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On a Class of Generalized Autoconvolution Equations of the Third Kind

L. Berg and L. v. Wolfersdorf

Dedicated to the memory of our teacher Prof. Hans Schubert

Abstract. Existence, uniqueness, smoothness, and asymptotics of solutions to a class of generalized autoconvolution equations with outer and inner coefficient are investigated. Further, holomorphic and asymptotic solutions of the equations are derived, and a numerical procedure for an approximate solving of the equations is tested by an example.

Keywords: *Quadratic integral equations, autoconvolution equations, asymptotic expansion of solutions, asymptotic solutions, smoothness of solutions, holomorphic solutions, numerical solution.*

MSC 2000: 45G10, 45D05, 45M05, 65R20

1. Introduction

Whereas for linear convolution equations of Volterra type and also for nonlinear convolution equations of Volterra-Hammerstein type there is a vast literature (cf. [7, 9]), quadratic autoconvolution equations of Volterra type have been studied only in recent times. Exceptions are the well-known integral equations for the Jacobian theta zero function and for the Mittag-Leffler function dealt with by F. Bernstein and G. Doetsch in the twenties of the last century by Laplace transform and Volterra methods [5, 6] (cf. also [12, 14]). The interest in equations of autoconvolution type now arises on one side from practical problems of spectroscopy where the autoconvolution equation of the first kind is treated in context of the theory of incorrectly posed problems (cf. [8]), and on the other side from identification problems for memory kernels in heat transfer and viscoelasticity where the memory kernels are determined as solutions to

L. Berg: Institut für Mathematik der Universität Rostock, D-18051 Rostock; lothar.berg@uni-rostock.de
L. v. Wolfersdorf: TU Bergakademie Freiberg, Fakultät für Mathematik und Informatik, D-09596 Freiberg; wolfersd@math.tu-freiberg.de

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nonlinear generalized autoconvolution equations of the second kind with a given free term (cf. [14] for a short overview). To obtain global in time solutions of these generalized autoconvolution equations the iteration method with weighted norms is used which can be applied to other nonlinear Volterra equations of autoconvolution type, too (cf. [11, 13, 14, 15]).

In the present paper we give a first thorough investigation of nontrivial solutions to a class of generalized autoconvolution equations of the third kind

$$k(x)y(x) = \int_0^x a(\xi)y(\xi)y(x-\xi)\,d\xi\,, \quad x > 0\,. \tag{1.1}$$

These equations are natural generalizations of the above-mentioned equations of Bernstein and Doetsch. A special case is considered in the monograph [3, p. 167] as an example for quadratic equations of convolution type with infinitely many solutions. Equation (1.1) has the peculiarity that with any solution y_0 also $e^{cx}y_0$ with arbitrary c is a solution. For $k(x) > 0$ and $a(x) \geq 0$ the equation has the further property that a solution which is positive for small x (in particular, a continuous solution with positive value at $x = 0$) remains positive for all x (as long as it exists).

In the present paper, we prove existence theorems for nontrivial continuous solutions of equation (1.1). A suitable ansatz reduces it to an equation with free term to which a theorem by J. Janno [10] can be applied. In one case we show that this ansatz is satisfied automatically. Further, applying a general theorem of [2] we determine asymptotic solutions at infinity to a special class of equations (1.1). Moreover, we prove the existence of the first and second derivatives of the continuous solutions to equation (1.1) under adequate smoothness assumptions on the coefficients k and a. Also holomorphic solutions to equation (1.1) with holomorphic k and $a(x) \equiv 1$ are studied by Cauchy's majorant method where the coefficients of the Taylor series of the solution can be recursively determined by the coefficients of the Laurent series of the function $1/k$. Finally, we deal with a numerical procedure for solving the equation and show its effectiveness by calculating approximately a solution of an equation arising from the theory of nonlinear (rational) difference equations [4]. Thereby some calculations are carried out by means of the DERIVE system.

Our existence theorems for nontrivial solutions to equation (1.1) can be compared with certain results for equations of Hammerstein-Volterra type with power-like nonlinearities (cf. [14]) which were obtained by different iteration methods. The global existence of solutions to equation (1.1) in spite of the quadratic nonlinearity here follows from the properties of the autoconvolution integral (and the vanishing of the coefficient k at zero).

To prove the existence theorems in the following we use a special case of a general existence theorem by J. Janno [10] on operator equations of the form

$$z(x) = f(x) + G[z](x) + L[z,z](x) \tag{1.2}$$

with a linear operator G and a bilinear operator L in $C[0,T]$, $0 < T < \infty$, where $C[0,T]$ is equipped with the exponentially weighted norms

$$\|z\|_\sigma = \|e^{-\sigma x} z(x)\| = \max_{0 \le x \le T} |e^{-\sigma x} z(x)| \,, \ \sigma \ge 0 \,,$$

equivalent to $\|z\| = \|z\|_0$. We state this theorem as

Lemma 1. *Let the linear operator $G : C[0,T] \to C[0,T]$ and the bilinear operator $L : C[0,T] \times C[0,T] \to C[0,T]$ satisfy the inequalities*

$$\|G[z]\|_\sigma \le M(\sigma)\|z\|_\sigma \ , \ \ \sigma \ge \sigma_0 > 0 \tag{1.3}$$

for any $z \in C[0,T]$ with a continuous function M satisfying $M(\sigma) \to 0$ as $\sigma \to \infty$, and

$$\|L[z_1, z_2]\|_\sigma \le N \|z_1\|_\sigma \|z_2\|_\sigma \,, \ \ \sigma \ge \sigma_0 > 0 \tag{1.4}$$

with a constant N and

$$\|L[z_1, z_2]\|_\sigma \le \begin{cases} \mu_1(\sigma)\|z_1\|\|z_2\|_\sigma \\ \mu_2(\sigma)\|z_1\|_\sigma\|z_2\| \end{cases} \tag{1.5}$$

with continuous functions μ_k, $k = 1,2$, satisfying $\mu_k(\sigma) \to 0$ as $\sigma \to \infty$ for any pair $z_1, z_2 \in C[0,T]$. Then equation (1.2) *has a uniquely determined solution $z \in C[0,T]$.*

In the following we choose $\sigma_0 > 1$ without loss of generality.

2. The superlinear case

Our main goal is to prove existence of (non-trivial) continuous solutions of the equation

$$k(x)y(x) = \int_0^x a(\xi)y(\xi)y(x-\xi)\,d\xi \ , \ \ x \in [0,T] \tag{2.1}$$

where $0 < T < \infty$. We begin with the case that the solution is superlinear for small x.

Theorem 1. *Let $k \in C[0,T]$, $0 < T < \infty$ with $k(x) > 0$ in $(0,T]$ and $k(0) = 0$ possessing the finite asymptotic expansion*

$$k(x) = Ax + Bx^{2+\delta} + o(x^{2+\delta}),\ \delta > 0 \quad \text{as } x \to 0, \tag{2.2}$$

where $A > 0$, $B \in \mathbb{R}$. Let further $a \in C[0,T]$ with $a(0) = 1$ possessing the asymptotic expansion

$$a(x) = 1 + \lambda\, x^{1+\delta_1} + o(x^{1+\delta_1}),\ \delta_1 > 0 \quad \text{as } x \to 0, \tag{2.3}$$

where $\lambda \in \mathbb{R}$. Then equation (2.1) has a solution $y \in C[0,T]$ of the form

$$y(x) = A + x^{1+\delta_0} z(x), \quad \delta_0 = \min(\delta, \delta_1) > 0 \tag{2.4}$$

with $z \in C[0,T]$ and

$$z(0) = \begin{cases} \frac{1}{\delta}\,[A\lambda - (2+\delta)B] & \text{if } \delta = \delta_1 \\ -\frac{1}{\delta}\,(2+\delta)B & \text{if } \delta < \delta_1 \\ \frac{1}{\delta_1}\,A\lambda & \text{if } \delta_1 < \delta\ . \end{cases} \tag{2.5}$$

The solution y is unique in the class of functions of type (2.4).

Corollary 1. *By the remark in the Introduction, there exists a one-parametric family of solutions $y = e^{cx} y_0$, $c \in \mathbb{R}$, with y_0 as the solution (2.4) which are of the form $y(x) = A + A_1 x + x^{1+\delta_0} z_1(x)$, $A_1 \in \mathbb{R}$, $z_1 \in C[0,T]$.*

Proof of Theorem 1. Inserting the ansatz (2.4) into (2.1) we get the equation

$$z(x) = f_0(x) + G_0[z](x) + L_0[z,z](x) \tag{2.6}$$

for z, where

$$f_0(x) = \frac{A}{x^{1+\delta_0} k(x)} \left[A \int_0^x a(\xi)\, d\xi - k(x)\right] \tag{2.7}$$

$$G_0[z](x) = \frac{A}{x^{1+\delta_0} k(x)} \int_0^x [\xi^{1+\delta_0} z(\xi) + (x-\xi)^{1+\delta_0} z(x-\xi)] a(\xi)\, d\xi$$

$$L_0[z_1,z_2](x) = \frac{1}{x^{1+\delta_0} k(x)} \int_0^x \xi^{1+\delta_0} (x-\xi)^{1+\delta_0} z_1(\xi) z_2(x-\xi) a(\xi)\, d\xi\ . \tag{2.8}$$

By (2.2) and (2.3) we have

$$A \int_0^x a(\xi)\, d\xi - k(x) = A\lambda\, \frac{x^{2+\delta_1}}{2+\delta_1} - B\, x^{2+\delta} + o(x^{2+\delta_0})$$

implying $f_0 \in C[0,T]$ with

$$f_0(0) = \begin{cases} \frac{A\lambda}{2+\delta} - B & \text{if } \delta = \delta_1 \\ -B & \text{if } \delta < \delta_1 \\ \frac{A\lambda}{2+\delta_1} & \text{if } \delta_1 < \delta \ . \end{cases} \tag{2.9}$$

We split the linear part $G_0[z]$ as follows:

$$G_0[z](x) = \frac{2}{x^{2+\delta_0}} \int_0^x \xi^{1+\delta_0} z(\xi)\, d\xi + G_1[z](x),$$

where $G_1[z] = J_1 + J_2$ with

$$\left.\begin{aligned} J_1(x) &= \frac{2}{x^{2+\delta_0}} \frac{Ax - k(x)}{k(x)} \int_0^x \xi^{1+\delta_0} z(\xi)\, d\xi \\ J_2(x) &= \frac{A}{x^{1+\delta_0} k(x)} \int_0^x \left[\xi^{1+\delta_0} z(\xi) + (x-\xi)^{1+\delta_0} z(x-\xi)\right][a(\xi)-1]d\xi, \end{aligned}\right\} \tag{2.10}$$

and write equation (2.6) in the form

$$z(x) - \frac{2}{x^{2+\delta_0}} \int_0^x \xi^{1+\delta_0} z(\xi)\, d\xi = g(x), \tag{2.11}$$

where $g = g(\cdot, z(\cdot))$ is given by

$$g(x) = f_0(x) + G_1[z](x) + L_0[z,z](x) \ .$$

For any $z \in C[0,T]$ we have $G_1[z] \in C[0,T]$ with $G_1[z](0) = 0$ and $L_0[z,z] \in C[0,T]$ with $L_0[z,z](0) = 0$, therefore $g \in C[0,T]$ with $g(0) = f_0(0)$.

