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Finite Truncations of Generalized One-Dimensional Discrete Convolution Operators and Asymptotic Behavior of the Spectrum. The Matrix Case.

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Abstract. In this paper we study the sequence $\{A_N(a)\}_{N\in\mathbb{N}}$ of finite truncations of a generalized discrete convolution operator, which have matrices of the form

$$A_N(a) \sim \left(a\left(\frac{n}{E(N)}, \frac{k}{E(N)}, n-k\right)\right)_{n,k=1,\dots,N},$$

where a is some function defined on $[0,+\infty)\times[0,+\infty)$, $E(\cdot)$ is defined on $\mathbb{N}$ and $E(N)\to\infty$, $\frac{N}{E(N)}\to\infty$ as $N\to\infty$. For this sequence we get a generalization of the Szegö limit theorem.

Keywords: *Szegö limit theorem, convolution operator, eigenvalues, Toeplitz operator*
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1. Introduction

The study of truncated convolution operators and the asymptotic behavior of their spectrum was started by G. Szegö [22]. The results, that he had received, were advanced subsequently by many authors. A thorough bibliography on this topic is contained in [1], pages 165 – 172, and in [2], pages 243 – 253. However, regular convolution operators were investigated in most cases. Generalized convolution operators have been studied in the papers [4], [5], [15], [18], [21], [23] and [25 – 30].

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The investigation presented in this paper is a continuation of the research, which the authors started in [26], [27] and [30] when studying the collective asymptotic behavior of the spectrum of the operators $A_N(a)$ $(N \in \mathbb{N})$, which are given by matrices of the form

$$\left(a\Big(\frac{n}{E(N)}, \frac{k}{E(N)}, n-k\Big)\right)_{n,k=1,\dots,N} .$$

Therein $a : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{Z} \to \mathbb{C}$ (where $\mathbb{R}^+ = [0, +\infty)$) is some function, which is uniformly continuous in the first two variables. The function $E : \mathbb{N} \to (0, +\infty)$ possesses the following properties:

$$E(N) \xrightarrow[N\to\infty]{} \infty , \quad \frac{N}{E(N)} \xrightarrow[N\to\infty]{} \infty .$$

This paper, as already stated, continues the previous research and generalizes previous results to the case where $a : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{Z} \to \mathbb{C}_{m\times m}$. Here $\mathbb{C}_{m\times m}$ $(m \in \mathbb{N})$ denotes the space of complex matrices of order $m \times m$. In [25], [26], [28] and [29] the authors investigated also operators of another form, which are given by a matrix

$$\left(a\Big(\frac{n}{N}, \frac{k}{N}, n-k\Big)\right)_{n,k=1,\dots,N} ,$$

where $a : [0,1] \times [0,1] \times \mathbb{Z} \to \mathbb{C}$ is a function, which is continuous in the first two variables.

The papers [4], [5], [15], [18], [21] and [23] contain other approaches to deriving analogs of the classical theorems in more general cases. It should be mentioned, that the probability theory and statistical mechanics are potential areas of applications for generalized convolution operators.

2. Main results

Let us introduce the following notations:

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ are the sets of natural, integer, real and complex numbers, respectively, $\mathbb{R}^+ = [0, +\infty)$; S is the unit circle in the complex plane; μ is the Lebesgue measure on S.

$l^2(U)$, $U \subset \mathbb{Z}$, is the Banach space of all complex-valued functions X defined on U, with the norm

$$\|X\|_{l^2(U)} = \Big(\sum_{n\in U} |X(n)|^2\Big)^{\frac{1}{2}} < \infty ,$$

$l^2(\emptyset) = \{0\}$ ($\emptyset$ is the empty set). Especially, $l^2 = l^2(\mathbb{Z})$, $\mathbb{C}_m = l^2(\{1,\dots,m\})$, $m \in \mathbb{N}$ (and elements of this spaces will be considered as vector-columns), $l^2_m(U)$, $m \in \mathbb{N}$, $U \subset \mathbb{Z}$, is the Banach space of all functions X defined on U with values in $\mathbb{C}_m$ and with the norm

$$\|X\|_{l^2_m(U)} = \left(\sum_{n\in U} \|X(n)\|^2_{\mathbb{C}_m}\right)^{\frac{1}{2}} < \infty \ .$$

Analogously $l^2_m(\emptyset) = \{0\}$ and $l^2_m = l^2_m(\mathbb{Z})$, $\mathbb{C}^N_m = l^2_m(\{1,\dots,N\})$, $N, m \in \mathbb{N}$.

$\mathrm{Hom}(K_1, K_2)$ is the Banach space of bounded linear operators from a Banach space K_1 to a Banach space K_2; $\mathrm{End}(K_1) = \mathrm{Hom}(K_1, K_1)$.

$P_{U,V} \in \mathrm{Hom}(l^2_m(U\cap\mathbb{Z}), l^2_m(V\cap\mathbb{Z}))$ for $U, V \subset \mathbb{R}$ and $(V\cap\mathbb{Z}) \subset (U\cap\mathbb{Z})$ is the *operator of truncation*; $J_{V,U} \in \mathrm{Hom}(l^2_m(V\cap\mathbb{Z}), l^2_m(U\cap\mathbb{Z}))$ is the *operator of continuation by zero* and $Q^V_U = J_{V,U}P_{U,V}$.

$E : \mathbb{N} \to \mathbb{R}^+\backslash\{0\}$ is a function having the following properties:

$$E(N) \xrightarrow[N\to\infty]{} \infty\,, \quad \frac{N}{E(N)} \xrightarrow[N\to\infty]{} \infty \ .$$

Definition 1. Let us denote by $\mathfrak{D}$ the Banach algebra of all functions $a : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{Z} \to \mathbb{C}$ possessing the properties:

1) $a(x,y,n)$ is uniformly continuous on $\mathbb{R}^+ \times \mathbb{R}^+$ for everyfixed $n \in \mathbb{Z}$;

2) the series $\sum_{n\in\mathbb{Z}} a(x,y,n)$ satisfies the Weierstrass condition of uniform convergence: $\sum_{n\in\mathbb{Z}} \sup_{(x,y)\in\mathbb{R}^+\times\mathbb{R}^+} |a(x,y,n)| < \infty$.

The multiplication in $\mathfrak{D}$ is defined as follows:

$$(a*b)(x,y,n) = \sum_{k\in\mathbb{Z}} a(x,y,n-k)b(x,y,k) \ .$$

The norm in $\mathfrak{D}$ is defined by

$$\|a\|_{\mathfrak{D}} = \sum_{n\in\mathbb{Z}} \sup_{(x,y)\in\mathbb{R}^+\times\mathbb{R}^+} |a(x,y,n)| \ .$$

Let $\mathbb{C}_{m\times m}$ be the space of all complex matrices of order $m \times m$.

Definition 2. Let us denote by $\mathfrak{D}_{m\times m}$ the Banach algebra of all matrix-valued functions $a : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{Z} \to \mathbb{C}_{m\times m}$ of the form

$$a(x,y,n) = (a_{ij}(x,y,n))_{i,j=1,\dots,m} \ , \quad a_{ij} \in \mathfrak{D} \ ,$$

with the usual definition of multiplication (for matrices whose entries are elements in a Banach algebra) and the norm

$$\|a\|_{\mathfrak{D}_{m\times m}} = \sum_{i,j=1}^{m} \|a_{ij}\|_{\mathfrak{D}} \ .$$

Below we introduce some more notation. If $a \in \mathfrak{D}_{m\times m}$, then:

$A_N(a) \in \operatorname{End}(\mathbb{C}_m^N)$ is a linear operator given by the matrix

$$\left(a\Big(\frac{n}{E(N)}, \frac{k}{E(N)}, n-k\Big)\right)_{n,k=1,\dots,N}$$

(it should be emphasized that entries of this matrix are matrices of order $m\times m$);

$a^\xi : \mathbb{Z} \to \mathbb{C}_{m\times m}$ for $\xi \in \mathbb{R}^+$ is defined by

$$a^\xi(n) = a(\xi,\xi,n)\ , \quad n \in \mathbb{Z}\ ;$$

$L\left(a^\xi\right) \in \operatorname{End}(l_m^2)$ is the operator with the matrix representation

$$(a^\xi(n-k))_{n,k\in\mathbb{Z}}\ ;$$

$L_U\left(a^\xi\right) \in \operatorname{End}(l_m^2(U\cap\mathbb{Z}))$ for $U \subset \mathbb{R}$ is defined by

$$L_U\left(a^\xi\right) = P_{\mathbb{Z},U} L\left(a^\xi\right) J_{U,\mathbb{Z}}\ .$$

We define

$$(\Lambda a)(x,y,t) = \sum_{n\in\mathbb{Z}} a(x,y,n)t^n\ , \quad x,y\in\mathbb{R}^+,\ t\in S$$
$$(\Lambda a^\xi)(t) = \sum_{n\in\mathbb{Z}} a^\xi(n)t^n\ , \qquad t \in S\ .$$

Moreover, let the function $\Phi_f : \operatorname{End}(K) \to \mathbb{C}$ be defined by

$$\Phi_f(A) = \frac{1}{\operatorname{Numb}(B)} \sum_{k\in B} f(\lambda_k)\ ,$$

where K is a finite-dimensional Banach space; f is a function defined on some open domain in the complex plane; $\{\lambda_k\}$ are the eigenvalues of the operator $A \in \operatorname{End}(K)$ taking multiplicities into account; B is the set of those indices k, for which λ_k belongs to the domain of the function f; $\operatorname{Numb}(B)$ is the cardinality of the set B.

Let $W(S)$ be the space of complex-valued functions on S which can be expanded into an absolutely converging Fourier series; let $W^+(S)(\subset W(S))$ be the subspace of those functions which allow an analytical continuation to $\{t\in\mathbb{C}, |t|<1\}$; let $W^-(S)(\subset W(S))$ be the subspace of functions that allow an analytical continuation to $\{t\in\mathbb{C}, |t|>1\}$ and having at infinity a finite limit.

