Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 24 (2005), No. 1, 179–187

Global Nonexistence for a Quasilinear Evolution Equation with a Generalized Lewis Function

Yong Zhou

Abstract. In this work we consider the following quasilinear parabolic equation

$$a(x,t)u_t - \operatorname{div}\left(|\nabla u|^{m-2}\nabla u\right) = f(u),$$

where $a(x,t) \geq 0$ is a generalized Lewis function. The main result is that the solution blows up in finite time if the initial datum $u(x,0)$ possesses suitable positive energy. Moreover, we have a precise estimate for the lifespan of the solution in this case. Blowup of solutions with vanishing initial energy is considered also.

Keywords: *Global nonexistence, quasilinear evolution equation*

MSC 2000: Primary 35B45, secondary 35K55

1. Introduction

We consider the following initial boundary value problem with generalized Lewis function $a(x,t)$ which depends on both spacial variable and time:

$$\begin{cases} a(x,t)u_t - \operatorname{div}\left(|\nabla u|^{m-2}\nabla u\right) = f(u) & x \in \Omega,\ t > 0 \\ u(x,t) = 0 & x \in \partial\Omega,\ t \geq 0 \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases} \tag{1}$$

where $m \geq 2$ ($m = 2$, then $\operatorname{div}(|\nabla u|^{m-2}\nabla u) = \Delta u$) , f is a continuous function, and Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 1$, with smooth boundary $\partial\Omega$.

Very recently, Tan [6] considered Problem (1) for $m = 2$, $a(x,t) = a(x) \geq 0$ and $f(u) = |u|^p$ with $p = 2^* - 1 = (n+2)/(n-2)$ (one can find a review of previous results in [6] and references therein, which are not list in this paper

Y. Zhou: Department of Mathematics, East China Normal University, Shanghai 200062, China; yzhou@math.ecnu.edu.cn

ISSN 0232-2064 / $ 2.50

just for concision). He proved that the solution to problem (1) blows up in finite time if $u_0 \in H_0^1(\Omega)$ satisfies the following initial conditions:

$$\begin{aligned} \frac{1}{2}\int_\Omega |\nabla u_0(x)|^2\, dx - \frac{1}{2^*}\int_\Omega |u_0(x)|^{2^*}\, dx < \frac{1}{n} S^{\frac{n}{2}} \\ \int_\Omega |u_0(x)|^{2^*}\, dx > S^{\frac{n}{2}}, \end{aligned} \tag{2}$$

where

$$S = \inf_{u\in H_0^1} \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^{2^*}}^2}.$$

In this paper, we generalize Tan's result extensively. First, $m \geq 2$, f is a general source term and $a(x,t) \geq 0$, almost every in Ω, is a generalized Lewis function. Second, we replace '$<$' by '$\leq$' in (2) (see (5) below for our case). Moreover, the lifespan of the solution can be bounded precisely.

The main theorem reads as follows.

Theorem 1. *Assume that $a(x,t)$ is a positive function which belongs to the space $W^{1,\infty}(0,\infty; L^\infty(\Omega))$ and that $a_t(x,t) \leq 0$ a.e. for $t \geq 0$. Let*

$$|f(u)| \leq C_0|u|^{p-1}, \quad p\, F(u) \leq u\, f(u), \tag{3}$$

where $C_0 > 0, p > m \geq 2$ and

$$F(u) = \int_0^u f(s)ds, \quad p \leq \frac{nm}{n-m} \quad \text{for } n > m.$$

If $u_0 \in W_0^{1,m}(\Omega)$ satisfies

$$\|u_0\|_{L^p} > \lambda_0 \equiv (C_0 B_0^m)^{\frac{-1}{p-m}} \tag{4}$$

and has positive energy

$$\begin{aligned} E(0) &= \frac{1}{m}\int_\Omega |\nabla u_0(x)|^m\, dx - \int_\Omega F(u_0(x))\, dx \\ &\leq E_0 \equiv \left(\frac{1}{m} - \frac{1}{p}\right)\left(C_0 B_0^p\right)^{-\frac{m}{p-m}}, \end{aligned} \tag{5}$$

