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A Regularity Result for the Heterogeneous Evolution Dam Problem

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Abstract. We consider a non steady-state fluid flow through a heterogeneous porous medium governed by a nonlinear Darcy law. Under a general condition on the permeability, we prove the L^p-continuity of the saturation for any $p \geq 1$.

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1. Formulation of the problem

We consider a porous medium supplied by several reservoirs of an incompressible fluid. It is represented by a bounded domain Ω of $\mathbb{R}^n$ with locally Lipschitz boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where Γ_1 is the impervious part of the boundary, Γ_2 is the part in contact with either air or the fluid reservoirs.

The fluid infiltrates through Ω obeying to the following generalized Darcy law (see [10, Chapter 3]):

$$v = -\mathcal{A}(x, \nabla(p + x_n)) \ ,$$

where $\mathcal{A}$ is a vector function defined in $\Omega \times \mathbb{R}^n$ with values in $\mathbb{R}^n$, $x = (x_1, ..., x_n)$, v is the fluid velocity and p its pressure.

We are concerned with the problem of finding the pressure p and the saturation χ of the fluid. For convenience we introduce the following functions : $u = p + x_n$, $g = 1 - \chi$ and $\psi = \phi + x_n$, where ϕ is a nonnegative Lipschitz function representing the exterior air or fluid pressure defined on $\overline{Q}$ with $Q = \Omega \times (0, T)$ and T a positive number.

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Using the mass conservation law, Darcy's law, the boundary conditions and the initial data, we obtain the following strong formulation for our problem (see [4]):

$$(\mathbf{SF})\quad \begin{cases} u \geq x_n,\ 0 \leq g \leq 1,\ g(u - x_n) = 0 & \text{in } Q \\ div(\mathcal{A}(x, \nabla u) - g\mathcal{A}(x, e)) + g_t = 0 & \text{in } Q \\ u = \psi & \text{on } \Sigma_2 \\ g(\cdot, 0) = g_0 & \text{in } \Omega \\ (\mathcal{A}(x, \nabla u) - g\mathcal{A}(x, e)) \cdot \nu = 0 & \text{on } \Sigma_1 \\ (\mathcal{A}(x, \nabla u) - g\mathcal{A}(x, e)) \cdot \nu \leq 0 & \text{on } \Sigma_4 \end{cases}$$

where $g_0 \in L^\infty(\Omega)$, ν is the outward unit normal vector to $\partial\Omega$, $e = (0, ..., 0, 1) \in \mathbb{R}^n$, and

$\Sigma_1 = \Gamma_1 \times (0, T)$: the impervious part
$\Sigma_2 = \Gamma_2 \times (0, T)$: the pervious part
$\Sigma_3 = \Sigma_2 \cap \{\phi > 0\}$: the part covered by fluid
$\Sigma_4 = \Sigma_4 \cap \{\phi = 0\}$: the part where the fluid flows outside Ω.

For $\mathcal{A}$, we assume the following with $q > 1$ and $0 < m \leq M < \infty$:

$$\left.\begin{array}{ll} (\mathbf{i}) & x \mapsto \mathcal{A}(x, \xi) \text{ is measurable for all } \xi \in \mathbb{R}^n \\ (\mathbf{ii}) & \xi \mapsto \mathcal{A}(x, \xi) \text{ is continuous for a.a. } x \in \Omega \\ (\mathbf{iii}) & \text{for all } \xi \in \mathbb{R}^n \text{ and for a.a. } x \in \Omega : \\ & \qquad \mathcal{A}(x, \xi).\xi \geq m|\xi|^q \quad \text{and} \quad |\mathcal{A}(x, \xi)| \leq M|\xi|^{q-1} \\ (\mathbf{iv}) & \text{for all } \xi, \zeta \in \mathbb{R}^n \text{ and for a.a. } x \in \Omega : \\ & \qquad (\mathcal{A}(x, \xi) - \mathcal{A}(x, \zeta)).(\xi - \zeta) \geq 0. \end{array}\right\} \tag{1.1}$$

Using the strong formulation, we are led to the following weak formulation with $A(x) = \mathcal{A}(x, e)$:

$$(\mathbf{P})\quad \begin{cases} \text{Find } (u, g) \in L^q(0, T, W^{1,q}(\Omega)) \times L^\infty(Q) \text{ such that :} \\ u \geq x_n,\ 0 \leq g \leq 1,\ g(u - x_n) = 0 \quad \text{a.e. in } Q \\ \qquad\qquad u = \psi \quad \text{on } \Sigma_2 \\ \displaystyle\int_Q \Big[\big(\mathcal{A}(x, \nabla u) - gA(x)\big) \cdot \nabla\xi + g\xi_t\Big] dx\, dt + \int_\Omega g_0(x)\xi(x, 0)\, dx \leq 0 \\ \forall \xi \in W^{1,q}(Q) :\ \xi = 0 \text{ on } \Sigma_3,\ \xi \geq 0 \text{ on } \Sigma_4,\ \xi(x, T) = 0 \text{ for a.a. } x \in \Omega. \end{cases}$$

Under the assumptions (1.1), the existence of a solution was proved in [11, Theorem 3.1] and also in [4, Theorem 5.1] for generalized boundary conditions. Here we are concerned with the L^p-continuity of the function g. We recall that it has been proved in [3, Proposition 1.6] in the case where $\mathcal{A}(x,\xi) = \xi$ that $g \in C^0([0,T], L^p(\Omega))$ for all $p \geq 1$ (see also [2, Theorem 2.4] for the compressible case). This result was improved in [11, Theorem 4.5] in the case where $A(x)$ is a constant vector.

Our objective in this paper is to extend this regularity result to the case where $A \in C^1(\bar{\Omega})$ and $div(A) \geq 0$. The main idea of the proof is based on a monotonicity result of g along the orbits of a differential equation. A similar monotonicity is proved in [5, Theorem 2.1] for the stationary case.

We recall the following results from [11].

Proposition 1.1. *For each solution* (u,g) *of* (P), *we have*

$$u \in L^\infty(Q), \quad g \in C^0\big([0,T], W^{-1,q'}(\Omega)\big) \tag{1.2}$$

$$div\big(\mathcal{A}(x,\nabla u) - gA(x)\big) + g_t = 0 \quad \text{in } \mathcal{D}'(Q). \tag{1.3}$$

Moreover if $div\big(A(x)\big) \geq 0$ *in* $\mathcal{D}'(\Omega)$, *we obtain*

$$div\big(gA(x)\big) - g_t = div\big(\mathcal{A}(x,\nabla u)\big) \geq 0 \quad \text{in } \mathcal{D}'(Q). \tag{1.4}$$

2. A monotonicity property of g

From now on, we assume that

$$A = \mathcal{A}(\cdot, e) = (a_1, ..., a_n) \in C^1(\bar{\Omega}, \mathbb{R}^n). \tag{2.1}$$

$$div\big(A(x)\big) \geq 0 \quad \text{in } C^0(\Omega). \tag{2.2}$$

From (1.1) iii), we have

$$m \leq a_n(x) \leq M, \ |a_i(x)| \leq M \qquad \forall x \in \bar{\Omega}, \ \forall i = 1, ..., n. \tag{2.3}$$

Using (2.1), it is easy to see that there exists a C^1 extension of A to $\mathbb{R}^n$ denoted also by A and satisfying (2.3) in $\mathbb{R}^n$, with possibly different constants that we still denote by m and M.