It is known (for references see, e.g., [9]), that a nonvanishing function $g \in W(S)$ can be represented in the form

$$g(t) = g_+(t) t^{\chi} g_-(t) \ ,$$

where $\chi \in \mathbb{Z}$; the function $g_+ \in W^+(S)$ has no zeros for $|t| \leqslant 1$; $g_- \in W^-(S)$ has no zeros for $|t| \geqslant 1$ and $g_-(\infty) \neq 0$. This representation is called the factorization of the function g. In this case the number χ is equal to the winding number of the function g (i.e., the increment of its argument when going round the unit circle in positive direction, divided by 2π):

$$\chi = \operatorname*{wind}_S g(t) = \frac{1}{2\pi} \operatorname*{arg}_{t \in S} g(t) \ .$$

Let $C(\mathbb{R}^+ \times \mathbb{R}^+ \times S)$ be the normed space of all bounded uniformly continuous complex-valued functions φ, defined on $\mathbb{R}^+ \times \mathbb{R}^+ \times S$, with the norm

$$\|\varphi\|_{C(\mathbb{R}^+ \times \mathbb{R}^+ \times S)} = \sup_{x,y \in \mathbb{R}^+,\, t \in S} |\varphi(x,y,t)| \ .$$

Definition 3. We denote by $\mathfrak{U}$ the set of all functions φ from $C(\mathbb{R}^+ \times \mathbb{R}^+ \times S)$, which are uniformly continuous on $\mathbb{R}^+ \times \mathbb{R}^+ \times S$ and satisfy the following conditions:

1) the function $\varphi(\xi, \xi, t)$ belongs to the space $W(S)$ for each fixed $\xi \in \mathbb{R}^+$;

2) the closure of the image of the function $\varphi(\xi, \xi, t)$ for all $\xi \in \mathbb{R}^+$, $t \in S$ does not contain zero;

3) the function $\varphi(0, 0, t)$ has winding number zero.

Let us remark, that (by Definition 3) the function $\varphi(\xi, \xi, t)$ ($\varphi \in \mathfrak{U}$) has the winding number zero for every $\xi \in \mathbb{R}^+$.

By $W_{m \times m}(S)$, $W^+_{m \times m}(S)$ and $W^-_{m \times m}(S)$ we denote the spaces of all matrices of order $m \times m$ whose entries are functions from $W(S)$, $W^+(S)$ and $W^-(S)$, respectively. By a right factorization of a matrix-valued function $g \in W_{m \times m}(S)$ having a nonvanishing determinant we mean a representation of the form

$$g(t) = g_{r-}(t) \begin{pmatrix} t^{\chi_1} & 0 & \dots & 0 \\ 0 & t^{\chi_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t^{\chi_m} \end{pmatrix} g_{r+}(t) \ ,$$

where $g_{r-} \in W^-_{m \times m}(S)$, $g_{r+} \in W^+_{m \times m}(S)$, the determinants of their continuations into the corresponding domains and also the determinant of g_{r-} at infinity do not vanish. The integer numbers $\chi_1 \leqslant \chi_2 \leqslant \ldots \leqslant \chi_m$ are called the right partial indices. It is a well known fact, that $\sum_{k=1}^{m} \chi_k = \operatorname{wind}_S \det g(t)$.

By a left factorization of an invertible matrix-valued function $g \in W_{m\times m}(S)$ we denote a representation of the form

$$g(t) = g_{l+}(t) \begin{pmatrix} t^{\nu_1} & 0 & \dots & 0 \\ 0 & t^{\nu_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t^{\nu_m} \end{pmatrix} g_{l-}(t) \ ,$$

where $g_{l+} \in W^{+}_{m\times m}(S)$, $g_{l-}(t) \in W^{-}_{m\times m}(S)$, the determinants of their continuations into the corresponding domains and the determinant of g_{r-} at infinity do not vanish. The integer numbers $\nu_1 \leqslant \nu_2 \leqslant \dots \leqslant \nu_m$ are called the left partial indices. In this case also $\sum_{k=1}^{m} \nu_k = \operatorname{wind}_S \det g(t)$.

It is also important to mention (see [2], [9]) that right and left partial indices are uniquely determined.

Let φ be a matrix-valued function on $\mathbb{R}^+ \times \mathbb{R}^+ \times S$ of the form

$$\varphi(x, y, t) = (\varphi_{ij}(x, y, t))_{i,j=1,\dots,m} \ ,$$

where $\varphi_{ij}(x, y, t)$ are bounded complex-valued functions defined and uniformly continuous on $\mathbb{R}^+ \times \mathbb{R}^+ \times S$. We denote by $\operatorname{Lim}_{\varphi}(S)$ the set of all matrix-valued functions ψ on S, which have the form

$$\psi(t) = (\psi_{ij}(t))_{i,j=1,\dots,m} \ ,$$

where the $\psi_{ij}(\cdot)$ are defined on S and continuous, and for which there exists such a sequence $\{x_n\}_{n\in\mathbb{N}}$, $x_n \in \mathbb{R}^+$, that $x_n \to +\infty$ as $n \to \infty$ and every function $\varphi_{ij}(x_n, x_n, t)$ converges uniformly to $\psi_{ij}(t)$ as $n \to \infty$. Let $C_{m\times m}(\mathbb{R}^+ \times \mathbb{R}^+ \times S)$ be the normed space of all matrix-valued functions of the form

$$\varphi(x, y, t) = (\varphi_{ij}(x, y, t))_{i,j=1,\dots,m} \ ,$$

where $\varphi_{ij} \in C_{m\times m}(\mathbb{R}^+ \times \mathbb{R}^+ \times S)$, with the norm

$$\|\varphi\|_{C_{m\times m}(\mathbb{R}^+\times\mathbb{R}^+\times S)} = \max_{i,j=1,\dots,m} \ \sup_{x,y\in\mathbb{R}^+,\, t\in S} |\varphi_{ij}(x, y, t)| \ .$$

Definition 4. Let us denote by $\mathfrak{U}_{m\times m}$ the set of all functions $\varphi(x, y, t)$ from $C_{m\times m}(\mathbb{R}^+ \times \mathbb{R}^+ \times S)$, which satisfy the following four conditions:

1) for every fixed $\xi \in \mathbb{R}^+$ the matrix-valued function $\varphi(\xi, \xi, t)$ belongs to the space $W_{m\times m}(S)$;

2) the closure of the image of the determinant of $\varphi(\xi, \xi, t)$ for all $\xi \in \mathbb{R}^+$, $t \in S$ does not contain zero;

3) $\varphi(0, 0, t)$ allows a right factorization with zero partial indices;

4) every matrix-valued function $\psi(t) \in \mathrm{Lim}_\varphi(S)$ allows a left factorization with zero partial indices.

Let us introduce another important notation $B_{\delta,N}(a)$. Let $\delta > 0$, $a \in \mathfrak{D}_{m\times m}$ and $\Lambda a \in \mathfrak{U}_{m\times m}$. Let us remark, that because of the last condition of Definition 4 and the stability criterion for partial indices (see [16]), there exists such an $N_0 \in \mathbb{N}$, that for each $N > N_0$ the matrix-valued function $\left(\Lambda a^{\frac{N}{E(N)}}\right)(t)$ allows the left factorization with zero partial indices (therefore, the operator $L_{(-\infty,N]}\left(a^{\frac{N}{E(N)}}\right)$ is invertible). For all $N \in \mathbb{N}$ such that $N > \max\{3\delta E(N), N_0\}$ (in this case the following expression has a meaning), we introduce

$$\begin{aligned} B_{\delta,N}(a) = {} & J_{[1,E(N)\xi_1),[1,N]} P_{[1,+\infty),[1,E(N)\xi_1)} L^{-1}_{[1,+\infty)}\left(a^0\right) \\ & \times J_{[1,E(N)\xi_2),[1,+\infty)} P_{[1,N],[1,E(N)\xi_2)} \\ & + \sum_{k=2}^{\eta(N)} J_{[E(N)\xi_{k-1},E(N)\xi_k),[1,N]} P_{\mathbb{Z},[E(N)\xi_{k-1},E(N)\xi_k)} L^{-1}\left(a^{\xi_{k-1}}\right) \\ & \times J_{[E(N)\xi_{k-2},E(N)\xi_{k+1}),\mathbb{Z}} P_{[1,N],[E(N)\xi_{k-2},E(N)\xi_{k+1})} \\ & + J_{[E(N)\xi_{\eta(N)},N],[1,N]} P_{(-\infty,N],[E(N)\xi_{\eta(N)},N]} L^{-1}_{(-\infty,N]}\left(a^{\frac{N}{E(N)}}\right) \\ & \times J_{[E(N)\xi_{\eta(N)-1},N],(-\infty,N]} P_{[1,N],[E(N)\xi_{\eta(N)-1},N]} \ , \end{aligned} \tag{1}$$

where $\xi_k = k\delta$ for $k = 0, 1, 2, \ldots$. Therein $\eta(N) \in \mathbb{N}$ satisfies the condition

$$\frac{1}{\delta}\frac{N}{E(N)} - 3 < \eta(N) \leqslant \frac{1}{\delta}\frac{N}{E(N)} - 2 \ .$$

For $N \in \mathbb{N}$, for which $N \leqslant \max\{3\delta E(N), N_0\}$, we define $B_{\delta,N}(a) = 0$.

It is necessary to mention that this notation is correct. The invertibility of the "extreme" operators $L_{[1,+\infty)}\left(a^0\right)$ and $L_{(-\infty,N]}\left(a^{\frac{N}{E(N)}}\right)$ in the scalar case (if $\Lambda a \in \mathfrak{U}$ is fulfilled) is shown in papers [6 – 8], [11 – 13], [17] and [19]. The corresponding results for the matrix case (if $\Lambda a \in \mathfrak{U}_{m\times m}$) can be found for example in [2] and [9]. The invertibility of the "middle" operators $L\left(a^{\xi_{k-1}}\right)$ is obvious:

$$L^{-1}\left(a^{\xi_{k-1}}\right) = L\left(\left(a^{\xi_{k-1}}\right)^{-1}\right) ,$$

where $\left(a^{\xi_{k-1}}\right)^{-1}$ are the Fourier coefficients of the function $\left(\Lambda a^{\xi_{k-1}}\right)^{-1}(\cdot)$.

Here is a theorem about the almost inverse operator.

Theorem 1. *Let $a \in \mathfrak{D}_{m\times m}$ and $\Lambda a \in \mathfrak{U}_{m\times m}$. Then:*

1) $\sup\limits_{\delta>0,N\in\mathbb{N}} \|B_{\delta,N}(a)\| < \infty$;

2) *for an arbitrary $\varepsilon > 0$ there exists such a $\delta_0 > 0$, that for every $\delta < \delta_0$ there is a corresponding $N(\delta) \in \mathbb{N}$ satisfying the following condition:*

$$\sup_{N > N(\delta)} \|B_{\delta,N}(a)A_N(a) - E_N\| < \varepsilon \ ,$$

where E_N is the identity operator in$\mathbb{C}_m^N$.