where B_0 is the optimal constant of Sobolev embedding

$$\|v\|_{L^p} \leq B_0 \|\nabla v\|_{L^m} \tag{6}$$

for $v \in W_0^{1,m}(\Omega)$. Then no global solutions of Problem (1) can exist for $u_0 \not\equiv 0$. Moreover, if $E(0) < E_0$, then the lifespan T^* of the solution to Problem (1) can be bounded above as

$$T^* \leq \frac{8\|\sqrt{a_0(x)}u_0(x)\|_{L^2}^2}{(p-2)^2(E_0 - E(0))},$$

where $a(x,0)$ is denoted by $a_0(x)$.

Remark 1. One can prove that the set of functions $u_0 \in W_0^{1,m}(\Omega)$ satisfying

$$\begin{cases} \frac{1}{m}\int_\Omega |\nabla u_0(x)|^m \, dx - \int_\Omega F(u_0(x)) \, dx < \left(\frac{1}{m} - \frac{1}{p}\right)\left(C_0 B_0^p\right)^{-\frac{m}{p-m}} \\ \qquad\qquad\qquad \|u_0\|_{L^p} = \left(C_0 B_0^m\right)^{-\frac{1}{p-m}} \end{cases}$$

is empty. In fact, one can compute directly (see (10) below) that

$$\frac{1}{m}\int_\Omega |\nabla u_0(x)|^m \, dx - \int_\Omega F(u_0(x)) \, dx \geq \left(\frac{1}{m} - \frac{1}{p}\right)\left(C_0 B_0^p\right)^{-\frac{m}{p-m}}$$

as $\|u_0\|_{L^p} = (C_0 B_0^m)^{-\frac{1}{p-m}}$.

Since the local existence of solution $u(x,t) \in C([0,T]; W_0^{1,m}(\Omega))$ to Problem (1) is well-known, we do not repeat it, for concision.

2. Proof of the main theorem

The method used here is not new but classical concavity method (see [2, 3, 4]). However, our argument is slightly different from previous arguments and it is more concise, in the author's opinion.

In order to prove the main theorem, we recall the following lemma in [1, Lemma 2.1].

Lemma 1. *Suppose that a positive, twice differential function $\psi(t)$ satisfies for $t \geq 0$ the inequality*

$$\psi''\psi - (1+\alpha)(\psi')^2 \geq 0,$$

where $\alpha > 0$. If $\psi(0) > 0$, $\psi'(0) > 0$, then

$$\psi \to \infty \quad as \quad t \to t_1 \leq t_2 = \frac{\psi(0)}{\alpha\psi'(0)}.$$

The corresponding energy to Problem (1) is given by

$$E(t) = \frac{1}{m}\int_\Omega |\nabla u(x,t)|^m \, dx - \int_\Omega F(u(x,t)) \, dx, \tag{7}$$

and one can find that $E(t) \leq E(0)$ easily from

$$\frac{dE(t)}{dt} = -\int_\Omega a(x,t) u_t^2(x,t) \, dx \leq 0. \tag{8}$$

The second lemma is a delicate estimate for the solution.

Lemma 2. *Suppose* $\|u_0\|_{L^p} > \lambda_0$ *and* $E(0) < E_0$. *Then*

$$\|u(x,t)\|_{L^p} > \lambda_0 \quad \text{and} \quad \|\nabla u(x,t)\|_{L^m} > \left(C_0 B_0^p\right)^{-\frac{1}{p-m}} \tag{9}$$

for all $t \geq 0$.

