Minimal Quasidifferential of a Piecewise Linear Function in $\mathbb{R}^3$

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Abstract. A quasidifferential of piecewise linear function in $\mathbb{R}^n$ can be a pair of polytopes (A, B). We prove that a minimal pair (C, D) of compact convex sets which is equivalent to (A, B) is a pair of polytopes for $n = 3$.

Keywords: *Convex analysis, minimal pairs of compact convex sets*

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1. Introduction

Quasidifferential calculus, which was introduced by Demyanov and Rubinov in [7], is useful in nonsmooth optimization. Quasidifferentiable functions include, in particular, convex functions and piecewise linear functions.

Let X be a normed space, X^* its dual space and $\mathcal{K}(X^*)$ the family of nonempty weak-*-compact convex subsets of X^*. For a function $f : X \to \mathbb{R}$ the directional derivative is defined by

$$\left.\frac{\partial f}{\partial g}\right|_{x_0} = \lim_{t \to 0^+} \frac{f(x_0 + tg) - f(x_0)}{t}$$

and its quasidifferential is defined by

$$\mathrm{D}f \mid_{x_0} = \left(\underline{\partial} f \mid_{x_0}, \overline{\partial} f \mid_{x_0}\right) \in \mathcal{K}(X^*) \times \mathcal{K}(X^*),$$

where

$$\left.\frac{\partial f}{\partial g}\right|_{x_0} = \max_{v \in \underline{\partial} f|_{x_0}} v(g) + \min_{w \in \overline{\partial} f|_{x_0}} w(g) = \max_{v \in \underline{\partial} f|_{x_0}} v(g) - \max_{w \in -\overline{\partial} f|_{x_0}} w(g).$$

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If we put $A = \underline{\partial} f|_{x_0}$ and $B = -\overline{\partial} f|_{x_0}$, then

$$\left.\frac{\partial f}{\partial g}\right|_{x_0} = p_A(g) - p_B(g),$$

where p_A and p_B are the support functions of the weak-*-compact convex sets A and B.

While the differential is an element of X^*, the quasidifferential is a pair of nonempty compact convex sets in X^*. More exactly, we can look at the quasidifferential as a quotient class in $\mathcal{K}(X^*) \times \mathcal{K}(X^*)$ with respect to the equivalence class " $\sim$ ", where $(A, B) \sim (C, D)$ if and only if $A + D = B + C$ with the Minkowski sum $A + B = \{ a + b \mid a \in A \text{ and } b \in B \}$.

The study of pairs of compact convex sets produced a number of results: the existence of minimal pairs [12], uniqueness of minimal pairs up to translation for $\dim X = 2$ ([9, 16]), and many others ([4, 13, 17] and others). Most of these results are gathered in [14]. Pairs of compact convex sets are applied in the numerical evaluation of the Aumann-Integral ([1, 2] and others) and in combinatorial convexity ([8, 11]).

It is proved in [10] and [5, 6] that a minimal pair equivalent to a pair of polytopes in $\mathbb{R}^2$ is a pair of polytopes. In [14] the question is posed: whether for any pair of polytopes in $\mathbb{R}^n$, $n > 2$, there exists an equivalent minimal pair of compact convex sets that are polytopes? We give the positive answer to the question for $n = 3$ in Theorem 2. Theorem 3 states that a minimal quasidifferential of a piecewise linear function in $\mathbb{R}^3$ is a pair of polytopes.

2. Main body

Let A, B, C be nonempty compact convex sets in $X = \mathbb{R}^3$. By $A \vee B$ we denote the convex hull of $A \cup B$. For $f \in (\mathbb{R}^3)^* \setminus \{0\}$ we define the face $H_f A$ of A by

$$H_f A = \{a \in A \mid f(a) = \max_{x \in A} f(x)\}.$$

We say that A is a *summand* of B and B is an *anti-summand* of A if $A + C = B$ for some C. We say that C is a *minimal common anti-summand* (m.c.a.s.) of A and B if C is an anti-summand of A and B and for any anti-summand D of A and B the inclusion $D \subset C$ implies $D = C$.

From now on we assume that A and B are convex polytopes with nonempty interiors, that A and B have only coparallel edges (that is $H_f A$ is an interval if and only if $H_f B$ is an interval) and that polytopes A and B are simple (that is each vertex of A and each vertex of B belongs to exactly three edges of A or B).

In the following, let C be a m.c.a.s. of A and B.

Definition 1.

(i) We say that an interval k is an *edge* of C if $k = H_f C$ and $H_f A$ is an interval for some $f \in (\mathbb{R}^3)^* \setminus \{0\}$.

(ii) We say that an interval k is an *extreme edge* of C if $k = H_g H_f C$ and $H_g H_f A$ is an interval for some $f, g \in (\mathbb{R}^3)^* \setminus \{0\}$.

For an edge k of C we denote $\overline{k} = H_f A,\ \overline{\overline{k}} = H_f B$. By $| k |$ we denote the length of the interval k. For an extreme edge k of C we denote $\overline{k} = H_g H_f A,\ \overline{\overline{k}} = H_g H_f B$.

There exists a similarity between edges and exposed points, extreme edges and extreme points. In particular, x is an extreme point of C if and only if every neighborhood of x contains an exposed point of C. In a similar way, the following proposition holds true.

The following obvious proposition characterizes extreme edges of C.

Proposition 1. *Let $k \subset C$. Then k is an extreme edge of C if and only if k is not properly contained in any interval $k' \subset C$ and every ϵ-neighborhood $B(k, \epsilon)$ of k contains some edge of C.*

Proposition 2. *Let k be an extreme edge of C. Then $| k | \geq \max (| \overline{k} |, | \overline{\overline{k}} |)$.*

Proposition 2 follows from Theorem 3.2.8 in [15] and Theorem 2.6 in [10]. In fact, C is an anti-summand of both A and B if and only if the inequality $| k | \geq \max (| \overline{k} |, | \overline{\overline{k}} |)$ holds true for all edges k of C.

