Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 24 (2005), No. 1, 117–135

Asymptotic Behavior of Discontinuous Solutions to Thermoelastic Systems with Second Sound

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Abstract. We consider the Cauchy problem for linear and semilinear thermoelastic systems with second sound in one space dimension with discontinuous initial data. Due to Cattaneo's law, replacing Fourier's law for heat conduction, the system is strictly hyperbolic. We investigate the behavior of discontinuous solutions as the relaxation parameter tends to zero, which corresponds to a formal convergence of the system to the hyperbolic-parabolic type of classical thermoelasticity. We obtain that the jump of the temperature goes to zero while the jumps of the gradient of the displacement and the spatial derivative of the temperature are propagated along the characteristic curves of the elastic fields when the relaxation parameter vanishes. Moreover, when certain growth conditions are imposed on the nonlinear functions, we deduce that these jumps decay exponentially when the time goes to infinity, more rapidly for small heat conduction coefficient.

Keywords: *semilinear, hyperbolic thermoelasticity, jump*

MSC 2000: 35B40, 74F05

1. Introduction

We consider the semilinear system of thermoelasticity of second sound

$$u_{tt} - \alpha^2 u_{xx} + \beta\theta_x = f(u, u_x, u_t, \theta) \tag{1.1}$$

$$\theta_t + \gamma q_x + \delta u_{tx} = g(u, u_x, u_t, \theta) \tag{1.2}$$

$$\tau q_t + q + \kappa\theta_x = 0 \tag{1.3}$$

in $(0, \infty) \times \mathbb{R}$, with initial conditions

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad \theta(0, x) = \theta_0(x), \quad q(0, x) = q_0(x) \tag{1.4}$$

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ISSN 0232-2064 / $ 2.50

which may have jumps at $x = 0$. Here u, θ, q represent the elastic displacement, the temperature difference and the heat flux, respectively, and are functions of $(t, x) \in (0, \infty) \times \mathbb{R}$. Moreover, $\alpha, \beta, \gamma, \delta, \tau, \kappa$ are positive constants, cp. [2, 7]. Equation (1.3) is Cattaneo's law for heat conduction and turns, as $\tau \to 0$, into Fourier's law

$$q + \kappa\theta_x = 0. \tag{1.5}$$

The system (1.1), (1.2), (1.5) is the hyperbolic-parabolic system of classical thermoelasticity, cp. [5], while we shall study (1.1) – (1.3), which is a strictly hyperbolic system.

Some results on the existence and large time behavior of smooth solutions to system of thermoelasticity of second sound in one space dimension or high space dimensions have been obtained in [7, 8, 11]. There is some literature devoted to the study of the propagation of weak singularities of solutions to the hyperbolic-parabolic system of classical thermoelasticity, and it is observed that the weak singularities will be propagated mainly by the hyperbolic characteristic fields while the parabolic impact exists, cp. [6, 9, 12, 13] and references therein. The question how to describe the behavior of discontinuous solutions to the hyperbolic-parabolic system of thermoelasticity is open. The purpose of the present work is to study this problem by investigating the asymptotic behavior of these jumps as $\tau \to 0$, thus obtaining interesting information about the relation between classical thermoelasticity ($\tau = 0$) and the system considered here.

Assuming that $u_0 \in W^{1,\infty}(\mathbb{R}), (u_1, \theta_0, q_0) \in L^\infty(\mathbb{R})$, and u_0', u_1, θ_0, q_0 have jumps at $x = 0$, we know there exists a unique local solution (u, θ, q) to the semilinear hyperbolic problem (1.1) – (1.4), and it exists globally in time when certain growth conditions are imposed on nonlinear functions f and g, moreover the discontinuities of the initial data will be propagated along the characteristics of the system (1.1) – (1.3) by using the classical theory of hyperbolic equations ([1, 10]). In this paper, we obtain that the jump of θ goes to zero while the jumps of $\nabla_{t,x}u$ and $\partial_x\theta$ are propagated along the characteristic curves $\{x \pm \alpha t = 0\}$ when $\tau \to 0$, which shows a regularizing effect of the parabolic operator for the limit. Moreover, when f and g are independent of (u_t, u_x) or have certain growth restriction with respect to (u_t, u_x) (see Theorem 4.1 for details.), one deduces that the jumps of $\nabla_{t,x}u$ on $\{x \pm \alpha t = 0\}$ decay exponentially when $t \to \infty$, more rapidly for small heat conduction coefficient, which is similar to the phenomenon observed by Hoff ([4]) for the discontinuous solutions to the compressible Navier-Stokes equations. For simplicity of the presentation, we shall study the system (1.1) – (1.3) only for the constant coefficient case, but it is not difficult to see that our all discussion can be extended to the case of variable coefficients.

The remainder of the paper is arranged as follows: In Section 2 we shall study the homogeneous linearized system, i.e. $f = g = 0$ in (1.1), (1.2). The special semilinear system with f and g only depending on (u, θ) will be considered in Section 3, and the general case (1.1) – (1.3) will be investigated in Section 4.

2. The linearized system

First we consider the homogeneous linearized equations corresponding to the system (1.1) – (1.3), i.e. where $f = g = 0$,

$$u_{tt} - \alpha^2 u_{xx} + \beta\theta_x = 0 \tag{2.1}$$

$$\theta_t + \gamma q_x + \delta u_{tx} = 0 \tag{2.2}$$

$$\tau q_t + q + \kappa\theta_x = 0 \tag{2.3}$$

Let

$$U := \begin{pmatrix} u_t + \alpha u_x \\ u_t - \alpha u_x \\ \theta \\ q \end{pmatrix},$$

then U satisfies

$$\partial_t U + A_1 \partial_x U + A_0 U = 0, \tag{2.4}$$

where

$$A_1 := \begin{pmatrix} -\alpha & 0 & \beta & 0 \\ 0 & \alpha & \beta & 0 \\ \frac{\delta}{2} & \frac{\delta}{2} & 0 & \gamma \\ 0 & 0 & \frac{\kappa}{\tau} & 0 \end{pmatrix}, \qquad A_0 := \text{diag}\left[0, 0, 0, \frac{1}{\tau}\right].$$

The characteristic polynomial for A_1 equals

$$det(\lambda \cdot Id - A_1) = \lambda^4 - (\alpha^2 + \delta\beta + \frac{\kappa\gamma}{\tau})\lambda^2 + \alpha^2 \frac{\kappa\gamma}{\tau}$$

hence the eigenvalues are

$$\lambda_{1,2,3,4} = \pm\sqrt{\frac{\alpha^2 + \delta\beta + \frac{\kappa\gamma}{\tau} \pm \sqrt{(\alpha^2 + \delta\beta + \frac{\kappa\gamma}{\tau})^2 - 4\alpha^2\frac{\kappa\gamma}{\tau}}}{2}},$$

and (2.1) – (2.3) is strictly hyperbolic, cp. [7]. Since

$$\sqrt{(\alpha^2 + \delta\beta + \frac{\kappa\gamma}{\tau})^2 - 4\alpha^2\frac{\kappa\gamma}{\tau}} = \frac{\kappa\gamma}{\tau} + \delta\beta - \alpha^2 + 2\tau\frac{\alpha^2\delta\beta}{\kappa\gamma} + O(\tau^2)$$

we obtain

$$\lambda_{1,2} = \mp\alpha(1 - \frac{\delta\beta}{2\kappa\gamma}\tau) + O(\tau^2), \quad \lambda_{3,4} = \mp\sqrt{\frac{\kappa\gamma}{\tau}} + O(\sqrt{\tau}). \tag{2.5}$$