Proof. Thanks to (3) and (6), one has for any $t \geq 0$

$$\begin{aligned} E(t) &= \frac{1}{m}\int_\Omega |\nabla u(x,t)|^m \, dx - \int_\Omega F\left(u(x,t)\right) \, dx \\ &\geq \frac{1}{mB_0^m}\|u(\cdot,t)\|_{L^p}^m - \frac{C_0}{p}\|u(\cdot,t)\|_{L^p}^p. \end{aligned}$$

Now if we let

$$g(s) = \frac{1}{mB_0^m}s^m - \frac{C_0}{p}s^p, \ \ s \geq 0,$$

then

$$\left.\begin{array}{l} g(s) \text{ is strictly increasing on } [0,\lambda_0) \\ g(s) \text{ takes its maximum value } E_0 \text{ at } \lambda_0 \\ g(s) \text{ is strictly decreasing on } (\lambda_0,\infty). \end{array}\right\} \tag{10}$$

Since $E_0 > E(0) \geq E(t) \geq g\left(\|u(\cdot,t)\|_{L^p}\right)$ for all time $t \geq 0$, it follows from (10) that there is no time t^* such that $\|u(\cdot,t^*)\|_{L^p} = \lambda_0$. Then by the continuity of $\|u(\cdot,t)\|_{L^p}$-norm with respect to time variable and the initial condition that $E(0) < E_0$ and $\|u_0\|_{L^p} > \lambda_0$, one has

$$\|u(\cdot,t)\|_{L^p} > \lambda_0 = (C_0 B_0^m)^{-\frac{1}{p-m}},$$

for all time $t \geq 0$, and consequently

$$\|\nabla u(\cdot,t)\|_{L^m} \geq B_0^{-1}\|u(\cdot,t)\|_{L^p} > B_0^{-1}\lambda_0 = (C_0 B_0^p)^{-\frac{1}{p-m}}.$$

This finishes the proof. ∎

After the above preparation, we go to the proof of the main theorem.

Proof of Theorem 1. We consider two cases.

`Case 1:` $E(0) < E_0$. The goal is to construct a suitable function ψ which satisfies the conditions in Lemma 1. Following the arguments of [5, Theorem 1.1] (see [8] also), it is not difficult to find the following one is suitable for our purpose,

$$\begin{aligned} \psi(t) = &\int_0^t \int_\Omega a(x,\tau)u^2(x,\tau) \, dx \, d\tau + \int_0^t \int_\Omega (\tau - t)a_t(x,\tau)u^2(x,\tau) \, dx \, d\tau \\ &+ (T_0 - t)\int_\Omega a_0(x)u_0^2(x) \, dx + \beta(t+t_0)^2, \quad t < T_0, \end{aligned} \tag{11}$$

where t_0, T_0 and β are positive constants to be determined later. Then using equation (1) and integration by parts, one obtains

$$\begin{aligned}\psi'(t) &= \int_\Omega a(x,t)u^2(x,t)\,dx - \int_0^t \int_\Omega a_t(x,\tau)u^2(x,\tau)\ dx\,d\tau \\ &\quad - \int_\Omega a_0(x)u_0^2(x)\ dx + 2\beta(t+t_0) \qquad (12)\\ &= 2\int_0^t \int_\Omega a(x,\tau)u(x,\tau)u_t(x,\tau)\ dx\,d\tau + 2\beta(t+t_0)\end{aligned}$$

$$\psi''(t) = 2\int_\Omega a(x,t)u(x,t)u_t(x,t)\,dx + 2\beta. \qquad (13)$$

Then, due to (3) and (9)