Definition 2.

(i) We call an extreme edge k of C *minimal* if $| k | = \max (| \overline{k} |, | \overline{\overline{k}} |)$.

(ii) We call $(k_1, \ldots, k_n)$ a *chain of extreme edges* if $k_i \neq k_j$ for $i \neq j$, and k_i and k_{i+1} have a common vertex for $i = 1, \ldots, n - 1$.

(iii) We call a chain of extreme edges $(k_1, \ldots, k_n)$ a *maximal chain* if it is not a part of any longer chain of extreme edges of C.

(iv) We call two chains of extreme edges $(k_1, \ldots, k_n)$ and $(k'_1, \ldots, k'_n)$ *coparallel* if $\overline{k_i} = \overline{k'_i},\ i = 1, \ldots, n$.

The following proposition gives an upper bound for the length of any chain of minimal edges.

Proposition 3. *Let p be the number of all edges of A and let $(k_1, \ldots, k_n)$ be a chain of minimal edges of C. Then $n < (p + 1)^{p+1} + p + 1$.*

Proof. Let us assume that $n = (p+1)^{p+1} + p + 1$. Then $(\overline{k_1}, \ldots, \overline{k_{p+1}})$, $\ldots$, $(\overline{k_{n-p}}, \ldots, \overline{k_n})$ are $(p+1)^p + 1$ sequences of the length $p+1$ of edges of A. Then $(\overline{k_r}, \ldots, \overline{k_{r+p}}) = (\overline{k_s}, \ldots, \overline{k_{s+p}})$ for some $r, s = 1, \ldots, n - p - 1$ with $r + p < s$. Moreover, $\overline{k_{r+i}} = \overline{k_{r+j}} = \overline{k_{s+i}} = \overline{k_{s+j}}$ for some $i, j = 0, \ldots, p$, $i < j$. Then the chains $(k_{r+i}, \ldots, k_{r+j})$ and $(k_{s+i}, \ldots, k_{s+j})$ are coparallel with respective segments of the same length. Hence the set $E = k_{r+i} \vee k_{r+j} \vee k_{s+i} \vee k_{s+j}$ is a paralellopiped with nonempty interior.

Let $k_{r+i} \subset H_{f_1}C$, $k_{r+j} \subset H_{f_2}C$, $k_{s+i} \subset H_{f_3}C$, $k_{s+j} \subset H_{f_4}C$, where $f_1, f_2, f_3, f_4 \in (\mathbb{R}^3)^* \setminus \{0\}$. Then $k_{r+i} \subset H_{f_1}E$, $k_{r+j} \subset H_{f_2}E$, $k_{s+i} \subset H_{f_3}E$, $k_{s+j} \subset H_{f_4}E$. Let us take any $x \in \mathbb{R}^3, x \neq 0$. For some $t > 0$ and some $y \in k_{r+i} \cup k_{r+j} \cup k_{s+i} \cup k_{s+j}$ we have $y - tx \in E$. Then for some $i = 1, 2, 3, 4$ we have $f_i(y) = \max \ f_i(E) \geq f_i(y - tx)$. Hence $f_i(x) \geq 0$.

We have just proved that $\max (f_1, f_2, f_3, f_4) \geq 0$. We also have $k_{r+i} \subset H_{f_1}A \cap H_{f_2}A \cap H_{f_3}A \cap H_{f_4}A$. We can assume that $0 \in \overline{k_{r+i}}$. Then

$$A \subset f_1^{-1}\big((-\infty, 0]\big) \cap f_2^{-1}\big((-\infty, 0]\big) \cap f_3^{-1}\big((-\infty, 0]\big) \cap f_4^{-1}\big((-\infty, 0]\big)$$
$$= \big(\max (f_1, f_2, f_3, f_4)\big)^{-1}(0)$$
$$\subset f^{-1}(0).$$

Hence A is contained in a two-dimensional subspace. This contradicts our assumption that A has a nonempty interior. Then $n < (p+1)^{p+1} + p + 1$. ∎

The next proposition and corollary characterize extreme points of C.

Proposition 4. *Let x be an extreme point of C. For any $\epsilon > 0$ there exists an edge k of C such that* $\mathrm{dist}\,(k, x) < \epsilon$ *and* $\mid k \mid - \max (\mid \overline{k} \mid, \mid \overline{\overline{k}} \mid) < \epsilon$.

Proof. Let $H_f C = \{x\}$ and $C_\delta = \{c \in C \mid f(c) \leq f(x) - \delta\}$. For small enough $\delta > 0$ we have $\mathrm{diam}\,(C \setminus C_\delta) < \min (\epsilon, \min \{\mid \overline{k} \mid \mid \ \overline{k}$ is an edge of $A\})$. If $H_g A$ is an edge of A and $H_g C_\delta$ is a one-point set or an interval parallel to $H_g A$, then $k = H_g C$ is an edge of C and $H_g C_\delta = k \cap C_\delta$. Hence $\mid H_g C_\delta \mid \geq \mid k \mid - \epsilon$. If $\mid k \cap C_\delta \mid \geq \max (\mid \overline{k} \mid, \mid \overline{\overline{k}} \mid)$ for all edges k of C then, according to Theorem 2.6 in [10], C_δ is a common summand of A and B. Therefore, $\mid k \cap C_\delta \mid < \max (\mid \overline{k} \mid, \mid \overline{\overline{k}} \mid)$ for some edge k of C. Then $\mid k \mid < \max (\mid \overline{k} \mid, \mid \overline{\overline{k}} \mid) + \mid k \setminus C_\delta \mid$, and $k \setminus C_\delta \neq \emptyset$ and $k \setminus C_\delta \subset B(x, \epsilon)$.