Computing the right eigenvectors $r_k = (r_{1k}, r_{2k}, r_{3k}, r_{4k})'$ with $(\lambda_k \cdot Id - A_1)r_k = 0$ for $k = 1, 2, 3, 4$, we get

$$r_1 = \begin{pmatrix} 1 \\ \frac{\lambda_1+\alpha}{\lambda_1-\alpha} \\ \frac{\lambda_1+\alpha}{\beta} \\ \frac{\kappa(\lambda_1+\alpha)}{\tau\beta\lambda_1} \end{pmatrix}, \; r_2 = \begin{pmatrix} \frac{\lambda_2-\alpha}{\lambda_2+\alpha} \\ 1 \\ \frac{\lambda_2-\alpha}{\beta} \\ \frac{\kappa(\lambda_2-\alpha)}{\tau\beta\lambda_2} \end{pmatrix}, \; r_k = \begin{pmatrix} \frac{\beta}{\lambda_k+\alpha} \\ \frac{\beta}{\lambda_k-\alpha} \\ 1 \\ \frac{\kappa}{\tau\lambda_k} \end{pmatrix} \text{ for } k = 3, 4. \tag{2.6}$$

The left eigenvectors $l_k = (l_{k1}, l_{k2}, l_{k3}, l_{k4})$ with $l_k(\lambda_\kappa \cdot Id - A_1) = 0$ for $k = 1, 2, 3, 4$ are given by

$$\begin{aligned} l_1 &= c_1\left(1, \frac{\lambda_1+\alpha}{\lambda_1-\alpha}, \frac{2(\lambda_1+\alpha)}{\delta}, \frac{2\gamma(\lambda_1+\alpha)}{\lambda_1\delta}\right) \\ l_2 &= c_2\left(\frac{\lambda_2-\alpha}{\lambda_2+\alpha}, 1, \frac{2(\lambda_2-\alpha)}{\delta}, \frac{2\gamma(\lambda_2-\alpha)}{\lambda_2\delta}\right) \\ l_k &= c_k\left(\frac{\delta}{2(\lambda_k+\alpha)}, \frac{\delta}{2(\lambda_k-\alpha)}, 1, \frac{\gamma}{\lambda_k}\right) \qquad \text{for } k = 3, 4 \end{aligned} \tag{2.7}$$

for constants $\{c_k\}_{k=1}^4$ satisfying the normalization

$$l_i r_k = \delta_{ik} = \begin{cases} 1, & i = k \\ 0, & i \neq k. \end{cases}$$

By a simple computation, we can choose

$$c_1 = 1 + O(\tau), \quad c_2 = 1 + O(\tau), \quad c_3 = \frac{1}{2} + O(\tau), \quad c_4 = \frac{1}{2} + O(\tau) \tag{2.8}$$

in (2.7) to have the above normalization.

We study the thermoelastic system (2.1) – (2.3) with initial data

$$u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ \theta(0,x) = \theta_0(x),\ q(0,x) = q_0(x), \tag{2.9}$$

where u_0, u_1, θ_0 and q_0 are piecewise smooth with possible jumps at $x = 0$. Let

$$L := \begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{pmatrix}, \quad R := (r_1, r_2, r_3, r_4)$$

with $l_j = (l_{j1}, \ldots, l_{j4})$ and $r_j = (r_{1j}, \ldots, r_{4j})'$, then $V = LU$ satisfies

$$\partial_t V + \Lambda \partial_x V + \hat{A}_0 V = 0 \tag{2.10}$$

$$V(0,x) \equiv V_0(x) \equiv \begin{cases} V_0^-(x), & x < 0 \\ V_0^+(x), & x > 0 \ , \end{cases} \tag{2.11}$$

where $\Lambda = diag[\lambda_1, \lambda_2, \lambda_3, \lambda_4]$,

$$\hat{A}_0 := LA_0R = \frac{1}{\tau}\begin{pmatrix} l_{14}r_{41} & \ldots & l_{14}r_{44} \\ \vdots & & \vdots \\ l_{44}r_{41} & \cdots & l_{44}r_{44} \end{pmatrix}$$

and $V_0(x) = L(u_1 + \alpha u_0', u_1 - \alpha u_0', \theta_0, q_0)'$. By

$$\Sigma_k := \{(t,x) \mid x - \lambda_k t = 0\}, \quad 1 \le k \le 4$$

we denote the characteristic lines for (2.10). If $V = (V_1, V_2, V_3, V_4)'$ has a jump on Σ_k for some fixed $k \in \{1,2,3,4\}$, then for $i \in \{1,2,3,4\}$, from the differential equations (2.10), we know that

$$(\partial_t + \lambda_i \partial_x) V_i = -\frac{1}{\tau} \sum_{j=1}^{4} l_{i4} r_{4j} V_j \tag{2.12}$$

should be locally bounded everywhere. However, for any $i \neq k, X_i := \partial_t + \lambda_i \partial_x$ is transversal to Σ_k. We obtain

$$(\partial_t + \lambda_i \partial_x)\ V_{i|\Sigma_k} = \text{const} \cdot [V_i]\ _{\Sigma_k} \widetilde{\delta}_{\Sigma_k}, \tag{2.13}$$

where $\widetilde{\delta}_{\Sigma_k}$ is the Dirac measure supported on Σ_k. Here and afterward we denote by $[V_i]_{\Sigma_k}$ the jump of V_i on Σ_k, i.e. at (t^*, x^*) with $x^* = \lambda_k t^*$:

$$[V_i]_{\Sigma_k}(t^*, x^*) := \lim_{\substack{(t,x)\to(t^*,x^*) \\ x > \lambda_k t}} V_i(t,x) - \lim_{\substack{(t,x)\to(t^*,x^*) \\ x < \lambda_k t}} V_i(t,x).$$

From (2.12) and (2.13), one gets

$$[V_i]\ _{\Sigma_k} = 0, \qquad i \neq k, \tag{2.14}$$

that is, V_i has no jump on Σ_k if $i \neq k$.

Moreover, by (2.10)

$$(\partial_t + \lambda_k \partial_x)\ V_k + \frac{1}{\tau} \sum_{j=1}^{4} l_{k4}\ r_{4j}\ V_j = 0,$$

which implies

$$(\partial_t + \lambda_k \partial_x)\, [V_k]_{\Sigma_k} + \frac{1}{\tau}\, l_{k4}\, r_{4k}\, [V_k]_{\Sigma_k} = 0 \tag{2.15}$$

by using (2.14) and noting that $\partial_t + \lambda_k \partial_x$ is tangential to Σ_k.