The auxiliary equation (2.11) for known $g \in C[0,T]$ has the unique continuous solution

$$z(x) = g(x) + \frac{2}{x^{\delta_0}} \int_0^x g(\xi)\xi^{\delta_0 - 1}\, d\xi \ .$$

Hence we obtain instead of (2.6) the equivalent equation

$$z(x) = f(x) + G[z](x) + L[z,z](x), \tag{2.12}$$

where

$$f(x) = f_0(x) + \frac{2}{x^{\delta_0}} \int_0^x f_0(\xi)\xi^{\delta_0 - 1}\, d\xi \in C[0,T] \tag{2.13}$$

with $f(0) = \frac{1}{\delta_0}(2+\delta_0)f_0(0)$, and

$$G[z](x) = G_1[z](x) + \frac{2}{x^{\delta_0}} \int_0^x G_1[z](\xi)\xi^{\delta_0 - 1}\, d\xi \tag{2.14}$$

$$L[z_1, z_2](x) = L_0[z_1, z_2](x) + \frac{2}{x^{\delta_0}} \int_0^x L_0[z_1, z_2](\xi)\xi^{\delta_0-1}\, d\xi\ . \tag{2.15}$$

For any $z \in C[0,T]$ we have $G[z] \in C[0,T]$ with $G[z](0) = 0$, and for any pair $z_1, z_2 \in C[0,T]$ we have $L[z_1, z_2] \in C[0,T]$ with $L[z_1, z_2](0) = 0$. Hence $z(0) = f(0)$ for the solution z of (2.12).

In applying Lemma 1 to equation (2.12) we have to show that the inequalities (1.3) - (1.5) hold. To prove inequality (1.3) for (2.14) we estimate

$$|e^{-\sigma x} G[z](x)| \le |e^{-\sigma x} G_1[z](x)| + \frac{2}{x^{\delta_0}} \int_0^x \xi^{\delta_0 - 1}\, d\xi\ \|G_1[z]\|_\sigma$$

implying

$$\|G[z]\|_\sigma \le (1 + \frac{2}{\delta_0})\|G_1[z]\|_\sigma\ . \tag{2.16}$$

Further, in view of (2.2) for the first expression J_1 in (2.10) we obtain

$$\begin{aligned} |e^{-\sigma x} J_1(x)| &\le \text{ Const } \frac{x^{2+\delta}}{x^{3+\delta_0}} \int_0^x \xi^{1+\delta_0} e^{-\sigma(x-\xi)}\, d\xi\ \|z\|_\sigma \\ &\le \text{ Const } T^\delta \frac{1}{\sigma}\ \|z\|_\sigma\ , \end{aligned}$$

i.e., $\|J_1\|_\sigma \le$ Const $\frac{1}{\sigma}\ \|z\|_\sigma$. Analogously, in view of (2.2) and (2.3) we estimate the second expression J_2 in (2.10)

$$\begin{aligned} |e^{-\sigma x} J_2(x)| &\le \text{ Const } \frac{1}{x^{2+\delta_0}} \int_0^x \left[\xi^{2+\delta_0+\delta_1} + \xi^{1+\delta_0} x^{1+\delta_1}\right] e^{-\sigma(x-\xi)}\, d\xi\ \|z\|_\sigma \\ &\le \text{ Const } T^{\delta_1} \frac{1}{\sigma}\ \|z\|_\sigma\ , \end{aligned}$$

i.e., $\|J_2\|_\sigma \le$ Const $\frac{1}{\sigma}\ \|z\|_\sigma$. Hence the inequality (1.3) for $G[z]$ with $M(\sigma) =$ Const $\frac{1}{\sigma}$ follows.

To prove the inequalities (1.4) and (1.5) for (2.15), analogously to (2.16), we again have

$$\|L[z_1, z_2]\|_\sigma \le (1 + \frac{2}{\delta_0})\|L_0[z_1, z_2]\|_\sigma\, . \tag{2.17}$$

Further, since a is a bounded function we obtain from (2.8)

$$\begin{aligned}|e^{-\sigma x}L_0[z_1,z_2](x)| &\le \text{ Const } \frac{1}{x^{2+\delta_0}}\int_0^x \xi^{1+\delta_0}(x-\xi)^{1+\delta_0}\,d\xi\ \|z_1\|_\sigma\|z_2\|_\sigma \\ &\le \text{ Const } T^{1+\delta_0}\|z_1\|_\sigma\|z_2\|_\sigma\ ,\end{aligned}$$

i.e., $\|L_0[z_1,z_2]\|_\sigma \le N_0\|z_1\|_\sigma\|z_2\|_\sigma$, and by (2.17) we have inequality (1.4) with $N=(1+\frac{2}{\delta_0})N_0$. Moreover,

$$\begin{aligned}|e^{-\sigma x}L_0[z_1,z_2](x)| &\le \text{ Const } \frac{1}{x^{2+\delta_0}}\int_0^x \xi^{1+\delta_0}(x-\xi)^{1+\delta_0}e^{-\sigma\xi}\,d\xi\ \|z_1\|\|z_2\|_\sigma \\ &\le \text{ Const } T^{\delta_0}\frac{1}{\sigma}\|z_1\|\|z_2\|_\sigma\ ,\end{aligned}$$

and analogously with z_1 and z_2 interchanged. I.e., the inequality (1.5) is fulfilled with functions $\mu_k=$ Const $\frac{1}{\sigma}$. Finally, from $z(0)=f(0)=\frac{1}{\delta_0}(2+\delta_0)f_0(0)$ and (2.9) the relations (2.5) follow. Theorem 1 is proved. ∎

3. The sublinear case

We continue with the case that the solution is sublinear for small x.

Theorem 2. *Let $k\in C[0,T]$, $0<T<\infty$ with $k(x)>0$ in $(0,T]$ and $k(0)=0$ possessing the asymptotic expansion*

$$k(x)=Ax+Bx^{1+\delta}+o(x^{1+\delta}),\ \delta>0\ \text{ as }\ x\to 0, \tag{3.1}$$

where $A>0$, $B\in\mathbb{R}$. Let further $a\in C[0,T]$ with $a(0)=1$ possessing the asymptotic expansion

$$a(x)=1+\lambda x^{\delta_1}+o(x^{\delta_1}),\ \delta_1>0\ \text{ as }\ x\to 0, \tag{3.2}$$

where $\lambda\in\mathbb{R}$. Let $\delta_0=\min(\delta,\delta_1)$ satisfy the inequality $\frac{1}{2}<\delta_0<1$.
Then equation (2.1) has a one-parametric family of solutions $y_c\in C[0,T]$, $c\in\mathbb{R}$, of the form

$$y_c(x)=A+x^{\delta_0}z_c(x) \tag{3.3}$$

with $z_c\in C[0,T]$ and the value

$$z_c(0)=\begin{cases}\frac{1}{1-\delta}\,[(1+\delta)B-A\lambda] & \text{if } \delta=\delta_1\\ \frac{1+\delta}{1-\delta}\,B & \text{if } \delta<\delta_1\\ -\frac{A\lambda}{1-\delta_1} & \text{if } \delta_1<\delta\ ,\end{cases} \tag{3.4}$$

which is independent of c. The solutions y_c are unique in the class of functions of type (3.3).

As parameter $c \in \mathbb{R}$ one can take the free parameter C in the solution of equation (3.7) or the parameter in a representation of the family of solutions $y = e^{cx} y_0$ with a particular solution y_0.

Proof. The proof goes along the lines of the above proof of Theorem 1. Again we obtain an equation for $z = z_c$ of the form

$$z(x) - \frac{2}{x^{1+\delta_0}} \int_0^x \xi^{\delta_0} z(\xi)\, d\xi = f_0(x) + g_0(x), \tag{3.5}$$

where

$$f_0(x) = \frac{A}{x^{\delta_0} k(x)} \left[A \int_0^x a(\xi)\, d\xi - k(x) \right] \in C[0,T] \tag{3.6}$$

and

$$g_0(x) = G_1[z](x) + L_0[z,z](x)$$

with analogous expressions G_1 and L_0 as in the proof of Theorem 1, where only δ_0 is replaced by $\delta_0 - 1$. For $z \in C[0,T]$ we have $g_0 \in C[0,T]$ with $g_0 = O(x^{\delta_0})$ as $x \to 0$. Observing $\frac{1}{2} < \delta_0 < 1$ the general continuous solution of equation (3.5) can be given by the expression

$$z(x) = Cx^{1-\delta_0} + f_0(x) + g_0(x) - 2x^{1-\delta_0} \int_x^T \frac{f_0(\xi)}{\xi^{2-\delta_0}}\, d\xi + 2x^{1-\delta_0} \int_0^x \frac{g_0(\xi)}{\xi^{2-\delta_0}}\, d\xi$$

with arbitrary $C \in \mathbb{R}$. This means, equation (3.5) is equivalent to the family of equations

$$z(x) = f(x) + G[z](x) + L[z,z](x), \tag{3.7}$$

where

$$f(x) = Cx^{1-\delta_0} + f_0(x) - 2x^{1-\delta_0} \int_x^T \frac{f_0(\xi)}{\xi^{2-\delta_0}}\, d\xi \in C[0,T],$$

and G and L are defined as in the proof of Theorem 1, also with $\delta_0 - 1$ instead of δ_0. Using now the inequality

$$\int_0^x e^{-\sigma\eta} d\eta = \frac{1}{\sigma}[1 - e^{-\sigma x}] \le \frac{1}{\sigma}(\sigma x)^\alpha$$

for $\alpha = 1 - \delta_0$ and for an $\alpha = \alpha_0$ with $\alpha_0 \in (0,1)$ and $\alpha_0 + 2\delta_0 > 2$, respectively, we can show that

$$\|G[z]\|_\sigma \le \text{Const}\, \frac{1}{\sigma^{1-\alpha_0}} \|z\|_\sigma$$

and analogously for $L[z_1, z_2]$ so that the inequalities (1.3) – (1.5) for $G[z]$ and $L[z_1, z_2]$ are satisfied. Therefore the existence of a family of solutions y_c of the form (3.3) with $c = C$ to equation (2.1) is proved. By the uniqueness assertion in the theorem in [10] it can also be written as $y_c = e^{cx}y_0$, $c \in \mathbb{R}$, with a particular solution y_0 of the equation. The joint value (3.4) of all z_c at zero easily follows as in the proof of Theorem 1. Theorem 2 is proved. ∎

Remark 1. The existence of continuous solutions to equation (2.1) in the case $0 < \delta_0 \le \frac{1}{2}$ in Theorem 2 is an open problem.

4. The logarithmic case

The last existence theorem deals with the limit case $\delta = \delta_1 = 0$ in Theorem 1 and $\delta = \delta_1 = 1$ in Theorem 2, respectively. Here, the solution contains a logarithmic term for small x, in general.