Since every extreme point of C is a limit of some sequence of exposed points the proposition holds true. ∎

Corollary 1. *Let x be an extreme point of C. There exists an extreme edge k of C such that x is one end of k.*

Proof. There exists a sequence $(k_n = H_{f_n}C = y_n \vee z_n)_n$ of edges of C such that $\mathrm{dist}\,(k_n, x)$ tends to 0, $f_n \in (\mathbb{R}^3)^*, \parallel f_n \parallel = 1, n \in \mathbb{N}$. We can choose a

subsequence $(k_{n_m})_m$ such that f_{n_m} tends to some f, y_{n_m} tends to y and z_{n_m} tends to z. Then $y \vee z \subset H_f C$.

If $H_f C$ is an interval let $k = H_f C$. If $\dim H_f C = 2$ then $y \vee z$ is contained in the relative boundary of $H_f C$. Hence, for some $g \in (\mathbb{R}^3)^*$ we have $y \vee z \subset H_g H_f C$, where $k = H_g H_f C$ is an interval. In both cases $x \in y \vee z \subset k$. ∎

Let V be a two-dimensional subspace of $\mathbb{R}^3$ orthogonal to an extreme edge k of C. Denote by $\mathrm{pr}_V : \mathbb{R}^3 \longrightarrow V$ the orthogonal projection on V. The arc $\bigcup \{\mathrm{pr}_V(H_f C) \mid f \in (\mathbb{R}^3)^* \setminus \{0\}, \overline{k} \subset H_f A\}$ is a part of the boundary of two-dimensional compact convex set $p_V(C)$.

Definition 3. We say that an extreme edge k *lies between extreme edges* k' and k'' if $\overline{k} = \overline{k'} = \overline{k''}$ if the point $\mathrm{pr}_V(k)$ lies between the points $\mathrm{pr}_V(k')$ and $\mathrm{pr}_V(k'')$ on the arc

$$\bigcup \{\mathrm{pr}_V(H_f C) \mid f \in (\mathbb{R}^3)^* \setminus \{0\}, \overline{k} \subset H_f A\} \subset V.$$

Remark 1. If an extreme edge k lies between extreme edges $k' \subset H_{f'} C$ and $k'' \subset H_{f''} C$, then $k \subset H_f C$ for some $f = tf' + (1-t)f''$, where $t \in [0,1]$. Note that $t = 0$ (resp. $t = 1$) if and only if k and k'' (k and k') are two parallel sides of two-dimensional face $H_f C$.

Remark 2. Let $\{k_1, k_1', k_1''\}$ and $\{k_2, k_2', k_2''\}$ be two sets of coparallel extreme edges of C such that k_1 lies between k_1' and k_1'' and $k_1 \cap k_2 = \{x\}$, $k_1' \cap k_2' = \{x'\}$, $k_1'' \cap k_2'' = \{x''\}$. There exist linear functionals h and f_1 in $(\mathbb{R}^3)^* \setminus \{0\}$ such that $\mathrm{card}\, h(k_1 \cup k_2) = 1$ and $k_1 \subset H_{f_1} C$. Let us denote

$$\begin{aligned} l_1^t &= h^{-1}(t) \cap \bigcup \{H_f C \mid f \in (\mathbb{R}^3)^* \setminus \{0\}, \overline{k_1} \subset H_f A\} \\ &= H_{f_1}(h^{-1}(t) \cap C), \end{aligned}$$

where $t \in (h(k_1'), h(k_1''))$. Then l_1^t is an interval parallel to k_1. The interval l_1^t is an extreme edge of C or l_1^t is a maximal interval parallel to k_1 contained in a two-dimensional face of C, a face with two sides being extreme edges coparallel to k_1. Hence $\mid l_1^t \mid \geq \max (\mid \overline{k_1} \mid, \mid \overline{\overline{k_1}} \mid)$.

Now we need a somewhat technical proposition concerning extreme edges coparallel to three edges of A with common endpoint.

Proposition 5. *Let* k_1, k_2 *and* k_3 *be extreme edges of* C, $k_1 \cap k_2 = \{x\}$, $\overline{k_1} \cap \overline{k_2} \cap \overline{k_3} = \{\overline{x}\}$, $\overline{k_3} = \overline{x} \vee \overline{y}$, $h \in (\mathbb{R}^3)^* \setminus \{0\}$, $h(k_1 \cup k_2) = \{h(x)\}$ *and* $h(\overline{y}) > h(\overline{x})$. *Then* $\min h(k_3) \geq h(x)$.

Proof. Let $k_1 = H_{g_1}H_{f_1}C, k_2 = H_{g_2}H_{f_2}C$ and $k_1 = H_{g_1}H_{f_1}C$. Let us assume that $\overline{x} = 0$. Then $\Lambda := f_1^{-1}((-\infty, 0]) \cap f_2^{-1}((-\infty, 0]) \cap f_3^{-1}([0, -\infty))$ is a cone. Let $z \in k_3$ and denote $\overline{z} = z - x$. Then $\overline{z} \in \Lambda$. Now we assume that $h(\overline{z}) < 0$. We introduce such coordinates in $\mathbb{R}^3$ that $\overline{k_1} = (0,0,0) \vee (1,0,0)$, $\overline{k_2} = (0,0,0) \vee (0,1,0)$ and $\overline{k_3} = (0,0,0) \vee (0,0,1)$. Then $f_1(u) = a_2u_2 + a_3u_3$, $f_2(u) = b_1u_1 + b_3u_3$, $f_3(u) = c_1u_1 + c_2u_2$, $h(u) = du_3$ for $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ and for some $a_2, a_3, b_1, b_3, c_1, c_2 \leq 0$ and $d > 0$.