If we can show that

$$l_{k4}\, r_{4k} \geq C > 0 \tag{2.16}$$

uniformly as $\tau > 0$, then (2.15) will imply that the jump of V_k on Σ_k decays exponentially as $\tau \to 0$. Rel. (2.16) will be true for $k = 3, 4$ uniformly in τ, and $l_{k4}\, r_{4k}$ are of order τ for $k = 1, 2$. Indeed, from (2.6) and (2.7) we have

$$l_{34}\, r_{43} = \frac{1}{2} + o(1) \tag{2.17}$$

as $\tau \to 0$. By (2.15), (2.17) we conclude

$$[V_3]_{\Sigma_3} = [V_3]_{\Sigma_3(0)}\; e^{\,-\frac{t}{\tau}\, l_{34}\, r_{43}} \longrightarrow 0 \quad (\tau \to 0), \tag{2.18}$$

where $[V_3]_{\Sigma_3(0)}$ denotes the initial jump of V_3, that is, the jump of V_3 along Σ_3 decays exponentially as $\tau \to 0$. Similarly we obtain for V_4 along Σ_4

$$[V_4]_{\Sigma_4} = [V_4]_{\Sigma_4(0)}\; e^{\,-\frac{t}{\tau}\, l_{44}\, r_{44}} \longrightarrow 0 \quad (\tau \to 0) \tag{2.19}$$

with

$$l_{44}\, r_{44} = \frac{1}{2} + o(1). \tag{2.20}$$

Knowing for classical thermoelasticity that there is in general no smoothing effect, but that singularities are propagated essentially as far as the pure elastic part (wave equation for the displacement u) (cp. [6, 9, 12, 5]), we cannot hope for a similar behavior of the remaining components $[V_1]_{\Sigma_1}, [V_2]_{\Sigma_2}$. Instead we obtain, observing

$$\lambda_{1,2} = \mp\, \alpha\, (1 - \frac{\delta\beta}{2\kappa\gamma}\, \tau) + O\,(\tau^2),$$

that

$$l_{14}\, r_{41} = \frac{\beta\delta}{2\kappa\gamma}\, \tau + O\,(\tau^2), \quad l_{24}\, r_{42} = \frac{\beta\delta}{2\kappa\gamma}\, \tau + O\,(\tau^2).$$

Hence, by (2.15), for $k = 1, 2$

$$(\partial_t + \lambda_k \partial_x)\, [V_k]_{\Sigma_k} + \left(\frac{\beta\delta}{2\kappa\gamma} + O(\tau) \right) [V_k]_{\Sigma_k} = 0$$

implying

$$[V_k]_{\Sigma_k} = [V_k]_{\Sigma_k(0)}\; e^{-\frac{\beta\delta}{2\kappa\gamma}\, t \,+\, O(\tau)\, t}, \tag{2.21}$$

that is, these jumps will not disappear as $\tau \to 0$, but for fixed small τ, are decaying exponentially as $t \to \infty$. Returning from V to the variable

$$U = (u_t + \alpha u_x,\ u_t - \alpha u_x,\ \theta,\ q)' = RV = \sum_{j=1}^{4} V_j\, r_j,$$

we obtain for (u, θ), using (2.18), (2.19), (2.21),

$$\begin{aligned} u_t + \alpha\, u_x &= V_1 + \frac{\lambda_2 - \alpha}{\lambda_2 + \alpha}\, V_2 + \frac{\beta}{\lambda_3 + \alpha}\, V_3 + \frac{\beta}{\lambda_4 + \alpha}\, V_4 \\ &= V_1 +\ O(\tau)V_2\ +\ O(\sqrt{\tau})V_3\ +\ O(\sqrt{\tau})V_4 \end{aligned} \tag{2.22}$$

showing a persistent jump in V_1 as $\tau \to 0$, similarly for

$$\begin{aligned} u_t - \alpha\, u_x &= \frac{\lambda_1 + \alpha}{\lambda_1 - \alpha}\, V_1 + V_2 + \frac{\beta}{\lambda_3 - \alpha}\, V_3 + \frac{\beta}{\lambda_4 - \alpha}\, V_4 \\ &= O\ (\tau)\ V_1 + V_2 + O\ (\sqrt{\tau})V_3\ +\ O\ (\sqrt{\tau})V_4 \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} \theta &= \frac{\lambda_1 + \alpha}{\beta}\, V_1 + \frac{\lambda_2 - \alpha}{\beta}\, V_2 + V_3 + V_4 \\ &= O\ (\tau)\ V_1 + O\ (\tau)\ V_2 + V_3 + V_4, \end{aligned} \tag{2.24}$$

that is, the jumps of $u_t \pm \alpha u_x$ persist while the jump in the temperature θ disappears as $\tau \to 0$, which is the "final" parabolic effect being present.

Finally, we shall demonstrate that jumps in the first spatial derivative of θ will persist on Σ_1 and Σ_2. For this, we need to study the behavior of $[\partial_x V_k]_{\Sigma_j}$ $(k = 1, \ldots, 4; j = 1, 2)$.

The differential equation (2.10) yields

$$\Big(\partial_t + \lambda_k \partial_x\Big)\ V_k + \frac{1}{\tau} \sum_{j=1}^{4} l_{k4}\ r_{4j}\ V_j = 0, \quad k = 1, 2, 3, 4. \tag{2.25}$$

Observing (2.14) we conclude

$$\Big[(\partial_t + \lambda_k \partial_x)\ V_k\Big]_{\Sigma_j} + \frac{1}{\tau}\ l_{k4}\ r_{4j}\ [V_j]_{\Sigma_j} = 0, \quad j \neq k \tag{2.26}$$

by noting that $[V_k]_{\Sigma_j} = 0$ and $\partial_t + \lambda_j \partial_x$ tangential to Σ_j imply

$$[(\partial_t + \lambda_j \partial_x) V_k]_{\Sigma_j} = 0, \quad j \neq k. \tag{2.27}$$

Since $\partial_t + \lambda_k \partial_x$ is transversal to Σ_j $(j \neq k)$ we can obtain $[\nabla_{(t,x)} V_k]_{\Sigma_j} (j \neq k)$ from $[(\partial_t + \lambda_k\, \partial_x) V_k]_{\Sigma_j}$.

In detail, for the case $(k, j) = (1, 2)$, we have

$$\Big[(\partial_t + \lambda_1 \partial_x)\; V_1\Big]_{\Sigma_2} + \frac{1}{\tau}\,\big(1 + O(\tau)\big)\frac{2\kappa\gamma(\lambda_1 + \alpha)(\lambda_2 - \alpha)}{\lambda_1 \lambda_2 \tau \beta \delta}\;[V_2]_{\Sigma_2} = 0$$

hence

$$\Big[(\partial_t + \lambda_1 \partial_x)\; V_1\Big]_{\Sigma_2} + \Big(\frac{\beta\delta}{2\kappa\gamma} + O(\tau)\Big)[V_2]_{\Sigma_2} = 0$$

which implies, using (2.21),

$$\Big[(\partial_t + \lambda_1 \partial_x)\; V_1\Big]_{\Sigma_2} = -\frac{\beta\delta}{2\kappa\gamma}[V_2]_{\Sigma_2(0)}\; e^{-\frac{\beta\delta}{2\kappa\gamma}t} + O(\tau). \tag{2.28}$$

Similarly, we obtain

$$\Big[(\partial_t + \lambda_2 \partial_x)\; V_2\Big]_{\Sigma_1} = -\frac{\beta\delta}{2\kappa\gamma}[V_1]_{\Sigma_1(0)}\; e^{-\frac{\beta\delta}{2\kappa\gamma}t} + O(\tau) \tag{2.29}$$

$$\Big[(\partial_t + \lambda_3 \partial_x)\; V_3\Big]_{\Sigma_j} = -\frac{\delta}{4\sqrt{\gamma\kappa\tau}}\big(1 + O(\tau)\big)[V_j]_{\Sigma_j}, \qquad j = 1, 2 \tag{2.30}$$

$$\Big[(\partial_t + \lambda_4 \partial_x)\; V_4\Big]_{\Sigma_j} = \frac{\delta}{4\sqrt{\kappa\gamma\tau}}\big(1 + O(\tau)\big)[V_j]_{\Sigma_j}, \qquad j = 1, 2. \tag{2.31}$$