Theorem 3. *Let $k \in C[0,T]$, $0 < T < \infty$ with $k(x) > 0$ in $(0,T]$ and $k(0) = 0$ possessing the expansion*

$$k(x) = Ax + Bx^2 + C(x) \qquad (A > 0), \tag{4.1}$$

where $B \in \mathbb{R}$, $C(x) = o(x^2)$ as $x \to 0$ with $\int_0^T \frac{|C(x)|}{x^3}\,dx < \infty$. Let further $a \in C[0,T]$ with $a(0) = 1$ possessing the expansion

$$a(x) = 1 + \beta x + \gamma(x), \tag{4.2}$$

where $\beta \in \mathbb{R}$, $\gamma(x) = o(x)$ as $x \to 0$ with $\int_0^T \frac{|\gamma(x)|}{x^2}\,dx < \infty$. Then equation (2.1) has a solution $y \in C[0,T]$ of the form

$$y(x) = A + \mu x \ln x + xz(x)\ , \quad \mu = A\beta - 2B \tag{4.3}$$

with $z \in C[0,T]$ and $z(0) = 0$. The solution y is unique in the class of functions of type (4.3).

Corollary 2. *There exists a one-parametric family of solutions $y = e^{cx}y_0$, $c \in \mathbb{R}$ with y_0 as the solution (4.3) which have the form*

$$y(x) = A + \mu x \ln x + xz_1(x)\ , \quad z_1 \in C[0,T]\ .$$

Proof of Theorem 3. Inserting (4.3) into (2.1) we get the equation

$$z(x) = f_0(x) + G_0[z](x) + L_0[z,z](x) \tag{4.4}$$

for z, where

$$f_0(x) = \frac{1}{xk(x)}\Big\{A^2\int_0^x a(\xi)\,d\xi + A\mu\int_0^x \xi\ln\xi\; a(\xi)\,d\xi$$
$$+ A\mu\int_0^x (x-\xi)\ln(x-\xi)a(\xi)\,d\xi \tag{4.5}$$
$$+ \mu^2\int_0^x \xi(x-\xi)\ln\xi\ln(x-\xi)a(\xi)\,d\xi - k(x)[A+\mu x\ln x]\Big\}$$

and

$$G_0[z](x) = \frac{1}{xk(x)}\int_0^x \{\xi[A+\mu(x-\xi)\ln(x-\xi)]z(\xi)$$
$$+ (x-\xi)[A+\mu\xi\ln\xi]z(x-\xi)\}a(\xi)\,d\xi$$
$$L_0[z_1,z_2](x) = \frac{1}{xk(x)}\int_0^x \xi(x-\xi)a(\xi)z_1(\xi)z_2(x-\xi)\,d\xi. \tag{4.6}$$

By (4.1) and (4.2) we have

$$x\,k(x)f_0(x) = \frac{\mu^2}{2}\,x^3\ln x - A\mu\beta\,\frac{x^3}{4} - AC(x) - \mu\,C(x)x\ln x$$
$$+ A^2\int_0^x \gamma(\xi)\,d\xi + A\mu\int_0^x [\gamma(\xi)+\gamma(x-\xi)]\xi\ln\xi\,d\xi$$
$$+ \mu^2\int_0^x \xi(x-\xi)\ln\xi\ln(x-\xi)a(\xi)\,d\xi$$

from which $f_0 \in C[0,T]$ with $f_0(0)=0$ and $\int_0^T \frac{|f_0(x)|}{x}\,dx < \infty$ follow.

We split up

$$G_0[z](x) = \frac{2}{x^2}\int_0^x \xi\,z(\xi)\,d\xi + G_1[z](x),$$

where

$$G_1[z](x) = \frac{2}{x^2}\,\frac{Ax-k(x)}{k(x)}\int_0^x \xi\,z(\xi)\,d\xi$$
$$+ \frac{A}{xk(x)}\int_0^x [\xi\,z(\xi)+(x-\xi)z(x-\xi)][a(\xi)-1]\,d\xi \tag{4.7}$$
$$+ \frac{\mu}{xk(x)}\int_0^x \xi(x-\xi)[\ln(x-\xi)z(\xi)+\ln\xi\;z(x-\xi)]a(\xi)\,d\xi\,,$$

and write equation (4.4) in the form

$$z(x) - \frac{2}{x^2}\int_0^x \xi\,z(\xi)\,d\xi = g(x), \tag{4.8}$$

where $g = g(\cdot, z(\cdot))$ is given by

$$g(x) = f_0(x) + G_1[z](x) + L_0[z,z](x) \ .$$

Observing (4.1) and (4.2), we estimate in (4.7)

$$\begin{aligned}|G_1[z](x)| \leq \text{Const}\Bigg\{ &\int_0^x \left[\frac{\xi}{x} + \frac{\xi^2}{x^2}\right] |z(\xi)|\, d\xi + \frac{1}{x^2}\int_0^x \xi(x-\xi)|z(x-\xi)|\, d\xi \\ &+ \frac{1}{x^2}\int_0^x \xi(x-\xi)[|\ln(x-\xi)||z(\xi)| + |\ln\xi||z(x-\xi)|]\, d\xi\Bigg\} \\ &\leq \text{Const} \;\; x[1 + |\ln x|]\, \|z\|.\end{aligned}$$

This estimate implies $G_1[z] \in C[0,T]$ for any $z \in C[0,T]$ with $G_1[z](0) = 0$ and $\int_0^T \frac{|G_1[z](x)|}{x}\, dx < \infty$. Further, from

$$\begin{aligned}|L_0[z,z](x)| &\leq \text{Const} \;\; \frac{1}{x^2}\int_0^x \xi(x-\xi)|z(\xi)||z(x-\xi)|\, d\xi \\ &\leq \text{Const} \;\; x\, \|z\|^2\end{aligned}$$

follow the assertions $L_0[z,z] \in C[0,T]$ for any $z \in C[0,T]$ with $L_0[z,z](0) = 0$ and $\int_0^T \frac{|L_0[z,z](x)|}{x}\, dx < \infty$. Therefore, we also have $g \in C[0,T]$ with $g(0) = 0$ and $\int_0^T \frac{|g(x)|}{x}\, dx < \infty$. Then equation (4.4) with $z(0) = 0$ is equivalent to the equation

$$z(x) = f(x) + G[z](x) + L[z,z](x), \tag{4.9}$$

where

$$f(x) = f_0(x) + 2\int_0^x \frac{f_0(\xi)}{\xi}\; d\xi \in C[0,T]$$

with $f(0) = 0$ and

$$G[z](x) = G_1[z](x) + 2\int_0^x G_1[z](\xi)\; \frac{d\xi}{\xi} \tag{4.10}$$

$$L[z_1,z_2](x) = L_0[z_1,z_2](x) + 2\int_0^x L_0[z_1,z_2](\xi)\; \frac{d\xi}{\xi}\ . \tag{4.11}$$

Again, for any $z \in C[0,T]$ we have $G[z] \in C[0,T]$ with $G[z](0) = 0$, and for any pair $z_1, z_2 \in C[0,T]$ we have $L[z_1,z_2] \in C[0,T]$ with $L[z_1,z_2](0) = 0$. Hence $z(0) = f(0) = 0$ for the solution z of (4.9).

We prove the inequalities (1.3) – (1.5) for $G[z]$ and $L[z_1, z_2]$. At first by (4.1) and (4.2) we estimate in (4.7), observing $\sigma > 1$,

$$\begin{aligned}
|e^{-\sigma x} G_1[z](x)| &\le \text{Const } \left\{ e^{-\sigma x} \int_0^x \left[\frac{\xi}{x} + \frac{\xi^2}{x^2} + \frac{\xi(x-\xi)}{x^2} \right] |z(\xi)|\, d\xi \right. \\
&\quad \left. + \frac{e^{-\sigma x}}{x^2} \int_0^x \xi(x-\xi)[|\ln(x-\xi)||z(\xi)| + |\ln\xi||z(x-\xi)|]\, d\xi \right\} \\
&\le \text{Const } \|z\|_\sigma \left[\int_0^x e^{-\sigma(x-\xi)}\, d\xi + \int_0^x e^{-\sigma(x-\xi)} |\ln(x-\xi)|\, d\xi \right] \\
&\le \text{Const } \frac{1}{\sigma} [1 + \ln\sigma] \|z\|_\sigma \, .
\end{aligned}$$

Further,

$$\begin{aligned}
&\left| e^{-\sigma x} \int_0^x G_1[z](\xi) \frac{d\xi}{\xi} \right| \\
&\qquad \le \text{Const } e^{-\sigma x} \int_0^x \frac{1}{\xi} \int_0^\xi [1 + |\ln(\xi-\eta)|] |z(\eta)| d\eta\, d\xi \\
&\qquad \le \text{Const } \|z\|_\sigma \left[\int_0^x e^{-\sigma(x-\xi)}\, d\xi + \int_0^x \frac{1}{\xi} e^{-\sigma(x-\xi)} \int_0^\xi e^{-\sigma\eta} |\ln\eta| d\eta\, d\xi \right] \\
&\qquad \le \text{Const } \frac{1}{\sigma} [1 + \ln\sigma] \|z\|_\sigma \, .
\end{aligned}$$

Hence by (4.10) we have

$$\|G[z]\|_\sigma \le \text{ Const } \frac{1}{\sigma} [1 + \ln\sigma] \|z\|_\sigma \, ,$$

and (1.3) is proved.

To prove (1.4) and (1.5) we estimate in (4.6)

$$\begin{aligned}
|e^{-\sigma x} L_0[z_1, z_2](x)| &\le \text{Const } \frac{1}{x^2} \int_0^x \xi(x-\xi)\, d\xi\ \|z_1\|_\sigma \|z_2\|_\sigma \\
&\le \text{Const } x \|z_1\|_\sigma \|z_2\|_\sigma
\end{aligned}$$

and

$$\begin{aligned}
|e^{-\sigma x} L_0[z_1, z_2](x)| &\le \text{Const } \frac{1}{x^2} \int_0^x \xi(x-\xi) e^{-\sigma\xi}\, d\xi\ \|z_1\| \|z_2\|_\sigma \\
&\le \text{Const } \frac{1}{\sigma} \|z_1\| \|z_2\|_\sigma \, .
\end{aligned}$$

Further,

$$\begin{aligned}
&\left| e^{-\sigma x} \int_0^x L_0[z_1, z_2](\xi) \frac{d\xi}{\xi} \right| \\
&\qquad \le \text{Const } e^{-\sigma x} \int_0^x \frac{1}{\xi^3} \int_0^\xi \eta(\xi - \eta)|z_1(\eta)||z_2(\xi - \eta)|d\eta\, d\xi \\
&\qquad \le \text{Const } e^{-\sigma x} \int_0^x \frac{1}{\xi} \int_0^\xi |z_1(\eta)||z_2(\xi - \eta)|d\eta\, d\xi \\
&\qquad \le \text{Const } x\|z_1\|_\sigma \|z_2\|_\sigma
\end{aligned}$$

and

$$\begin{aligned}
\left| e^{-\sigma x} \int_0^x L_0[z_1, z_2](\xi) \frac{d\xi}{\xi} \right| &\le \text{Const } \int_0^x \frac{1}{\xi} e^{-\sigma(x-\xi)} \int_0^\xi e^{-\sigma\eta} d\eta \ d\xi \ \|z_1\| \|z_2\|_\sigma \\
&\le \text{Const } \frac{1}{\sigma} \|z_1\| \|z_2\|_\sigma \ .
\end{aligned}$$

By (4.11) we obtain

$$\begin{aligned}
\|L[z_1, z_2]\|_\sigma &\le \text{Const } \|z_1\|_\sigma \|z_2\|_\sigma \\
\|L[z_1, z_2]\|_\sigma &\le \text{Const } \frac{1}{\sigma} \|z_1\| \|z_2\|_\sigma
\end{aligned}$$

and analogously with z_1 and z_2 interchanged. This proves (1.4) and (1.5). Theorem 3 is proved. ∎

5. Asymptotic behaviour at zero

It remains the question whether there exist further solutions than those determined before. Concerning Theorem 1 we shall show that this does not come true, where we begin with a special case.

