ the Sobolev-Slobodeckij spaces of fractional order. We have $H^{-1} = V'$ and $H^0 = H$ with the notation already introduced.

Corollary 2.4. *Let us consider the function*

$$f_{\alpha,k,l}(x,t) = \sum_{n=1}^{\infty} n^{-l} e^{(-n^\alpha t)} \sin n^k \pi x.$$

This function satisfies

- **(i)** $f_{\alpha,k,l} \notin C([0,T];H)$ *for* $\alpha > 0$ *and* $l \le \frac{1}{2}$,
- **(ii)** $f_{\alpha,k,l} \in L^p(0,T;V)$ *for* $p < \frac{\alpha}{k-l+1}$,
- **(iii)** $\frac{\mathrm{d}}{\mathrm{d}t} f_{\alpha,k,l} \in L^q(0,T;V')$ *for* $q < \frac{\alpha}{\alpha-k-l+1}$,
- **(iv)** $f_{\alpha,k,l} \in C([0,T];H^{-s})$ *for* $s > \frac{1-l}{k}$,
- **(v)** $f_{\alpha,k,l} \notin C([0,T];H^{-s})$ *for* $s < \frac{\frac{1}{2}-l}{k}$ *and* $l < \frac{1}{2}$.

Proof. The points (i) – (iii) can be shown similarly to Lemma 2.2. To prove (iv) and (v) we use interpolation theory. Given $v \in H$, we have

$$|v|_{H^{-s}} \le c_1 \, |v|_{V'}^{\theta_1} \, |v|_H^{1-\theta_1}$$

with $\theta_1 = s$. For a function $v \in V$, we obtain

$$|v|_H \le c_2 \, |v|_{H^{-s}}^{\theta_2} |v|_V^{1-\theta_2} \tag{2}$$

with $\theta_2 = \frac{1}{1+s}$. Hence, for $v_n(x) = \sin n^k \pi x$ we find

$$|v_n|_{H^{-s}} \le c_1 \, |v_n|_{V'}^{s} |v_n|_H^{1-s} = \frac{c_1}{\sqrt{2}} \pi^{-s} n^{-sk} =: c_3 \, n^{-sk}. \tag{3}$$

Let us denote by $f_{\alpha,k,l}^N$ the finite series

$$f_{\alpha,k,l}^N(x,t) = \sum_{n=1}^{N} n^{-l} e^{(-n^\alpha t)} \sin n^k \pi x.$$

Now, we are ready to prove (iv). We want to show that $f_{\alpha,k,l}$ is in the space $C([0,T],H^{-s})$. To this aim, let $t \in [0,T]$ be given. Let the pre-requisite $s > \frac{1-l}{k}$ be fulfilled. We derive using (3)

$$\|f_{\alpha,k,l}(t)\|_{H^{-s}} = \|\sum_{n=1}^{\infty} n^{-l} e^{(-n^\alpha t)} v_n\|_{H^{-s}} \leq \sum_{n=1}^{\infty} \|n^{-l} v_n\|_{H^{-s}} \leq c_3 \sum_{n=1}^{\infty} n^{-l-sk}.$$

By assumption, we have $-l - sk < -1$ for the exponent. Therefore, we obtain uniform convergence of $f_{\alpha,k,l}(t)$. This uniform convergence and the fact $n^{-l}e^{(-n^\alpha t)}v_n \in C([0,T];H^{-s})$ for each n and fixed s ensures the continuity of the abstract function.

To prove (v) we start from (2) and get

$$\|f^N_{\alpha,k,l}\|_{H^{-s}} \geq c_2^{-s-1} \|f^N_{\alpha,k,l}\|_H^{s+1} \|f^N_{\alpha,k,l}\|_V^{-s},$$

since obviously $f^N_{\alpha,k,l} \neq 0$ holds. For sufficiently large N and $l < \frac{1}{2}$, we estimate the H-Norm of $f^N_{\alpha,k,l}$ by

$$\|f^N_{\alpha,k,l}\|_H^2 = \frac{1}{2}\sum_{n=1}^{N} n^{-2l} \geq \frac{1}{2}\int_1^N x^{-2l}\,\mathrm{d}x = \frac{1}{2}\frac{1}{1-2l}(N^{1-2l}-1) \geq \frac{1}{4}\frac{1}{1-2l}N^{1-2l}.$$

Note, that the estimate is also correct for negative values of l.