Combining (2.30) and (2.27) for the case $(k, j) = (3, 1)$, it follows

$$[\partial_x V_3]_{\Sigma_1} = \frac{\delta}{4\kappa\gamma}\big(1 + O(\tau)\big)[V_1]_{\Sigma_1}. \tag{2.32}$$

Similarly, we get

$$[\partial_x V_4]_{\Sigma_1} = \frac{\delta}{4\kappa\gamma}\big(1 + O(\tau)\big)[V_1]_{\Sigma_1} \tag{2.33}$$

$$[\partial_x V_2]_{\Sigma_1} = -\left(\frac{\beta\delta}{4\alpha\kappa\gamma} + O(\tau)\right)[V_1]_{\Sigma_1}. \tag{2.34}$$

Observing (2.27) we obtain

$$[\partial_t V_3]_{\Sigma_1} = \frac{\delta\alpha}{4\kappa\gamma}\big(1 + O(\tau)\big)[V_1]_{\Sigma_1} \tag{2.35}$$

$$[\partial_t V_4]_{\Sigma_1} = \frac{\delta\alpha}{4\kappa\gamma}\big(1 + O(\tau)\big)[V_1]_{\Sigma_1} \tag{2.36}$$

$$[\partial_t V_2]_{\Sigma_1} = -\left(\frac{\beta\delta}{4\kappa\gamma} + O(\tau)\right)[V_1]_{\Sigma_1}. \tag{2.37}$$

In order to estimate $[\nabla_{(t,x)}V_1]_{\Sigma_1}$ we notice that $(\lambda_1\partial_t - \partial_x)$ is transversal to Σ_1, hence, applied to (2.25) for $k=1$, we get

$$0 = \Big[(\partial_t + \lambda_1\partial_x)(\lambda_1\partial_t - \partial_x)\ V_1 + \frac{1}{\tau}\sum_{j=1}^{4} l_{14}\ r_{4j}(\lambda_1\partial_t - \partial_x)\ V_j\Big]_{\Sigma_1}$$

implying

$$\begin{aligned}(\partial_t + \lambda_1\partial_x)\Big[(\lambda_1\partial_t - \partial_x)V_1\Big]_{\Sigma_1} + \frac{1}{\tau}\ l_{14}\ r_{41}\Big[(\lambda_1\partial_t - \partial_x)\ V_1\Big]_{\Sigma_1} \\ = -\frac{1}{\tau}\sum_{j=2}^{4} l_{14}\ r_{4j}\Big[(\lambda_1\partial_t - \partial_x)\ V_j\Big]_{\Sigma_1}.\end{aligned} \tag{2.38}$$

Substituting (2.32) – (2.37) into (2.38) it follows

$$\begin{aligned}(\partial_t + \lambda_1\ \partial_x)\Big[(\lambda_1\partial_t - \partial_x)\ V_1\Big]_{\Sigma_1} + \frac{\beta\delta}{2\kappa\gamma}\big(1 + O(\tau)\big)\Big[(\lambda_1\partial_t - \partial_x)\Big]_{\Sigma_1} \\ = -\frac{\beta^2\delta^2(\alpha^2+1)}{8\kappa^2\gamma^2\alpha}\ \big(1 + O(\sqrt{\tau})\big)\Big[V_1\Big]_{\Sigma_1}.\end{aligned}$$

This implies, using (2.21),

$$\begin{aligned}&\Big[(\lambda_1\partial_t - \partial_x)\ V_1\Big]_{\Sigma_1} \\ &\quad = \Big[(\lambda_1\partial_t - \partial_x)\ V_1\Big]_{\Sigma_1(0)} e^{-\frac{\beta\delta}{2\kappa\gamma}\big(1+O(\tau)\big)t} \\ &\quad\quad - \frac{\beta^2\delta^2(\alpha^2+1)}{8\kappa^2\gamma^2\alpha}\ \big(1 + O(\sqrt{\tau})\big)\int_0^t e^{-\frac{\beta\delta}{2\kappa\gamma}\big(1+O(\tau)\big)(t-s)}\ \Big[V_1\Big]_{\Sigma_1(s)} ds \\ &\quad = \Big[(\lambda_1\partial_t - \partial_x)\ V_1\Big]_{\Sigma_1(0)} e^{-\frac{\beta\delta}{2\kappa\gamma}\big(1+O(\tau)\big)t} \\ &\quad\quad - \frac{\beta^2\delta^2(\alpha^2+1)}{8\kappa^2\gamma^2\alpha}\ \Big(1 + O(\sqrt{\tau})\Big)te^{-\frac{\beta\delta}{2\kappa\gamma}\big(1+O(\tau)\big)t}\ \Big[V_1\Big]_{\Sigma_1(0)}\end{aligned} \tag{2.39}$$

which decays exponentially for fixed τ as $t \to \infty$. On the other hand, from (2.14) and the equation

$$(\partial_t + \lambda_1\ \partial_x)V_1 = -\frac{1}{\tau}\sum_{j=1}^{4} l_{14}\ r_{4j}\ V_j$$

we get

$$\Big[(\partial_t + \lambda_1\partial_x)\ V_1\Big]_{\Sigma_1} = -\frac{1}{\tau}\ l_{14}\ r_{41}\Big[V_1\Big]_{\Sigma_1} = -\frac{\beta\delta}{2\kappa\gamma}\big(1 + O(\tau)\big)\Big[V_1\Big]_{\Sigma_1}. \tag{2.40}$$

Combining (2.40) with (2.39) we get

$$(\lambda_1^2+1)[\partial_x V_1]_{\Sigma_1} = \lambda_1\Big[(\partial_t+\lambda_1\,\partial_x)V_1\Big]_{\Sigma_1} - \Big[(\lambda_1\partial_t-\partial_x)\,V_1\Big]_{\Sigma_1}$$

or

$$\begin{aligned}[\partial_x V_1]_{\Sigma_1} = {} & \frac{\beta\delta\alpha}{2\kappa\gamma(\alpha^2+1)}\big(1+O(\tau)\big)\Big[V_1\Big]_{\Sigma_1} \\ & + \frac{\beta^2\delta^2}{8\kappa^2\gamma^2\alpha}\big(1+O(\sqrt{\tau})\big)te^{-\frac{\beta\delta}{2\kappa\gamma}\left(1+O(\tau)\right)t}\Big[V_1\Big]_{\Sigma_1(0)} \\ & - \frac{1}{\alpha^2+1}\Big[(\lambda_1\partial_t-\partial_x)\,V_1\Big]_{\Sigma_{1(0)}} e^{-\frac{\beta\delta}{2\kappa\gamma}\left(1+O(\tau)\right)t}. \end{aligned} \tag{2.41}$$

Therefore, by using (2.24), (2.32), (2.33), (2.34) and (2.41) we conclude

$$\begin{aligned}[\partial_x\,\theta]_{\Sigma_1} &= \Big[O(\tau)\partial_x V_1 + O(\tau)\partial_x V_2 + \partial_x V_3 + \partial_x V_4\Big]_{\Sigma_1} \\ &= \frac{\delta}{2\kappa\gamma}\big(1+O(\tau)\big)\Big[V_1\Big]_{\Sigma_1} + O(\tau) \\ &\xrightarrow{\tau\to 0} \frac{\delta}{2\kappa\gamma}\lim_{\tau\to 0}\Big[V_1\Big]_{\Sigma_1}, \end{aligned} \tag{2.42}$$

that is, the jump in $\partial_x\,\theta$ will not disappear in general.