Theorem 4. *Let the conditions of Theorem* 1 *be satisfied with* $a \equiv 1$ *and* $\delta < 1$. *Then every continuous solution of*

$$k(x)y(x) = \int_0^x y(x - \xi)y(\xi)\, d\xi \tag{5.1}$$

which does not vanish, identically, has the asymptotic expansion

$$y(x) = A + \beta x - \left(1 + \frac{2}{\delta}\right) Bx^{1+\delta} + o(x^{1+\delta}) \qquad (x \to +0) \tag{5.2}$$

with a certain β.

Proof. By means of the substitution $\xi = xt$, equation (5.1) turns into

$$\frac{1}{x}k(x)y(x) = \int_0^1 y(x(1-t))y(xt)\,dt\,,$$

and for a continuous solution we obtain as $x \to 0$ the relation $Ay(0) = y^2(0)$, i.e., either $y(0) = 0$ or $y(0) = A$. In the first case we choose $\varepsilon > 0$ so small that $m = \max_{0\le x\le\varepsilon} |y(x)| < k_0$ with $k_0 = \min_{0\le x\le T} \frac{1}{x}k(x)$. Then, for $0 \le x \le \varepsilon$, (5.1) implies the inequality

$$|y| \le \frac{m^2}{k_0}$$

which is a contradiction to the definition of m when $m > 0$. Hence, y vanishes identically for $0 \le x \le \varepsilon$ and, according to (5.1), also for $0 \le x \le T$.

In the second case $y(0) = A$ we put $y(x) = A + \varphi(x)$ and obtain from (5.1) and $k(x) = (A + B(x))x$ with $B(x) = Bx^{1+\delta} + o(x^{1+\delta})$ that

$$\varphi(x) - \frac{2}{x}\int_0^x \varphi(\xi)\,d\xi = g(x) \tag{5.3}$$

with

$$g(x) = \frac{1}{Ax}\int_0^x \varphi(x-\xi)\varphi(\xi)\,d\xi - B(x)\left(1 + \frac{1}{A}\varphi(x)\right). \tag{5.4}$$

We exclude the trivial case that φ vanishes identically which belongs to the case $B(x) \equiv 0$. Equation (5.3) is a special case of (3.5) with $\delta_0 = 0$, so that (5.3) is equivalent to

$$\varphi(x) = \alpha x + g(x) - 2x\int_x^T \frac{g(t)}{t^2}\,dt \tag{5.5}$$

with an arbitrary $\alpha \in \mathbb{R}$. We introduce the increasing function

$$\psi(x) = \max_{0\le t\le x} |\varphi(t)|$$

so that

$$\left|\frac{1}{x}\int_0^x \varphi(x-\xi)\varphi(\xi)\,d\xi\right| \le \psi^2(x)$$

and $g(x) = O(\psi^2(x)) + O(x^{1+\delta})$. Moreover, we introduce the auxiliary functions

$$f(x) = \int_x^T \frac{\psi^2(t)}{t^2}\,dt$$

and $h(x) = xf(x)$ which are nonnegative and continuously differentiable. Hence, (5.5) implies

$$\varphi(x) = O(x) + O(\psi^2(x)) + O(h(x))\,. \tag{5.6}$$

Since

$$h(x) = \int_1^{\frac{T}{x}} \frac{\psi^2(x\xi)}{\xi^2}\, d\xi$$

converges to $\psi^2(0) = 0$ as $x \to 0$, we have $h(0) = h(T) = 0$ and, according to Rolle's theorem, there exists a zero of h' in $(0, T)$. Let the derivative $h'(x) = f(x) - \frac{1}{x}\psi^2(x)$ have two zeros $h'(x_1) = h'(x_2) = 0$ with $0 < x_1 < x_2 < T$. Then

$$\int_{x_1}^{x_2} \frac{\psi^2(t)}{t^2}\, dt = f(x_1) - f(x_2) = \frac{\psi^2(x_1)}{x_1} - \frac{\psi^2(x_2)}{x_2},$$

and the second mean value theorem yields

$$\int_{x_1}^{x_2} \frac{\psi^2(t)}{t^2}\, dt = \psi^2(x_1)\left(\frac{1}{x_1} - \frac{1}{\xi}\right) + \psi^2(x_2)\left(\frac{1}{\xi} - \frac{1}{x_2}\right)$$

with a certain ξ satisfying $x_1 \le \xi \le x_2$. From these two relations it follows $\psi^2(x_1) = \psi^2(x_2)$ and, by the monotony of ψ, $\psi(x) = \text{const}$ and therefore $f(x) = \frac{\psi^2}{x}$, i.e., $h'(x) \equiv 0$ for all $x \in (x_1, x_2)$. This means that the derivative h' has either exactly one zero or exactly one interval where it vanishes identically. In view of $f(T) = 0$ it is $h'(x) < 0$ in a neighbourhood of T. Hence, according to $\int_0^T h'(t)\, dt = 0$, there exists a positive number x_0 such that $h'(x) > 0$ for $x \in (0, x_0)$, i.e. that $\psi^2(x) < h(x)$ for $x \in (0, x_0)$, where x_0 can be chosen arbitrarily small. The function $h(x)$ is increasing in $[0, x_0]$, so that (5.6) implies $\psi(x) = O(x) + O(h(x))$, i.e.,

$$\psi(x) \le ax + bh(x) \tag{5.7}$$

for two positive constants a, b. Since $\psi^2(x) = -x^2 f'(x)$, it follows $-f' \le (a + bf)^2$, and by integration from x to x_0

$$\frac{1}{a + bf_0} - \frac{1}{a + bf} \le b(x_0 - x) \le bx_0$$

with $f_0 = f(x_0)$. In view of $h(0) = 0$ we can choose x_0 so small that $b(ax_0 + bh(x_0)) < 1$, i.e. that

$$f \le \frac{f_0 + ax_0(a + bf_0)}{1 - bx_0(a + bf_0)}$$

with positive denominator. Hence, $f = O(1)$, and (5.7) yields $\psi(x) = O(x)$, i.e., $\varphi(x) = O(x)$ as $x \to 0$. Using this estimate and $B(x) = Bx^{1+\delta} + o(x^{1+\delta})$, we find from (5.4) that $g(x) = -Bx^{1+\delta} + o(x^{1+\delta})$, and (5.5) written in the form

$$\varphi(x) = \beta x + g(x) + 2x \int_0^x \frac{g(t)}{t^2}\, dt$$

with $\beta = \alpha - 2\int_0^T \frac{g(t)}{t^2}\, dt$ implies (5.2). ∎

Remark 2. The solution (2.4) arises from (5.2) for $\beta = 0$, since in (2.5) we have the case $\delta < \delta_1$ with arbitrary large δ_1. Conversely, (5.2) arises from the solution (2.4) by multiplication with $e^{(\beta/A)x}$, cf. Corollary 1.

By analogous arguments, formula (5.2) can easily be transferred to the case $\delta = 1$ where it must read

$$y(x) = A + \beta x + \left(\frac{\beta^2}{2A} - 3B\right) x^2 + o(x^2) \quad (x \to +0)\,. \tag{5.8}$$

Finally, also the general case (2.3) can be treated. If we write $a(x) = 1+b(x)$ with $b(x) \sim \lambda x^{1+\delta_1}$, then to the right-hand side of (5.4) the term

$$\frac{1}{Ax}\int_0^x b(\xi)(A + \varphi(x-\xi))(A + \varphi(\xi))\,d\xi$$

must be added which is asymptotically equal to $\frac{A\lambda}{2+\delta_1}x^{1+\delta_1}$ and, after some calculations, we arrive at the asymptotic expansion

$$y(x) = A + \beta x + z(0)x^{1+\delta_0} + o(x^{1+\delta_0}) \tag{5.9}$$

with $z(0)$ from (2.5) so far as $\delta_0 < 1$. For $\beta = 0$ it coincides with (2.4), and for $\delta_0 = 1$ the term $\frac{\beta^2}{2A}x^2$ must be added as in (5.8).

6. Asymptotic solutions at infinity

In the case that the hypotheses of the Theorems 1 – 3 are satisfied for arbitrarily large T there arises the question concerning the asymptotic behaviour of the solution as $x \to \infty$, where we restrict ourselves to the special case

$$k(x)y(x) = \int_0^x e^{-\xi}y(x-\xi)y(\xi)\,d\xi, \tag{6.1}$$

i.e., to the case $a(x) = e^{-x}$ in (1.1), and we look for so called asymptotic solutions, cf. [2, p.73], i.e., (in changed notations) for functions y satisfying only

$$k(s)y(s) \sim \int_0^s e^{-\xi}y(s-\xi)y(\xi)\,d\xi \tag{6.2}$$

as $s \to \infty$. The asymptotic representation (6.2) means that the quotient of both sides converges to 1 (here we have used the variable s since x shall get a new meaning). It remains an open problem to show that an asymptotic solution yields the asymptotic representation of a genuine solution. For the investigation we need the following result from [2, §§20 – 21].

Theorem 5. *Let a, b, x, ω be real functions of s with $\omega > 0$ and $a + \omega < x < b - \omega$. Assume that $g(s,t)$ is a real function with the following properties: For sufficiently large s the second partial derivative $g_{tt}(s,t)$ is positive when $a(s) < t < b(s)$, and it satisfies*

$$g_t(s,x) = o\left(\sqrt{g_{tt}(s,x)}\right) \tag{6.3}$$

$$\omega\sqrt{g_{tt}(s,x)} \to +\infty \tag{6.4}$$

$$g_{tt}(s, x + \vartheta\omega) \sim g_{tt}(s,x) \tag{6.5}$$

uniformly in ϑ with $|\vartheta| \leq 1$ as $s \to \infty$. Then it holds

$$\int_a^b e^{-g(s,t)}\,dt \sim \sqrt{\frac{2\pi}{g_{tt}(s,x)}}e^{-g(s,x)} \qquad (s \to \infty)\,. \tag{6.6}$$

The asymptotic representation (6.6) is due to Laplace, whereas the condition $g_{tt}(s,t) > 0$ is due to H. Schubert, cf. [1].