If $a_3 < 0$ and $b_3 < 0$ then $\overline{z} \in \Lambda$ implies that $a_2\overline{z}_2 + a_3\overline{z}_3 \leq 0, b_1\overline{z}_1 + b_3\overline{z}_3 \leq 0, c_1\overline{z}_1 + c_2\overline{z}_2 \geq 0$. Then $a_3\overline{z}_3 > 0, b_3\overline{z}_3 > 0$. Hence $a_2\overline{z}_2 < 0, b_1\overline{z}_1 < 0$. Then $\overline{z}_2 > 0, \overline{z}_1 > 0$, and $f_3(\overline{z}) < 0$, which contradicts our assumption that $\overline{z} \in \Lambda$.

If $a_3 < 0, b_3 = 0$ then $a_3\overline{z}_3 > 0$. Hence $a_2\overline{z}_2 < 0, b_1\overline{z}_1 \leq 0$, and $\overline{z}_2 > 0, \overline{z}_1 \geq 0$. Then $c_2\overline{z}_2 \leq 0, c_1\overline{z}_1 \leq 0$. Since $c_2\overline{z}_2 + c_1\overline{z}_1 \geq 0$ then $c_2\overline{z}_2 = 0$, and $c_2 = 0$. Hence f_2, f_3 are linearly dependent and $H_{f_2}C = H_{f_3}C$. Let us notice that k_2, k_3 are contained in the relative boundary of $H_{f_2}C$. Since $H_{-h}H_{f_2}A = \overline{k_2}$ then $H_{-h}H_{f_2}C = k_2$. Hence $h(z) \in h(k_3) \subset h(H_{f_2}C)$ and $\min h(k_3) \geq h(k_2) = h(x)$, which contradicts the assumption that $h(\overline{z}) < 0$.

If $a_3 = 0, b_3 < 0$, then f_1, f_3 are linearly dependent and we repeat the previous reasoning.

If $a_3 = b_3 = 0$, then

$$f_3 = \frac{c_1}{b_1}f_2 + \frac{c_2}{a_2}f_1.$$

Since $x \in H_{f_1}C \cap H_{f_2}C$ then $x \in H_{f_3}C = k_3$. Hence $k_3 \subset H_{f_2}C$ and we repeat the reasoning. ∎

The following proposition shows that the middle one of three coparallel chains of exteme edges is not maximal.

Proposition 6. *Let* $(k_1, \ldots, k_n), (k_1', \ldots, k_n'), (k_1'', \ldots, k_n'')$ *be three coparallel chains of extreme edges, let* k_1 *lie between* k_1' *and* k_1'' *and* $a \in k_1, b \in k_n, a \neq b$ *be the end points of the chain* $(k_1, \ldots, k_n)$. *Then for any* $\epsilon > 0$ *there exists an edge* k *of* C *such that* $| k | < \max (| \overline{k} |, | \overline{\overline{k}} |) + \epsilon$ *and* $\text{dist}(a, k) < \epsilon$, *where* k *is not parallel to* k_1, *or* $\text{dist}(b, k) < \epsilon$, *where* k *is not parallel to* k_n. *Hence the chain* $(k_1, \ldots, k_n)$ *is not maximal.*

Proof. Let $h_1, \ldots, h_{n-1} \in (\mathbb{R}^3)^* \setminus \{0\}$ such that $h_i(k_i \cup k_{i+1})$ is a one-point set and $h_i(k_i'') < h_i(k_i) < h_i(k_i')$, $i = 1, \ldots, n-1$. Let $k_i \subset H_{f_i}C, i = 1, \ldots, n$. Let $0 < \delta < \min (h_1(k_1') - h_1(k_1), h_1(k_1) - h_1(k_1''))$. Let us denote

$$\alpha_1 = H_{f_1}\big(C \cap h_1^{-1}(h_1(k_1) + \delta)\big), \quad \beta_1 = H_{f_1}\big(C \cap h_1^{-1}(h_1(k_1) - \delta)\big)$$
$$\alpha_i = H_{f_i}\big(C \cap h_{i-1}^{-1}(h_{i-1}(\alpha_{i-1}))\big), \quad \beta_i = H_{f_i}\big(C \cap h_{i-1}^{-1}(h_{i-1}(\beta_{i-1}))\big)$$

for $i = 2, \ldots, n$. For all $i = 1, \ldots, n$, the intervals α_i, β_i are parallel to $\overline{k_i}$ and $| \alpha_i |, | \beta_i | \geq \max (| \overline{k_i} |, | \overline{\overline{k_i}} |)$. Let $g_1, \ldots, g_n \in (\mathbb{R}^3)^* \setminus \{0\}$ such that $g_i(\alpha_i \cup \beta_i)$ is a one-point set and $g_i(k_i) > g_i(\alpha_i), i = 1, \ldots, n$.

There exists a $\theta > 0$ such that $B(k_i, \theta) \cap B(k_j, \theta) = \emptyset$ for all i, j and such that k_i and k_j are parallel. Since all extreme edges of C are parallel to a finite amount of edges of A we can assume that θ is so small that the length of the interval $k \cap B(k_i, \theta)$ is less than ϵ/n for all $k_i, i = 1, \ldots, n$ and all nonparallel extreme edges k of C.

There exists a $\delta > 0$ small enough that $C \cap g_i^{-1}((g_i(\alpha_i, \infty))) \subset B(k_i, \theta)$. Let us denote

$$C_\delta = C \cap \bigcap_{i=1}^n g_i^{-1}((-\infty, g_i(\alpha_i)]).$$

Let $g \in (\mathbb{R}^3)^*$ and $H_g A$ be an edge of A. If $\max \ g(C) > \max \ g(C_\delta)$, then $H_g C$ contains an extreme edge k such that $k \subset \bigcup_{i=1}^n B(k_i, \theta)$.