Let $h_0 \in \mathbb{R}$ such that Ω is located above the hyperplane $x_n = h_0$ with empty intersection. We consider for each $\omega \in \mathbb{R}^{n-1}$ the differential system

$$\big(E(\omega)\big) \begin{cases} x'(s,\omega) = A(x(s,\omega)) \\ x(0,\omega) = (\omega, h_0). \end{cases}$$

Then we have

Proposition 2.1. *There exists a unique maximal solution $x(\cdot,\omega)$ of $\big(E(\omega)\big)$ defined on $(-\infty,\infty)$. Moreover x is of class C^1 with respect to ω, C^2 with respect to s, and we have*

$$\lim_{s\to\pm\infty} x_n(s,\omega)=\pm\infty. \tag{2.4}$$

Proof. By the classical theory of ordinary differential equations there exists a unique maximal solution $x(\cdot,\omega)$ of $\big(E(\omega)\big)$ defined on $(\alpha_-(\omega),\alpha_+(\omega))$. Moreover since A is of class C^1, x is of class C^1 with respect to ω and C^2 with respect to s.

Assume for example that $\alpha_-(\omega)>-\infty$. Since $A\in L^\infty(\mathbb{R}^n)$, $x(\cdot,\omega)$ is uniformly Lipschitz continuous in $\mathbb{R}^n$ and therefore $\lim_{s\to\alpha_-(\omega)} x(s,\omega)$ exists and is finite. It follows that we can extend $x(\cdot,\omega)$ to the left of $\alpha_-(\omega)$ which is impossible. Similarly we obtain a contradiction if $\alpha_+(\omega)<\infty$.

Moreover since

$$x_n(s,\omega)=h_0+\int_0^s a_n(x(\sigma,\omega))d\sigma$$

and a_n satisfies (2.3), we obtain $h_0+ms\le x_n(s,\omega)\le h_0+Ms$ if $s>0$ and $h_0+Ms\le x_n(s,\omega)\le h_0+ms$ if $s<0$ which leads to (2.4). ∎

We consider the mappings $\mathcal{T}:\mathbb{R}^n\to\mathbb{R}^n$ and $\mathcal{S}:\mathbb{R}^n\to\mathbb{R}^n$ defined by

$$\mathcal{T}(s,\omega)=x(s,\omega)\quad\text{and}\quad \mathcal{S}(s,\omega)=(\omega,L(s,\omega))=(\omega,\tau),$$

where

$$L(s,\omega)=\int_0^s |A(x(\sigma,\omega))|d\sigma=\int_0^s |x'(\sigma,\omega)|d\sigma$$

represents the arc length of the curve $x(\cdot,\omega)$ from the point (ω,h_0) to the point $x(s,\omega)$. Then we have

Theorem 2.2. *$\mathcal{T}$ and $\mathcal{S}$ are C^1-diffeomorphisms from $\mathbb{R}^n$ into $\mathbb{R}^n$. Moreover*

$$\mathcal{J}\mathcal{S}(s,\omega)=(-1)^{n+1}|A(x(s,\omega))|\neq 0$$
$$Y(s,\omega)=\mathcal{J}\mathcal{T}(s,\omega)=(-1)^{n+1}a_n(\omega,h_0)\cdot\exp\Big(\int_0^s (divA)(x(\sigma,\omega))d\sigma\Big)\neq 0,$$

where $\mathcal{J}$ denotes the Jacobian.

Proof. Since x is C^1 and $|A(x(s,\omega))|\ge m>0$, clearly $\mathcal{T}$ and $\mathcal{S}$ are C^1 mappings.

`Case:` $\mathcal{JS}(s,\omega) = (-1)^{n+1}|A(x(s,\omega))|$.
We have

$$\mathcal{JS}(s,\omega) = \begin{vmatrix} 0 & 1 & 0 & . & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & . & 0 \\ . & . & 0 & . & . & & & . \\ . & . & . & . & . & . & & . \\ . & . & . & & . & . & . & . \\ . & . & . & & & . & 1 & 0 \\ 0 & 0 & . & . & . & . & 0 & 1 \\ |A(x(s,\omega))| & \dfrac{\partial L}{\partial \omega_1} & \dfrac{\partial L}{\partial \omega_2} & . & . & . & \dfrac{\partial L}{\partial \omega_{n-2}} & \dfrac{\partial L}{\partial \omega_{n-1}} \end{vmatrix}$$

which leads to $\mathcal{JS}(s,\omega) = (-1)^{n+1}|A(x(s,\omega))| \neq 0$.

`Case:` $\mathcal{S}(\mathbb{R}^n) = \mathbb{R}^n$.
Let $x_0 = (\omega_0, \tau_0) \in \mathbb{R}^n$ with $\omega_0 = (\omega_{01}, ..., \omega_{0n-1}) \in \mathbb{R}^{n-1}$ and $\tau_0 \in \mathbb{R}$. We have $\frac{\partial L}{\partial s}(s,\omega_0) = |A(x(s,\omega_0))| > 0$. So $L(.,\omega_0)$ is an increasing function on $(-\infty,\infty)$. Moreover by (2.3) we have

$$ms \leq L(s,\omega_0) \leq Ms \quad \text{if } s > 0 \quad \text{and } Ms \leq L(s,\omega_0) \leq ms \quad \text{if } s < 0.$$

So $\lim_{s\to\pm\infty} L(s,\omega_0) = \pm\infty$ and therefore $L(.,\omega_0)$ is one to one from $\mathbb{R}$ to $\mathbb{R}$.

We deduce that there exists a unique $s_0 \in \mathbb{R}$ such that $L(s_0,\omega_0) = \tau_0$. Hence we have proved that $\mathcal{S}(s_0,\omega_0) = (\omega_0, L(s_0,\omega_0)) = (\omega_0,\tau_0) = x_0$.

`Case:` $\mathcal{T}(\mathbb{R}^n) = \mathbb{R}^n$.
Let $x_0 \in \mathbb{R}^n$. Let z be the unique maximal solution of the following differential system

$$\begin{cases} z'(s) = A(z(s)) \\ z(0) = x_0. \end{cases}$$

As for the equation $\big(E(\omega)\big)$, one can verify that the solution z is defined on $(-\infty,\infty)$ and that $\lim_{s\to\pm\infty} z_n(s) = \pm\infty$. Moreover $z_n'(s) = a_n(z(s)) > 0$. It follows that z_n is one to one from $\mathbb{R}$ to $\mathbb{R}$. So there exists a unique $s_0 \in \mathbb{R}$ such that $z_n(s_0) = h_0$.

Now if we set $\omega_0 = (z_1(s_0), ..., z_{n-1}(s_0))$, we obtain $z(s_0) = (\omega_0, h_0)$. Finally, it is easy to check that $x(s,\omega_0) = z(s+s_0)$ and that $\mathcal{T}(-s_0,\omega_0) = x(-s_0,\omega_0) = z(0) = x_0$.