Summarizing, essentially (2.14), (2.18) – (2.24), (2.42), we have proved the following theorem.

Theorem 2.1. *Let (u,θ,q) be the solution to (2.1) – (2.3) with initial data satisfying the following: u_0', u_1, θ_0, q_0 are piecewise smooth with a possible jump at $x=0$. Then, along the characteristic curves, we have as $\tau\longrightarrow 0$*

$$[\nabla_{(t,x)}u]_{\Sigma_{3,4}} \longrightarrow 0, \quad [\theta]_{\Sigma_{3,4}} \to 0, \quad [\partial_x\,\theta]_{\Sigma_{3,4}} \longrightarrow 0$$

exponentially in τ, and $[\theta]_{\Sigma_{1,2}} = O(\tau)$ for any fixed (t,x). The jumps in the gradient of u and $\partial_x\,\theta$ do not disappear as $\tau\to 0$ on $\Sigma_{1,2}$, but, for fixed small τ, they decay exponentially as $t\to\infty$. Moreover, the rates of exponential decay are given explicitly in terms of the coefficients of the differential equations (2.1) – (2.3).

Remark 2.2. From (2.21), (2.22), (2.23) and (2.42) we see that the jumps on $\Sigma_{1,2}$ in $\nabla_{t,x}u$ and $\partial_x\theta$ decay more rapidly for small heat conduction coefficient $\kappa\gamma$ when $t\to+\infty$, which is similar to the phenomenon, observed in David Hoff [3, 4], that for the discontinuous solutions of the compressible Navier-Stokes equations, the jumps of the fluid density and velocity gradient across the particle path decay exponentially in time, more rapidly for small viscosities.

3. The semilinear system (I)

We now turn to the system (1.1) – (1.3) for the special semilinear case where f and g depend at most on u and θ. Again the initial data are assumed to be piecewise smooth with possible jumps at $x = 0$; additionally f and g are assumed to be smooth and globally Lipschitz in their arguments for simplicity. These assumptions are not optimal certainly, e.g., the global Lipschitz property of (f, g) is used in deriving estimates (3.8), (3.9), (3.12) and (3.13) below. When the initial data satisfy $u_0 \in W^{1,\infty}$ and $(u_1, \theta_0, q_0) \in L^\infty$, then it is easy to show that the solution to the problem (1.1) – (1.3) is bounded locally in time, and it even can be bounded globally in time when certain growth conditions are imposed on (f, g). If this is true, then it is not necessary to require (f, g) to be Lipschitz, obviously.

Denoting by

$$u_\pm := (\partial_t \pm \alpha\partial_x)u$$

and

$$U := (u, u_+, u_-, \theta, q)' \equiv (U_0, U_1, U_2, U_3, U_4)'$$

we obtain the following first order system for U:

$$\partial_t U + B_1\partial_x U + B_0 U = F(U) \tag{3.1}$$

$$U(0, x) \equiv U_0(x) \equiv \begin{cases} U_0^-(x), & x < 0 \\ U_0^+(x), & x > 0\,, \end{cases} \tag{3.2}$$

where $U_0(x) = (u_0, u_1 + \alpha u_0', u_1 - \alpha u_0', \theta_0, q_0)'$,

$$B_1 := \begin{pmatrix} \alpha & 0 \\ 0 & A_1 \end{pmatrix}, \quad B_0 := \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & & & & \\ 0 & & & & \\ 0 & & & A_0 & \\ 0 & & & & \end{pmatrix}$$

with A_1, A_0 being given in (2.4) and

$$F(U) := \Big(0,\ f(U_0, U_3),\ f(U_0, U_3),\ g(U_0, U_3),\ 0\Big)'.$$

By $r_1, ..., r_4, l_1, ..., l_4$ we denote the right resp. left eigenvectors to A_1 as given in (2.6), (2.7), but also their natural extension to $\mathbb{R}^5$ adding a leading zero:

$$r_1 = \Big(0,\ 1,\ \frac{\lambda_1 + \alpha}{\lambda_1 - \alpha},\ \frac{\lambda_1 + \alpha}{\beta},\ \frac{\kappa(\lambda_1 + \alpha)}{\tau\beta\lambda_1}\Big)'$$

and so on. If

$$\lambda_0 := \alpha, \quad r_0 := (1, 0, 0, 0, 0)', \quad l_0 := (1, 0, 0, 0, 0),$$

then $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$ are the eigenvalues of B_1 with right and left eigenvectors $(r_j, l_j),\ j = 0, 1, 2, 3, 4$. Let now

$$R := (r_0, ..., r_4), \quad L := (l_0', ..., l_4')', \quad V := LU,$$

then

$$\partial_t V + \widetilde{\Lambda}\, \partial_x V + \widetilde{B}_0 V = \widetilde{F}(V) \tag{3.3}$$

$$V(0, x) = V_0(x), \tag{3.4}$$

where

$$\widetilde{\Lambda} := \operatorname{diag}[\lambda_0, ..., \lambda_4], \quad \widetilde{B}_0 := \begin{pmatrix} 0 & -r_{11} & -r_{12} & -r_{13} & -r_{14} \\ 0 & & & & \\ 0 & & \hat{A}_0 & & \\ 0 & & & & \\ 0 & & & & \end{pmatrix} \tag{3.5}$$

with $\hat{A}_0$ from (2.10), and

$$\begin{aligned} \widetilde{F}(V) := LF(RV) = (0,\ & l_{11}f + l_{12}f + l_{13}g,\ l_{21}f + l_{22}f + l_{23}g, \\ & l_{31}f + l_{32}f + l_{33}g,\ l_{41}f + l_{42}f + l_{43}g)'. \end{aligned} \tag{3.6}$$

Denote by $\Sigma_k := \{(t, x)| x - \lambda_k t = 0\}$ the characteristic lines, $k = 0, 1, 2, 3, 4$. Then V_j does not jump on Σ_k when $j \neq k$, which can be obtained as in Section 2. Moreover, under the assumption that u_0 is continuous on $\mathbb{R}$, V_0 does not jump at all which immediately follows from the differential equation in (3.3).