We can assume that ϵ is less than the length of the shortest edge of A. If no k_i is parallel to k, then $0 < \max \ (|\ \overline{k}\ |, |\ \overline{\overline{k}}\ |) - \epsilon \leq |\ k\ | - \epsilon \leq |\ k \cap C\delta\ |$ and $\max \ g(C) = \max \ g(C_\delta)$ which contradicts our assumption. Hence some k_i is parallel to k and $g_i(k) > g_i(\alpha_i)$. Then $H_g C_\delta$ is equal to α_i or β_i or it is a two-dimensional face of C_δ containing α_i and β_i.

If $\max \ g(C) = \max \ g(C_\delta)$ and $k = H_g C$ is an edge of C, then $H_g C_\delta = k \cap C_\delta$ and $|\ k \cap C_\delta\ | \geq \max \ (|\ \overline{k}\ |, |\ \overline{\overline{k}}\ |) - \epsilon$. If $\max \ g(C) = \max \ g(C_\delta)$ and $k = H_g C$ is a two-dimensional face of C while $H_g C_\delta$ is an interval, then $H_g C_\delta = \alpha_i$ or β_i for some $i = 1, \ldots, n$.

Applying Theorem 2.6 in [10] we see that for some edge k of C we have $\max \ (|\ \overline{k}\ |, |\ \overline{\overline{k}}\ |) - \epsilon \leq |\ k \cap C_\delta\ | \leq \max \ (|\ \overline{k}\ |, |\ \overline{\overline{k}}\ |)$.

Since ϵ can be arbitrarily small there exists a sequence $(k^{(q)} = x_q \vee y_q)_q$ of edges such that both $\text{dist}(k^{(q)}, k_1 \cup \ldots \cup k_n)$ and $|\ k^{(q)}\ | - \max \ (\overline{k^{(q)}}, \overline{\overline{k^{(q)}}})$ tend to 0. We can choose a subsequence $(k^{(q_m)})_m$ such that x_{q_m} tends to some x, y_{q_m} tends to some y and $x \vee y$ is contained in some extreme edge of C. Then $(x \vee y) \cap \bigcup_{i=1}^n k_i \neq \emptyset$ and x or y is an end point of some k_i. Let x be this endpoint. If $\{x\} = k_i \cap k_{i+1}$ for some i then, applying Proposition 5, either $h_i(k) \geq h_i(k_i')$ or $h_i(k) \leq h_i(k_i'')$. Hence x does not belong to k_i. Therefore, $x \in \{a, b\}$ and $\text{dist}\ (k^{q_m}, \{a, b\})$ tends to 0. ∎

Another proposition shows that the smaller is the distance between middle segments in coparallel chains of extreme edges the smaller is the difference between the lengths of these segments.

Proposition 7. *For any $\epsilon > 0$ there exists a $\delta > 0$ such that for any two coparallel chains $(k_1, k_2, k_3), (k_1', k_2', k_3')$ of extreme edges of C holds: if $\text{dist}\ (k_2, k_2') < \delta$, then $|\ k_2'\ | > |\ k_2\ | - \epsilon$.*

Proof. Let us assume the opposite statement. For some $\epsilon > 0$ there exist two sequences of coparallel chains $(k_1^{(n)}, k_2^{(n)}, k_3^{(n)}), (k_1^{(n)\prime}, k_2^{(n)\prime}, k_3^{(n)\prime}), k_1^{(n)} = a_n \vee$

$b_n, k_2^{(n)} = b_n \vee c_n, k_3^{(n)} = c_n \vee d_n, k_1^{(n)\prime} = a'_n \vee b'_n, k_2^{(n)\prime} = b'_n \vee c'_n, k_3^{(n)\prime} = c'_n \vee d'_n$ such that dist $(k_2^{(n)}, k_2^{(n)\prime})$ tends to 0 and $\mid k_2^{(n)\prime} \mid \leq \mid k_2^{(n)} \mid - \epsilon$. We can choose a sequence $(n_m)_m, m \in \mathbb{N}$ such that all chains $(k_1^{(n_m)}, k_2^{(n_m)}, k_3^{(n_m)})$ and $(k_1^{(n_m)\prime}, k_2^{(n_m)\prime}, k_3^{(n_m)\prime})$ are coparallel, a_{n_m} tends to some a, b_{n_m} to b, c_{n_m} to c, a'_{n_m} to a', b'_{n_m} to b' and c'_{n_m} to c'.

The intervals $a \vee b$, $b \vee c$, $c \vee d$, $a' \vee b'$, $b' \vee c'$ and $c' \vee d'$ are contained in respective extreme edges of C. The interval $a \vee b$ is parallel to $a' \vee b'$, $b \vee c$ to $b' \vee c'$ and $c \vee d$ to $c' \vee d'$. Then $b \vee c$ and $b' \vee c'$ are parallel extreme edges and dist $(b \vee c, b' \vee c') = 0$, and $b = b', c = c'$. Hence

$$0 = \| c - b \| - \| c' - b' \| = \lim_{m \to \infty} (\mid k_2^{(n_m)} \mid, \mid k_2^{(n_m)\prime} \mid) \geq \epsilon > 0,$$

which contradicts our assumption.   ∎

The following technical proposition that we need is similar to the previous one.

Proposition 8. *For any $\epsilon > 0$ there exists a $\delta > 0$ such that for any two coparallel chains $(k_1, k_2), (k'_1, k'_2)$ of extreme edges of C holds: if $\mathrm{dist}\,(a, k'_1) < \delta$, where a is the endpoint of k_1 that does not belong to k_2, then $\mid k'_1 \mid > \mid k_1 \mid - \epsilon$.*

The proof of the proposition is very similar to the proof of Proposition 7. We also need the following proposition on closed chains of extreme edges.