`Case:` $Y(s,\omega) = \mathcal{JT}(s,\omega) = (-1)^{n+1}a_n(\omega,h_0)\cdot\exp\big(\int_0^s (divA)(x(\sigma,\omega))d\sigma\big)$.
We need the following Lemma (see [13, Lemma 2.7] for the proof).

Lemma 2.3. *Let U be an open set of $\mathbb{R}^n$, $f \in C^2(U,\mathbb{R}^n)$ and $v \in C^1(\mathbb{R}^n,\mathbb{R}^n)$. Then we have*

$$(div(v))(f(x))\mathcal{J}f(x) = div(D_f(v))(x) \qquad \forall x \in U, \tag{2.5}$$

where $D_f(v) = (D_f(v)_j)$ *and* $D_f(v)_j$ *is the determinant of the matrix obtained from* Df *by replacing the* j^{th} *column by* $v(f)$.

Let $\delta > 0$, ρ_δ a mollifier and $A_\delta = \rho_\delta * A$. For each $\omega \in \mathbb{R}^{n-1}$, let $x_\delta(\cdot, \omega)$ be the solution of the differential equation

$$\begin{cases} x'_\delta(s,\omega) = A(x_\delta(s,\omega)) \\ x_\delta(0,\omega) = (\omega, h_0). \end{cases}$$

We denote by $\mathcal{T}_\delta$ the diffeomorphism defined in the same way as $\mathcal{T}$.

Since A_δ and $\mathcal{T}_\delta$ are C^∞ functions, we can apply (2.5) with $U = \mathbb{R}^n$, $f = \mathcal{T}_\delta$ and $v = A_\delta$. We obtain

$$(div(A_\delta))(\mathcal{T}_\delta(s,\omega))Y_\delta(s,\omega) = div(D_{\mathcal{T}_\delta}(A_\delta))(s,\omega) \quad \forall (s,\omega) \in \mathbb{R}^n \tag{2.6}$$

where $Y_\delta = \mathcal{J}\mathcal{T}_\delta$.

Using the notations of Lemma 2.3, we claim that $D_{\mathcal{T}_\delta}(A_\delta)_1 = \mathcal{J}\mathcal{T}_\delta(s,\omega)$ and that $D_{\mathcal{T}_\delta}(A_\delta)_j = 0$ for all $j \geq 2$. Indeed, $D_{\mathcal{T}_\delta}(A_\delta)_j$ is the determinant of the matrix M_j obtained from $D\mathcal{T}_\delta(s,\omega)$ by replacing the j^{th} column by

$$A_\delta(\mathcal{T}_\delta(s,\omega)) = A_\delta(x_\delta(s,\omega)) = x'_\delta(s,\omega) = \frac{\partial \mathcal{T}_\delta}{\partial s}(s,\omega)$$

which is the 1^{st} column of $D\mathcal{T}_\delta(s,\omega)$. It follows that the matrix M_1 and $D\mathcal{T}_\delta(s,\omega)$ are identical. This leads to $D_{\mathcal{T}_\delta}(A_\delta)_1 = \mathcal{J}\mathcal{T}_\delta(s,\omega)$.

For $j \geq 2$, the 1^{st} and j^{th} columns of M_j are exactly the same and therefore $D_{\mathcal{T}_\delta}(A_\delta)_j = 0$ for all $j \geq 2$. Hence (2.6) becomes

$$(div(A_\delta))(\mathcal{T}_\delta(s,\omega))Y_\delta(s,\omega) = \frac{\partial}{\partial s}\Big(D_{\mathcal{T}_\delta}(A_\delta)_1\Big)(s,\omega) = \frac{\partial Y_\delta}{\partial s}(s,\omega) \quad \forall (s,\omega) \in \mathbb{R}^n$$

which leads to

$$Y_\delta(s,\omega) = Y_\delta(0,\omega)\exp\Big(\int_0^s (divA_\delta)(x_\delta(\sigma,\omega))d\sigma\Big) \qquad \forall (s,\omega) \in \mathbb{R}^n.$$

Letting δ go to zero, we obtain

$$Y(s,\omega) = Y(0,\omega)\exp\Big(\int_0^s (divA)(x(\sigma,\omega))d\sigma\Big) \qquad \forall (s,\omega) \in \mathbb{R}^n.$$

Moreover we have

$$Y(0,\omega) = \begin{vmatrix} a_1(\omega,h_0) & 1 & 0 & . & . & . & . & 0 \\ a_2(\omega,h_0) & 0 & 1 & 0 & . & . & . & 0 \\ . & . & 0 & . & . & & & . \\ . & . & . & . & . & . & & . \\ . & . & . & & . & . & . & . \\ . & . & . & & & . & . & 0 \\ . & 0 & . & . & . & . & 0 & 1 \\ a_n(\omega,h_0) & 0. & . & . & . & . & 0 & 0 \end{vmatrix} = (-1)^{n+1}a_n(\omega,h_0) \neq 0.$$

Thus the result follows. ∎

Remark 2.4. Let $D = \mathcal{T}^{-1}(\Omega)$. Then D is a domain of $\mathbb{R}^n$ and $\mathcal{T} : D \to \Omega$ and $\mathcal{S} : D \to \mathcal{S}(D)$ are C^1-diffeomorphisms.

The following monotonicity result generalizes the fact that $g_{x_n} - g_t \geq 0$ in $\mathcal{D}'(Q)$ when $\mathcal{A}(x,\xi) = \xi$ (see [2], [3]). It will play a major role in the proof of the continuity of g.

Theorem 2.5. *Let* (u,g) *be a solution of problem* (P), $\lambda(\omega,\tau) = |A(\mathcal{T} \circ \mathcal{S}^{-1}(\omega,\tau))|^{-1}$ *and* $f(\omega,\tau,t) = g(\mathcal{T} \circ \mathcal{S}^{-1}(\omega,\tau),t) \cdot |Y \circ \mathcal{S}^{-1}(\omega,\tau)|$. *Then we have*

$$f_\tau - \lambda f_t \geq 0 \quad \text{in} \quad \mathcal{D}'(\mathcal{S}(D) \times (0,T)). \tag{2.7}$$

Proof. Let $\phi \in \mathcal{D}(\mathcal{S}(D) \times (0,T))$, $\phi \geq 0$. Then $\widetilde{\phi}(x,t) = \phi(\mathcal{S} \circ \mathcal{T}^{-1}(x),t) \in C_0^1(\mathcal{T}(D) \times (0,T)) = C_0^1(\Omega \times (0,T))$, $\widetilde{\phi} \geq 0$ and by (1.4) and (2.2), we have

$$\int_{\Omega \times (0,T)} gA(x) \cdot \nabla\widetilde{\phi} - g\widetilde{\phi}_t dxdt \leq 0$$

which can be written as

$$\int_{\Omega \times (0,T)} \Big[g(x,t)A(x)\nabla(\phi(\cdot,t) \circ \mathcal{S} \circ \mathcal{T}^{-1}) \\ - g(x,t)\phi_t(\mathcal{S} \circ \mathcal{T}^{-1}(x),t)\Big] dx\, dt \leq 0. \tag{2.8}$$