Theorem 6. *Assume that $k(s) \sim e^{\alpha s}$ as $s \to \infty$ with $\alpha > 0$. Let p be the positive solution of*

$$p^{\alpha+1} = p + 1 \tag{6.7}$$

and define $\beta = 1/\ln p$ and $\gamma = \sqrt{\beta/(2\pi)}$. Then the function

$$y(s) = \frac{\gamma}{\sqrt{s}}e^{-\beta s \ln s} \tag{6.8}$$

satisfies (6.2).

Proof. By means of the substitution $\xi = st$ we write the right-hand side of (6.2) in the form

$$s\int_0^1 e^{-st}y(s(1-t))y(st)\,dt\,.$$

This integral obtains the form of the integral in (6.6), if we insert (6.8) and choose

$$g(s,t) = st + \beta st\ln(st) + \beta s(1-t)\ln(s(1-t)) + \frac{1}{2}\ln(t(1-t)) - 2\ln\gamma$$

so that

$$g_t(s,t) = s(1 + \beta\ln t - \beta\ln(1-t)) + \frac{1}{2}\left(\frac{1}{t} - \frac{1}{1-t}\right)$$

$$g_{tt}(s,t) = \beta s\left(\frac{1}{t} + \frac{1}{1-t}\right) - \frac{1}{2}\left(\frac{1}{t^2} + \frac{1}{(1-t)^2}\right)\,.$$

The condition $g_{tt}(s,t) > 0$ is satisfied, if we choose $a = 1/(2\beta s)$ and $b = 1 - a$ in (6.6). Moreover, we choose x according to $1 + \beta \ln x = \beta \ln(1-x)$, i.e., $x = 1/\left(e^{1/\beta} + 1\right)$, and $\omega = s^{-1/3}$. Then we find

$$\begin{aligned} g(s,x) &= \beta s \ln s + \left(\beta s + \frac{1}{2}\right) \ln(1-x) + \frac{1}{2} \ln x - 2 \ln \gamma \\ g_t(s,x) &= \frac{1-2x}{2x(1-x)} \\ g_{tt}(s,x) &\sim \frac{\beta x}{x(1-x)}, \end{aligned}$$

and we see that also the conditions (6.3)-(6.5) are satisfied.

According to (6.6) we have proved

$$\int_{\frac{1}{2\beta s}}^{1-\frac{1}{2\beta s}} e^{-g(s,t)}\, dt \sim \gamma^2 \sqrt{\frac{2\pi}{\beta s}} e^{-\beta s \ln s - \beta s \ln(1-x)}. \tag{6.9}$$

In view of the elementary estimate

$$\begin{aligned} \int_0^{\frac{1}{2\beta s}} e^{-g(s,t)}\, dt &= O\left(\int_0^{\frac{1}{2\beta s}} e^{-\beta s(1-t) \ln(s(1-t))} \frac{dt}{\sqrt{t}} \right) \\ &= O\left(\frac{1}{\sqrt{s}} e^{-\beta s \left(1-\frac{1}{2\beta s}\right) \left(\ln s + \ln(1-\frac{1}{2\beta s})\right)} \right) \\ &= O(e^{-\beta s \ln s}), \end{aligned}$$

an analogous estimate for the second remainder integral, and according to $\beta \ln(1-x) < 0$, the remainders have a smaller order than the right-hand side of (6.9), and the integral in (6.9) can be taken from 0 to 1. With the choice of β and γ as in the theorem it immediately follows that the right-hand side of (6.9) is asymptotically equal to $e^{\alpha s} y(s)$ with (6.8). ∎

Let us mention that the assumption on k in Theorem 6 can easily be modified in different ways, but we resign from such modifications which require only standard arguments.

7. Smoothness of solutions in the superlinear case

At first we complement the existence theorems for continuous solutions to equation (2.1) by statements on the smoothness of the solutions.

Theorem 7. *Let k and a fulfill the assumptions of Theorem 1 and, in addition, let be $k \in C^1[0,T]$ with*

$$k'(x) = A + B(2+\delta)x^{1+\delta} + o(x^{1+\delta}) \quad as \;\; x \to 0 \, . \tag{7.1}$$

Then the solution y of equation (2.1), given by (2.4), is in $C^1[0,T]$, where

$$y'(x) = x^{\delta_0} w(x) \;, \quad \delta_0 = \min(\delta, \delta_1) > 0 \;, \tag{7.2}$$

with $w \in C[0,T]$, and the function z, defined in (2.4) can be represented as

$$z(x) = \frac{1}{x^{1+\delta_0}} \int_0^x \xi^{\delta_0} w(\xi)\, d\xi \, . \tag{7.3}$$

If, in addition, $k \in C^2[0,T]$ with (7.1) and

$$k''(x) = B(1+\delta)(2+\delta)x^{\delta} + o(x^{\delta}) \quad as \;\; x \to 0 \;, \tag{7.4}$$

and $a \in C^1[0,T]$ with

$$a'(x) = \lambda(1+\delta_1)x^{\delta_1} + o(x^{\delta_1}) \quad as \;\; x \to 0 \;, \tag{7.5}$$

then the solution of equation (2.1) is in $C^2(0,T]$, where

$$y''(x) = x^{\delta_0 - 1} v(x), \;\; \delta_0 = \min(\delta, \delta_1) > 0 \;, \tag{7.6}$$

with $v \in C[0,T]$, and besides (7.3) there holds the relation

$$w(x) = \frac{1}{x^{\delta_0}} \int_0^x \xi^{\delta_0 - 1} v(\xi)\, d\xi \, . \tag{7.7}$$

Proof for $k \in C^1$. The proof consists of two parts. In the first part we derive a (linear) integral equation for y' and a linear integral equation for z. Then in the second part we prove the statements (7.2) and (7.3) starting from the equation for the continuous function w in (7.2).

First part. Differentiating equation (2.1) we obtain formally

$$k'(x)y(x) + k(x)y'(x) = A\, a(x)y(x) + \int_0^x y(x-\xi)a(x-\xi)y'(\xi)\, d\xi \;, \tag{7.8}$$

i.e., the function $u = y'$ must satisfy the linear integral equation

$$u(x) = \varphi(x) + \frac{1}{k(x)} \int_0^x K_0(x-\xi)u(\xi)\, d\xi, \tag{7.9}$$

where

$$\varphi(x) = \frac{A\,a(x) - k'(x)}{k(x)}\,y(x)\,, \quad K_0(x) = a(x)y(x)\,.$$

Then the function $w = x^{-\delta_0}y'$ fulfills the equation

$$w(x) = \psi_0(x) + \frac{1}{x^{\delta_0}k(x)}\int_0^x K_0(x-\xi)\xi^{\delta_0}w(\xi)\,d\xi, \tag{7.10}$$

where $\psi_0 = x^{-\delta_0}\varphi$. In view of (2.2) – (2.4) we have $\psi_0 \in C[0,T]$ and

$$K_0(x) = A + x^{1+\delta_0}\,M_0(x)$$

with $M_0 \in C[0,T]$. Writing equation (7.10) in the form

$$w(x) - \frac{1}{x^{1+\delta_0}}\int_0^x \xi^{\delta_0}w(\xi)\,d\xi = \psi(x), \tag{7.11}$$

where

$$\psi(x) = \psi_0(x) + \frac{1}{x^{\delta_0}}\int_0^x \left[\frac{K_0(x-\xi)}{k(x)} - \frac{1}{x}\right]\xi^{\delta_0}w(\xi)\,d\xi \,\in\, C[0,T]\,,$$

we get

$$w(x) = \psi(x) + \frac{1}{x^{\delta_0}}\int_0^x \psi(\xi)\xi^{\delta_0-1}\,d\xi$$

or

$$\begin{aligned} w(x) = \psi_1(x) &+ \frac{1}{x^{\delta_0}}\int_0^x \left[\frac{K_0(x-\xi)}{k(x)} - \frac{1}{x}\right]\xi^{\delta_0}w(\xi)\,d\xi \\ &+ \frac{1}{x^{\delta_0}}\int_0^x \frac{1}{\xi}\int_0^\xi \left[\frac{K_0(\xi-\eta)}{k(\xi)} - \frac{1}{\xi}\right]\eta^{\delta_0}w(\eta)d\eta\,d\xi, \end{aligned} \tag{7.12}$$

where

$$\psi_1(x) = \psi_0(x) + \frac{1}{x^{\delta_0}}\int_0^x \psi_0(\xi)\xi^{\delta_0-1}\,d\xi \in C[0,T]\,.$$

Equation (7.12) can be shown to have a bounded continuous kernel. Therefore, for given $y \in C[0,T]$ equation (7.10) has a unique solution in $C[0,T]$.

The function z, defined in (2.4), is the unique continuous solution of the quadratic integral equation (2.6). Hence the linear Volterra integral equation

$$Z(x) = f_0(x) + G_0[Z](x) + L_0[z,Z](x) \tag{7.13}$$

has the solution $Z = z$. Equation (7.13) can be written as a Volterra equation with bounded continuous kernel so that it is uniquely solvable in $C[0,T]$.

Therefore, $Z = z$ is the unique continuous solution of (7.13). And it is the unique continuous solution of the augmented linear Volterra equation

$$Z(x) = f_0(x) + G_0[Z](x) + L_0[z, Z](x) + H_0[Z](x) \tag{7.14}$$

with the additional linear term $H_0[Z] = H[Z - z]$, where

$$H[Z](x) = \frac{1}{x^{1+\delta_0} k(x)} \int_0^x [k'(\xi) - A\, a(\xi)]\xi^{1+\delta_0} Z(\xi)\, d\xi$$

has a bounded continuous kernel.

Second part. We newly define w as the unique continuous solution of equation (7.10) for the given solution $y \in C[0,T]$ of equation (2.1) which exists by Theorem 1. Equation (7.10) for w is equivalent to equation (7.9) for $u = x^{\delta_0} w$. In turn, by (7.8) this is equivalent to the equation

$$k'(x)y(x) + k(x)u(x) = A\, a(x)y(x) + \int_0^x y(\xi)a(\xi)u(x-\xi)\, d\xi\, . \tag{7.15}$$

Integrating this equation, we obtain

$$\int_0^x [k'(\xi)y(\xi) + k(\xi)u(\xi)]\, d\xi = A \int_0^x a(\xi)y(\xi)\, d\xi + \int_0^x \int_\eta^x u(\xi - \eta)\, d\xi\, y(\eta)a(\eta) d\eta,$$

where we changed the order of integration in the iterated integral. Introducing the primitive function $U(x) = \int_0^x u(\xi)\, d\xi$ of u and integrating the second integral on the left-hand side by parts, we get

$$\begin{aligned} k(x)U(x) - \int_0^x k'(\xi)U(\xi)\, d\xi = A \int_0^x a(\xi)y(\xi)\, d\xi &- \int_0^x k'(\xi)y(\xi)\, d\xi \\ &+ \int_0^x U(x-\xi)y(\xi)a(\xi)\, d\xi\, . \end{aligned}$$

Inserting here the expression (2.4) for y and introducing the function $Z = x^{-(1+\delta_0)}U$, after some calculations we arrive at equation (7.14) for Z. By uniqueness of the solution z of this equation we have $z = Z$, i.e., the function z can be represented in the form

$$z(x) = \frac{1}{x^{1+\delta_0}} \int_0^x \xi^{\delta_0} w(\xi)\, d\xi$$

with the above defined unique continuous solution w of equation (7.10), and the solution (2.4) of (2.1) in the form

$$y(x) = A + \int_0^x \xi^{\delta_0} w(\xi)\, d\xi\, .$$

Obviously, the last function is in $C^1[0,T]$ and relation (7.2) with (7.3) holds.