Proposition 9. *Let $n \in \mathbb{N}$. Then the amount of all closed chains of n extreme edges of C is finite.*

Proof. Let us assume that the amount of all closed chains of n extreme edges of C is infinite. Then there exists an infinite set $\{(k_1^i, \ldots, k_n^i)\}_{i \in I}$ of coparallel closed chains of extreme edges. Let $f_j^i \in (\mathbb{R}^3)^*, \| f_j^i \| = 1, k_j^i \subset H_{f_j^i} C$. Choosing an appropriate subsequence we can assume that $\lim_{i \to \infty} f_j^i = f_j$ for some $f_j \in (\mathbb{R}^3)^*, j = 1, \ldots, n$, and f_1^i tends to f_1 monotonously from one side, i.e., if $i < i'$, then $f_1^{i'} = \alpha f_1^i + \beta f_1$ for some $\alpha, \beta \geq 0$. Hence f_j^i tends to f_j monotonously from one side for all $j = 1, \ldots, n$. There exist extreme edges $k_1, \ldots, k_n$ such that $k_j \subset H_{f_j} C$ for $j = 1, \ldots, n$. Hence for any $\epsilon > 0$ and any $j = 1, \ldots, n$ we have $k_j^l \subset B(k_j, \epsilon)$ for almost all $l \in \mathbb{N}$. Then $(k_1, \ldots, k_n)$ is a closed chain of extreme edges coparallel to $(k_1^l, \ldots, k_n^l)$.

Let $g_j^l = \alpha_j^l f_j^l + \beta_j^l f_j, \alpha_j^l, \beta_j^l > 0$ be such that $g_j^l(k_j^l) = g_j^l(k_j), j = 1, \ldots, n$. Let $K_j^l = \{x \in C \mid g_j^l(x) > g_j^l(k_j)\}$. For any $\epsilon > 0$ and any $j = 1, \ldots, n$ we have $\overline{K_j^l} \subset B(k_j, \epsilon)$ for almost all $l \in \mathbb{N}$. For sufficiently large l we have $\overline{K_j^l} \cap \overline{K_{j'}^l} \neq \emptyset$ if and only if $\mid j - j' \mid \leq 1$ or $\{j, j'\} = \{1, n\}$. Let us denote $C_l = C \setminus \bigcup_{j=1}^n K_j^l$. Notice that $H_{g_j^l} C_l = k_j^l \vee k_j$. Then $H_{\alpha g_j^l + \beta f_j} C_l = k_j$ for $\alpha, \beta > 0$, and $H_{\alpha g_j^l + \beta f_j^l} C_l = k_j^l$ for $\alpha, \beta > 0$.

Consider an extreme edge $k \subset H_f C$ such that $f = \alpha f_j + \beta f_j^l$ for no $j = 1, \ldots, n$ and no $\alpha, \beta \geq 0$. If k and k_j are parallel then $k \cap K_j^l = \emptyset$. The set $K^l = \partial C \cap \bigcup_{j=1}^{n} K_j^l$ is open in the boundary ∂C of C. Then the relative boundary of K^l in ∂C is equal to $\bigcup_{j=1}^{n} k_j \cup \bigcup_{j=1}^{n} k_j^l$. If ϵ is sufficiently small then k is not contained in

$$\bigcup \{ B(k_j, \epsilon) \mid j = 1, \ldots, n, k_j \text{ is not parallel to } k \}.$$

Hence k is not contained in $\overline{K^l}$. Then $k \cap K^l \neq \emptyset$ implies that some internal point of of the interval k belongs to the relative boundary of K^l in ∂C. Hence some internal point of the interval k belongs to k_j or k_j^l for some j. This contradicts the fact that k_j and k_j^l are extreme edges.

Then C_l is a common summand of A and B which is properly contained in C. This contradicts the fact that C is a minimal common summand and thus proves our proposition. ∎

Now we are prepared to prove our main result.

Theorem 1. *Let A and B be two simple polytopes in $\mathbb{R}^3$ with nonempty interiors such that A and B have only coparallel edges. Assume that the compact convex set C is a minimal common anti-summand (m.c.a.s.) of A and B. Then C is a polytope.*

Proof. The set C is the convex hull of all extreme edges of C. Let us assume that C is not a polytope. Then the amount of all extreme edges of C is infinite. There exists an edge $\bar{l}$ of A such that the amount of all extreme edges k of C with $\bar{k} = \bar{l}$ is infinite.

Applying successively Proposition 6 we can prove that for any $n \in \mathbb{N}$ the amount of all chains $(k_1, \ldots, k_n)$ of n extreme edges is infinite and there exists a sequence $\bar{l}_1, \ldots, \bar{l}_n$ of edges of A such that the amount of all chains $(k_1, \ldots, k_n)$ of extreme edges with $\overline{k_1} = \overline{l_1}, \ldots, \overline{k_n} = \overline{l_n}$ is infinite.