$$\begin{aligned}\psi''(t) &= 2\int_\Omega -|\nabla u(x,t)|^m\,dx + 2\int_\Omega u(x,t)f(u(x,t))\,dx + 2\beta \\ &\ge 2\left(\frac{p}{m}-1\right)\int_\Omega |\nabla u(x,t)|^m\ dx - 2pE(t) + 2\beta \\ &= 2\left(\frac{p}{m}-1\right)\int_\Omega |\nabla u(x,t)|^m\ dx - 2pE(0) \\ &\quad + 2\beta + 2p\int_0^t \int_\Omega a(x,\tau)(u_t(x,\tau))^2\ dx\,d\tau \\ &\ge 2\left(\frac{p}{m}-1\right)(C_0B_0^p)^{\frac{-m}{p-m}} - 2pE(0) \\ &\quad + 2\beta + 2p\int_0^t \int_\Omega a(x,\tau)(u_t(x,\tau))^2\ dx\,d\tau \\ &= 2p(E_0 - E(0)) + 2\beta + 2p\int_0^t \int_\Omega a(x,\tau)(u_t(x,\tau))^2\ dx\,d\tau.\end{aligned}$$

Now, let $\beta = 2(E_0 - E(0)) > 0$, and note that $p > 2$, then

$$\psi''(t) \ge (p+2)\beta + (p+2)\int_0^t \int_\Omega a(x,\tau)(u_t(x,\tau))^2\ dx\,d\tau. \qquad (14)$$

From (11) (12), (13) and (14), we have

$$\begin{cases}\psi(0) = T_0\displaystyle\int_\Omega a_0(x)u_0^2(x)\ dx + \beta t_0^2 > 0 \\ \psi'(0) = 2\beta t_0 > 0 \\ \psi''(t) \ge (p+2)\beta > 0 \ \text{ for all } t \ge 0.\end{cases}$$

Therefore ψ and ψ' are both positive. Since $a_t(x,t) \le 0$, for all $x \in \Omega$ and $t \ge 0$, by the construction of ψ, it is clearly that

$$\psi(t) \ge \int_0^t \int_\Omega a(x,\tau)u^2(x,\tau)\,dx\,d\tau + \beta(t+t_0)^2. \tag{15}$$

Thus for all $(\xi,\eta) \in \mathbb{R}^2$, from (11), (12), (13), (14) and (15) follows

$$\begin{aligned}
\psi(t)\xi^2 + \psi'(t)\xi\eta + \frac{\psi''(t)}{p+2}\eta^2 \\
&\ge \left(\int_0^t \int_\Omega a(x,\tau)|u(x,\tau)|^2\,dx\,d\tau + \beta(t+t_0)^2\right)\xi^2 \\
&\quad + 2\xi\eta \int_0^t \int_\Omega a(x,\tau)u(x,\tau)u_t(x,\tau)\,dx\,d\tau + 2\beta(t+t_0)\xi\eta \\
&\quad + \beta\eta^2 + \eta^2 \int_0^t \int_\Omega a(x,\tau)(u_t(x,\tau))^2\,dx\,d\tau \\
&\ge 0,
\end{aligned}$$

which implies that

$$\psi(t)\frac{\psi''(t)}{p+2} - \left(\frac{\psi'(t)}{2}\right)^2 \ge 0,$$

that is

$$\psi(t)\psi''(t) - \frac{p+2}{4}\big(\psi'(t)\big)^2 \ge 0.$$

Then using Lemma 1, one obtains that $\psi(t)$ goes to ∞ as t tends to

$$\frac{2(T_0\|\sqrt{a_0}u_0\|_{L^2}^2 + \beta t_0^2)}{(p-2)\beta t_0}.$$

The remaining thing is to choose suitable t_0 and T_0. Let t_0 be any number which depends only on p, $E_0 - E(0)$ and $\|u_0\|_{L^2}$ as

$$t_0 > \frac{\|\sqrt{a_0}u_0\|_{L^2}^2}{(p-2)(E_0 - E(0))}.$$

Fix t_0, then T_0 can be chosen as

$$T_0 = \frac{2(T_0\|\sqrt{a_0}u_0\|_{L^2}^2 + \beta t_0^2)}{(p-2)\beta t_0} = \frac{2(E_0 - E(0))t_0^2}{(p-2)(E_0 - E(0))t_0 - \|\sqrt{a_0}u_0\|_{L^2}^2}.$$

Therefore the lifespan of the solution $u(x,t)$ is bounded by

$$T^* < \inf_{t\geq t_0} \frac{2(E_0 - E(0))t^2}{(p-2)(E_0 - E(0))t - \|\sqrt{a_0}u_0\|_{L^2}^2} = \frac{8\|\sqrt{a_0}u_0\|_{L^2}^2}{(p-2)^2(E_0 - E(0))}.$$

This finishes the proof for Case 1.