Using the change of variables $\mathcal{S} \circ \mathcal{T}^{-1}(x) = (\omega,\tau)$, we obtain

$$\begin{aligned} &\int_\Omega g(x,t)\phi_t(\mathcal{S} \circ \mathcal{T}^{-1}(x),t)dx \\ &\quad = \int_{\mathcal{S}(D)} g(\mathcal{T} \circ \mathcal{S}^{-1}(\omega,\tau),t) \\ &\qquad \cdot \phi_t(\omega,\tau,t) \cdot |\mathcal{J}\big(\mathcal{T} \circ \mathcal{S}^{-1}\big)(\omega,\tau)|\, d\omega\, d\tau \\ &\quad = \int_{\mathcal{S}(D)} g(\mathcal{T} \circ \mathcal{S}^{-1}(\omega,\tau),t) \cdot \phi_t(\omega,\tau,t) \\ &\qquad \cdot |\mathcal{J}\mathcal{T}\big(\mathcal{S}^{-1}(\omega,\tau)\big)| \cdot |\mathcal{J}\mathcal{S}^{-1}(\omega,\tau)|\, d\omega\, d\tau \\ &\quad = \int_{\mathcal{S}(D)} |A(\mathcal{T} o \mathcal{S}^{-1}(\omega,\tau))|^{-1} \cdot g(\mathcal{T} o \mathcal{S}^{-1}(\omega,\tau),t) \\ &\qquad \cdot |Y\big(\mathcal{S}^{-1}(\omega,\tau)\big)| \cdot \phi_t(\omega,\tau,t)\, d\omega\, d\tau \\ &\quad = \int_{\mathcal{S}(D)} \lambda(\omega,\tau) \cdot f(\omega,\tau,t) \cdot \phi_t(\omega,\tau,t)\, d\omega\, d\tau \end{aligned} \tag{2.9}$$

Using the change of variables $\mathcal{T}(s,\omega) = x$, we get

$$\begin{aligned}\int_\Omega g(x,t)A(x)\cdot\nabla(\phi(\cdot,t)\circ\mathcal{S}\circ\mathcal{T}^{-1})\,dx \\ &= \int_D g(\mathcal{T}(s,\omega),t)A(\mathcal{T}(s,\omega)) \\ &\quad\cdot\nabla(\phi(\cdot,t)\circ\mathcal{S}\circ\mathcal{T}^{-1})\circ\mathcal{T}(s,\omega)\cdot|Y(s,\omega)|\,ds\,d\omega \\ &= \int_D g(\mathcal{T}(s,\omega),t)\cdot|Y(s,\omega)|\cdot\frac{\partial}{\partial s}\big(\phi(\mathcal{S}(s,\omega),t)\big)\,ds\,d\omega\end{aligned} \tag{2.10}$$

since

$$\begin{aligned}\frac{\partial}{\partial s}\big(\phi(\mathcal{S}(s,\omega),t)\big) &= \frac{\partial}{\partial s}\big(\phi(\mathcal{S}\circ\mathcal{T}^{-1}\circ\mathcal{T}(s,\omega),t)\big) \\ &= \big(\nabla(\phi(\cdot,t)\circ\mathcal{S}\circ\mathcal{T}^{-1})\circ\mathcal{T}(s,\omega)\big)\frac{\partial\mathcal{T}}{\partial s}(s,\omega) \\ &= A(\mathcal{T}(s,\omega))\cdot\big(\nabla(\phi(\cdot,t)\circ\mathcal{S}\circ\mathcal{T}^{-1})\circ\mathcal{T}(s,\omega)\big).\end{aligned}$$

Using the change of variables $\mathcal{S}^{-1}(\omega,\tau) = (s,\omega)$ in (2.10), we obtain

$$\begin{aligned}\int_D g(\mathcal{T}(s,\omega),t)\cdot|Y(s,\omega)|\cdot\frac{\partial}{\partial s}\big(\phi(\mathcal{S}(s,\omega),t)\big)\,ds\,d\omega \\ &= \int_{\mathcal{S}(D)} g(\mathcal{T}\circ\mathcal{S}^{-1}(\omega,\tau),t)\cdot|Y\circ\mathcal{S}^{-1}(\omega,\tau)| \\ &\quad\cdot\Big(\frac{\partial}{\partial s}(\phi(\cdot,t)\circ\mathcal{S})\Big)(\mathcal{S}^{-1}(\omega,\tau))\cdot|\mathcal{J}\mathcal{S}^{-1}(\omega,\tau)|\,d\omega\,d\tau \\ &= \int_{\mathcal{S}(D)} f(\omega,\tau,t)\cdot\Big(\frac{\partial}{\partial s}(\phi(\cdot,t)\circ\mathcal{S})\Big)\big(\mathcal{S}^{-1}(\omega,\tau)\big) \\ &\quad\cdot|A(\mathcal{T}\circ\mathcal{S}^{-1}(\omega,\tau))|^{-1}\,d\omega\,d\tau \\ &= \int_{\mathcal{S}(D)} f(\omega,\tau,t)\cdot\frac{\partial\phi}{\partial\tau}(\omega,\tau,t)\,d\omega\,d\tau\end{aligned} \tag{2.11}$$

since

$$\begin{aligned}\frac{\partial}{\partial s}\big(\phi(\cdot,t)\circ\mathcal{S}\big)(\mathcal{S}^{-1}(\omega,\tau)) &= \big(\nabla\phi(\cdot,t)\circ\mathcal{S}\big)(\mathcal{S}^{-1}(\omega,\tau))\cdot\frac{\partial\mathcal{S}}{\partial s}(\mathcal{S}^{-1}(\omega,\tau)) \\ &= \nabla\phi(\omega,\tau,t)\cdot\frac{\partial\mathcal{S}}{\partial s}(\mathcal{S}^{-1}(\omega,\tau)) \\ &= \frac{\partial\phi}{\partial\tau}(\omega,\tau)\cdot|A(\mathcal{T}\circ\mathcal{S}^{-1}(\omega,\tau))| \\ &= |A(\mathcal{T}\circ\mathcal{S}^{-1}(\omega,\tau))|\cdot\frac{\partial\phi}{\partial\tau}(\omega,\tau,t)\cdot\end{aligned}$$

Taking into account (2.8) - (2.11), we obtain

$$\int_{\mathcal{S}(D)\times(0,T)} \Big(f(\omega,\tau,t)\cdot\frac{\partial\phi}{\partial\tau} - \lambda(\omega,\tau)\cdot f(\omega,\tau,t)\cdot\frac{\partial\phi}{\partial t}\Big)\, d\omega\, d\tau\, dt \le 0 ,$$

which is (2.7). ∎

As a consequence of Theorem 2.5, we have the following monotonicity results.