Proof of the additional statement for $k \in C^2$. In view of (7.9) the first derivative y' is given by

$$y'(x) = \varphi(x) + \frac{1}{k(x)} \int_0^x a(x-\xi) y(x-\xi) y'(\xi)\, d\xi, \tag{7.16}$$

where again

$$\varphi(x) = \frac{A\, a(x) - k'(x)}{k(x)}\, y(x)\ .$$

Since the right-hand side of (7.16) is continuously differentiable in $(0,T]$, the left-hand side y' has the continuous derivative y'' in $(0,T]$. Differentiating (7.8), we get the formula

$$\begin{aligned} y''(x) = \frac{1}{k(x)} \Big\{ & y(x)[A\, a'(x) - k''(x)] + y'(x)[A\, a(x) + A - 2k'(x)] \\ & + \int_0^x [a(x-\xi) y'(x-\xi) + a'(x-\xi) y(x-\xi)] y'(\xi)\, d\xi \Big\}\ . \end{aligned} \tag{7.17}$$

Observing the assumptions (2.2), (2.3), (7.1), (7.4), (7.5) and (2.4), (7.2), from (7.17) we easily infer that

$$y''(x) = \frac{1}{x}\,[A(1+\delta_1)x^{\delta_1} - B(1+\delta)(2+\delta)x^{\delta}] + o(x^{\delta_0 - 1}) \qquad \text{as } x \to 0\ ,$$

i.e., y'' has the representation (7.6). Finally, from (7.2) and (7.6) relation (7.7) follows. Theorem 7 is proved. ∎

Remark 3. From the relation (7.17) for y'' under adequate smoothness assumptions on k and a we can successively obtain the existence of derivatives of higher order of y as long as the occurring integrals in differentiating (7.17) will exist.

8. Smoothness of solutions in the logarithmic case

We continue the investigation of the smoothness of the solution dealing now with the important logarithmic case.

Theorem 8. *Let k and a fulfill the assumptions of Theorem 3, and, in addition, let be $k \in C^1[0,T]$ with*

$$k'(x) = A + 2Bx + D(x) \tag{8.1}$$

where $D = o(x)$ as $x \to 0$ and $\int_0^T \frac{|D(x)|}{x^2}\, dx < \infty$. Then the solution y of equation (2.1) given by (4.3) with $z(0) = 0$ is in $C^1(0,T]$, where

$$y'(x) = \mu \ln x + w(x) \tag{8.2}$$

with μ *from* (4.3), $w \in C[0,T]$, *and there holds the relation*

$$z(x) = -\mu + \frac{1}{x}\int_0^x w(\xi)\,d\xi \tag{8.3}$$

with $w(0) = \mu$.

If, in addition, $k \in C^2[0,T]$ *with* (8.1) *and*

$$k''(x) = 2B + E(x), \tag{8.4}$$

where $E = o(1)$ *as* $x \to 0$ *and* $\int_0^T \frac{|E(x)|}{x}\,dx < \infty$, *and* $a \in C^1[0,T]$ *with*

$$a'(x) = \beta + \delta(x), \tag{8.5}$$

where $\delta = o(1)$ *as* $x \to 0$ *and* $\int_0^T \frac{|\delta(x)|}{x}\,dx < \infty$, *then the solution of equation* (2.1) *is in* $C^2(0,T]$, *where*

$$y''(x) = \frac{\mu}{x} + \frac{1}{x}\,V(x) \tag{8.6}$$

with $V \in C[0,T]$ *and* $V(0) = 0$, $\int_0^T \frac{|V(x)|}{x}\,dx < \infty$, *and besides* (8.3) *there holds the relation*

$$w(x) = \mu + \int_0^x \frac{V(\xi)}{\xi}\,d\xi \tag{8.7}$$

implying $w \in C^1(0,T]$.

Proof for $k \in C^1$. The proof follows the lines of the proof of Theorem 7. Again $u = y'$ satisfies the corresponding equation (7.9) and $w = y' - \mu \ln x$ the equation

$$w(x) = \psi_0(x) + \frac{1}{k(x)}\int_0^x K_0(x-\xi)w(\xi)\,d\xi, \tag{8.8}$$

where again $K_0(x) = a(x)y(x)$ and

$$\psi_0(x) = \varphi(x) - \mu \ln x + \frac{\mu}{k(x)}\int_0^x K_0(x-\xi)\ln\xi\;d\xi$$

with φ as in the proof of Theorem 7. In view of (4.1) – (4.3) we have

$$K_0(x) = A + \mu\, x \ln x + x\, M_0(x)$$

with $M_0 \in C[0,T]$ and $\psi_0 \in C[0,T]$ with $\psi_0(0) = 0$ and $\int_0^T \frac{|\psi_0(x)|}{x}\,dx < \infty$. We have to look for a continuous solution w of (8.8) satisfying $w(0) = \mu$, since only this condition guarantees that (8.2) is compatible with (4.3) and $z(0) = 0$.

We write equation (8.8) in the form

$$w(x) - \frac{1}{x}\int_0^x w(\xi)\, d\xi = \psi(x), \tag{8.9}$$

where

$$\psi(x) = \psi_0(x) + \int_0^x \left[\frac{K_0(x-\xi)}{k(x)} - \frac{1}{x}\right] w(\xi)\, d\xi \in C[0,T]$$

satisfies $\psi(0) = 0$ and $\int_0^T \frac{|\psi(x)|}{x}\, dx < \infty$. Observing $w(0) = \mu$, we further obtain the equation

$$\begin{aligned} w(x) &= \mu + \psi(x) + \int_0^x \frac{\psi(\xi)}{\xi}\, d\xi \\ &= \mu + \psi_0(x) + \int_0^x \frac{\psi_0(\xi)}{\xi}\, d\xi + \int_0^x \left[\frac{K_0(x-\xi)}{k(x)} - \frac{1}{x}\right] w(\xi)\, d\xi \\ &\quad + \int_0^x \frac{1}{\xi}\int_0^\xi \left[\frac{K_0(\xi-\eta)}{k(\xi)} - \frac{1}{\xi}\right] w(\eta) d\eta\, d\xi\ . \end{aligned} \tag{8.10}$$

There holds

$$\psi_0(x) + \int_0^x \frac{\psi_0(\xi)}{\xi}\, d\xi \in C[0,T]$$

with value zero at $x = 0$, and the Volterra integral operator in (8.10) has a logarithmic singular kernel and acts in $C[0,T]$. Therefore, for given $y \in C[0,T]$ and given value μ at $x = 0$ equation (8.8) has a unique solution in $C[0,T]$.

Analogously as before, the function $Z = z$ is the unique continuous solution of the augmented linear Volterra equation (7.14) satisfying the additional condition $Z(0) = 0$, where f_0, $G_0[Z]$ and $L_0[z, Z]$ are given by the formulas (4.5) – (4.6) in Theorem 3 and $H_0[Z] = H[Z - z]$ with

$$H[Z](x) = \frac{1}{xk(x)}\int_0^x \xi\{k'(\xi) - a(\xi)[A + \mu(x-\xi)\ln(x-\xi)]\}Z(\xi)\, d\xi\ . \tag{8.11}$$

The Volterra integral operator in (7.14) at present has a logarithmic singular kernel, too.

We now redefine w as the unique continuous solution of equation (8.8) for the given solution $y \in C[0,T]$ of the form (4.3) from Theorem 3 with the additional condition $w(0) = \mu$. Then the function $u = w + \mu \ln x$ satisfies equation (7.15) from which by integrating and inserting the expression (4.3) for y, after some longer elementary calculations, equation (7.14) for the function

$$Z(x) = -\mu + \frac{1}{x}\int_0^x w(\xi)\, d\xi$$

with $Z(0) = 0$ arises. By uniqueness of the solution to this equation we obtain $z = Z$, and (4.3) implies

$$y(x) = A + \mu\, x \,\ln x - \mu x + \int_0^x w(\xi)\, d\xi\,.$$

That means, y is in $C^1(0,T]$ and the relations (8.2) and (8.3) with $w(0) = \mu$ hold.

Proof of the additional statement for $k \in C^2$. As in the proof of Theorem 7, formulas (7.15) for y' and (7.16) for y'' hold true yielding $y \in C^2(0,T]$. In view of the assumptions (4.1), (4.2), (8.1), (8.4), (8.5) on k, a and (4.3), (8.2) for y, y' from (7.16) we conclude that (8.6) for y'' is valid. Further, the representations (8.2) and (8.6) imply relation (8.7). Theorem 8 is proved. ∎

We still investigate the solution y under the stronger assumptions

$$\left.\begin{aligned} k(x) &= Ax + Bx^2 + o(x^3 \ln\, x) \\ k'(x) &= A + 2Bx + o(x^2 \ln\, x) \\ k''(x) &= 2B + o(x \ln\, x) \\ a(x) &= 1 + \beta x + o(x^2 \ln\, x) \\ a'(x) &= \beta + o(x \ln\, x) \end{aligned}\right\} \tag{8.12}$$

as $x \to 0$. Then from (7.16) the representation

$$y''(x) = \frac{\mu}{x} + \frac{\mu^2}{A}\, \ln^2 x + \ln x \cdot v_0(x) \tag{8.13}$$

where $v_0 \in C[0,T]$ with $v_0(0) = \frac{3\mu^2}{A}$, and the relation

$$w(x) = \mu + \frac{\mu^2}{A}\big[x\, \ln^2 x - 2x\, \ln x + 2x\big] + \int_0^x v_0(\xi) \ln \xi\, d\xi \tag{8.14}$$

follow. From (8.2) and (8.14) we get the asymptotic expansion

$$y'(x) = \mu\, \ln x + \mu + \frac{\mu^2}{A}\, x\, \ln^2 x + \frac{\mu^2}{A}\, x\, \ln x + o(x \ln x) \tag{8.15}$$

as $x \to 0$, and from (4.3), (8.3) and (8.14) we have

$$y(x) = A + \mu\, x \ln x + \frac{\mu^2}{2A}\, x^2 \ln^2 x + o(x^2 \ln x) \tag{8.16}$$

as $x \to 0$.