The completed proof of Theorem 1 is contained in Section 6. In Section 5 we prove first a special case of this statement for so-called truncations of the function a, which have the form

$$a_q(x,y,n) = \begin{cases} a(x,y,n), & x,y \in \mathbb{R}^+,\ |n| \leqslant q\,, \\ 0, & x,y \in \mathbb{R}^+,\ |n| > q\,, \end{cases}$$

where $q \in \mathbb{N}$ is large enough. Then, in Section 6, we derive the proof for the general situation while using the approximation of the function a by its truncations a_q.

Before we formulate the main result it makes sense to explain one important fact (see [3]). Let T be a linear bounded operator acting in a Banach space K; let $\sigma(T)$ be its spectrum; let f be an analytic function on a neighbourhood of $\sigma(T)$; let U be an open set, whose boundary Γ consists of a finite number of rectifiable Jordan curves (positively oriented). Let us suppose that $U \supset \sigma(T)$ and that $U \cup \Gamma$ is contained in the domain of analyticity of the function f. Then the operator $f(T)$ is defined by

$$f(T) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda,T)d\lambda \ ,$$

where $R(\lambda,T)$ is the resolvent of the operator T. By $\operatorname{tr} T$ we denote the trace of the operator T.

Theorem 2. *Let $a \in \mathfrak{D}_{m\times m}$; let $\mathbb{G}(a)$ be the set of all $\lambda \in \mathbb{C}$, for which $(\Lambda a)(x,y,t) - \lambda I_{m\times m} \in \mathfrak{U}_{m\times m}$, where $I_{m\times m}$ is the unit matrix of order $m \times m$; let $\mathbb{F}(a) = \mathbb{C}\backslash\mathbb{G}(a)$; let D be an open subset of $\mathbb{C}$ containing the set $\mathbb{F}(a)$; let f be an analytic function on D. Then:*

1) *the spectrum of the operator $A_N(a)$ is contained in D for $N \in \mathbb{N}$ large enough;*

2) *the following limit relation holds:*

$$\Phi_f(A_N(a)) - \frac{E(N)}{N}\int_0^{\frac{N}{E(N)}} \left\{\frac{1}{2\pi m}\int_S \operatorname{tr}\ f\left[(\Lambda a)(x,x,t)\right]d\mu\right\}dx \xrightarrow[N\to\infty]{} 0\ .$$

Theorem 2 is proved in Section 7 first for a case, when f is a rational function, and then for a function f, which can be approximated by rational functions.

3. The uniform boundedness of the operator $B_{\delta,N}(a)$

This section contains the proof of the first statement of Theorem 1.

Definition 5. Let us denote by $\widetilde{\mathfrak{D}}_{m\times m}$ the subalgebra of such functions in $\mathfrak{D}_{m\times m}$, which do not depend on x and y, i.e., $\widetilde{\mathfrak{D}}_{m\times m}$ is a Banach algebra of functions $c: \mathbb{Z} \to \mathbb{C}_{m\times m}$.

Lemma 1. *Let $a \in \mathfrak{D}_{m\times m}$, $\Lambda a \in \mathfrak{U}_{m\times m}$. Let also $a_q \in \mathfrak{D}_{m\times m}$ for $q \in \mathbb{N}$ be defined as*

$$a_q(x,y,n) = \begin{cases} a(x,y,n), & x,y \in \mathbb{R}^+,\ |n| \leqslant q \\ 0, & x,y \in \mathbb{R}^+,\ |n| > q\ . \end{cases}$$

Then there is such a $q_0 \in \mathbb{N}$, that for all $q > q_0$ and $\xi \in \mathbb{R}^+$ the operator $L\left(a_q^{\xi}\right)$ is invertible and the family of operators $\left\{L^{-1}\left(a_q^{\xi}\right)\right\}_{\xi\in\mathbb{R}^+,\, q>q_0}$ is precompact.

Proof. It is obvious, that we can find such a $q_0 \in \mathbb{N}$, that the closure of the set of all values of $\det\left(\Lambda a_q^{\xi}\right)(t)$ for $q > q_0$, $\xi \in \mathbb{R}^+$, $t \in S$ does not contain zero.

The sequences $\{a_q^{\xi}(n)\}_{n\in\mathbb{Z}}$, $\xi \in \mathbb{R}^+$, $q > q_0$, form a precompact set in the algebra $\widetilde{\mathfrak{D}}_{m\times m}$. We denote its closure by $\mathcal{B}$. Clearly, $\det(\Lambda b)(t)$ for $b \in \mathcal{B}$ does not vanish.

The convolution operators $L(b)$, $b \in \mathcal{B}$, form a compact set and are invertible. In this case the set of the inverse operators $L^{-1}(b)$, $b \in \mathcal{B}$, is also compact. The statement is true, since

$$\left\{L^{-1}\left(a_q^{\xi}\right)\right\}_{\xi\in\mathbb{R}^+,\, q>q_0} \subset \left\{L^{-1}(b)\right\}_{b\in\mathcal{B}}\ . \qquad \blacksquare$$

Lemma 2. *Let the conditions of Lemma 1 be fulfilled. Then there exist q_0, $N_0 \in \mathbb{N}$ such that for all $q > q_0$ and $N > N_0$*

the operator $L_{(-\infty,1]}\left(a_q^{\frac{N}{E(N)}}\right)$ is invertible and

the family of operators $\left\{L^{-1}_{(-\infty,1]}\left(a_q^{\frac{N}{E(N)}}\right)\right\}_{N>N_0,\, q>q_0}$ is precompact.

Proof. Analogously, the sequences $\left\{a_q^{\frac{N}{E(N)}}(n)\right\}_{n\in\mathbb{Z}}$ for $q, N \in \mathbb{N}$ form a precompact set in the algebra $\widetilde{\mathfrak{D}}_{m\times m}$. By $\mathcal{B}_{q_0,N_0}$ ($q_0, N_0 \in \mathbb{N}$) we denote the closure of the set of the sequences $\left\{a_q^{\frac{N}{E(N)}}(n)\right\}_{n\in\mathbb{Z}}$ for $q > q_0$, $N > N_0$. Using the properties of the space $U_{m\times m}$ and the stability criterion for partial indices (see [16]), we can show, that q_0 and N_0 can be fixed so that every matrix-valued function $(\Lambda b)(t)$, $b \in \mathcal{B}_{q_0,N_0}$, has a nonvanishing determinant and allows the left factorization with zero partial indices.

The set of operators $L_{(-\infty,1]}(b)$, $b \in \mathcal{B}_{q_0,N_0}$, is compact and they are invertible. The set of the inverse operators $L^{-1}_{(-\infty,1]}(b)$, $b \in \mathcal{B}_{q_0,N_0}$, is also compact, and the statement is true. $\blacksquare$

Let Numb(B) be a number of elements in the set B. If B is infinite, then we put Numb$(B) = +\infty$. By the overlapping rate of a family of sets u_α, $\alpha \in \mathcal{U}$, where $\mathcal{U}$ is some index set, we mean the value $\sup_{x \in u} r(x)$, where $u = \bigcup_{\alpha \in \mathcal{U}} u_\alpha$ and $r(x) = \mathrm{Numb}\{\alpha \in \mathcal{U} |\ x \in u_\alpha\}$. In the case $u = \emptyset$ we put the overlapping rate to be equal to zero. (Obviously, if the overlapping rate is equal to 1, then the sets u_α are mutually disjoint.)

Lemma 3. *Let $\{A_k\}_{k=1}^n$ be a family of operators from $\mathrm{End}(l_m^2(U))$, $U \subset \mathbb{Z}$; $c = \max_k \|A_k\|$; let $\{v^k\}_{k=1}^n$, $\{w^k\}_{k=1}^n$ be families of subsets of U with overlapping rates 1 and r, respectively. Then the following estimate is true:*

$$\left\| \sum_{k=1}^{n} J_{v^k,U} P_{U,v^k} A_k J_{w^k,U} P_{U,w^k} \right\| \leqslant rc\ .$$

Proof. This Lemma is proved in [20] (page 27). ∎

Proposition 1. *Let the conditions of Lemma 1 be fulfilled. Then there exists such a $q_0 \in \mathbb{N}$, that $\Lambda a_q \in \mathfrak{U}_{m\times m}$ for every $q > q_0$ and*

$$\sup_{\delta>0,\, N\in\mathbb{N},\, q>q_0} \|B_{\delta,N}(a_q)\| < \infty\ .$$

Proof. This statement follows from Lemmas 1, 2 and 3. ∎

4. The local estimate of the operator $A_N(a)$

This section deals with the local approximation of the operator $A_N(a)$ by the regular convolution operator. Let $\alpha M = \{\alpha x \,/\, x \in M\}$ for some set $M \subset \mathbb{R}$ and some $\alpha \in \mathbb{R}$. For $\xi \in \mathbb{R}^+$ we denote $U_\xi(\delta) = (\xi - \delta, \xi + \delta)$, where $\delta > 0$.

Lemma 4. *Let the function $a \in \mathfrak{D}_{m\times m}$ be of the form*

$$a(x, y, n) = \sum_{r=1}^{p} f^r(x, y) a^r(n)\ ,$$

where $p \in \mathbb{N}$, $f^1, \ldots, f^p$ are complex-valued functions defined and uniformly continuous on $\mathbb{R}^+ \times \mathbb{R}^+$, $a^1, \ldots, a^p \in \widetilde{\mathfrak{D}}_{m\times m}$. Then for every $\varepsilon > 0$ there exists a $\delta_0 > 0$ such that for any $\delta > \delta_0$ and $N \in \mathbb{N}$ the following inequality holds:

$$\sup_{\xi\in\mathbb{R}^+} \left\| Q_{[1,N]}^{E(N)U_\xi(\delta)\cap[1,N]} \left(A_N(a) - L_{[1,N]}\left(a^\xi\right)\right) Q_{[1,N]}^{E(N)U_\xi(\delta)\cap[1,N]} \right\| < \varepsilon\ .$$