Similarly, we have to derive a bound of the V-norm. For sufficiently large N, we obtain

$$\begin{aligned}\|f^N_{\alpha,k,l}\|_V^2 &= \frac{\pi^2}{2}\sum_{n=1}^{N} n^{-2l}n^{2k} \leq \frac{\pi^2}{2}\left(1 + \int_0^N (x+1)^{2k-2l}\,\mathrm{d}x\right)\\ &= \frac{\pi^2}{2}\left(1 + \frac{1}{1+2k-2l}\left((N+1)^{1+2k-2l} - 1\right)\right)\\ &\leq \frac{\pi^2}{1+2k-2l}N^{1+2k-2l}.\end{aligned}$$

Since by assumption $l < \frac{1}{2}$, it holds $1 + 2k - 2l > 0$ for $k > 0$. Altogether, we found

$$\|f^N_{\alpha,k,l}\|_{H^{-s}} \geq cN^{\frac{s+1}{2}(1-2l) - \frac{s}{2}(1+2k-2l)} = cN^{\frac{1}{2}-sk-l},$$

which tends to infinity if $s < \frac{\frac{1}{2}-l}{k}$. Hence, $f_{\alpha,k,l}$ cannot be a function continuous in time with values in such H^{-s} spaces. ∎

These examples show that under certain conditions it may happen that abstract functions are continuous with value in some space H^{-s} but discontinuous with values in spaces of integrable functions.

3. Application to an optimal control problem

In this section, we will consider optimal control of the instationary Navier-Stokes equations. As a model problem serves the minimization of the quadratic objective functional

$$\begin{aligned} J(y,u) = {} & \frac{\alpha_T}{2} \int_\Omega |y(x,T) - y_T(x)|^2 \, \mathrm{d}x \\ & + \frac{\alpha_Q}{2} \int_Q |y(x,t) - y_Q(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t \\ & + \frac{\alpha_R}{2} \int_Q |\operatorname{curl} y(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t + \frac{\gamma}{2} \int_Q |u(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t \end{aligned} \tag{4}$$

subject to the instationary Navier-Stokes equations

$$\begin{aligned} y_t - \nu \Delta y + (y \cdot \nabla) y + \nabla p &= u + f && \text{in } Q \\ \operatorname{div} y &= 0 && \text{in } Q \\ y &= 0 && \text{on } \Gamma \\ y(0) &= y_0 && \text{in } \Omega. \end{aligned} \tag{5}$$

The control u has to be in a set of admissible controls U_{ad},

$$u \in U_{ad}, \tag{6}$$

where U_{ad} is given by

$$U_{ad} = \{ u \in L^2(Q)^2 : \; u_{a,i}(x,t) \le u_i(x,t) \le u_{b,i}(x,t) \text{ a.e. on } Q, \; i = 1,2 \}.$$

Let us denote the optimization problem *minimize the functional* $J(y,u)$ *under the constraints* (5) *and* (6) by **(P)**.

Here, Ω is an open bounded subset of $\mathbb{R}^2$ with C^2-boundary Γ such that Ω is locally on one side of Γ, and Q is defined by $Q = (0,T) \times \Omega$. Further, functions $y_T \in L^2(\Omega)^2$, $y_Q \in L^2(Q)^2$, and $y_0 \in H \subset L^2(\Omega)^2$ are given. The source term f is required to belong to $L^2(0,T;V')$. The parameters γ and ν are positive real numbers. The constraints u_a, u_b are required to be in $L^2(Q)^2$ with $u_{a,i}(x,t) \le u_{b,i}(x,t)$ a.e. on Q, $i = 1,2$.

3.1. Notations and preliminary results. First, we introduce some notations and provide some results that we need later on.

To begin with, we define the solenoidal spaces

$$\begin{aligned} H &:= \{ v \in L^2(\Omega)^2 : \; \operatorname{div} v = 0 \} \\ V &:= \{ v \in H_0^1(\Omega)^2 : \; \operatorname{div} v = 0 \}. \end{aligned}$$

These spaces are Hilbert spaces with their norms denoted by $|\cdot|_H$, $|\cdot|_V$, and scalar products $(\cdot,\cdot)_H$, $(\cdot,\cdot)_V$, respectively. The dual of V with respect to the scalar product of H we denote by V' with the duality pairing $\langle\cdot,\cdot\rangle_{V',V}$.