Theorem 2.6. *Let* (u,g) *be a solution of problem* (P),

$$C := \mathcal{S}(D), \quad J_\epsilon := \Big[0, \frac{\min(m,1)}{4}\epsilon\Big], \quad C_\epsilon := \{(\omega,\tau)\in C \,/\, d\big((\omega,\tau),\partial C\big) > \epsilon\}.$$

Then for each $\epsilon > 0$ *small enough, for all* $k \in J_\epsilon$*, and for a.a.* $(\omega,\tau)\in C_\epsilon$*, we have*

$$f\Big(\omega,\tau-k,t+\int_{\tau-k}^{\tau}\lambda(\omega,\sigma)d\sigma\Big) \le f(\omega,\tau,t) \quad \forall t\in[0,T-\epsilon] \tag{2.12}$$

$$f\Big(\omega,\tau+k,t-\int_{\tau}^{\tau+k}\lambda(\omega,\sigma)d\sigma\Big) \ge f(\omega,\tau,t) \quad \forall t\in[\epsilon,T] . \tag{2.13}$$

To prove Theorem 2.6 we need the following two lemmas.

Lemma 2.7. *Let* (u,g) *be a solution of problem* (P),

$$P_\epsilon := C_\epsilon\times(0,T-\epsilon), \quad \vartheta_k(\omega,\tau,t) := \Big(\omega,\tau-k,t+\int_{\tau-k}^{\tau}\lambda(\omega,\sigma)\,d\sigma\Big).$$

Then for each $\xi\in\mathcal{D}(\overline{P}_\epsilon)$, $\xi\ge 0$, *the function*

$$F(k) = \int_{P_\epsilon} f\big(\vartheta_k(\omega,\tau,t)\big)\xi(\omega,\tau,t)\, d\omega\, d\tau\, dt$$

is nonincreasing on the interval J_ϵ.

Proof. Let $P_0 = C\times(0,T)$. We claim that

$$\vartheta_k(\overline{P}_\epsilon) \subset P_{\frac{3\epsilon}{4}} \subset P_0 \quad \forall k\in J_\epsilon .$$

Indeed we first have

$$P_{\frac{3\epsilon}{4}} = C_{\frac{3\epsilon}{4}} \times (0, T-\frac{3\epsilon}{4}) \subset C\times(0,T) = P_0.$$

Next if $k=0$, we have $\vartheta_k(\overline{P}_\epsilon) = \overline{P}_\epsilon \subset P_{\frac{3\epsilon}{4}}$. Let now $k\in J_\epsilon$, with $k>0$, and let $(\omega,\tau,t)\in\vartheta_k(\overline{P}_\epsilon)$. There exists $(\omega,\nu,s)\in\overline{P}_\epsilon$ such that $(\omega,\tau,t)=\vartheta_k(\omega,\nu,s)$. Since $(\omega,\nu)\in\overline{C}_\epsilon$, we have

$$\epsilon \le d\big((\omega,\nu),\partial C\big) \le d\big((\omega,\nu),(\omega,\tau)\big) + d\big((\omega,\tau),\partial C\big) = k + d\big((\omega,\tau),\partial C\big).$$

Since $k \le \frac{\min(m,1)}{4}\,\epsilon$, we deduce that

$$d\big((\omega,\tau),\partial C\big) > \epsilon - k \ge \epsilon - \frac{\epsilon}{4} = \frac{3\epsilon}{4}$$

and then $(\omega,\tau) \in C_{\frac{3\epsilon}{4}}$.

It remains to show that $t \in (0, T - \frac{3\epsilon}{4})$. We have $t = s + \int_{\nu-k}^{\nu} \lambda(\omega,\sigma)d\sigma > 0$ since $k > 0$. On the other hand one has $s < T - \epsilon$ and by (1.1) iii) $\lambda(\omega,\sigma) \le \frac{1}{m}$. Therefore, since $k \le \frac{\min(m,1)}{4}\,\epsilon$, we have

$$t = s + \int_{\nu-k}^{\nu} \lambda(\omega,\sigma)d\sigma < T - \epsilon + \frac{k}{m} \le T - \epsilon + \frac{1}{m}\cdot\frac{m}{4}\epsilon = T - \frac{3\epsilon}{4}.$$

It follows that

$$F(k) = \int_{\vartheta_k^{-1}\left(P_{\frac{3\epsilon}{4}}\right)} f(\vartheta_k(\omega,\tau,t))\xi(\omega,\tau,t)\,d\omega\,d\tau\,dt \quad \forall k \in J_\epsilon. \tag{2.14}$$

Moreover, ϑ_k is differentiable with $\mathcal{J}\vartheta_k(\omega,\tau,t) = 1$. Therefore we obtain from (2.14) by using the change of variables $\vartheta_k(\omega,\tau,t) = (\omega,\nu,s)$

$$\begin{aligned} F(k) &= \int_{P_{\frac{3\epsilon}{4}}} f(\omega,\nu,s)\xi\Big(\vartheta_k^{-1}(\omega,\nu,s)\Big)\,d\omega\,d\nu\,ds \\ &= \int_{P_{\frac{3\epsilon}{4}}} f(\omega,\nu,s)\xi\Big(\omega,\nu+k,s-\int_{\nu}^{\nu+k}\lambda(\omega,\sigma)d\sigma\Big)\,d\omega\,d\nu\,ds. \end{aligned} \tag{2.15}$$

From (2.15), we deduce that F is differentiable with

$$\begin{aligned} F'(k) &= \int_{P_{\frac{3\epsilon}{4}}} f(\omega,\nu,s)\Big\{\frac{\partial\xi}{\partial\tau} - \lambda(\omega,\nu+k)\frac{\partial\xi}{\partial t}\Big\}\Big(\vartheta_k^{-1}(\omega,\nu,s)\Big)\,d\omega\,d\nu\,ds \\ &= \int_{P_{\frac{3\epsilon}{4}}} f(\omega,\nu,s)\Big\{\frac{\partial\zeta}{\partial\nu} - \lambda(\omega,\nu)\frac{\partial\zeta}{\partial s}\Big\}(\omega,\nu,s)\,d\omega\,d\nu\,ds\ , \end{aligned}$$

where $\zeta(\omega,\nu,s) = \xi\big(\vartheta_k^{-1}(\omega,\nu,s)\big) = \xi\big(\omega,\nu+k,s-\int_{\nu}^{\nu+k}\lambda(\omega,\sigma)d\sigma\big)$. Now it is not difficult to verify that for each $k \in J_\epsilon$, we have $\zeta \in C_0^1(P_{\frac{\epsilon}{2}})$ and $\zeta \ge 0$. It follows then from Theorem 2.5 that $F'(k) \le 0$ for all $k \in J_\epsilon$. ∎

Lemma 2.8. *Let* (u,g) *be a solution of problem* (P), *and*

$$R_\epsilon := C_\epsilon \times (\epsilon, T), \qquad \Theta_k(\omega,\tau,t) := \Big(\omega,\ \tau+k,\ t - \int_{\tau}^{\tau+k}\lambda(\omega,\sigma)d\sigma\Big).$$

Then for each $\xi \in \mathcal{D}(\overline{R}_\epsilon)$, $\xi \ge 0$, *the function*

$$G(k) = \int_{R_\epsilon} f\Big(\omega,\tau+k,t-\int_{\tau}^{\tau+k}\lambda(\omega,\sigma)d\sigma\Big)\xi(\omega,\tau,t)\,d\omega\,d\tau\,dt$$

is nondecreasing on the interval J_ϵ.