Proof. Let $a^r \in \widetilde{\mathfrak{D}}_{m\times m}$ $(r = 1, \dots, p)$ have the form $a^r(n) = (a^r_{ij}(n))_{i,j=1,\dots,m}$, $n \in \mathbb{Z}$. Let also $c \in \widetilde{\mathfrak{D}}_{m\times m}$, $c(n) = (c_{ij}(n))_{i,j=1,\dots,m}$, where $c_{ij}(n) = \sum_{r=1}^{p} |a^r_{ij}(n)|$. Let us fix $\varepsilon > \varepsilon_1 > 0$. Because of the uniform continuity of the functions $f^1, \dots, f^p$ there exists a $\delta_0 > 0$ such that for any $\delta > \delta_0$ the following inequality is fulfilled:

$$\sup_{\xi\in\mathbb{R}^+} \sum_{r=1}^{p} \sup_{x,y\in U_\xi(\delta)\cap\mathbb{R}^+} |f^r(x,y) - f^r(\xi,\xi)| < \frac{\varepsilon_1}{\mathcal{C}} ,$$

where $\mathcal{C} = \|c\|_{\widetilde{\mathfrak{D}}_{m\times m}}$. Then

$$\sup_{\xi\in\mathbb{R}^+} \left\| Q_{[1,N]}^{E(N)U_\xi(\delta)\cap[1,N]} \left(A_N(a) - L_{[1,N]}\left(a^\xi\right)\right) Q_{[1,N]}^{E(N)U_\xi(\delta)\cap[1,N]} \right\|^2$$

$$= \sup_{\xi\in\mathbb{R}^+} \sup_{\|X\|=1} \sum_{n\in E(N)U_\xi(\delta)\cap[1,N]\cap\mathbb{N}} \sum_{i=1}^{m} \Bigg| \sum_{k\in E(N)U_\xi(\delta)\cap[1,N]\cap\mathbb{N}}$$
$$\sum_{j=1}^{m}\sum_{r=1}^{p} \left[f^r(\xi,\xi) - f^r\left(\frac{n}{E(N)}, \frac{k}{E(N)}\right)\right] a^r_{ij}(n-k)X_j(k) \Bigg|^2$$

$$\leqslant \sup_{\|X\|=1} \sum_{n\in E(N)U_\xi(\delta)\cap[1,N]\cap\mathbb{N}} \sum_{i=1}^{m} \Bigg(\sum_{k\in E(N)U_\xi(\delta)\cap[1,N]\cap\mathbb{N}} \sum_{j=1}^{m}$$
$$\sup_{\xi\in\mathbb{R}^+} \sum_{r=1}^{p} \left| f^r(\xi,\xi) - f^r\left(\frac{n}{E(N)}, \frac{k}{E(N)}\right)\right| |a^r_{ij}(n-k)\|X_j(k)| \Bigg)^2$$

$$\leqslant \frac{\varepsilon_1^2}{\mathcal{C}^2} \sup_{\|X\|=1} \sum_{n\in\mathbb{Z}} \sum_{i=1}^{m} \left(\sum_{k=1}^{N}\sum_{j=1}^{m} |c_{ij}(n-k)||X_j(k)| \right)^2$$

$$\leqslant \frac{\varepsilon_1^2}{\mathcal{C}^2} \sup_{\|X\|=1} \sum_{n\in\mathbb{Z}} \sum_{i=1}^{m} \left\{ \sum_{k=1}^{N}\sum_{j=1}^{m} |c_{ij}(n-k)| \right\} \left\{ \sum_{k=1}^{N}\sum_{j=1}^{m} |c_{ij}(n-k)||X_j(k)|^2 \right\}$$

$$\leqslant \frac{\varepsilon_1^2}{\mathcal{C}} \sup_{\|X\|=1} \sum_{k=1}^{N}\sum_{j=1}^{m} \left\{ \sum_{n\in\mathbb{Z}}\sum_{i=1}^{m} |c_{ij}(n-k)| \right\} |X_j(k)|^2$$

$$\leqslant \varepsilon_1^2 \sup_{\|X\|=1} \sum_{k=1}^{N} \|X(k)\|^2_{\mathbb{C}_m} = \varepsilon_1^2 < \varepsilon^2. \qquad \blacksquare$$

Lemma 5. *Suppose that for the function $a \in \mathfrak{D}_{m\times m}$ there exists $q \in \mathbb{N}$ such that $a(x,y,n) = 0$ for every $|n| > q$. Then for every $\varepsilon > 0$ there exists such a $\delta_0 > 0$, that for every $\delta > \delta_0$ and $N \in \mathbb{N}$ the following inequality holds:*

$$\sup_{\xi\in\mathbb{R}^+} \left\| Q_{[1,N]}^{E(N)U_\xi(\delta)\cap[1,N]} \left(A_N(a) - L_{[1,N]}\left(a^\xi\right)\right) Q_{[1,N]}^{E(N)U_\xi(\delta)\cap[1,N]} \right\| < \varepsilon .$$

Proof. The matrix-valued function $a(x,y,n)=(a_{ij}(x,y,n))_{i,j=1,\dots,m}$ can be represented in the form

$$a(x,y,n)=\sum_{k=-q}^{q}\sum_{i,j=1}^{m}a_{ij}(x,y,k)b^{k\,i\,j}(n)\ ,$$

where $b^{k\,i\,j}(n)=0$ for $n\neq k$, and for $n=k$ it is a matrix with entry 1 on the crossing of the j-th line and j-th column, the other entries of which are equal to zero. In accordance with Lemma 4 we get the necessary result. ∎

Proposition 2. *Let $a\in\mathfrak{D}_{m\times m}$. Then for every $\varepsilon>0$ there exists such a $\delta_0>0$, that for any $\delta>\delta_0$ and $N\in\mathbb{N}$ the following inequality holds:*

$$\sup_{\xi\in\mathbb{R}^+}\left\|Q_{[1,N]}^{E(N)U_\xi(\delta)\cap[1,N]}\left(A_N(a)-L_{[1,N]}\left(a^\xi\right)\right)Q_{[1,N]}^{E(N)U_\xi(\delta)\cap[1,N]}\right\|<\varepsilon\ .$$

Proof. Let us take an arbitrary $\varepsilon>0$. For $a(x,y,n)=(a_{ij}(x,y,n))_{i,j=1,\dots,m}$ let us denote

$$R_q=\sum_{|n|>q}\sum_{i,j=1}^{m}\sup_{x,y\in\mathbb{R}^+}|a_{ij}(x,y,n)|\ .$$

We take such a $q_0\in\mathbb{N}$, that $R_{q_0}<\varepsilon/3$. Let also for $x,y\in\mathbb{R}^+$

$$a_{q_0}(x,y,n)=\begin{cases}a(x,y,n), & |n|\leqslant q_0\\ 0, & |n|>q_0\ .\end{cases}$$

By Lemma 5 there exists such a $\delta_0>0$, that for any $\delta>\delta_0$, $N\in\mathbb{N}$ the following inequality is fulfilled:

$$\sup_{\xi\in\mathbb{R}^+}\left\|Q_{[1,N]}^{E(N)U_\xi(\delta)\cap[1,N]}\left(A_N(a_{q_0})-L_{[1,N]}\left(a_{q_0}^\xi\right)\right)Q_{[1,N]}^{E(N)U_\xi(\delta)\cap[1,N]}\right\|<\frac{\varepsilon}{3}\ .$$

We have to show the correctness of the estimates

$$\sup_{\xi\in\mathbb{R}^+}\left\|Q_{[1,N]}^{E(N)U_\xi(\delta)\cap[1,N]}\left(A_N(a)-A_N(a_{q_0})\right)Q_{[1,N]}^{E(N)U_\xi(\delta)\cap[1,N]}\right\|<\frac{\varepsilon}{3}\qquad(2)$$

$$\sup_{\xi\in\mathbb{R}^+}\left\|Q_{[1,N]}^{E(N)U_\xi(\delta)\cap[1,N]}\left(L_{[1,N]}\left(a^\xi\right)-L_{[1,N]}\left(a_{q_0}^\xi\right)\right)Q_{[1,N]}^{E(N)U_\xi(\delta)\cap[1,N]}\right\|<\frac{\varepsilon}{3}\ .\qquad(3)$$

Let us prove the inequality (2). It holds

$$\begin{aligned}&\left\|Q_{[1,N]}^{E(N)U_\xi(\delta)\cap[1,N]}(A_N(a)-A_N(a_{q_0}))Q_{[1,N]}^{E(N)U_\xi(\delta)\cap[1,N]}\right\|^2\\ &=\sup_{\|X\|=1}\sum_{n\in E(N)U_\xi(\delta)\cap[1,N]\cap\mathbb{N}}\sum_{i=1}^{m}\Bigg|\sum_{k\in E(N)U_\xi(\delta)\cap[1,N]\cap\mathbb{N}}\\ &\quad\sum_{j=1}^{m}\left[a_{ij}\Big(\frac{n}{E(N)},\frac{k}{E(N)},n-k\Big)-(a_{q_0})_{ij}\Big(\frac{n}{E(N)},\frac{k}{E(N)},n-k\Big)\right]X_j(k)\Bigg|^2\end{aligned}$$

$$\leqslant \sup_{\|X\|=1} \sum_{n\in\mathbb{N}} \sum_{i=1}^{m} \Bigg(\sum_{k=1}^{N}$$

$$\sum_{j=1}^{m} \sup_{x,y\in\mathbb{R}^+} |a_{ij}(x,y,n-k) - -(a_{q_0})_{ij}(x,y,n-k)||X_j(k)| \Bigg)^2$$

$$\leqslant \sup_{\|X\|=1} \sum_{n\in\mathbb{Z}} \sum_{i=1}^{m} \Bigg(\sum_{k=1}^{N} \sum_{j=1}^{m} \sup_{x,y\in\mathbb{R}^+} |a_{ij}(x,y,n-k) - (a_{q_0})_{ij}(x,y,n-k)| \Bigg)$$

$$\times \Bigg(\sum_{k=1}^{N} \sum_{j=1}^{m} \sup_{x,y\in\mathbb{R}^+} |a_{ij}(x,y,n-k) - (a_{q_0})_{ij}(x,y,n-k)||X_j(k)|^2 \Bigg)$$

$$\leqslant R_{q_0}^2 \sup_{\|X\|=1} \sum_{k=1}^{N} \sum_{j=1}^{m} |X_j(k)|^2 < \frac{\varepsilon^2}{9} \ .$$

Thus the correctness of inequality (2) is shown. Inequality (3) can be proved similarly. ∎

5. The function $\phi[A]$

Let $\rho(F_1, F_2)$ for $F_1, F_2 \subset \mathbb{R}$ be the quantity defined by

$$\rho(F_1, F_2) = \inf_{z_1\in F_1, z_2\in F_2} |z_1 - z_2| \ ,$$

and if one of the sets F_1, F_2 is empty, then we put $\rho(F_1, F_2) = +\infty$.