`Case 2:` $E(0) = E_0$. For this case, actually we consider the following claim.

Claim. *There exists* $\bar{t} > 0$ *such that* $E(\bar{t}) < E_0$.

Suppose Claim 1 is not true which means that $E(t) = E_0$ for all $t \geq 0$. Then by the continuity of $\|u(\cdot,t)\|_{L^p}$ there exists a t_ϵ, small enough, such that

$$E(t) = E_0 \quad \text{and} \quad \|u(x,t)\|_{L^p} > \lambda_0 \quad \text{for all } t \in [0, t_\epsilon].$$

Then we consider the solution of (1) on $[0, t_\epsilon]$,

$$0 = E(t) - E_0 = -\int_0^{t_\epsilon}\int_\Omega a(x,t)u_t^2(x,s)\,dxds,$$

which turns out to be

$$\int_\Omega a(x,t)u(x,t)u_t(x,t)\,dx = 0 \quad \text{a.e. on } [0, t_\epsilon].$$

And consequently, due to the equation (1),

$$\begin{aligned}\int_\Omega & a(x,t)u(x,t)u_t(x,t)\,dx \\ &= -\int_\Omega |\nabla u(x,t)|^m\,dx + \int_\Omega u(x,t)f(u(x,t))\,dx = 0\end{aligned} \tag{16}$$

a.e. on $(0, t_\epsilon]$. On the other hand,

$$\begin{aligned}E_0 = E(t) &= \frac{1}{m}\int_\Omega |\nabla u(x,t)|^m\,dx - \int_\Omega F(u(x,t))\,dx \\ &\geq \frac{1}{m}\int_\Omega |\nabla u(x,t)|^m\,dx - \frac{1}{p}\int_\Omega u(x,t)f(u(x,t))\,dx \\ &= \left(\frac{1}{m} - \frac{1}{p}\right)\int_\Omega |\nabla u(x,t)|^m\,dx \quad \big(\text{by (16)}\big) \\ &\geq \left(\frac{1}{m} - \frac{1}{p}\right) B_0^{-m}\|u(x,t)\|_{L^p}^m \\ &> \left(\frac{1}{m} - \frac{1}{p}\right) B_0^{-m}\lambda_0^m = E_0,\end{aligned}$$

which is a contradiction.

The proof of Theorem 1 is complete since one can apply the previous case (Case 1) after shifting the time origin to $\bar{t}$. ∎

Remark 2. Although Vitillaro [7] established abstract theorems of global nonexistence for a class of evolution equations (including (1)), the proof of Theorem 1 given here is simpler, clearer and more readable for this concrete model.

If the initial energy is nonpositive, then the solution blows up in finite time without the restriction (4). More precisely, we have

Theorem 2. *Assume that $a(x,t)$ is a positive function which belongs to the space $W^{1,\infty}(0,\infty;L^\infty(\Omega))$ and the derivative with respect to time $a_t(x,t)\geq 0$ a.e. for $t\geq 0$. If the nonzero initial datum $u_0\in W_0^{1,m}(\Omega)$ satisfies $p\,F(u)\leq uf(u)$ with $p>m>2$ and $E(0)\leq 0$, then the corresponding solution to* (1) *blows up in finite time.*

Theorem 2 is analogous to Theorem 1.2 in [6] for $m>2$, but here we show this result by a different argument, which is simpler than all the previous ones. Moreover, the lifespan of the solution can be given explicitly. For $m=2$, one can prove finite time blowup by the same method as that used for the proof of Theorem 1.