Proof. As in the previous lemma one can verify that

$$\Theta_k(\overline{R}_\epsilon) \subset R_{\frac{3\epsilon}{4}} \subset P_0 \qquad \forall k \in J_\epsilon \ .$$

This leads to

$$G(k) = \int_{\Theta_k^{-1}\left(R_{\frac{3\epsilon}{4}}\right)} f(\Theta_k(\omega,\tau,t))\xi(\omega,\tau,t)\, d\omega\, d\tau\, dt. \tag{2.16}$$

Moreover $\Theta_k = \vartheta_k^{-1}$, and therefore we obtain from (2.16) by using the change of variables $\Theta_k(\omega,\tau,t) = (\omega,\nu,s)$

$$\begin{aligned} G(k) &= \int_{R_{\frac{3\epsilon}{4}}} f(\omega,\nu,s)\xi\Big(\vartheta_k(\omega,\nu,s)\Big)\, d\omega\, d\nu\, ds \\ &= \int_{R_{\frac{3\epsilon}{4}}} f(\omega,\nu,s)\xi\Big(\omega,\nu-k,s+\int_{\nu-k}^{\nu}\lambda(\omega,\sigma)d\sigma\Big)\, d\omega\, d\nu\, ds. \end{aligned} \tag{2.17}$$

From (2.17), we deduce that G is differentiable with

$$\begin{aligned} G'(k) &= \int_{R_{\frac{3\epsilon}{4}}} f(\omega,\nu,s)\Big\{-\frac{\partial\xi}{\partial\tau} + \lambda(\omega,\nu-k)\frac{\partial\xi}{\partial t}\Big\}\Big(\vartheta_k(\omega,\nu,s)\Big)\, d\omega\, d\nu\, ds \\ &= \int_{R_{\frac{3\epsilon}{4}}} f(\omega,\nu,s)\Big\{-\frac{\partial\zeta}{\partial\nu} + \lambda(\omega,\nu)\frac{\partial\zeta}{\partial s}\Big\}(\omega,\nu,s)\, d\omega\, d\nu\, ds \ , \end{aligned}$$

where

$$\zeta(\omega,\nu,s) = \xi\big(\vartheta_k(\omega,\nu,s)\big) = \xi\big(\omega,\nu-k,s+\int_{\nu-k}^{\nu}\lambda(\omega,\sigma)d\sigma\big).$$

Finally for each $k \in J_\epsilon$, we have $\zeta \in C_0^1(R_{\frac{\epsilon}{2}})$ and $\zeta \geq 0$. Thus we obtain by Theorem 2.5 that $G'(k) \geq 0$ for all $k \in J_\epsilon$. ∎

Proof of Theorem 2.6. Since $g \in C^0([0,T], W^{-1,q'}(\Omega))$, we deduce that $f \in C^0([0,T], W^{-1,q'}(C))$ and therefore Theorem 2.6 follows immediately from Lemma 2.7 and Lemma 2.8. ∎

3. Continuity of g

The main result of this paper is the following theorem.

Theorem 3.1. *Let (u,g) be a solution of problem* (P). *Then*

$$g \in C^0([0,T], L^p(\Omega)) \qquad \forall p \geq 1 \ .$$

The proof of Theorem 3.1 is based on Theorem 2.6 and the following lemma.

Lemma 3.2. *Let* (u,g) *be a solution of* (P). *Then* $f\in C^0([0,T],L^2(C))$.

Proof. First of all we deduce from (1.2) and the fact that g is bounded

$$\forall t\in(0,T):\quad f(\cdot,t+\epsilon)\underset{\epsilon\to0}{\rightharpoonup} f(\cdot,t)\quad\text{weakly in } L^2(C) \tag{3.1}$$

$$f(\cdot,\epsilon)\underset{\epsilon\to0^+}{\rightharpoonup} f(\cdot,0)\quad\text{weakly in } L^2(C) \tag{3.2}$$

$$f(\cdot,T-\epsilon)\underset{\epsilon\to0^+}{\rightharpoonup} f(\cdot,T)\quad\text{weakly in } L^2(C). \tag{3.3}$$

Note that it is enough to show that for all $t\in(0,T)$

$$\lim_{\epsilon\to0}\int_C\big(f^2(\omega,\tau,t+\epsilon)-f^2(\omega,\tau,t)\big)\,d\omega\,d\tau=0 \tag{3.4}$$

$$\lim_{\epsilon\to0^+}\int_C\big(f^2(\omega,\tau,\epsilon)-f^2(\omega,\tau,0)\big)\,d\omega\,d\tau=0 \tag{3.5}$$

$$\lim_{\epsilon\to0^+}\int_C\big(f^2(\omega,\tau,T-\epsilon)-f^2(\omega,\tau,T)\big)\,d\omega\,d\tau=0. \tag{3.6}$$

We distinguish several cases.

`Case 1:` $t\in[0,T)$. Let $\epsilon>0$ small enough and let $k_\epsilon(\omega,\tau)$ be defined by $\epsilon=\int_{\tau-k_\epsilon(\omega,\tau)}^{\tau}\lambda(\omega,\sigma)d\sigma$. We would like to show that

$$\lim_{\epsilon\to0}\int_C\big(f^2(\omega,\tau,t+\epsilon)-f^2(\omega,\tau,t)\big)\,d\omega\,d\tau=0. \tag{3.7}$$

We first remark that for ϵ small enough one has

$$\begin{aligned}\Big|\int_C\big(f^2(\omega,\tau,t+\epsilon)&-f^2(\omega,\tau,t)\big)\,d\omega\,d\tau\Big|\\ &\le\Big|\int_{C_{2\sqrt{\epsilon}}}\big(f^2(\omega,\tau,t+\epsilon)-f^2(\omega,\tau,t)\big)\,d\omega\,d\tau\Big|\\ &\quad+\Big|\int_{C\setminus C_{2\sqrt{\epsilon}}}\big(f^2(\omega,\tau,t+\epsilon)-f^2(\omega,\tau,t)\big)\,d\omega\,d\tau\Big|\\ &=I_{\epsilon,1}+I_{\epsilon,2}.\end{aligned} \tag{3.8}$$

Note that since $f\in L^\infty(\mathrm{P})$, one has $I_{\epsilon,2}\le c|C\setminus C_{2\sqrt{\epsilon}}|$ (here and after we denote by c any positive constant) and therefore since $\lim\limits_{\epsilon\to0}|C\setminus C_{2\sqrt{\epsilon}}|=0$, we obtain

$$\lim_{\epsilon\to0}I_{\epsilon,2}=0. \tag{3.9}$$

Moreover

$$
\begin{aligned}
I_{\epsilon,1} &\le \Big| \int_{C_{2\sqrt{\epsilon}}} \big(f^2(\omega,\tau,t+\epsilon) - f^2(\omega,\tau+k_\epsilon,t)\big)\, d\omega\, d\tau \Big| \\
&\quad + \Big| \int_{C_{2\sqrt{\epsilon}}} \big(f^2(\omega,\tau+k_\epsilon,t) - f^2(\omega,\tau,t)\big)\, d\omega\, d\tau \Big| \qquad (3.10)\\
&= I_{\epsilon,3} + I_{\epsilon,4}\ .
\end{aligned}
$$