Definition 6. Let $U \subset \mathbb{Z}$, $A \in \mathrm{End}(l_m^2(U))$. By $\phi[A] : [0, +\infty) \to [0, +\infty)$ we denote the function, which is defined as follows:

$$(\phi[A])(d) = \sup_{F_1, F_2 \subset U,\, \rho(F_1,F_2)\geqslant d} \left\| Q_U^{F_1} A Q_U^{F_2} \right\| \ .$$

The function $\phi[A]$ was first introduced in the paper [14] as an auxiliary notation for obtaining necessary estimates. Propositions 3 and 4, which are given below, were proved in the same paper in the scalar (multidimensional) case, but they can easily be generalized to the case of matrix-valued functions.

Proposition 3. *The function $\phi[A]$ possesses the following properties:*

1) $(\phi[A])(d) \leqslant \|A\|$*;*
2) *if* $d_1 \leqslant d_2$*, then* $(\phi[A])(d_1) \geqslant (\phi[A])(d_2)$*;*
3) $(\phi[A+B])(d) \leqslant (\phi[A])(d) + (\phi[B])(d)$*;*
4) $(\phi[AB])(d) \leqslant \|A\|(\phi[B])(\lambda d) + \|B\|(\phi[A])((1-\lambda)d)$ *for every* $\lambda \in [0, 1]$*;*

5) *if the operator B is invertible, then for any $n \in \mathbb{N}$*

$$(\phi[B^{-1}])(d) \leqslant \frac{\|B^{-1}\|^2 \|B\|}{n} + 4\left\|B^{-1}\right\|^2 (\phi[B])\Big(\frac{d}{4n-1}\Big).$$

Proposition 4. *If the families of the operators $\{A_i\}_{i\in I}$ and $\{A_i^{-1}\}_{i\in I}$, acting in the space $l^2_m(U_i)$ $(U_i \subset \mathbb{Z})$, are bounded and if $(\phi[A_i])\,(d) \to 0$ uniformly in $i \in I$ for $d \to \infty$, then $\left(\phi[A_i^{-1}]\right)(d) \to 0$ uniformly in $i \in I$ for $d \to \infty$.*

Proof. This statement follows from the property 5) of Proposition 3. ∎

Proposition 5. *Let $a \in \mathfrak{D}_{m\times m}$. Then:*

1) $(\phi[A_N(a)])\,(d) \xrightarrow[d\to\infty]{} 0$ *uniformly in $N \in \mathbb{N}$;*

2) $\left(\phi\left[L_U\left(a^\xi\right)\right]\right)(d) \xrightarrow[d\to\infty]{} 0$ *uniformly in $U \subset \mathbb{R}$ and $\xi \in \mathbb{R}^+$.*

Proof. Let us prove the first statement. Let $a \in \mathfrak{D}_{m\times m}$ have the form $a(x,y,n) = (a_{ij}(x,y,n))_{i,j=1,\dots,m}$ and $c \in \widetilde{\mathfrak{D}}_{m\times m}$ such that $c(n) = (c_{ij}(n))_{i,j=1,\dots,m}$, where $c_{ij}(n) = \sup_{x,y\in\mathbb{R}^+} |a_{ij}(x,y,n)|$ for $n \in \mathbb{Z}$. Let also F_1, $F_2 \subset ([1,N]\cap\mathbb{N})$ such that $\rho(F_1,F_2) \geqslant d$. If one of this sets is empty, then $\phi[A_N(a)](d) = 0$. Otherwise

$$\begin{aligned}
&\phi[A_N(a)](d)\\
&\quad= \left\|Q^{F_1}_{[1,N]} A_N(a) Q^{F_2}_{[1,N]}\right\|\\
&\quad= \sup_{\|X\|=1} \sum_{n\in F_1}\sum_{i=1}^m \left|\sum_{k\in F_2}\sum_{j=1}^m a_{ij}\Big(\frac{n}{E(N)}, \frac{k}{E(N)}, n-k\Big) X_j(k)\right|^2\\
&\quad\leqslant \sup_{\|X\|=1} \sum_{n\in F_1}\sum_{i=1}^m \left(\sum_{k\in F_2}\sum_{j=1}^m c_{ij}(n-k)|X_j(k)|\right)^2\\
&\quad\leqslant \sup_{\|X\|=1} \sum_{n\in F_1}\sum_{i=1}^m \left(\sum_{k\in F_2}\sum_{j=1}^m c_{ij}(n-k)\right)\left(\sum_{k\in F_2}\sum_{j=1}^m c_{ij}(n-k)|X_j(k)|^2\right)\\
&\quad\leqslant R_d^2 \sup_{\|X\|=1} \sum_{k=1}\sum_{j=1}^m |X_j(k)|^2 \leqslant R_d^2\,,
\end{aligned}$$

where $R_d = \sum_{|n|\geqslant d}\sum_{i,j=1}^m c_{ij}(n) \to 0$ for $d \to \infty$, and the first statement is proved. The second statement can be proved analogously. ∎

The correctness of the following important statement follows from Propositions 4, 5 and Lemmas 1, 2.

Proposition 6. *Let $a \in \mathfrak{D}_{m\times m}$ and $\Lambda a \in \mathfrak{U}_{m\times m}$. Then:*

1) $\left(\phi\left[L^{-1}\left(a^\xi\right)\right]\right)(d) \xrightarrow[d\to\infty]{} 0$*; uniformly in $\xi \in \mathbb{R}^+$*

2) $\left(\phi\left[L^{-1}_{(-\infty,1]}\left(a^{\frac{N}{E(N)}}\right)\right]\right)(d) \xrightarrow[d\to\infty]{} 0$ *uniformly in $N \in \mathbb{N}$.*

6. A special case of Theorem 1

At the beginning of this section, the following remark should be made. Let $\tau_{-N} \in \mathrm{Hom}\big(l^2_m((-\infty, N] \cap\mathbb{Z}), l^2_m((-\infty,1] \cap \mathbb{Z})\big), N \in \mathbb{N}$, be a shift operator acting like

$$(\tau_{-N}X)(n) = X(n+N-1) \ .$$

Obviously

$$L_{(-\infty,N]}\big(a^{\frac{N}{E(N)}}\big) = (\tau_{-N})^{-1}L_{(-\infty,1]}\big(a^{\frac{N}{E(N)}}\big)\tau_{-N} \ ,$$

and if $\Lambda a \in \mathfrak{U}_{m\times m}$, then there exists such an $N_0 \in \mathbb{N}$, that for every $N > N_0$ the operator $L_{(-\infty,N]}\big(a^{\frac{N}{E(N)}}\big)$ is invertible and

$$L^{-1}_{(-\infty,N]}\big(a^{\frac{N}{E(N)}}\big) = (\tau_{-N})^{-1}L^{-1}_{(-\infty,1]}\big(a^{\frac{N}{E(N)}}\big)\tau_{-N} \ .$$

Therefore the norms of the operators $L^{-1}_{(-\infty,1]}\big(a^{\frac{N}{E(N)}}\big)$ and $L^{-1}_{(-\infty,N]}\big(a^{\frac{N}{E(N)}}\big)$ for $N > N_0$ are equal.

The following result is the special case of Theorem 1.

Proposition 7. *Let $a \in \mathfrak{D}_{m\times m}$ and suppose there is a $q \in \mathbb{N}$ such that $a(x,y,n) = 0$ for any $|n| > q$. Let, moreover, $\Lambda a \in \mathfrak{U}_{m\times m}$. Then for every $\varepsilon > 0$ there exists such a $\delta_0 > 0$, that for every $\delta < \delta_0$ there is an $N(\delta) \in \mathbb{N}$ satisfying the following condition:*

$$\sup_{N>N(\delta)} \|B_{\delta,N}(a)A_N(a) - E_N\| < \varepsilon \ .$$

Proof. Let us take an arbitrary $\varepsilon > 0$. By Lemma 2 we can find such an $N_0 \in \mathbb{N}$, that the operator $L_{(-\infty,1]}\big(a^{\frac{N}{E(N)}}\big)$ is invertible for each $N > N_0$ and the family of operators $\big\{L^{-1}_{(-\infty,1]}\big(a^{\frac{N}{E(N)}}\big)\big\}_{N>N_0}$ is precompact. In accordance with Proposition 2, the number $\delta_0 > 0$ can be chosen so that for any $\delta < \delta_0$, $\xi \in \mathbb{R}^+$ and $N \in \mathbb{N}$ the following estimate holds:

$$\left\| Q^{E(N)U_\xi(3\delta)\cap[1,N]}_{[1,N]} \left(A_N(a) - L_{[1,N]}\left(a^\xi\right)\right) Q^{E(N)U_\xi(3\delta)\cap[1,N]}_{[1,N]} \right\| \leqslant \frac{\varepsilon}{18\mathcal{A}'} \ , \qquad (4)$$

where

$$\mathcal{A}' = \max\left\{ \left\| L^{-1}_{[1,+\infty)}\left(a^0\right)\right\|, \sup_{\xi\in\mathbb{R}^+} \left\|L^{-1}\left(a^\xi\right)\right\|, \sup_{N>N_0}\left\|L^{-1}_{(-\infty,1]}\big(a^{\frac{N}{E(N)}}\big)\right\| \right\} .$$

Now we fix some $\delta < \delta_0$ and such an $N' \in \mathbb{N}$, that every $N > N'$ satisfies the inequality $N > \max\{3\delta E(N), N_0\}$. Let us consider three families of subsets of $\mathbb{R}^+$:

1) $\{u_k\}_{k\in\mathbb{N}}$, where $u_k = [(k-1)\delta, k\delta) \cap (0,+\infty)$;

2) $\{v_k\}_{k\in\mathbb{N}}$, where $v_k = [(k-2)\delta, (k+1)\delta) \cap (0, +\infty)$ (obviously, the overlapping rate of these sets is equal to 5);
3) $\{w_k\}_{k\in\mathbb{N}}$, where $w_k = [(k-3)\delta, (k+2)\delta) \cap (0, +\infty)$ (the overlapping rate is equal to 9).

It is useful to note that $u_k \subset v_k \subset w_k$. Let us denote for convenience $\mathcal{A} = \|a\|_{\mathfrak{D}_{m\times m}}$.