`(i)`: The behavior of $[V_k]_{\Sigma_k}$ for $k = 3, 4$.
The equation (3.3) yields

$$(\partial_t + \lambda_k\, \partial_x)[V_k]_{\Sigma_k} + \frac{1}{\tau}\, l_{k4} r_{4k}\, [V_k]_{\Sigma_k} = [\widetilde{F}_k\ (V)]_{\Sigma_k}$$

which implies

$$[V_k]_{\Sigma_k} = [V_k]_{\Sigma_k(0)} e^{-\frac{t}{\tau}\, l_{k4} r_{4k}} + \int_0^t [\widetilde{F}_k\ (V)]_{\Sigma_k(s)} e^{-\frac{t-s}{\tau}\, l_{k4} r_{4k}}\, ds. \tag{3.7}$$

On the other hand we have, using the expansions from Section 2,

$$\begin{aligned} \widetilde{F}_3(V) &= (l_{31}f + l_{32}f + l_{33}g)(RV) \\ &= (l_{31}f + l_{32}f + l_{33}g)\Big(V_0, \sum_{j=1}^{4} r_{3j} V_j\Big) \\ &= \Big(\frac{1}{2} + O(\tau)\Big)\Big(g + O(\tau^{\frac{1}{2}}) f\Big)\Big(V_0, \sum_{j=1}^{2} r_{3j} V_j + V_4 + V_3\Big) \end{aligned}$$

which implies

$$\big|[\widetilde{F}_3(V)]_{\Sigma_3(s)}\big| \leq c\,\big|[V_3]_{\Sigma_3(s)}\big| \tag{3.8}$$

for a constant $c > 0$. Similarly, we have

$$\widetilde{F}_4(V) = \Big(\frac{1}{2} + O(\tau)\Big)\Big(g + O(\tau^{\frac{1}{2}})f\Big)\Big(V_0, \sum_{j=1}^{3} r_{3j}V_j + V_4\Big)$$

which implies

$$\big|[\widetilde{F}_4(V)]_{\Sigma_4(s)}\big| \leq c\,\big|[V_4]_{\Sigma_4(s)}\big|. \tag{3.9}$$

Substituting (3.8) into (3.7) and using $l_{34}r_{43} = \frac{1}{2} + O(\tau)$ (see Section 2), we obtain

$$\big|[V_3]_{\Sigma_3}\big| \leq \big|[V_3]_{\Sigma_3(0)}\big|e^{-\frac{t}{\tau}l_{34}r_{43}} + c\int_0^t \big|[V_3]_{\Sigma(s)}\big|e^{-\frac{t-s}{\tau}l_{34}r_{43}}ds$$

which implies

$$\big|[V_3]_{\Sigma_3}\big| \leq \big|[V_3]_{\Sigma_3(0)}\big|e^{-\frac{t}{\tau}l_{34}r_{43}+ct} = \big|[V_3]_{\Sigma_3(0)}\big|e^{-\frac{t}{2\tau}(1+O(\tau))+ct}, \tag{3.10}$$

that is, $[V_3]_{\Sigma_3}$ decays again exponentially fast as $\tau \to 0$.

Similarly, we can obtain the exponential decay for $[V_4]_{\Sigma_4}$:

$$\big|[V_4]_{\Sigma_4}\big| \leq \big|[V_4]_{\Sigma_4(0)}\big|e^{-\frac{t}{2\tau}(1+O(\tau))+ct}. \tag{3.11}$$

`(ii):` The behavior of $[V_k]_{\Sigma_k}$ for $k = 1, 2$.
Equation (3.7) holds again for $k = 1, 2$, and we have

$$\widetilde{F}_1(V) = \big(1 + O(\tau)\big)\Big(f - \frac{\delta\beta}{4\kappa\gamma}\tau f + \frac{\alpha\beta}{\kappa\gamma}\tau g\Big)\Big(V_0, \sum_{j=1}^{4} r_{3j}V_j\Big),$$

which implies

$$\big|[\widetilde{F}_1(V)]_{\Sigma_1(s)}\big| \leq c\,\big|r_{31}\big|\big|[V_1]_{\Sigma_1(s)}\big| \leq c\tau\,\big|[V_1]_{\Sigma_1(s)}\big| \tag{3.12}$$

Analogously,

$$\widetilde{F}_2(V) = \big(1 + O(\tau)\big)\Big(f - \frac{\delta\beta}{4\kappa\gamma}\tau f - \frac{\alpha\beta}{\kappa\gamma}\tau g\Big)\Big(V_0, \sum_{j=1}^{4} r_{3j}V_j\Big)$$

and

$$\big|[\widetilde{F}_2(V)]_{\Sigma_2(s)}\big| \leq c\,\big|r_{32}\big|\big|[V_2]_{\Sigma_2(s)}\big| \leq c\tau\,\big|[V_2]_{\Sigma_2(s)}\big|. \tag{3.13}$$

Substituting (3.12) and (3.13), respectively, into (3.7) we get, using $l_{k4}\ r_{4k} = \frac{\beta\delta}{2\kappa\gamma}\tau + O(\tau^2)$ for $k = 1, 2$,

$$\begin{aligned} \big|[V_k]_{\Sigma_k}\big| &\le \big|[V_k]_{\Sigma_k(0)}\big| e^{(-\frac{\beta\delta}{2\kappa\gamma}+O(\tau))t+c\tau t} \\ &\overset{\tau\to 0}{\longrightarrow} \lim_{\tau\to 0} \big|[V_k]_{\Sigma_k(0)}\big| e^{-\frac{\beta\delta}{2\kappa\gamma}\ t}. \end{aligned} \tag{3.14}$$

On the other hand, we have from (3.7)

$$\big|[V_k]_{\Sigma_k}\big| \ge \big|[V_k]_{\Sigma_k(0)}\big| e^{-\frac{\beta\delta}{2\kappa\gamma}\ t+O(\tau)t} - c\tau \int_0^t \big|[V_k]_{\Sigma_k(s)}\big| e^{(-\frac{\beta\delta}{2\kappa\gamma}+O(\tau))(t-s)}\, ds$$

for $k = 1, 2$, which implies

$$\begin{aligned} \big|[V_k]_{\Sigma_k}\big| &\ge \big|[V_k]_{\Sigma_k(0)}\big| e^{-\frac{\beta\delta}{2\kappa\gamma}\ t+O(\tau)t} - \big|[V_k]_{\Sigma_k(0)}\big| e^{-\frac{\beta\delta}{2\kappa\gamma}\ t+O(\tau)t}(e^{c\tau t} - 1) \\ &\overset{\tau\to 0}{\longrightarrow} \lim_{\tau\to 0} \big|[V_k]_{\Sigma_k(0)}\big| e^{-\frac{\beta\delta}{2\kappa\gamma}\ t}. \end{aligned} \tag{3.15}$$

Combining (3.14) and (3.15) we conclude

$$\lim_{\tau\to 0} \big|[V_k]_{\Sigma_k}\big| = \lim_{\tau\to 0} \big|[V_k]_{\Sigma_k(0)}\big| e^{-\frac{\beta\delta}{2\kappa\gamma}\ t}, \quad k = 1, 2, \tag{3.16}$$

which means that, when $\tau \to 0$, the jumps of $[V_1]_{\Sigma_1}$ and $[V_2]_{\Sigma_2}$, respectively, persist and decay exponentially as $t \to \infty$, more rapidly for small heat conduction coefficient $\kappa\gamma$ as noted in Remark 2.2.