From (1.1) *iii)* and the definition of λ, we have

$$\frac{1}{M} \le \lambda(\omega,\tau) \le \frac{1}{m} \qquad \forall (\omega,\tau) \in C.$$

This leads easily to

$$m\epsilon \le k_\epsilon(\omega,\tau) \le M\epsilon \qquad \forall (\omega,\tau) \in C.$$

It follows that for $\epsilon \le \big(\frac{\min(m,1)}{4M}\big)^2$

$$k_\epsilon(\omega,\tau) \in J_{\sqrt{\epsilon}} \qquad \forall (\omega,\tau) \in C. \tag{3.11}$$

Indeed it is enough to verify that $M\epsilon \le \frac{\min(m,1)}{4}\sqrt{\epsilon}$ which is equivalent to $\epsilon \le \big(\frac{\min(m,1)}{4M}\big)^2$. We also have for $\epsilon < \frac{1}{M^2}$

$$(\omega, \tau + k_\epsilon(\omega,\tau)) \in C_{\sqrt{\epsilon}} \qquad \forall (\omega,\tau) \in C_{2\sqrt{\epsilon}}. \tag{3.12}$$

Indeed let $(\omega,\tau) \in C_{2\sqrt{\epsilon}}$. Then

$$
\begin{aligned}
2\sqrt{\epsilon} &< d\big((\omega,\tau), \partial C\big) \\
&\le d\big((\omega,\tau), (\omega,\tau+k_\epsilon)\big) + d\big((\omega,\tau+k_\epsilon), \partial C\big) \\
&= k_\epsilon + d\big((\omega,\tau+k_\epsilon), \partial C\big).
\end{aligned}
$$

So

$$d\big((\omega,\tau+k_\epsilon),\partial C\big) - \sqrt{\epsilon} > \sqrt{\epsilon} - k_\epsilon \ge \sqrt{\epsilon} - M\epsilon = \sqrt{\epsilon} M\Big(\frac{1}{M} - \sqrt{\epsilon}\Big) > 0.$$

Moreover for $\epsilon < (T-t)^2$ we have $0 \le t < T - \sqrt{\epsilon}$. Taking into account (3.11) and (3.12), we can use (2.12) for ϵ small enough $\big(\epsilon < \min\big(\big(\frac{\min(m,1)}{4M}\big)^2, (T-t)^2\big)\big)$ to obtain

$$
\begin{aligned}
f(\omega,\tau,t+\epsilon) &= f(\omega,\tau+k_\epsilon - k_\epsilon, t + \int_{\tau-k_\epsilon}^{\tau} \lambda(\omega,\sigma)\, d\sigma) \\
&\le f(\omega,\tau+k_\epsilon,t)
\end{aligned}
$$

for a.a. $(\omega,\tau)\in C_{2\sqrt{\epsilon}}$. After using the fact that

$$|f^2(\omega,\tau,t+\epsilon)-f^2(\omega,\tau+k_\epsilon,t)|\le c|f(\omega,\tau,t+\epsilon)-f(\omega,\tau+k_\epsilon,t)|,$$

we get

$$\begin{aligned}
I_{\epsilon,3} &\le c\int_{C_{2\sqrt{\epsilon}}}|f(\omega,\tau,t+\epsilon)-f(\omega,\tau+k_\epsilon,t)|\,d\omega\,d\tau\\
&= c\int_{C_{2\sqrt{\epsilon}}}\big(f(\omega,\tau+k_\epsilon,t)-f(\omega,\tau,t+\epsilon)\big)\,d\omega\,d\tau\\
&= c\int_{C_{2\sqrt{\epsilon}}}\big(f(\omega,\tau,t)-f(\omega,\tau,t+\epsilon)\big)\,d\omega\,d\tau\\
&\quad + c\int_{C_{2\sqrt{\epsilon}}}\big(f(\omega,\tau+k_\epsilon,t)-f(\omega,\tau,t)\big)\,d\omega\,d\tau\\
&= cI_{\epsilon,5}+cI_{\epsilon,6}
\end{aligned}\tag{3.13}$$

Now

$$\begin{aligned}
|I_{\epsilon,5}| &\le \Big|\int_C\big(f(\omega,\tau,t)-f(\omega,\tau,t+\epsilon)\big)d\omega d\tau\Big|\\
&\quad+\Big|\int_{C\setminus C_{2\sqrt{\epsilon}}}\big(f(\omega,\tau,t)-f(\omega,\tau,t+\epsilon)\big)\,d\omega\,d\tau\Big|\\
&\le \Big|\int_C\big(f(\omega,\tau,t)-f(\omega,\tau,t+\epsilon)\big)\,d\omega\,d\tau\Big|+c\,|C\setminus C_{2\sqrt{\epsilon}}|.
\end{aligned}$$

It follows from (3.1) - (3.2) that

$$\lim_{\epsilon\to 0} I_{\epsilon,5}=0. \tag{3.14}$$

Regarding $I_{\epsilon,6}$, we use the change of variables $(\omega,\tau)\to G_\epsilon(\omega,\tau)=(\omega,\tau+k_\epsilon(\omega,\tau))=(\omega',\tau')$. A simple calculation shows that for ϵ small enough

$$\mathcal{J}G_\epsilon(\omega,\tau)=1+\frac{\partial k_\epsilon}{\partial\tau}(\omega,\tau)=\frac{2\lambda(\omega,\tau-k_\epsilon(\omega,\tau))-\lambda(\omega,\tau)}{\lambda(\omega,\tau-k_\epsilon(\omega,\tau))}>0$$

which leads to

$$\begin{aligned}
I_{\epsilon,6}=&\int_{G_\epsilon(C_{2\sqrt{\epsilon}})}f(\omega',\tau',t)\frac{\lambda(\omega',\tau'-2k_\epsilon(\omega,\tau))}{2\lambda(\omega',\tau'-2k_\epsilon(\omega,\tau))-\lambda(\omega',\tau'-k_\epsilon(\omega,\tau))}\,d\omega'\,d\tau'\\
&-\int_{C_{2\sqrt{\epsilon}}}f(\omega,\tau,t)\,d\omega\,d\tau.
\end{aligned}$$

Therefore it is clear that since $\lim_{\epsilon\to 0} k_\epsilon(\omega,\tau) = 0$ and

$$\lim_{\epsilon\to 0} C_{2\sqrt{\epsilon}} = \lim_{\epsilon\to 0} G_\epsilon(C_{2\sqrt{\epsilon}}) = C \ ,$$

we have

$$\lim_{\epsilon\to 0} I_{\epsilon,6} = 0. \tag{3.15}$$

In the same way we prove that

$$\lim_{\epsilon\to 0} I_{\epsilon,4} = 0. \tag{3.16}$$

Taking into account (3.8) - (3.10) and (3.13) - (3.16), we get (3.7). In particular we have proved (3.5) which leads to the continuity of f at $t = 0$.