Furthermore, since the asymptotic expansion (8.16) for y contains logarithmic terms, it makes sense to include such terms also in the expansions of k and a. In particular, under the assumptions

$$\left.\begin{aligned} k(x) &= Ax + Bx^2 + q_2x^3\ln^2 x + q_1x^3\ln x + o(x^3\ln x) \\ k'(x) &= A + 2Bx + 3q_2x^2\ln^2 x + [3q_1 + 2q_2]x^2\ln x + o(x^2\ln x) \\ k''(x) &= 2B + 6q_2x\,\ln^2 x + [6q_1 + 10q_2]x\,\ln x + o(x\,\ln x) \\ a(x) &= 1 + \beta x + p_2x^2\ln^2 x + p_1x^2\ln x + o(x^2\ln x) \\ a'(x) &= \beta + 2p_2x\,\ln^2 x + [2p_1 + 2p_2]x\,\ln x + o(x\,\ln x) \end{aligned}\right\} \tag{8.17}$$

as $x \to 0$, the asymptotic expansions

$$\begin{aligned} y(x) = A + \mu\,x\,\ln x &+ \left[\frac{\mu^2}{2A} + Ap_2 - 3q_2\right]x^2\ln^2 x \\ &+ \left[Ap_1 - 3q_1 - 2Ap_2 + 4q_2\right]x^2\ln x + o(x^2\ln x) \end{aligned} \tag{8.18}$$

$$\begin{aligned} y'(x) = \mu\,\ln x + \mu &+ \left[\frac{\mu^2}{A} + 2Ap_2 - 6q_2\right]x\,\ln^2 x \\ &+ \left[\frac{\mu^2}{A} + 2Ap_1 - 6q_1 - 2Ap_2 + 2q_2\right]x\,\ln x + o(x\,\ln x) \end{aligned} \tag{8.19}$$

$$\begin{aligned} y''(x) = \frac{\mu}{x} &+ \left[\frac{\mu^2}{A} + 2Ap_2 - 6q_2\right]\ln^2 x \\ &+ \left[\frac{3\mu^2}{A} + 2Ap_1 - 6q_1 + 2Ap_2 - 10q_2\right]\ln x + o(\ln x) \end{aligned} \tag{8.20}$$

as $x \to 0$ hold true. If k/x and a also contain terms of the form $x^2\ln^n x,\ n > 2$, such terms occur in the expansions for y, too. We summarize the last results in

Corollary 3. *Under the additional assumptions* (8.12) *on* k *and* a *the asymptotic expansions* (8.16), (8.15) *and* (8.13) *for* y, y' *and* y'' *and the relation* (8.14) *for* w *are valid. Under the assumptions* (8.17) *on* k *and* a *the asymptotic expansions* (8.18) – (8.20) *for* y, y' *and* y'' *hold.*

9. Holomorphic solutions

Next, we strengthen the hypotheses of the Theorems 1 and 7 in such a way that the solutions become holomorphic functions (in a neighbourhood of zero). As at the beginning of Section 5 we restrict ourselves to the case $a \equiv 1$, i.e., to the integral equation

$$y(x) = \frac{1}{k(x)}\int_0^x y(x-\xi)y(\xi)\,d\xi\,, \tag{9.1}$$

where in this section x shall be a complex variable, and we begin with some preparations.

Lemma 2. *The function*

$$f(x) = 2\frac{1-x}{x-2}\ln(1-x) \tag{9.2}$$

has the Taylor series

$$f(x) = x - \sum_{n=3}^{\infty} e_n x^n \quad (|x| < 1) \tag{9.3}$$

with positive coefficients

$$e_n = \sum_{\nu=1}^{n-1} \frac{1}{2^\nu}\frac{1}{n-\nu} - \frac{1}{n} \quad (n \geq 3)\,, \tag{9.4}$$

and the reciprocal has the Laurent series

$$\frac{1}{f(x)} = \frac{1}{x} + \sum_{n=2}^{\infty} d_n x^{n-1} \quad (0 < |x| < 1) \tag{9.5}$$

with $d_n > 0$ *for all* n.

Proof. It can easily be checked that the function (9.2) can be expanded into the form (9.3) with (9.4). The assertion $e_n > 0$ for $n \geq 3$ follows immediately from

$$\sum_{\nu=0}^{n-1} \frac{1}{2^\nu}\frac{n}{n-\nu} > \sum_{\nu=1}^{n-1} \frac{1}{2^\nu}\left(1 + \frac{\nu}{n}\right) = 1 + \frac{2}{n} - \frac{4}{2^n} - \frac{2}{n2^n} > 1\,.$$

Writing (9.3) in the form $f(x) = x(1 - x^2 g(x))$ we find

$$\frac{1}{f(x)} = \frac{1}{x}\sum_{n=0}^{\infty} \left(x^2 g(x)\right)^n, \tag{9.6}$$

and the last assertion follows from the fact that $g(x)$ has a Taylor series at $x = 0$ with only positive coefficients. ∎

The first coefficients in (9.3) and (9.5), respectively, are

$$e_3 = e_4 = \frac{1}{6}\,, \quad e_5 = \frac{2}{15}\,, \quad e_6 = \frac{1}{10}\,, \quad e_7 = \frac{31}{420}$$
$$d_2 = d_3 = \frac{1}{6}\,, \quad d_4 = \frac{29}{180}\,, \quad d_5 = \frac{7}{45}\,, \quad d_6 = \frac{1139}{7560}\,, \quad d_7 = \frac{11}{120}\,.$$

By means of elementary calculations it follows

Lemma 3. *Assume that* $k(x) = Arf\left(\frac{x}{r}\right)$ *with* f *from* (9.2) *and* $r > 0$. *Then* $y(x) = \frac{Ae^{-x/r}}{1-x/r}$ *is the solution of* (9.1) *satisfying* (2.4) *with the Taylor series*

$$y(x) = A\left(1 + \sum_{n=2}^{\infty} c_n \left(\frac{x}{r}\right)^n\right) \quad (|x| < r)\,, \tag{9.7}$$

and positive coefficients

$$c_n = \sum_{\nu=0}^{n} \frac{(-1)^\nu}{\nu!} \quad (n \geq 2)\,.$$

In the following we use the notations

$$\frac{1}{k(x)} = \sum_{n=0}^{\infty} b_n x^{n-1} \quad (x \neq 0) \tag{9.8}$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n\,. \tag{9.9}$$

Theorem 9. *Let* k *be a holomorphic function for small* $|x|$ *with* $k(0) = k''(0) = 0$ *and* $k'(0) = A > 0$. *Then the integral equation* (9.1) *has a uniquely determined holomorphic solution* y *for small* $|x|$ *with* $y'(0) = 0$, *and the coefficients in* (9.9) *are recursively determined by* $a_0 = A$, $a_1 = 0$ *and*

$$a_n = \frac{n+1}{n-1} \sum_{\mu+\nu+\kappa=n}{}' \frac{\mu!\nu!}{(n-\kappa+1)!}\, a_\mu a_\nu b_\kappa \quad (n \geq 2), \tag{9.10}$$

where the dash at the sum shall mean that the triples $(n, 0, 0)$ *and* $(0, n, 0)$ *are excluded for* (μ, ν, κ).

Proof. Inserting the series (9.8), (9.9) into (9.1), we get by comparing coefficients

$$a_n = \sum_{\mu+\nu+\kappa=n} \frac{\mu!\nu!}{(\mu+\nu+1)!}\, a_\mu a_\nu b_\kappa\,.$$

For $n = 0$ and $a_0 \neq 0$ this equation implies $a_0 = 1/b_0 = A$, for $n = 1$ it is an identity in view of $a_1 = b_1 = 0$, and for $n \geq 2$ we obtain (9.10). In order to show the convergence of (9.9) for small $|x|$ we use Cauchy's method of majorants. Since the right-hand side of (9.10) has only positive coefficients, we see for positive x: If (9.8) is replaced by a majorant with the same $b_0 = 1/A$, then the corresponding solution is a majorant of the original solution. We shall show that there exists an $r > 0$ such that (9.8) has the majorant $1/(Arf(x/r))$. Then the solution $Ae^{-x/r}/(1 - x/r)$ mentioned in Lemma 3 is a majorant of the solution $y(x)$, and therefore also the latter one is a holomorphic function for $|x| < r$.

Obviously, (9.8) has a non-vanishing radius of convergence ϱ, so that $\varrho^{-1} = \overline{\lim} \sqrt[n]{|b_n|}$ is finite, and we can define a positive number r by

$$\frac{1}{r} = \sqrt{6} \sup_{n \geq 2} \sqrt[n]{A|b_n|} . \tag{9.11}$$

For $|x| < r$ this implies

$$\sum_{n=2}^{\infty} |b_n x^n| \leq \frac{1}{A} \sum_{n=2}^{\infty} \frac{|x|^n}{(r\sqrt{6})^n} \leq \frac{1}{A} \sum_{n=0}^{\infty} \left[\left(\frac{|x|}{r} \right)^2 g\left(\frac{|x|}{r} \right) \right]^n ,$$

since $(x^2 g(x))^n > \left(\frac{1}{6}x^2(1+x)\right)^n > \frac{x^{2n}}{6^n}\left(1 + \frac{x}{\sqrt{6}}\right)$ for positive x.

According to (9.6) we have found the wanted majorant, and the theorem is proved. ∎

Remark 4. The determination of r by means of (9.11) is not optimal, since the proof yields the estimate $r \leq \frac{\varrho}{\sqrt{6}}$, whereas we conjecture that $r \geq \varrho$. The example $k(x) = \sin(\omega x)/(\omega J_0(\omega x))$, $\omega \neq 0$, with the corresponding solution $y(x) = J_0(\omega x)$ of (9.1) shows that (9.9) can even be an entire function, though (9.8) has only a finite radius of convergence. The recursions (9.10) show that y is always an even function when k is an odd one as in the just mentioned example.

10. Example

As a further concrete example we consider the special case $k(x) = 2 \sinh x$ of (6.1) , i.e., the integral equation

$$y(x) = \frac{1}{2 \sinh x} \int_0^x e^{-\xi} y(x-\xi) y(\xi) \, d\xi \qquad (0 \leq x). \tag{10.1}$$

According to Theorem 3 it has a continuous solution for $0 \leq x < \infty$, and this solution is positive, cf. the Introduction. Moreover, it has the asymptotic expansion

$$y(x) = 2 - 2x \ln x + x^2 \ln^2 x + \left(1 - \frac{\pi^2}{6}\right) x^2 + o(x^2) \qquad (x \to 0) , \tag{10.2}$$

where the first three terms on the right-hand side are those from (8.16), and the fourth term can be calculated directly out of (10.1). In particular, it is $y(0) = 2$.