Let us fix $n = 2(p+1)^{p+1} + 2p + 5$. Let Λ be an infinite family of coparallel chains of n extreme edges. There exists $(k_1, \ldots, k_n) \in \Lambda$ such that for any $\epsilon > 0$ there exists another chain $(k_1', \ldots, k_n') \in \Lambda$ such that $\mathrm{dist}\,(k_1, k_1') < \epsilon$. Let us denote by Λ_ϵ the family of all chains $(k_1', \ldots, k_n') \in \Lambda$ of extreme edges not equal to $(k_1, \ldots, k_n)$ such that $\mathrm{dist}\,(k_i, k_i') < \epsilon$ and if an extreme edge k lies between k_i and k_i' then $\mathrm{dist}\,(k, k_i) < \epsilon$. It is quite easy to prove that Λ_ϵ is not empty for all $\epsilon > 0$. Thanks to Proposition 3 we can choose $i, j \in \mathbb{N}$ such that

$$1 < i < (p+1)^{p+1} + 2p + 3 < j < 2(p+1)^{p+1} + 2p + 5,$$

where $\mid k_i \mid \, > \max\,(\mid \overline{k_i} \mid, \mid \overline{\overline{k_i}} \mid)$ and $\mid k_j \mid \, > \max\,(\mid \overline{k_j} \mid, \mid \overline{\overline{k_j}} \mid)$.

Let

$$\epsilon = \frac{1}{2} \min\left(| \, k_i \, | - \max\left(| \, \overline{k_i} \, |, | \, \overline{\overline{k_i}} \, |\right), | \, k_j \, | - \max\left(| \, \overline{k_j} \, |, | \, \overline{\overline{k_j}} \, |\right)\right).$$

Let us fix a $\delta > 0$ satisfying Propositions 7 and 8. Let $\{a\} = k_i \cap k_{i+1}$, $\{b\} = k_{j-1} \cap k_j$. There exists an extreme edge k_i'' such that $\mathrm{dist}\,(a, k_i'') < \delta$, $| \, k_i'' \, | < \max\left(| \, \overline{k_i} \, |, | \, \overline{\overline{k_i}} \, |\right) + \delta$, k_i'' is coparallel to k_i and k_i'' lies between k_i and k_i' for some $(k_1', \ldots, k_n') \in \Lambda_{\delta/2}$. There exists no extreme edge k_{i-1}'' such that the chain (k_{i-1}'', k_i'') is coparallel to (k_{i-1}, k_i). Proposition 6 implies the existence of extreme edges $k_{i+1}'', \ldots, k_n''$ such that the chain $(k_i'', \ldots, k_n'')$ is coparallel to $(k_i, \ldots, k_n)$ and lies between $(k_i, \ldots, k_n)$ and $(k_i', \ldots, k_n')$.

Let us denote $\{b''\} = k_{j-1}'' \cap k_j''$. There exists a sequence $(k_j^{(m)})_m$ of extreme edges such that

$$\lim_{m \to \infty} \mathrm{dist}\,(b'', k_j^{(m)}) = 0, \quad \lim_{m \to \infty} | \, k_j^{(m)} \, | = \max\left(| \, \overline{k_j} \, |, | \, \overline{\overline{k_j}} \, |\right),$$

all $k_j^{(m)}$ are coparallel to k_j and lie between k_j and k_j'. For sufficiently large m we have $\mathrm{dist}\,(b, k_j^{(m)}) < \delta$ and $| \, k_j^{(m)} \, | < \max\left(| \, \overline{k_j} \, |, | \, \overline{\overline{k_j}} \, |\right) + \delta$. Then there exists no extreme edge $k_{j+1}^{(m)}$ such that the chain $(k_j^{(m)}, k_{j+1}^{(m)})$ is coparallel to (k_j, k_{j+1}).

Due to Proposition 6, there exist extreme edges $k_1^{(m)}, \ldots, k_{j-1}^{(m)}$ such that the chain $(k_1^{(m)}, \ldots, k_j^{(m)})$ is coparallel to $(k_1, \ldots, k_j)$. Then $\mathrm{dist}\,(k_i'', k_i^{(m)})$ tends to 0 and for sufficiently large m we have $| \, k_i^{(m)} \, | > | \, k_i \, | - \epsilon > \max\left(| \, \overline{k_i} \, |, | \, \overline{\overline{k_i}} \, |\right) + \epsilon$. Then k_i'' is contained in a longer interval which is contained in C and this is impossible. Therefore, the compact convex set C is a polytope. ∎

In the following corollaries we repeat the statement of Theorem 1, gradually removing all the unnecessary assumptions.

Corollary 2. *Let A and B be two polytopes in $\mathbb{R}^3$ with nonempty interiors such that A and B have only coparallel edges. Let the compact convex set C be a minimal common anti-summand (m.c.a.s.) of A and B. Then C is a polytope.*

Proof. Let $a_1, \ldots, a_p$ be all the vertices of A. Let $f_1, \ldots, f_p \in (\mathbb{R}^3)^*$, $H_{f_1} A = \{a_1\}, \ldots, H_{f_p} A = \{a_p\}$. Let us define

$$A_\epsilon^i = \{x \in A \mid f_i(x) > f_i(a_i) - \epsilon\}, \quad i = 1, \ldots, p, \, \epsilon > 0.$$

Let us choose $\epsilon > 0$ such that $\overline{A_\epsilon^i} \cap \overline{A_\epsilon^j} = \emptyset$ for all $i, j = 1, \ldots, p$, $i \neq j$. Let $A_\epsilon = A \setminus \bigcup_{i=1}^p A_\epsilon^i$. Then $A + A_\epsilon$ and $B + A_\epsilon$ are simple polytopes and $C + A_\epsilon$ is a m.c.a.s. of $A + A_\epsilon$ and $B + A_\epsilon$.