We have to consider the following two products $\mathfrak{I}_N$ and $\mathfrak{I}'_N$ as $N > N'$:

$$\begin{aligned}\mathfrak{I}_N &= B_{\delta,N}(a)A_N(a)\\ &= J_{E(N)u_1,[1,N]}P_{[1,+\infty),E(N)u_1}L^{-1}_{[1,+\infty)}\left(a^0\right)J_{E(N)v_1,[1,+\infty)}P_{[1,N],E(N)v_1}A_N(a)\\ &\quad+\sum_{k=2}^{\eta(N)} J_{E(N)u_k,[1,N]}P_{\mathbb{Z},E(N)u_k}L^{-1}\left(a^{\xi_{k-1}}\right)J_{E(N)v_k,\mathbb{Z}}P_{[1,N],E(N)v_k}A_N(a)\\ &\quad+J_{[E(N)\xi_{\eta(N)},N],[1,N]}P_{(-\infty,N],[E(N)\xi_{\eta(N)},N]}L^{-1}_{(-\infty,N]}\left(a^{\frac{N}{E(N)}}\right)\\ &\quad\times J_{[E(N)\xi_{\eta(N)-1},N],(-\infty,N]}P_{[1,N],[E(N)\xi_{\eta(N)-1},N]}A_N(a)\end{aligned}$$

and

$$\begin{aligned}\mathfrak{I}'_N &= J_{E(N)u_1,[1,N]}P_{[1,+\infty),E(N)u_1}L^{-1}_{[1,+\infty)}\left(a^0\right)J_{E(N)v_1,[1,+\infty)}P_{[1,N],E(N)v_1}L_{[1,N]}\left(a^0\right)\\ &\quad+\sum_{k=2}^{\eta(N)} J_{E(N)u_k,[1,N]}P_{\mathbb{Z},E(N)u_k}L^{-1}\left(a^{\xi_{k-1}}\right)J_{E(N)v_k,\mathbb{Z}}\\ &\quad\times P_{[1,N],E(N)v_k}L_{[1,N]}\left(a^{\xi_{k-1}}\right)Q^{E(N)w_k}_{[1,N]}\\ &\quad+J_{[E(N)\xi_{\eta(N)},N],[1,N]}P_{(-\infty,N],[E(N)\xi_{\eta(N)},N]}L^{-1}_{(-\infty,N]}\left(a^{\frac{N}{E(N)}}\right)\\ &\quad\times J_{[E(N)\xi_{\eta(N)-1},N],(-\infty,N]}P_{[1,N],[E(N)\xi_{\eta(N)-1},N]}L_{[1,N]}\left(a^{\frac{N}{E(N)}}\right).\end{aligned}$$

Let us denote $\alpha_1 = \|\mathfrak{I}_N - \mathfrak{I}'_N\|$, $\alpha_2 = \|\mathfrak{I}'_N - E_N\|$.

`Case 1:` We consider first α_2 and get

$$\begin{aligned}\mathfrak{I}'_N &= J_{E(N)u_1,[1,N]}P_{[1,+\infty),E(N)u_1}L^{-1}_{[1,+\infty)}\left(a^0\right)Q^{E(N)v_1}_{[1,+\infty)}L_{[1,+\infty)}\left(a^0\right)J_{[1,N],[1,+\infty)}\\ &\quad+\sum_{k=2}^{\eta(N)} J_{E(N)u_k,[1,N]}P_{\mathbb{Z},E(N)u_k}L^{-1}\left(a^{\xi_{k-1}}\right)\\ &\quad\times Q^{E(N)v_k}_{\mathbb{Z}}L\left(a^{\xi_{k-1}}\right)J_{E(N)w_k,\mathbb{Z}}P_{[1,N],E(N)w_k}\\ &\quad+J_{[E(N)\xi_{\eta(N)},N],[1,N]}P_{(-\infty,N],[E(N)\xi_{\eta(N)},N]}L^{-1}_{(-\infty,N]}\left(a^{\frac{N}{E(N)}}\right)\\ &\quad\times Q^{[E(N)\xi_{\eta(N)-1},N]}_{(-\infty,N]}L_{(-\infty,N]}\left(a^{\frac{N}{E(N)}}\right)J_{[1,N],(-\infty,N]}\end{aligned}$$

and hence

$$\begin{aligned}\mathfrak{I}'_N = E_N &- J_{E(N)u_1,[1,N]} P_{[1,+\infty),E(N)u_1} L^{-1}_{[1,+\infty)}\left(a^0\right) \\ &\times Q^{[1,+\infty)\setminus E(N)v_1}_{[1,+\infty)} L_{[1,+\infty)}\left(a^0\right) J_{[1,N],[1,+\infty)} \\ &- \sum_{k=2}^{\eta(N)} J_{E(N)u_k,[1,N]} P_{\mathbb{Z},E(N)u_k} L^{-1}\left(a^{\xi_{k-1}}\right) \\ &\times Q^{\mathbb{Z}\setminus E(N)v_k}_{\mathbb{Z}} L\left(a^{\xi_{k-1}}\right) J_{E(N)w_k,\mathbb{Z}} P_{[1,N],E(N)w_k} \\ &- J_{[E(N)\xi_{\eta(N)},N],[1,N]} P_{(-\infty,N],[E(N)\xi_{\eta(N)},N]} L^{-1}_{(-\infty,N]}\left(a^{\frac{N}{E(N)}}\right) \\ &\times Q^{(-\infty,N]\setminus[E(N)\xi_{\eta(N)-1},N]}_{(-\infty,N]} L_{(-\infty,N]}\left(a^{\frac{N}{E(N)}}\right) J_{[1,N],(-\infty,N]}\ .\end{aligned}$$

Applying Lemma 3 (to the middle items) we get the following estimate for α_2:

$$\begin{aligned}\alpha_2 \leqslant \mathcal{A}&\left(\phi\left[L^{-1}_{[1,+\infty)}\left(a^0\right)\right]\right)\Big(\rho\left(E(N)u_1,[1,+\infty)\setminus E(N)v_1\right)\Big) \\ &+ 9\mathcal{A}\max_{2\leqslant k\leqslant\eta(N)}\left(\phi\left[L^{-1}\left(a^{\xi_{k-1}}\right)\right]\right)\Big(\rho\left(E(N)u_k,\mathbb{Z}\setminus E(N)v_k\right)\Big) \\ &+ \mathcal{A}\left(\phi\left[L^{-1}_{(-\infty,N]}\left(a^{\frac{N}{E(N)}}\right)\right]\right)\Big(\rho\left([E(N)\xi_{\eta(N)},N],(-\infty,E(N)\xi_{\eta(N)-1})\right)\Big)\ .\end{aligned}$$

It is clear, that

$$\rho\left(E(N)u_1,[1,+\infty)\setminus E(N)v_1\right) \xrightarrow[N\to\infty]{} \infty$$
$$\rho\left(E(N)u_k,\mathbb{Z}\setminus E(N)v_k\right) \xrightarrow[N\to\infty]{} \infty \qquad (k=2,\dots,\eta(N))$$
$$\rho\left([E(N)\xi_{\eta(N)},N],(-\infty,E(N)\xi_{\eta(N)-1})\right) \xrightarrow[N\to\infty]{} \infty\ .$$

Hence, in accordance with Proposition 6 there is such an $N'' \in \mathbb{N}$ ($> N'$), that for every $N > N''$ the inequality $\alpha_2 < \varepsilon/2$ holds.

`Case 2`: Let us consider now α_1. We have $\alpha_1 \leqslant \alpha_1' + \alpha_1''$, where

$$\begin{aligned}\alpha_1' = \Big\| & J_{E(N)u_1,[1,N]} P_{[1,+\infty),E(N)u_1} L^{-1}_{[1,+\infty)}\left(a^0\right) J_{[1,N],[1,+\infty)} \\ &\times Q^{E(N)v_1}_{[1,N]}\Big(A_N(a) - L_{[1,N]}\left(a^0\right)\Big) Q^{E(N)w_1}_{[1,N]} \\ &+ \sum_{k=2}^{\eta(N)} J_{E(N)u_k,[1,N]} P_{\mathbb{Z},E(N)u_k} L^{-1}\left(a^{\xi_{k-1}}\right) J_{[1,N],\mathbb{Z}} \\ &\times Q^{E(N)v_k}_{[1,N]}\Big(A_N(a) - L_{[1,N]}\left(a^{\xi_{k-1}}\right)\Big) Q^{E(N)w_k}_{[1,N]} \\ &+ J_{[E(N)\xi_{\eta(N)},N],[1,N]} P_{(-\infty,N],[E(N)\xi_{\eta(N)},N]} L^{-1}_{(-\infty,N]}\left(a^{\frac{N}{E(N)}}\right) J_{[1,N],(-\infty,N]} \\ &\times Q^{[E(N)\xi_{\eta(N)-1},N]}_{[1,N]}\Big(A_N(a) - L_{[1,N]}\left(a^{\frac{N}{E(N)}}\right)\Big) Q^{[E(N)\xi_{\eta(N)-2},N]}_{[1,N]}\Big\|\end{aligned}$$

and

$$\alpha_1'' = \Big\| J_{E(N)u_1,[1,N]} P_{[1,+\infty),E(N)u_1} L^{-1}_{[1,+\infty)}\left(a^0\right) J_{[1,N],[1,+\infty)} \\ \times Q^{E(N)v_1}_{[1,N]} \Big(A_N(a) - L_{[1,N]}\left(a^0\right) \Big) Q^{[1,N]\setminus E(N)w_1}_{[1,N]} \\ + \sum_{k=2}^{\eta(N)} J_{E(N)u_k,[1,N]} P_{\mathbb{Z},E(N)u_k} L^{-1}\left(a^{\xi_{k-1}}\right) J_{[1,N],\mathbb{Z}} Q^{E(N)v_k}_{[1,N]} A_N(a) Q^{[1,N]\setminus E(N)w_k}_{[1,N]} \\ + J_{[E(N)\xi_{\eta(N)},N],[1,N]} P_{(-\infty,N],[E(N)\xi_{\eta(N)},N]} L^{-1}_{(-\infty,N]}\big(a^{\frac{N}{E(N)}}\big) J_{[1,N],(-\infty,N]} \\ \times Q^{[E(N)\xi_{\eta(N)-1},N]}_{[1,N]} \Big(A_N(a) - L_{[1,N]}\big(a^{\frac{N}{E(N)}}\big) \Big) Q^{[1,N]\setminus[E(N)\xi_{\eta(N)-2},N]}_{[1,N]} \Big\| \,.$$

With the help of inequality (4) and Lemma 3, we get $\alpha_1' < \varepsilon/2$. Further the following estimate holds:

$$\alpha_1'' \leqslant \mathcal{A}'\big(\phi\left[A_N(a) - L_{[1,N]}\left(a^0\right)\right]\big)\big(\rho(E(N)v_1, [1,N]\setminus E(N)w_1)\big) \\ + \mathcal{A}' \sum_{k=2}^{\eta(N)} \big(\phi\left[A_N(a)\right]\big)\big(\rho(E(N)v_k, [1,N]\setminus E(N)w_k)\big) \\ + \mathcal{A}'\Big(\phi\big[A_N(a) - L_{[1,N]}\big(a^{\frac{N}{E(N)}}\big)\big]\Big)\Big(\rho([E(N)\xi_{\eta(N)-1}, N], [1, E(N)\xi_{\eta(N)-2}))\Big) \,.$$

Since $a(x,y,n) = 0$ for $|n| > q$, then there exists such an $N(\delta) \in \mathbb{N}$ ($> N''$), that $\alpha_1'' = 0$ for $N > N(\delta)$.