Lemma 4. *Let m be a positive constant and $M(x)$ a function such that*

$$y(x) \leq M(x) \leq M(m) \leq e^m \qquad (0 \leq x \leq m)\,. \tag{10.3}$$

Then, with $M = M(m)$, the solution of (10.1) satisfies the estimate

$$y(x) \leq \exp\left\{2x - \frac{x}{\ln 2}\ln\frac{x}{\ln M}\right\} \qquad (\ln M \leq x)\,. \tag{10.4}$$

Proof. Estimating y on the right-hand side of (10.1) by M where $M \geq 2$ we find

$$y(x) \leq \frac{M^2}{e^x + 1} \tag{10.5}$$

for $0 \leq x \leq m$. According to (10.3) this means that $y(m) < M$ additionally to $y(x) \leq M$ for $0 \leq x < m$. In view of the continuity of y we even have (10.5) not only for $0 \leq x \leq m + \varepsilon$ and some $\varepsilon > 0$, but for all $x \geq 0$, i.e., $y(x) \leq M$ for all $x \geq 0$.

Assume that $y \leq M_n e^{-nx}$ for $x \geq 0$, then (10.1) implies

$$y(x) \leq \frac{e^{-x}}{1 - e^{-2x}} M_n^2 e^{-nx} \int_0^x e^{-\xi}\, d\xi \leq M_n^2 e^{-(n+1)x}\,,$$

i.e., we can choose $M_{n+1} = M_n^2$. In view of $M_0 = M$ then we have $M_n = M^{2^n}$, so that we obtain

$$y(x) \leq M^{2^n} e^{-nx} \tag{10.6}$$

for $x \geq 0$ and all $n \in \mathbb{N}_0$. According to $M^{2^n} e^{-nx} = M^{2^{n-1}} e^{-(n-1)x}$ when $e^x = M^{2^{n-1}}$ we use (10.6) only for $M^{2^{n-1}} \leq e^x \leq M^{2^n}$, $n \in \mathbb{N}$, so that (10.6) implies (10.4). ∎

The conditions of Lemma 4 are satisfied for

$$M(x) = 1 + (1 - x \ln x)^2$$

and the real solution of $m \ln m = 2 + \frac{1}{e}$, so that $M = 1 + (1 + \frac{1}{e})^2$ (approximately it is $m = 2.5401$ and $M = 2.8711$). The inequality $y(x) \leq M(x)$ follows from (10.2) when x is small, and from the inequality

$$\frac{1}{2\sinh x} \int_0^x e^{-\xi} M(x - \xi) M(\xi)\, d\xi < M(x) \tag{10.7}$$

for $0 < x \leq 3$ which is valid in view of Figure 1 (in the interval $[0, m]$ the function $M(x)$ has two equal maxima at the points $\frac{1}{e}$ and m). After knowing

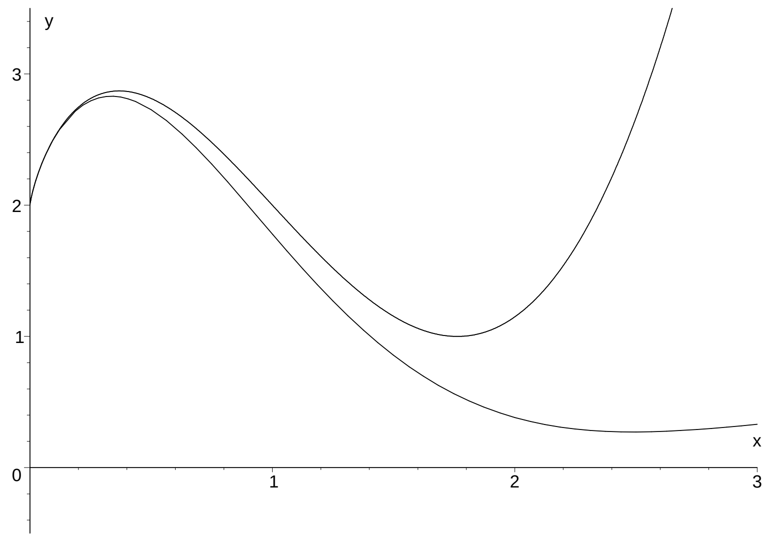

Figure 1: The function $M(x)$ (above) and its estimate in (10.7) (below)

that y is bounded we can apply Lemma 4 once more with the constant function $M(x) = \max_{0\leq\xi} y(\xi)$ and sufficiently large m whereby (10.4) is improved.

Equation (10.1) can be transformed in different ways. By means of the Laplace transform $Y(p) = \mathcal{L}\{y(x)\}$ it turns into the difference equation

$$Y(p-1) = Y(p+1)(1+Y(p))\,,$$

and for the solution $Y(p-c) = \mathcal{L}\{e^{cx}y(x)\}$ with $c = 1-C$, where C is Euler's constant, the asymptotic expansion (10.2) yields the result

$$Y(p-c) = \frac{2}{p} + \frac{2}{p^2}\ln p + \frac{2}{p^3}(\ln^2 p - \ln p) + o\left(\frac{1}{p^3}\right) \qquad (p \to +\infty)\,,$$

which sharpens the corresponding asymptotic expansion in [4, p. 1064].

By means of the substitutions $x = \ln\frac{1}{s}$, $\xi = \ln\frac{1}{t}$ and $y(x) = \eta(s)$ equation (10.1) turns into the integral equation

$$\eta(s) = \frac{s}{1-s^2}\int_s^1 \eta\left(\frac{s}{t}\right)\eta(t)\,dt \qquad (0 \leq s \leq 1). \tag{10.8}$$

This equation has the property that with a solution η also $s^{-c}\eta$ is a solution where c is an arbitrary constant, cf. Section 1. All solutions have the boundary values $\eta(0) = 0$ and $\eta(1) = 2$.

One of these solutions shall be visualized by means of a numerical approximation, where we use the collocation method, cf. [7, Vol. 4, p.196]. In order to do this we choose an arbitrary continuous function $\eta_0(s)$ with n parameters, and n different points $s_i \in (0,1)$, $i = 1, \ldots, n$. After calculating

$$\eta_1(s) = \frac{s}{1-s^2} \int_s^1 \eta_0\left(\frac{s}{t}\right) \eta_0(t)\, dt$$

we have to determine the parameters from the system

$$\eta_0(s_j) = \eta_1(s_j) \qquad (j = 1, \ldots, n)\,. \tag{10.9}$$

According to $\eta(0) = 0$ and $\eta(1) = 2$ we additionally choose the boundary values

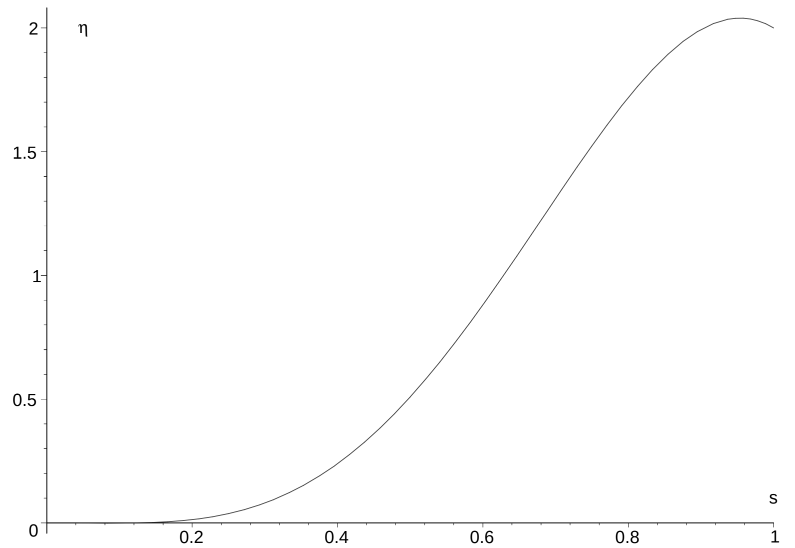

Figure 2: The collocation function $\eta_1(s)$

$\eta_0(0) = 0$ and $\eta_0(1) = 2$ which imply $\eta_1(0) = 0$ and $\eta_1(1) = 2$. We try this method in the case $n = 2$, $\eta_0(s) = as + bs^3 + (2 - a - b)s^6$ and $s_1 = \frac{1}{2}$, $s_2 = \frac{3}{4}$. After eliminating a^2 out of (10.9) we get

$$a = -\frac{3(1449b^2 - 4814b - 9928)}{9019b + 24492}\,, \tag{10.10}$$

and after inserting this expression into one of the equations (10.9) we obtain a polynomial in b of degree 4 with the (numerical) solutions

$$b_1 = -5.4508\,, \quad b_2 = -1.3278\,, \quad b_3 = 5.25706\,, \quad b_4 = 1160.55\,.$$

It proves that the best result arises for the parameter $b = b_3$, for which (10.10) yields $a_3 = -0.200683$. The corresponding function η_1 is illustrated in Figure 2,

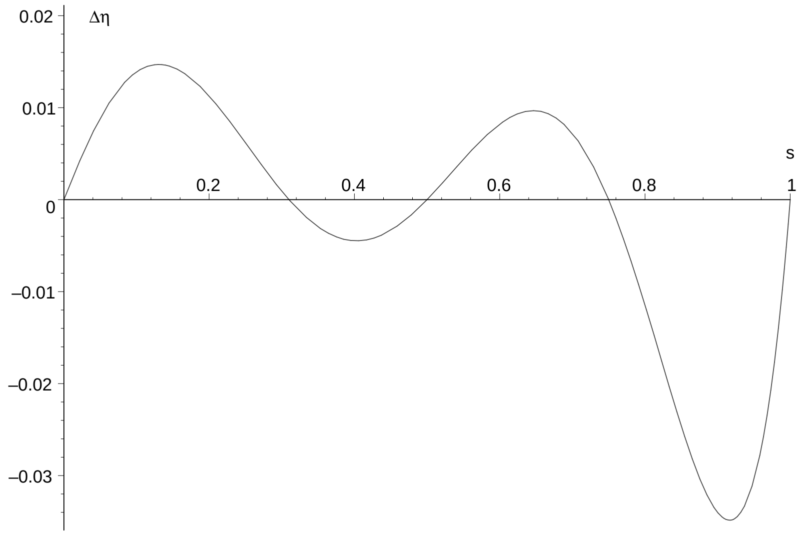

Figure 3: The error $\eta_1(s) - \eta_0(s)$

and the error $\triangle\eta = \eta_1 - \eta_0$ in Figure 3. Since the error is sufficiently small, we resign from improving it. From Figure 3 it can be seen that there is a further collocation point which lies by $s = 0.30997$.

We get a second approximation, if we split the interval of integration $(s, 1)$ in (10.8) into the sub-intervals $(s, \sqrt{s})$ and $(\sqrt{s}, 1)$, and apply to both sub-intervals the trapezoidal rule. In this way we get as an approximation the equation

$$\eta(s) = \frac{s}{2}\eta^2(\sqrt{s})$$

with the explicit solution

$$\eta(s) = 2s^{-c} \exp\left\{\frac{\ln s}{\ln 2} \ln\left(-\frac{\ln s}{\ln 2}\right)\right\}, \tag{10.11}$$

where c is an arbitrary constant. Though we only can expect that the approximation (10.11) of the solution of (10.8) is good in a neighbourhood of 1, it shows as $s \to +0$, i.e., after the substitution $s = e^{-x}$ as $x \to \infty$, an analogous behaviour as in (6.8) (with x instead of s) and (10.4).

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