Returning to the variables u_+, u_-, θ, q we have as in Section 2

$$\begin{aligned} u_+ &= V_1 + O(\tau)\ V_2 + O(\sqrt{\tau})\ V_3 + O(\sqrt{\tau})\ V_4 \\ u_- &= O(\tau)V_1 + V_2 + O(\sqrt{\tau})V_3 + O(\sqrt{\tau})V_4 \end{aligned}$$

$$\begin{aligned} \theta &= O(\tau)V_1 + O(\tau)V_2 + V_3 + V_4 \\ q &= -\frac{\delta}{2\gamma}(1 + O(\tau))V_1 - \frac{\delta}{2\gamma}(1 + O(\tau))V_2 \\ &\quad - \sqrt{\frac{\kappa}{\gamma\tau}}(1 + O(\tau))V_3 + \sqrt{\frac{\kappa}{\gamma\tau}}(1 + O(\tau))V_4, \end{aligned}$$

which now implies

$$[\theta]_{\Sigma_k} \longrightarrow 0 \quad \text{as} \quad \tau \to 0 \tag{3.17}$$

exponentially on $\Sigma_{3,4}$, of order $O(\tau)$ on $\Sigma_{1,2}$,

$$[u_\pm]_{\Sigma_{3,4}} \to 0 \quad \text{exponentially,} \quad \text{as} \quad \tau \to 0 \tag{3.18}$$

$$[u_{+(-)}]_{\Sigma_{2(1)}} \to 0 \quad \text{of order} \quad O(\tau) \tag{3.19}$$

$$\lim_{\tau\to 0}[u_{+(-)}]_{\Sigma_{1(2)}} = \lim_{\tau\to 0}[u_{+(-)}]_{\Sigma_{1(2)}(0)}\ e^{-\frac{\beta\delta}{2\kappa\gamma}t} \quad \text{is kept.} \tag{3.20}$$

We also notice

$$[q]_{\Sigma_{3,4}} \to 0 \quad \text{exponentially}, \quad \text{as} \quad \tau \to 0, \tag{3.21}$$

but

$$[q]_{\Sigma_{1,2}} \quad \text{is kept as} \quad \tau \to 0. \tag{3.22}$$

Summarizing essentially (3.17) – (3.22), we have proved

Theorem 3.1. *Let* (u, θ, q) *be the solution to* (1.1) – (1.3) *where* $f = f(u, \theta)$, $g = g(u, \theta)$ *are smooth and globally Lipschitz in* u *and* θ, *with initial data satisfying:* u_0', u_1, θ_0, q_0 *are piecewise smooth with a possible jump at* $x = 0$. *Then, along the characteristic curves, we have as* $\tau \to 0$, *the asymptotic behavior described in* (3.17) – (3.22).

Remark 3.2. Due to the nonlinearity of the system (1.1) – (1.3) with (f, g) depending on (u, θ), in general one could not obtain the precise behavior of the jump of $\partial_x\theta$ on Σ_k $(1 \leq k \leq 4)$ in constrast to the linear case, Theorem 2.1. However, here we can obtain the asymptotic behavior (3.21), (3.22) of $[q]_{\Sigma_k}$ as $\tau \to 0$, which is expected to have the same behavior as for $-\kappa[\theta_x]_{\Sigma_k}$ when $\tau \to 0$ formally from (1.3).

4. The semilinear case (II)

Finally, we discuss the general semilinear system (1.1) – (1.3) with smooth functions f and q depending on (u_t, u_x) as well, and initial data as before. With the notations from Section 3 for

$$U = (u, u_+, u_-, \theta, q)' = (U_0, U_1, U_2, U_3, U_4)'$$

the characteristic eigenvalues λ_k and curves Σ_k, $k = 0, .., 4$, we have

$$\partial_t U + B_1 \partial_x U + B_0 U = F(U),$$

where now F is given by $F(U) = (0, f, f, g, 0)'(U_0, U_1, U_2, U_3)$, without loss of generality taking (u_+, u_-) instead of (u_t, u_x).

Similarly, with $V = LU$

$$\partial_t V + \widetilde{\Lambda} \partial_x V + \widetilde{B}_0 V = \widetilde{F}(V), \tag{4.1}$$

where

$$\widetilde{F}(V) = LF\Big(V_0, \sum_{k=1}^{4} r_{1k} V_k, \sum_{k=1}^{4} r_{2k} V_k, \sum_{k=1}^{4} r_{3k} V_k\Big).$$

As in Section 2 one deduces

- V_j does not jump on Σ_k if $j \neq k$
- V_0 does not jump at all
- $[V_3]_{\Sigma_3}$ and $[V_4]_{\Sigma_4}$ decay exponentially as $\tau \to 0$.

Taking account of the dependency of $\tilde{F}(V)$ on $\sum_{k=1}^4 r_{1k}V_k$ and $\sum_{k=1}^4 r_{2k}V_k$ for the right hand side of (4.1), the estimates (3.12), (3.13) have to be modified in the present case into

$$\left|[\widetilde{F}_1(V)]_{\Sigma_1(s)}\right| \leq c\left|[V_1]_{\Sigma_1(s)}\right|$$
$$\left|[\widetilde{F}_2(V)]_{\Sigma_2(s)}\right| \leq c\left|[V_2]_{\Sigma_2(s)}\right|,$$

respectively, which does not allow for estimate (3.16) in general. Hence it remains open whether the jumps of $[V_1]_{\Sigma_1}$ and $[V_2]_{\Sigma_2}$ persist when $\tau \to 0$.

For θ and $\nabla_{(t,x)}u$ we obtain

$$\theta = O(\tau)V_1 + O(\tau)V_2 + V_3 + V_4,$$

hence

$$[\theta]_{\Sigma_k} \to 0 \quad \text{as } \tau \to 0, \quad k = 1,2,3,4 \tag{4.2}$$

of order $O(\tau)$ for $k = 1,2$, and exponentially for $k = 3,4$. Moreover,

$$u_+ = V_1 + O(\tau)\, V_2 + O(\sqrt{\tau})\, V_3 + O(\sqrt{\tau})\, V_4$$
$$u_- = O(\tau)V_1 + V_2 + O(\sqrt{\tau})V_3 + O(\sqrt{\tau})V_4$$

as before, hence

$$[\partial_t u]_{\Sigma_k}, \quad [\partial_x u]_{\Sigma_k} \to 0 \quad \text{as} \quad \tau \to 0, \quad k = 3,4, \tag{4.3}$$

exponentially, that is, possible discontinuities are preserved on $\Sigma_{1,2} = \{(t,x)|\ x = \pm\alpha t\}$, which are the final "hyperbolic" characterizing curves also for classical thermoelasticity, cp. [9].

Using a Taylor expansion of f and g, we can refine the asymptotics as follows:

$$\begin{aligned}
&\left|[\widetilde{F}_1(V)]_{\Sigma_1}\right| \\
&\leq (1+O(\tau))\Big(\ \|f_2'\|_{L^\infty} + \frac{\delta\beta}{4\kappa\gamma}\tau\|f_3'\|_{L^\infty} + \frac{\delta\alpha}{2\kappa\gamma}\tau\|f_4'\|_{L^\infty}\Big)\left|[V_1]_{\Sigma_1}\right| \\
&\quad + \Big(\frac{\alpha\beta}{\kappa\gamma}\tau + O(\tau^2)\Big)\Big(\ \|g_2'\|_{L^\infty} + \frac{\delta\beta}{4\kappa\gamma}\tau\|g_3'\|_{L^\infty} + \frac{\delta\alpha}{2\kappa\gamma}\tau\|g_4'\|_{L^\infty}\Big)\left|[V_1]_{\Sigma_1}\right|
\end{aligned} \tag{4.4}$$

where f_j' and g_j' denote the derivatives of f and g with respect to their j-th argument. If f and g are globally Lipschitz continuous in (u_+, u_-, θ), respectively, we conclude from (3.7), (4.4)

$$\begin{aligned} \big|[V_1]_{\Sigma_1}\big| &\le \big|[V_1]_{\Sigma_1(0)}\big| e^{-\frac{\beta\delta}{2\kappa\gamma}\,t + O(\tau)\,t} \\ &\quad + \Big(\,\|\,f_2'\|_{L^\infty} + O(\tau)\Big) \int\limits_0^t \big|[V_1]_{\Sigma_1(s)}\big| e^{(-\frac{\beta\delta}{2\kappa\gamma} + O(\tau))(t-s)}\,ds \end{aligned} \tag{4.5}$$

$$\begin{aligned} \big|[V_1]_{\Sigma_1}\big| &\ge \big|[V_1]_{\Sigma_1(0)}\big| e^{-\frac{\beta\delta}{2\kappa\gamma}\,t + O(\tau)\,t} \\ &\quad - \Big(\,\|\,f_2'\|_{L^\infty} + O(\tau)\Big) \int\limits_0^t \big|[V_1]_{\Sigma_1(s)}\big| e^{(-\frac{\beta\delta}{2\kappa\gamma} + O(\tau))(t-s)}\,ds. \end{aligned} \tag{4.6}$$