Theorem 1 implies that $C + A_\epsilon$ is a polytope. Therefore, C is a polytope. ∎

Corollary 3. *Let A and B be two polytopes in $\mathbb{R}^3$ with nonempty interiors. Let the compact convex set C be a minimal common anti-summand of A and B. Then C is a polytope.*

Proof. The polytopes $2A + B$ and $A + 2B$ have only coparallel edges, and the set $C + A + B$ is a m.c.a.s. of $2A + B$ and $A + 2B$. The previous corollary implies that $C + A + B$ is a polytope. ∎

Corollary 4. *Let A and B be two polytopes in $\mathbb{R}^3$. Let the compact convex set C be a minimal common anti-summand of A and B. Then C is a polytope.*

Proof. Let D be a polytope in $\mathbb{R}^3$ with nonempty interior. Then the polytopes $A + D$ and $B + D$ are polytopes with nonempty interior, and the set $C + D$ is a m.c.a.s.of $A + D$ and $B + D$. Corollary 3 implies that $C + D$ is a polytope. ∎

Theorem 2. *Let (A, B) be a minimal pair of nonempty compact convex sets in $\mathbb{R}^3$. If there exists a pair (C, D) of polytopes equivalent to (A, B) then A and B are polytopes.*

Proof. The set $A + D = B + C$ is a m.c.a.s. of C and D. Applying Corollary 4 we prove the theorem. For the details the reader is refered to Proposition 2.2 in [10]. ∎

Theorem 3. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a piecewise linear function and $x_0 \in \mathbb{R}^3$. Then a minimal quasidifferential $\mathrm{D}f \mid_{x_0}$ is a pair of polytopes.*

Proof. Since the function f is piecewise linear, in some neighbohood of 0 the function $g(x) = f(x - x_0) - f(x_0)$ is equal to $\frac{\partial f}{\partial \cdot}\big|_{x_0}$ which is piecewise linear and positively homogenous. Hence $\frac{\partial f}{\partial \cdot}\big|_{x_0}$ is a continuous selection of a finite number of some linear functions $f_1, \dots f_n$. Due to Theorem 2.2 in [3], we have

$$\frac{\partial f}{\partial \cdot}\bigg|_{x_0} = \min_{j \in \{1,\dots,k\}} \max_{i \in I_j} f_i(\cdot)$$

for some subsets $I_1, \dots, I_k$ of $\{1, \dots, n\}$.

For nonempty compact convex sets A, B, C and D we have

$$\max (p_A - p_B, p_C - p_D) = p_{(A+D)\vee(B+C)} - p_{B+D}$$

and

$$\min (p_A - p_B, p_C - p_D) = p_{(A+C)} - p_{(A+D)\vee(B+C)}.$$

Since every linear function f_i is equal to a difference of support functions of some sngletons then $\frac{\partial f}{\partial \cdot}\big|_{x_0} = p_E(\cdot) - p_F(\cdot)$ for some polytopes E and F. Theorem 2 implies that a minimal quasidifferential $\mathrm{D}f \mid_{x_0}$ is a pair of polytopes. ∎

The following example gives four different m.c.a.s. of two simple polytopes having only coparallel edges. Theorem 1 implies that they all must be polytopes.

Example. Let E be a regular octahedron and F be a paralellopiped containing E, with all facets being identical rhombi. Let A and B be the two polytopes defined by $A = 2E + F$ and $B = E + 2F$.

E

F

$A = 2E + F$

$B = E + 2F$

Notice that both polytopes A and B are simple, have nonempty interior and only coparallel edges.

C_1

C_2

The polytopes C_1, C_2, C_3 and C_4 are only four of a continuum of minimal common anti-summands of A and B. The thick lines represent front edges and thin lines represent back edges. Double lines represent minimal edges.

C_3 $\qquad\qquad\qquad\qquad\qquad$ C_4
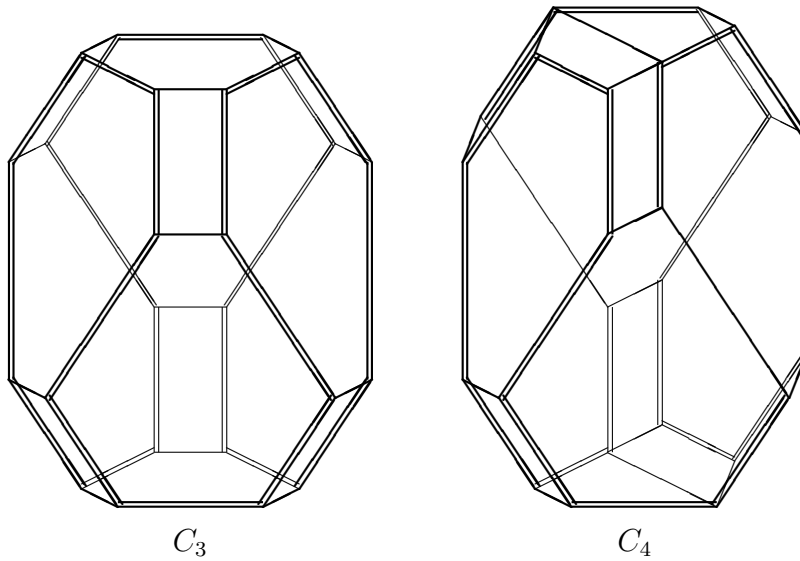
We know from Proposition 4 and Theorem 1 that each vertex of any m.c.a.s. C of A and B is an end point of some minimal edge. Therefore, minimal edges seem crucial in constructing m.c.a.s. of any two polytops, and, in consequence, in finding minimal quasi-differencials of piecewise linear functions.

Open questions. There are several questions arising from Theorem 1:

1. The notion of 'betweenness' from Definition 7 is essential in our proof of Theorem 1. This is why we were not able to generalize our proof to spaces $X = \mathbb{R}^n$, $n > 3$. How can we avoid it?

2. The amount of vertices of an m.c.a.s. C of A and B is bounded by some function of the amount of vertices of A and B. What is this function?

3. How can we construct effectively a m.c.a.s. C of A and B? How can we construct all m.c.a.s.'s of A and B?

Questions 2 and 3 can be formulated in terms of minimal pairs of compact convex sets and in terms of minimal quasidifferentials.

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