`Case 2:` $t \in (0,T]$. Let $\epsilon > 0$ small enough and let k_ϵ be defined by $\epsilon = \int_\tau^{\tau+k_\epsilon(\omega,\tau)} \lambda(\omega,\sigma)\,d\sigma$. We would like to show that

$$\lim_{\epsilon\to 0} \int_C \big(f^2(\omega,\tau,t-\epsilon) - f^2(\omega,\tau,t)\big)\,d\omega\,d\tau = 0. \tag{3.17}$$

As in the first case we remark that for ϵ small enough one has

$$\begin{aligned}
\Big|\int_C \big(f^2(\omega,\tau,t-\epsilon) - f^2(\omega,\tau,t)\big)\,d\omega\,d\tau\Big| \\
&\le \Big|\int_{C_{2\sqrt{\epsilon}}} \big(f^2(\omega,\tau,t-\epsilon) - f^2(\omega,\tau,t)\big)\,d\omega\,d\tau\Big| \\
&\quad + \Big|\int_{C\setminus C_{2\sqrt{\epsilon}}} \big(f^2(\omega,\tau,t-\epsilon) - f^2(\omega,\tau,t)\big)\,d\omega\,d\tau\Big| \\
&= I_{\epsilon,7} + I_{\epsilon,8}.
\end{aligned} \tag{3.18}$$

We have $I_{\epsilon,8} \le c|C \setminus C_{2\sqrt{\epsilon}}|$ and therefore

$$\lim_{\epsilon\to 0} I_{\epsilon,8} = 0. \tag{3.19}$$

Moreover

$$\begin{aligned}
I_{\epsilon,7} &\le \Big|\int_{C_{2\sqrt{\epsilon}}} \big(f^2(\omega,\tau,t-\epsilon) - f^2(\omega,\tau-k_\epsilon,t)\big)\,d\omega\,d\tau\Big| \\
&\quad + \Big|\int_{C_{2\sqrt{\epsilon}}} \big(f^2(\omega,\tau-k_\epsilon,t) - f^2(\omega,\tau,t)\big)\,d\omega\,d\tau\Big| \\
&= I_{\epsilon,9} + I_{\epsilon,10}.
\end{aligned} \tag{3.20}$$

As in the previous case we have

$$m\epsilon \le k_\epsilon(\omega,\tau) \le M\epsilon \qquad \forall(\omega,\tau) \in C\ .$$

Then it is not difficult to verify that for $\epsilon < \min\big(\big(\frac{\min(m,1)}{4M}\big)^2, t^2\big)$, we have

$$\begin{aligned} k_\epsilon(\omega,\tau) \in J_{\sqrt{\epsilon}} \qquad & \forall(\omega,\tau)\in C \\ (\omega,\tau-k_\epsilon(\omega,\tau)) \in C_{\sqrt{\epsilon}} \qquad & \forall(\omega,\tau)\in C_{2\sqrt{\epsilon}}\ . \end{aligned}$$

Therefore we obtain by using (2.13) for ϵ small enough

$$\begin{aligned} f(\omega,\tau,t-\epsilon) &= f(\omega,\tau-k_\epsilon+k_\epsilon, t-\int_\tau^{\tau+k_\epsilon}\lambda(\omega,\sigma)d\sigma) \\ &\geq f(\omega,\tau-k_\epsilon,t) \end{aligned}$$

for a.a. $(\omega,\tau)\in C_{2\sqrt{\epsilon}}$. Using the fact that

$$|f^2(\omega,\tau,t-\epsilon)-f^2(\omega,\tau-k_\epsilon,t)| \leq c|f(\omega,\tau,t-\epsilon)-f(\omega,\tau-k_\epsilon,t)|,$$

we get

$$\begin{aligned} I_{\epsilon,9} &\leq c\int_{C_{2\sqrt{\epsilon}}} |f(\omega,\tau,t-\epsilon)-f(\omega,\tau-k_\epsilon,t)|\,d\omega\,d\tau \\ &= c\int_{C_{2\sqrt{\epsilon}}} \big(f(\omega,\tau,t-\epsilon)-f(\omega,\tau-k_\epsilon,t)\big)\,d\omega\,d\tau \\ &= c\int_{C_{2\sqrt{\epsilon}}} \big(f(\omega,\tau,t-\epsilon)-f(\omega,\tau,t)\big)\,d\omega\,d\tau \\ &\quad + c\int_{C_{2\sqrt{\epsilon}}} \big(f(\omega,\tau,t)-f(\omega,\tau-k_\epsilon,t)\big)\,d\omega\,d\tau \\ &= cI_{\epsilon,11}+cI_{\epsilon,12}. \end{aligned} \tag{3.21}$$

We have

$$\begin{aligned} |I_{\epsilon,11}| &\leq \Big|\int_C \big(f(\omega,\tau,t-\epsilon)-f(\omega,\tau,t)\big)\,d\omega\,d\tau\Big| \\ &\quad + \Big|\int_{C\setminus C_{2\sqrt{\epsilon}}} \big(f(\omega,\tau,t-\epsilon)-f(\omega,\tau,t)\big)\,d\omega\,d\tau\Big| \\ &\leq \Big|\int_C \big(f(\omega,\tau,t-\epsilon)-f(\omega,\tau,t)\big)\,d\omega\,d\tau\Big| + c|C\setminus C_{2\sqrt{\epsilon}}|. \end{aligned}$$

It follows from (3.1) and (3.3) that

$$\lim_{\epsilon\to 0} I_{\epsilon,11} = 0. \tag{3.22}$$

Arguing as for $I_{\epsilon,4}$ and $I_{\epsilon,6}$, we prove

$$\lim_{\epsilon\to 0} I_{\epsilon,10} = \lim_{\epsilon\to 0} I_{\epsilon,12} = 0. \tag{3.23}$$

Taking into account (3.18) - (3.23), we get (3.17). In particular we have proved (3.6) which leads to the continuity of f at $t = T$.

Combining (3.7) and (3.17) for $t \in (0, T)$, we obtain the continuity of f at $t \in (0, T)$, and therefore the lemma is proved. ∎

Proof of Theorem 3.1. Since $|Y \circ \mathcal{S}^{-1}|$ is positive, uniformly bounded and independent of t, we deduce from Lemma 3.2 that

$$g \circ \mathcal{T} \circ \mathcal{S}^{-1} = \frac{f}{|Y \circ \mathcal{S}^{-1}|} \in C^0\big([0,T], L^2(\mathcal{S}(D))\big).$$

Moreover by using the change of variables $\mathcal{T} \circ \mathcal{S}^{-1}$ it follows that

$$g \in C^0([0,T], L^2(\mathcal{T}(D))) = C^0\big([0,T], L^2(\Omega)\big). \tag{3.24}$$

Using the imbedding $L^2(\Omega) \subset L^p(\Omega)$, we obtain $g \in C^0([0,T], L^p(\Omega))$ for $p \in [1, 2]$. Now for $p > 2$, we obtain the result from

$$\begin{aligned} &|g(\omega,\tau,t+\epsilon) - g(\omega,\tau,t)|^p \\ &\qquad = |g(\omega,\tau,t+\epsilon) - g(\omega,\tau,t)|^{p-2}|g(\omega,\tau,t+\epsilon) - g(\omega,\tau,t)|^2 \\ &\qquad \leq c|g(\omega,\tau,t+\epsilon) - g(\omega,\tau,t)|^2. \end{aligned}$$ ∎

Remark 3.3. All results of this paper are clearly valid for the evolution dam problem with leaky boundary conditions ([12]).

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