We shall work in the standard space of abstract functions from $[0,T]$ to a real Banach space X, $L^p(0,T;X)$, endowed with its natural norm

$$\|y\|_{L^p(X)} := \|y\|_{L^p(0,T;X)} = \left(\int_0^T |y(t)|_X^p \,\mathrm{d}t\right)^{\frac{1}{p}}, \quad 1 \le p < \infty$$
$$\|y\|_{L^\infty(X)} := \operatorname*{vrai\,max}_{t\in(0,T)} |y(t)|_X.$$

In the sequel, we will identify the spaces $L^p(0,T;L^p(\Omega)^2)$ and $L^p(Q)^2$ for $1 < p < \infty$, and denote their norm by $\|u\|_p := \|u\|_{L^p(Q)^2}$. The usual $L^2(Q)^2$-scalar product we denote by $(\cdot,\cdot)_Q$ to avoid ambiguity.

In all what follows, $\|\cdot\|$ stands for norms of abstract functions, while $|\cdot|$ denotes norms of "stationary" spaces like H and V.

To deal with the time derivative in (5), we introduce the common spaces of functions y whose time derivatives y_t exist as abstract functions:

$$W^\alpha(0,T;V) := \{y \in L^2(0,T;V) : \; y_t \in L^\alpha(0,T;V')\}$$
$$W(0,T) := W^2(0,T;V),$$

where $1 \le \alpha < \infty$. Endowed with the norm

$$\|y\|_{W^\alpha} := \|y\|_{W^\alpha(0,T;V)} = \|y\|_{L^2(V)} + \|y_t\|_{L^\alpha(V')},$$

these spaces are Banach spaces. Every function of $W(0,T)$ is, up to changes on sets of zero measure, equivalent to a function of $C([0,T],H)$, and the imbedding $W(0,T) \hookrightarrow C([0,T],H)$ is continuous, cf. [1, 12]. As we saw above, there is *no* imbedding $W^\alpha(0,T;V)$ in $C([0,T],H)$ for $\alpha < 2$. However, the space $W(0,T)$ enjoys the following imbedding property:

Lemma 3.1. *The space $W(0,T)$ is continuously imbedded in $L^4(Q)^2$.*

Proof. For $v \in V$, the interpolation inequality

$$|v|_4 \le c|v|_H^{\frac{1}{2}}\,|v|_V^{\frac{1}{2}}$$

holds, cf. [13]. Let $v \in W(0,T)$ be given. Then, we can readily estimate

$$\|v\|_4^4 \le \int_0^T |v(t)|_4^4 \,\mathrm{d}t \le c\int_0^T |v|_H^2\,|v|_V^2 \,\mathrm{d}t \le c\|v\|_{L^\infty(H)}^2 \|v\|_{L^2(V)}^2 \le c\|v\|_W^4,$$

which proves the claim. ∎

For convenience, we define the trilinear form $b : V \times V \times V \mapsto \mathbb{R}$ by

$$b(u,v,w) = ((u\cdot\nabla)v,w)_2 = \int_\Omega \sum_{i,j=1}^{2} u_i \frac{\partial v_j}{\partial x_i}\, w_j \,\mathrm{d}x$$

together with

$$b_Q(u,v,w) = \int_0^T b(u(t),v(t),w(t))\,\mathrm{d}t.$$

An important property of b is given by the fact that for $u \in V$ and sufficient regular v, w it holds

$$b(u,v,w) = -b(u,w,v). \tag{7}$$

There are several estimates of b respectively b_Q available. We mention only the following one which we will need in the sequel. For detailed discussions consult [3, 13, 15].

Lemma 3.2. *Let $u, w \in L^4(Q)^2$ and $v \in L^2(0,T;V)$ be given. Then there is a constant $c > 0$ independently of u, v, w such that*

$$|b_Q(u,v,w)| \le c\,\|u\|_4\,\|v\|_{L^2(0,T;V)}\|w\|_4$$

holds.

To specify the problem setting, we introduce a linear operator A that maps continuously $L^2(0,T;V)$ to $L^2(0,T;V')$ by

$$\int_0^T \big\langle (Ay)(t),\, v(t)\big\rangle_{V',V}\,\mathrm{d}t := \int_0^T \big(y(t),v(t)\big)_V\,\mathrm{d}t,$$

and a nonlinear operator B by

$$\int_0^T \big\langle \big(B(y)\big)(t),\, v(t)\big\rangle_{V',V}\,\mathrm{d}t := \int_0^T b\big(y(t),y(t),v(t)\big)\,\mathrm{d}t.$$

As a conclusion of Lemma 3.1, eq. (7), and Lemma 3.2, we find that B is continuous as an operator from $W(0,T)$ to $L^2(0,T;V')$.