Proof. Suppose D_0 is the optimal constant of Poincaré's inequality $\|v\|_{L^m}\leq D_0\|\nabla v\|_{L^m}$ for $v\in W_0^{1,m}(\Omega)$. For any solution $u(x,t)$, let

$$\varphi(t)=\frac{1}{2}\int_\Omega a(x,t)u^2(x,t)\,dx$$

then

$$\begin{aligned}
\varphi'(t)&=\int_\Omega a\,u\,u_t\,dx+\int_\Omega a_t u^2\,dx\\
&\geq\int_\Omega u\left(\mathrm{div}(|\nabla u|^{m-2}\nabla u)+f(u)\right)\,dx\\
&=-\int_\Omega\left(|\nabla u|^m-uf(u)\right)\,dx\\
&\geq-\int_\Omega\left(|\nabla u|^m-pF(u)\right)\,dx\\
&=-\frac{p}{m}E(t)+\left(\frac{p}{m}-1\right)\int_\Omega|\nabla u|^m\,dx\quad\left(\text{by }(7)\right)\\
&\geq-\frac{p}{m}E(0)+\left(\frac{p}{m}-1\right)\int_\Omega|\nabla u|^m\,dx\quad\left(\text{by }(8)\right)\\
&\geq\left(\frac{p}{m}-1\right)|\Omega|^{1-\frac{m}{2}}D_0^{-m}\|u(\cdot,t)\|_{L^2}^m\\
&\geq\left(\frac{p}{m}-1\right)|\Omega|^{1-\frac{m}{2}}D_0^{-m}\left(\frac{2}{M}\right)^{\frac{m}{2}}\varphi^{\frac{m}{2}}(t),
\end{aligned}$$

where $M \geq a(x,t)$. The above inequality tells us that $\|u(\cdot,t)\|_{L^2} \to \infty$ as t goes to

$$\frac{mD_0^m|\Omega|^{\frac{m}{2}-1}M^{\frac{m}{2}}}{(m-2)(p-m)\|\sqrt{a_0(x)}u_0(x)\|_{L^2}^{m-2}}.$$

This finishes the proof. ∎

References

[1] Kalantarov, V. and O. A. Ladyzhenskaya: *The occurence of collapse for quasilinear equation of paprabolic and hyperbolic types.* J. Sov. Math. 10 (1978), 53 – 70.

[2] Levine, H. A.: *Some nonexistence and instability theorems for solutions of formally parabolic equations of the form* $Pu_t = -Au + F(u)$. Arch. Rational Mech. Anal. 51 (1973), 371 – 386.

[3] Payne, L. E.: *Improperly posed problems in partial differential equations.* Regional Conference Series in Applied Mathematics, No. 22. Philadelphia: Society for Industrial and Applied Mathematics 1975.

[4] Payne, L. E. and D. H. Sattinger: *Saddle points and instability of nonlinear hyperbolic equations.* Israel J. Math. 22 (1975)(3-4), 273 – 303.

[5] Pucci P. and J. Serrin: *Global nonexistence for abstract evolution equations with positive initial energy.* J. Differential Equations 150 (1998)(1), 203 – 214.

[6] Tan, Z.: *The reaction-diffusion equation with Lewis function and critical Sobolev exponent.* J. Math. Anal. Appl. 272 (2002)(2), 480 – 495.

[7] Vitillaro, E.: *Global nonexistence theorems for a class of evolution equations with dissipation.* Arch. Ration. Mech. Anal. 149 (1999)(2), 155 – 182.

[8] Zhou, Y.: *Global nonexistence for a quasilinear evolution equation with critical lower energy.* Arch. Inequal. Appl. 2 (2004)(1), 41 – 47.

Received 27.01.2004