Thus, $\|B_{\delta,N}(a)A_N(a) - E_N\| < \varepsilon$ for every $N > N(\delta)$, and the statement is proved. ∎

7. Proof of Theorem 1

In the previous section we have proved Proposition 7, the special case of the Theorem 1. Now we can prove Theorem 1 applying this auxiliary result.

Proof. Let $\varepsilon > 0$. Let us denote for $q \in \mathbb{N}$ and $x, y \in \mathbb{R}^+$

$$a_q(x,y,n) = \begin{cases} a(x,y,n), & |n| \leqslant q \\ 0, & |n| > q\,. \end{cases}$$

We can fix now such a $q_0 \in \mathbb{N}$, that $\Lambda a_q \in \mathfrak{U}_{m\times m}$ for every $q > q_0$. Let $B = \sup_{\delta>0, N\in\mathbb{N}, q>q_0} \|B_{\delta,N}(a_q)\|$ (this quantity is finite by Proposition 1) and $A = \|a\|_{\mathfrak{D}_{m\times m}}$. It is easy to show, that $q \in \mathbb{N}(> q_0)$ and $N_0 \in \mathbb{N}$ can be fixed such that for all $N > N_0$ and $\xi \in \mathbb{R}^+$ the following inequalities hold:

$$\|A_N(a_q) - A_N(a)\| < \frac{\varepsilon}{3B}$$

$$\left\| L^{-1}_{[1,+\infty)}\left(a^0_q\right) - L^{-1}_{[1,+\infty)}\left(a^0\right) \right\| < \frac{\varepsilon}{15A} \tag{5}$$

$$\left\| L^{-1}\left(a^\xi_q\right) - L^{-1}\left(a^\xi\right) \right\| < \frac{\varepsilon}{15A} \tag{6}$$

$$\left\| L^{-1}_{(-\infty,N]}\left(a_q^{\frac{N}{E(N)}}\right) - L^{-1}_{(-\infty,N]}\left(a^{\frac{N}{E(N)}}\right) \right\| < \frac{\varepsilon}{15A}\ . \tag{7}$$

In accordance with Proposition 7, there is such a $\delta_0 > 0$, that for every $\delta < \delta_0$ one can select $N(\delta) \in \mathbb{N}(> N_0)$ satisfying the condition:

$$\sup_{N > N(\delta)} \|B_{\delta,N}(a_q)A_N(a_q) - E_N\| < \frac{\varepsilon}{3}\ .$$

In consequence of Lemma 3 and inequalities (5), (6), (7) with q as above, we get $\|B_{\delta,N}(a_q) - B_{\delta,N}(a)\| < \varepsilon/3A$. Finally we have:

$$\begin{aligned} \|B_{\delta,N}(a)A_N(a) - E_N\| \leqslant{}& \|B_{\delta,N}(a)A_N(a) - B_{\delta,N}(a_q)A_N(a)\| \\ &+ \|B_{\delta,N}(a_q)A_N(a) - B_{\delta,N}(a_q)A_N(a_q)\| \\ &+ \|B_{\delta,N}(a_q)A_N(a_q) - E_N\|. \end{aligned}$$

It is clear, that the inequality $\|B_{\delta,N}(a)A_N(a) - E_N\| < \varepsilon$ is fulfilled for every $N > N(\delta)$. ∎

It can be helpful for the subsequent part of this paper to remind the definition of the sets $\mathbb{G}(a)$ and $\mathbb{F}(a)$. For $a \in \mathfrak{D}_{m\times m}$ we denote by $\mathbb{G}(a)$ the set of all $\lambda \in \mathbb{C}$, for which $(\Lambda a)(x, y, t) - \lambda I_{m\times m} \in \mathfrak{U}_{m\times m}$, where $I_{m\times m}$ is the unit matrix of order $m \times m$. We set $\mathbb{F}(a) = \mathbb{C}\backslash\mathbb{G}(a)$.

The next result follows from Theorem 1 and Proposition 1, which have been already proved.

Corollary 1. *If $a \in \mathfrak{D}_{m\times m}$ and $\Lambda a \in \mathfrak{U}_{m\times m}$, then there exists such an $N_0 \in \mathbb{N}$, that for all $N > N_0$ the operator $A_N(a)$ is invertible.*

Let us denote the unity in $\mathfrak{D}_{m\times m}$ by the symbol e. It is useful to note the fact, that $A_N(a - \lambda e) = A_N(a) - \lambda E_N$, where E_N is the identity operator in $\mathbb{C}^N_m$, $\lambda \in \mathbb{C}$.

Corollary 2. *If $a \in \mathfrak{D}_{m\times m}$ and $\lambda \in \mathbb{G}(a)$, then there exists an $N_0 \in \mathbb{N}$ such that for all $N > N_0$ the operator $A_N(a - \lambda e)$ is invertible.*

Corollary 3. *Let $a \in \mathfrak{D}_{m\times m}$, $\lambda \in \mathbb{G}(a)$. Then for an arbitrary $\varepsilon > 0$ there is a $\delta_0 > 0$ such that for each $\delta < \delta_0$ one can find an $N(\delta) \in \mathbb{N}$ satisfying the condition*

$$\sup_{N > N(\delta)} \|B_{\delta,N}(a - \lambda e)A_N(a - \lambda e) - E_N\| < \varepsilon\ .$$

The following Corollary 4 can be obtained from the Corollaries 2 and 3.

Corollary 4. *Let* $a \in \mathfrak{D}_{m\times m}$, $\lambda \in \mathbb{G}(a)$. *Then for every* $\varepsilon > 0$ *there is such a* $\delta_0 > 0$, *that for each* $\delta < \delta_0$ *one can find an* $N(\delta) \in \mathbb{N}$ *satisfying the condition*

$$\sup_{N>N(\delta)} \|B_{\delta,N}(a-\lambda e) - A_N^{-1}(a-\lambda e)\| < \varepsilon \ .$$

8. Proof of Theorem 2

In this section we prove Theorem 2 firstly for the case when f is a rational function. Then we prove the general case applying the well-known statement about the approximation of an analytic function by rational functions.

Proposition 8. *Let* $a \in \mathfrak{D}_{m\times m}$, $\Lambda a \in \mathfrak{U}_{m\times m}$. *Moreover, let the function* f *be defined on* $\mathbb{C}\backslash\{0\}$ *by* $f(z) = z^{-1}$. *Then the following limit relation holds:*

$$\Phi_f(A_N(a)) - \frac{E(N)}{N}\int_0^{\frac{N}{E(N)}} \left\{\frac{1}{2\pi m}\int_S \operatorname{tr} f\left[(\Lambda a)(x,x,t)\right] d\mu\right\} dx \xrightarrow[N\to\infty]{} 0 \ .$$

Remark. The quantity $\Phi_1(A)$ for any operator $A \in \operatorname{End}(\mathbb{C}_m^N)$ is the matrix trace of operator A divided by mN.

Proof. Let us denote for brevity

$$\mathfrak{T}_N = \frac{E(N)}{N}\int_0^{\frac{N}{E(N)}} \left\{\frac{1}{2\pi m}\int_S \operatorname{tr} f\left[(\Lambda a)(x,x,t)\right] d\mu\right\} dx \ .$$

Here we also use the notation introduced in the proof of Proposition 7. Let us fix some $\varepsilon > 0$. We have to show that there exists such an $N'' \in \mathbb{N}$, that for every $N > N''$ the inequality $|\Phi_f(A_N(a)) - \mathfrak{T}_N| < \varepsilon$ holds. It is clear, that $\Phi_f(A_N(a)) = \Phi_1\left(A_N^{-1}(a)\right)$.

Let us consider the almost inverse operator $B_{\delta,N}(a)$ (for $N \in \mathbb{N}$ large enough). As a consequence of Theorem 1 there is such a $\delta_0 > 0$, that for each $\delta < \delta_0$ one can choose $N(\delta) \in \mathbb{N}$ satisfying

$$\sup_{N>N(\delta)} \left|\Phi_1(B_{\delta,N}(a)) - \Phi_1\left(A_N^{-1}(a)\right)\right| < \frac{\varepsilon}{2} \ . \tag{8}$$

The integral $\int_S \operatorname{tr} f\left[(\Lambda a)(x,x,t)\right] d\mu$ is uniformly continuous as a function of x. Thus, one can fix $\delta < \delta_0$ and $N(\delta) \in \mathbb{N}$ such that inequality (8) and the condition

$$\left|\int_S \operatorname{tr} f\left[(\Lambda a)(x',x',t)\right] d\mu - \int_S \operatorname{tr} f\left[(\Lambda a)(x'',x'',t)\right] d\mu\right| < \frac{\varepsilon}{4} \ , \tag{9}$$

are fulfilled for any $x', x'' \in \mathbb{R}^+$, $|x' - x''| \leqslant \delta$. Let us estimate the term $|\Phi_1(B_{\delta,N}(a)) - \mathfrak{T}_N|$. It holds

$$
\begin{aligned}
&|\Phi_1(B_{\delta,N}(a)) - \mathfrak{T}_N| \\
&\quad = \Big| \Phi_1 \left(J_{E(N)u_1,[1,N]} P_{[1,+\infty),E(N)u_1} L^{-1}_{[1,+\infty)} \left(a^0\right) J_{E(N)v_1,[1,+\infty)} P_{[1,N],E(N)v_1} \right) \\
&\qquad + \sum_{k=2}^{\eta(N)} \Phi_1 \Big(J_{E(N)u_k,[1,N]} P_{\mathbb{Z},E(N)u_k} L^{-1} \left(a^{\xi_{k-1}}\right) J_{E(N)v_k,\mathbb{Z}} P_{[1,N],E(N)v_k} \Big) \\
&\qquad + \Phi_1 \Big(J_{[E(N)\xi_{\eta(N)},N],[1,N]} P_{(-\infty,N],[E(N)\xi_{\eta(N)},N]} L^{-1}_{(-\infty,N]} \big(a^{\frac{N}{E(N)}}\big) \\
&\qquad \times J_{[E(N)\xi_{\eta(N)-1},N],(-\infty,N]} P_{[1,N],[E(N)\xi_{\eta(N)-1},N]} \Big) - \mathfrak{T}_N \Big| \,.
\end{aligned}
$$