Testing system (5) by divergence-free functions, one obtains the solenoidal form of the Navier-Stokes equations

$$\begin{aligned} y_t + \nu Ay + B(y) &= u + f \\ y(0) &= y_0, \end{aligned}$$

where the first equation has to be understood in the sense of $L^2(0,T;V')$. It is well-known that for all initial values $y_0 \in H$ and source terms $u, f \in L^2(0,T;V')$ there exists a unique weak solution $y \in W(0,T)$ of (5), cf. [3, 13].

We introduce the linearized equation by

$$\begin{aligned} y_t + \nu A y + B'(\bar{y}) y &= u \\ y(0) &= y_0. \end{aligned} \tag{8}$$

Here, $\bar{y}$ is a given state $y \in W(0,T)$. This equation is solvable for all $u \in L^2(0,T;V')$ and $y_0 \in H$. Its unique solution y is in $W(0,T)$.

3.2. Optimality condition. Now, we return to the optimization problem **(P)**. We will call a control $u \in U_{ad}$ *locally optimal*, if there exists a $\rho > 0$ such that

$$J(\bar{y}, \bar{u}) \leq J(y, u)$$

for all $u \in U_{ad}$ with $\|u - \bar{u}\|_2 \leq \rho$. Here, $\bar{y}$ and y denote the states associated with $\bar{u}$ and u, respectively.

A first-order necessary condition for local optimality is stated in the next theorem.

Theorem 3.3. *Let $\bar{u}$ be a locally optimal control with associated state $\bar{y} = y(\bar{u})$. Then there exists a unique solution $\bar{\lambda} \in W^{4/3}(0,T;V)$ of the adjoint equation*

$$\begin{aligned} -\bar{\lambda}_t + \nu A \bar{\lambda} + B'(\bar{y})^* \bar{\lambda} &= \alpha_Q(\bar{y} - y_Q) + \alpha_R \overrightarrow{\operatorname{curl}} \operatorname{curl} \bar{y} \\ \bar{\lambda}(T) &= \alpha_T(\bar{y}(T) - y_T). \end{aligned} \tag{9}$$

Moreover, the variational inequality

$$(\gamma \bar{u} + \bar{\lambda}, u - \bar{u})_{L^2(Q)^2} \geq 0 \quad \forall u \in U_{ad} \tag{10}$$

is satisfied.

Proofs can be found in [6, 7, 14]. The regularity of $\bar{\lambda}$ is proven in [10, Proposition 3.3] for homogeneous initial conditions $\bar{\lambda}(T) = 0$.

The adjoint state λ is the solution of a linearized adjoint equation backward in time. So it is natural, to look for its dependance of the given data. For convenience, we denote by g the right-hand side of (9), and by λ_T the initial value $\alpha_T(\bar{y}(T) - y_T)$.

Theorem 3.4. *Let $\lambda_T \in H$, $g \in L^2(0,T;V')$, and $\bar{y} \in L^2(0,T;V) \cap L^\infty(0,T;H)$ be given. Then there exists a unique weak solution λ of (9) satisfying $\lambda \in W^{4/3}(0,T)$. The mapping $(g, \lambda_T) \mapsto \lambda$ is continuous in the mentioned spaces.*

Proof. At first, denote by w the weak solution of

$$\begin{aligned} -w_t + \nu A w &= g \\ w(T) &= \lambda_T. \end{aligned}$$

Its existence and regularity $w \in W(0,T)$ follows from the solvability of the instationary Stokes-equation, cf. [13]. Moreover, we get the continuity estimate

$$\|w\|_W \le c\left\{\|g\|_{L^2(V')} + |\lambda_T|_H\right\}. \tag{11}$$

Further, let z be the weak solution of

$$\begin{aligned} -z_t + \nu Az + B'(\bar{y})^* z &= -B'(\bar{y})^* w \\ z(T) &= 0. \end{aligned}$$

Since $\bar{y}$ and w are in $W(0,T)$, we get $B'(\bar{y})^* w \in L^{4/3}(0,T;V') \cap W^*(0,T)$ as follows. We write for $v \in W(0,T)$

$$\begin{aligned} \left|[B'(\bar{y})^* w]v\right| &= \left|\int_0^T b(\bar{y},v,w) + b(v,\bar{y},w)\,\mathrm{d}t\right| \\ &\le c\left\{\|\bar{y}\|_4\,\|v\|_{L^2(V)}\|w\|_4 + \|v\|_4\,\|\bar{y}\|_{L^2(V)}\|w\|_4\right\}. \end{aligned}$$