Inequality (4.5) implies

$$\begin{aligned} \big|[V_1]_{\Sigma_1}\big| &\le \big|[V_1]_{\Sigma_1(0)}\big| e^{-\frac{\beta\delta}{2\kappa\gamma}\,t + O(\tau)\,t} e^{(\|f_2'\|_{L^\infty} + O(\tau))t} \\ &\overset{\tau\to 0}{\longrightarrow} \lim_{\tau\to 0} \big|[V_1]_{\Sigma_1(0)}\big| e^{(\|f_2'\|_{L^\infty} - \frac{\beta\delta}{2\kappa\gamma})t}. \end{aligned} \tag{4.7}$$

Substituting (4.7) into (4.6) it follows

$$\begin{aligned} \big|[V_1]_{\Sigma_1}\big| &\ge \big|[V_1]_{\Sigma_1(0)}\ \big|\ e^{-\frac{\beta\delta}{2\kappa\gamma}\,t + O(\tau)\,t} \\ &\quad - \big|[V_1]_{\Sigma_1(0)}\ \big|\ e^{-\frac{\beta\delta}{2\kappa\gamma}\,t + O(\tau)\,t} \Big(e^{(\|f_2'\|_{L^\infty} + O(\tau))t} - 1 \Big) \\ &\overset{\tau\to 0}{\longrightarrow} \lim_{\tau\to 0} \big|[V_1]_{\Sigma_1(0)}\ \big|\ e^{-\frac{\beta\delta}{2\kappa\gamma}\,t} (2 - e^{\|f_2'\|_{L^\infty} t}). \end{aligned} \tag{4.8}$$

Similarly, we can deduce

$$\lim_{\tau\to 0} \big|[V_2]_{\Sigma_2}\big| \le \lim_{\tau\to 0} \big|[V_2]_{\Sigma_2(0)}\big| e^{(\|f_3'\|_{L^\infty} - \frac{\beta\delta}{2\kappa\gamma})t} \tag{4.9}$$

$$\lim_{\tau\to 0} \big|[V_2]_{\Sigma_2}\big| \ge \lim_{\tau\to 0} \big|[V_2]_{\Sigma_1(0)}\ \big|\ e^{-\frac{\beta\delta}{2\kappa\gamma}\,t} (2 - e^{\|f_3'\|_{L^\infty} t}). \tag{4.10}$$

Using again the representations

$$\begin{aligned} u_+ &= (\partial_t + \alpha\partial_x)u = V_1 + O(\tau)\ V_2 + O(\sqrt{\tau})\ V_3 + O(\sqrt{\tau})\ V_4 \\ u_- &= (\partial_t - \alpha\partial_x)u = O(\tau)V_1 + V_2 + O(\sqrt{\tau})V_3 + O(\sqrt{\tau})V_4 \\ \theta &= O(\tau)V_1 + O(\tau)V_2 + V_3 + V_4 \\ q &= -\frac{\delta}{2\gamma}(1 + O(\tau))V_1 - \frac{\delta}{2\gamma}(1 + O(\tau))V_2 \\ &\quad - \sqrt{\frac{\kappa}{\gamma\tau}}(1 + O(\tau))V_3 + \sqrt{\frac{\kappa}{\gamma\tau}}(1 + O(\tau))V_4 \end{aligned}$$

we conclude

Theorem 4.1.

(1) *In addition to the assumptions of Theorem* 3.1, *we also suppose that*

$$f = f(u, u_t + \alpha u_x, u_t - \alpha u_x, \theta)$$
$$g = g(u, u_t + \alpha u_x, u_t - \alpha u_x, \theta)$$

are smooth in their arguments. Then, for $k = 1, 2$,

$$[\theta]_{\Sigma_k} \to 0, \qquad [q]_{\Sigma_k} \to 0$$
$$[\partial_t u]_{\Sigma_k} \to 0, \quad [\partial_x u]_{\Sigma_k} \to 0$$

exponentially as $\tau \to 0$ *for* $k = 3, 4$, *and as* $\tau \to 0$,

$$[\theta]_{\Sigma_k} = O(\tau).$$

(2) *Additionally, if* f *and* g *are globally Lipschitz in their last three arguments, then* $u_t + \alpha u_x$ *and* $u_t - \alpha u_x$ *have jumps on* Σ_1 *and* Σ_2 *respectively. The jumps of* $u_t + \alpha u_x$ *and* $u_t - \alpha u_x$ *on* Σ_2 *and* Σ_1, *respectively, vanish of order* $O(\tau)$ *at least when* $\tau \to 0$, *and* $[q]_{\Sigma_k}$ *is kept as* $\tau \to 0$ *for* $k = 1, 2$. *Moreover, we have that*

(i) *if* f *is independent of* $u_t + \alpha u_x$ *(*$u_t - \alpha u_x$ *resp.), then the jump of* $u_t + \alpha u_x$ *(*$u_t - \alpha u_x$ *resp.) will persist for all* $t > 0$ *with the same rate as in the linear case (see Section* 2*).*

(ii) *if* f *depends on* $u_+ = u_t + \alpha u_x$ *(*$u_- = u_t - \alpha u_x$ *resp.), and satisfies* $\|\frac{\partial f}{\partial u_+}\|_{L^\infty} < \frac{\beta\delta}{2\kappa\gamma}$ $(\|\frac{\partial f}{\partial u_-}\|_{L^\infty} < \frac{\beta\delta}{2\kappa\gamma}$ *resp.), then the jump of* $u_t + \alpha u_x$ *(*$u_t - \alpha u_x$ *resp.) decays exponentially when* $t \to \infty$.

Remark 4.2. For smooth data the convergence of the solutions $(u, \theta, q) \equiv (u^\tau, \theta^\tau, q^\tau)$ of (1.1) – (1.3) with $f = g = 0$ to the solutions of the corresponding classical thermoelastic system $(u^0, \theta^0, q^0) = (u^0, \theta^0, -\kappa\theta^0_x)$ has been proved in [7], provided $q_0 = -\kappa\theta_{0,x}$. The same remains open for the discontinuous solutions discussed here.

Acknowledgement. The second author would like to express his gratitude for the hospitality of Department of Mathematics and Statistics in the University of Konstanz, Germany, when he visited there in July 2003. His research was also partially supported by the NSFC grant 10131050, the Educational Ministry of China and the Shanghai Science and Technology Committee grant 03QMH1407.

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Received 11.03.2004