It is clear, that the matrix trace of the first and the last term of the construction of the almost inverse operator is $o(N)$ as $N \to \infty$. Then there is such an $N' \in \mathbb{N}$ $(> N(\delta))$, that for $N > N'$ the following estimate holds:

$$
\begin{aligned}
&|\Phi_1(B_{\delta,N}(a)) - \mathfrak{T}_N| \\
&\quad \leqslant \Bigg| \sum_{k=2}^{\eta(N)} \Phi_1 \left(J_{E(N)u_k,[1,N]} P_{\mathbb{Z},E(N)u_k} L \left(\left(a^{\xi_{k-1}}\right)^{-1} \right) J_{E(N)v_k,\mathbb{Z}} P_{[1,N],E(N)v_k} \right) \\
&\qquad - \frac{E(N)}{N} \int_{\xi_1}^{\xi_{\eta(N)}} \left\{ \frac{1}{2\pi m} \int_S \operatorname{tr} f \left[(\Lambda a)(x,x,t) \right] d\mu \right\} dx \Bigg| + \frac{\varepsilon}{8} \,,
\end{aligned}
$$

where

$$
\begin{aligned}
&\Phi_1 \Big(J_{E(N)u_k,[1,N]} P_{\mathbb{Z},E(N)u_k} L \Big(\left(a^{\xi_{k-1}}\right)^{-1} \Big) J_{E(N)v_k,\mathbb{Z}} P_{[1,N],E(N)v_k} \Big) \\
&\qquad = \frac{\operatorname{Numb}(E(N)u_k \cap \mathbb{Z})}{mN} \, \frac{1}{2\pi} \int_S \operatorname{tr} \left[\left(\Lambda a^{\xi_{k-1}}\right)(t) \right]^{-1} d\mu \,.
\end{aligned}
$$

We note that $|\operatorname{Numb}(E(N)u_k \cap \mathbb{Z}) - E(N)(\xi_k - \xi_{k-1})| \leqslant 1$. Let us denote $K = \sup_{k \geqslant 2} \frac{1}{2\pi} \int_S \operatorname{tr} \left[\left(\Lambda a^{\xi_{k-1}}\right)(t) \right]^{-1} d\mu$. Then

$$
\begin{aligned}
&\Phi_1 \Big(J_{E(N)u_k,[1,N]} P_{\mathbb{Z},E(N)u_k} L \left(\left(a^{\xi_{k-1}}\right)^{-1} \right) J_{E(N)v_k,\mathbb{Z}} P_{[1,N],E(N)v_k} \Big) \\
&\qquad = \frac{E(N)(\xi_k - \xi_{k-1})}{mN} \frac{1}{2\pi} \int_S \operatorname{tr} \left[\left(\Lambda a^{\xi_{k-1}}\right)(t) \right]^{-1} d\mu + \alpha_k(N) \,,
\end{aligned}
$$

where $|\alpha_k(N)| \leqslant K/mN$ for every $k \geqslant 2$, and therefore $\sum_{k=2}^{\eta(N)} \alpha_k(N) \to 0$ as $N \to \infty$. Hence, there exists such an $N''(> N')$, that $\left| \sum_{k=2}^{\eta(N)} \alpha_k(N) \right| < \varepsilon/8$ for every $N > N''$. Taking into account (9), we obtain

$$\begin{aligned}
&|\Phi_1(B_{\delta,N}(a)) - \mathfrak{T}_N| \\
&\qquad \leqslant \Bigg| \sum_{k=2}^{\eta(N)} \frac{E(N)}{N} \frac{(\xi_k - \xi_{k-1})}{2\pi m} \int_S \operatorname{tr} \left[\left(\Lambda a^{\xi_{k-1}} \right)(t) \right]^{-1} d\mu \\
&\qquad\quad - \frac{E(N)}{N} \int_{\xi_1}^{\xi_{\eta(N)}} \left\{ \frac{1}{2\pi m} \int_S \operatorname{tr} f\left[(\Lambda a)(x,x,t) \right] d\mu \right\} dx \Bigg| + \frac{\varepsilon}{4} \\
&\qquad = \Bigg| \sum_{k=2}^{\eta(N)} \frac{E(N)}{N} \frac{1}{2\pi m} \int_{\xi_{k-1}}^{\xi_k} \int_S \operatorname{tr} f\left[(\Lambda a)\left(\xi_{k-1}, \xi_{k-1}, t\right) \right] d\mu dx \\
&\qquad\quad - \sum_{k=2}^{\eta(N)} \frac{E(N)}{N} \frac{1}{2\pi m} \int_{\xi_{k-1}}^{\xi_k} \int_S \operatorname{tr} f\left[(\Lambda a)(x,x,t) \right] d\mu dx \Bigg| + \frac{\varepsilon}{4} \\
&\qquad < \frac{\varepsilon}{2} \,.
\end{aligned}$$

Thus, $|\Phi_1(B_{\delta,N}(a)) - \mathfrak{T}_N| < \varepsilon/2$ for every $N > N''$. Finally we obtain, that for every $N > N''$

$$|\Phi_f(A_N(a)) - \mathfrak{T}_N| \leqslant \left|\Phi_1(B_{\delta,N}(a)) - \Phi_1\left(A_N^{-1}(a)\right)\right| + |\Phi_1(B_{\delta,N}(a)) - \mathfrak{T}_N| < \varepsilon,$$

that proves the proposition. ∎

Corollary 5. *Let $a \in \mathfrak{D}_{m\times m}$, $\lambda \in \mathbb{G}(a)$. Moreover, let the function f be defined on $\mathbb{C}\backslash\{\lambda\}$ by $f(z) = (z-\lambda)^{-1}$. Then the following limit relation holds:*

$$\Phi_f(A_N(a)) - \frac{E(N)}{N} \int_0^{\frac{N}{E(N)}} \left\{ \frac{1}{2\pi m} \int_S \operatorname{tr} f\left[(\Lambda a)(x,x,t)\right] d\mu \right\} dx \xrightarrow[N\to\infty]{} 0 \,.$$

Lemma 6. *Let M be a closed bounded subset of the complex plane $\mathbb{C}$, and let f be an analytic function defined on M. Then for each $\varepsilon > 0$ there are $r \in \mathbb{N}$, $c_j \in \mathbb{C}$ and $z_j \in \mathbb{C}\backslash M$ $(j = 1, \ldots, r)$ such that*

$$\sup_{z\in M} \left| f(z) - \sum_{j=1}^{r} c_j (z - z_j)^{-1} \right| < \varepsilon \,.$$

Proof. The lemma is a corollary e.g. from Theorem 8 in [24, page 28]. ∎

Now the proof of Theorem 2 follows.

Proof of Theorem 2. We note, that the first statement of Theorem 2 is a direct corollary of Theorem 1. We have to prove the second statement.

Let $\varepsilon > 0$. We are going to show that there is such an $N_0 \in \mathbb{N}$, that for each $N > N_0$ the following inequality is fulfilled:

$$\left|\Phi_f(A_N(a)) - \frac{E(N)}{N}\int_0^{\frac{N}{E(N)}}\left\{\frac{1}{2\pi m}\int_S \operatorname{tr} f\left[(\Lambda a)\,(x,x,t)\right]d\mu\right\}dx\right| < \varepsilon\ . \tag{10}$$

Let D_1 be such an open subset of $\mathbb{C}$, that $\mathbb{F}(a) \subset D_1$, $\overline{D_1} \subset D$, where $\overline{D_1}$ is a closure of the set D_1. Then in accordance with Lemma 6 we can find such a function $g(z) = \sum_{j=1}^{r} c_j(z-z_j)^{-1}$, where $r \in \mathbb{N}$, $c_j \in \mathbb{C}$, $z_j \in \mathbb{C}\backslash\overline{D_1}$, that

$$\sup_{z\in\overline{D_1}} |f(z) - g(z)| < \frac{\varepsilon}{3}$$

$$\sup_{x\in\mathbb{R}^+,\, t\in S} |\operatorname{tr} f[(\Lambda a)(x,x,t)] - \operatorname{tr} g[(\Lambda a)(x,x,t)]| < \frac{\varepsilon}{3}\ .$$

Hence

$$\begin{aligned}
&\left|\Phi_f(A_N(a)) - \frac{E(N)}{N}\int_0^{\frac{N}{E(N)}}\left\{\frac{1}{2\pi m}\int_S \operatorname{tr} f\left[(\Lambda a)(x,x,t)\right]d\mu\right\}dx\right| \\
&\qquad \leqslant |\Phi_f(A_N(a)) - \Phi_g(A_N(a))| \\
&\qquad\quad + \left|\Phi_g(A_N(a)) - \frac{E(N)}{N}\int_0^{\frac{N}{E(N)}}\left\{\frac{1}{2\pi m}\int_S \operatorname{tr} g\left[(\Lambda a)(x,x,t)\right]d\mu\right\}dx\right| \\
&\qquad\quad + \left|\frac{E(N)}{N}\int_0^{\frac{N}{E(N)}}\left\{\frac{1}{2\pi m}\int_S \operatorname{tr} g\left[(\Lambda a)(x,x,t)\right]d\mu\right\}dx\right. \\
&\qquad\quad \left. - \frac{E(N)}{N}\int_0^{\frac{N}{E(N)}}\left\{\frac{1}{2\pi m}\int_S \operatorname{tr} f\left[(\Lambda a)(x,x,t)\right]d\mu\right\}dx\right|\ .
\end{aligned}$$

By the Corollaries 2 and 8 there is such an $N_0 \in \mathbb{N}$, that for $N > N_0$ and each $z_j \in \mathbb{C}\backslash\overline{D_1}$ $(j = 1,\dots,r)$ the operator $A_N(a - z_j e)$ is invertible and

$$\left|\Phi_g(A_N(a)) - \frac{E(N)}{N}\int_0^{\frac{N}{E(N)}}\left\{\frac{1}{2\pi m}\int_S \operatorname{tr} g\left[(\Lambda a)(x,x,t)\right]d\mu\right\}dx\right| < \frac{\varepsilon}{3}\ .$$

Finally, for every $N > N_0$ it follows (10), and the statement is proved. ∎

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