By Lemma 3.2, we conclude

$$\left|[B'(\bar{y})^* w]v\right| \le c\|\bar{y}\|_W\|w\|_W\left\{\|v\|_{L^2(V)} + \|v\|_4\right\}.$$

Since $\|v\|_4 \le c\|v\|_W$, we get $B'(\bar{y})^* w \in W^*(0,T)$. The space V is continuously imbedded in $L^4(\Omega)^2$, which allows us to conclude $B'(\bar{y})^* w \in L^{4/3}(0,T;V')$. Therefore, we arrive at

$$\|B'(\bar{y})^* w\|_{W^*} + \|B'(\bar{y})^* w\|_{L^{4/3}(V')} \le c\|\bar{y}\|_W\|w\|_W. \tag{12}$$

Now, Proposition 2.2.1 in [9] respectively Proposition 2.4 in [10] imply the existence of z together with the regularity $z \in W^{4/3}(0,T)$ and the estimate

$$\|z\|_{W^{4/3}} \le c\left\{\|B'(\bar{y})^* w\|_{L^{4/3}(V')} + \|B'(\bar{y})^* w\|_{W^*}\right\} \le c\|\bar{y}\|_W\|w\|_W. \tag{13}$$

We construct a solution of the inhomogeneous adjoint equation (9) by $\lambda = z + w$. Using (11) and (13),

$$\|\lambda\|_{W^{4/3}} \le \|z\|_{W^{4/3}} + \|w\|_W \le c\,(1 + \|\bar{y}\|_W)\left\{\|g\|_{L^2(V')} + |\lambda_T|_H\right\}$$

is found, and the claim is proven. ∎

Observe, that the conditions of the previous theorem requires the initial value to be in H, whereas the regularity $\lambda \in W^{4/3}(0,T)$ does not guarantee $\lambda(t) \to \lambda_T$ in H for $t \to T$.

If the data are more regular, then the things are much easier. If for instance $f \in L^2(Q)^2$ and $y_0 \in V$ is given together with $y_T \in V$, then the state y and the adjoint λ admit the same regularity: it holds that λ belongs to a space $H^{2,1}$ which is continuously imbedded in $C([0,T],V)$, confer [9, 13].

3.3. Example. In this last section, we will answering the question: can the adjoint state be represented by a continuous abstract function? Clearly, if $\lambda_t \in L^{4/3}(0,T;V')$ together with $\lambda(T) \in V'$ hold, then it is obvious that λ is a continuous function with values in V'. Nevertheless, we are looking for a sharper imbedding result.

Let $\varepsilon > 0$ and integer $k > \frac{3}{2} + 3\varepsilon$ be given. Set $l = 1 - \frac{k}{3} + \varepsilon$ and $\alpha = \frac{8}{3}k$. Notice that by the definition of k and l we have $l < \frac{1}{2}$. Then the function $f := f_{\alpha,k,l}$ introduced in Section 2 fulfills:

(i) $f \in L^p(0,T;V)$ for $p < 2 + \frac{6\varepsilon}{4k-3\varepsilon}$

(ii) $\frac{\mathrm{d}}{\mathrm{d}t} f \in L^q(0,T;V')$ for $q < \frac{4}{3} + \frac{4\varepsilon}{6k-3\varepsilon}$

(iii) $f \in C([0,T],H^{-s})$ for $s > \frac{1}{3} + \frac{\varepsilon}{k}$

(iv) $f \notin C([0,T],H^{-s})$ for $s < \frac{1}{3} - \frac{\frac{1}{2}+\varepsilon}{k}$.

Here, we observe that $f \in W(2,4/3;V,V')$ for all possible k and ε. Thus, f has the same regularity as the adjoint state λ. And we can say that the space $W(2,4/3;V,V')$ is not continuous imbedded in $C([0,T],H^{-s})$ for $s < \frac{1}{3}$.

However, there is a positive result available.

Theorem 3.5. *The space $W(p,q;V,V')$ is* compactly *imbedded in the space $C([0,T],H^{-s})$ for*

$$s > \frac{\frac{1}{p} + \frac{1}{q} - 1}{1 + \frac{1}{p} - \frac{1}{q}}.$$

Proof. For the proof and detailed discussions, we refer to Amann [2]. The notation is adapted to that one used in the present article.

We combine these conclusions to

Corollary 3.6. *The space $W(2,4/3;V,V')$ is continuously imbedded in the space $C([0,T],H^{-s})$ for $s > \frac{1}{3}$. If $s < \frac{1}{3}$ holds, then $W(2,4/3;V,V')$ can not be imbedded in $C([0,T],H^{-s})$.*

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