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The Schwarz Problem for Polyanalytic Functions

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Abstract. The analytic Schwarz kernel function the real part of which is the harmonic Poisson kernel was generalized to a polyanalytic Schwarz kernel in a joint work with G. N. Hile [Rocky Mountain J. Math. 27 (1997), 669 – 706], see also H. Begehr [Singapore: World Scientific 1994]. It is here used to give some higher order Cauchy-Schwarz-Pompeiu representations and to solve the Schwarz problem for the Poisson, the inhomogeneous Bitsadze and the inhomogeneous polyanalytic equation.

Keywords: *Schwarz kernel, Poisson equation, Bitsadze equation, polyanalytic equation, Schwarz boundary value problem*

MSC 2000: 30E25, 30G20, 31A30, 35J40

1. Introduction

One of the basic boundary value problems in complex analysis is the Schwarz problem. For a given continuous real function on the unit circle $|z| = 1$ of the complex plane $\mathbb{C}$ an analytic function is sought in the unit disc $\mathbb{D} = \{|z| < 1\}$ the boundary values of the real part of which on the unit circle $\partial\mathbb{D} = \{|z| = 1\}$ coincide with the prescribed function. This problem turns out as a particular and simplest case of the Riemann-Hilbert problem posed by Riemann in his thesis [11], where an analytic function is sought in a domain D of $\mathbb{C}$ attaining a given linear combination of its real and imaginary parts on the boundary ∂D. Schwarz [12] has solved this particular problem long before the general Riemann-Hilbert problem was successfully treated [1, 9, 10, 13]. He has modified the Cauchy kernel, and the real part of his Schwarz kernel turns out to coincide with the Poisson kernel for harmonic functions [1]. Here the Schwarz problem will be solved for the inhomogeneous polyanalytic equation $\partial_{\bar{z}}^n w = f$.

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2. Cauchy-Schwarz-Pompeiu representation

Applying the Gauss theorem

$$\frac{1}{2i}\int_{\partial D} w(z)dz = \int_D w_{\bar z}(z)\,dx\,dy$$

for $w \in C^1(D;\mathbb{C}) \cap C(\bar D;\mathbb{C})$ with a regular domain $D \subset \mathbb{C}$ leads to the Cauchy-Pompeiu formula

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_D w_{\bar\zeta}(\zeta)\frac{d\xi\,d\eta}{\zeta - z}\ . \tag{1}$$

In the particular case of $D = \mathbb{D}$ the Gauss theorem also gives

$$0 = \frac{1}{2\pi i}\int_{|\zeta|=1} w(\zeta)\frac{\bar z d\zeta}{1 - \bar z\zeta} - \frac{1}{\pi}\int_{|\zeta|<1} w_{\bar\zeta}(\zeta)\frac{\bar z}{1 - \bar z\zeta}\,d\xi\,d\eta$$

for $|z| < 1$. Adding the complex conjugate of this equation to the Cauchy-Pompeiu formula for $D = \mathbb{D}$ shows

$$\begin{aligned} w(z) = {} & \frac{1}{2\pi i}\int_{|\zeta|=1} \operatorname{Re} w(\zeta)\,\frac{\zeta + z}{\zeta - z}\,\frac{d\zeta}{\zeta} + \frac{1}{2\pi}\int_{|\zeta|=1} \operatorname{Im} w(\zeta)\,\frac{d\zeta}{\zeta} \\ & - \frac{1}{\pi}\int_{|\zeta|<1}\left(\frac{w_{\bar\zeta}(\zeta)}{\zeta - z} + \frac{z\overline{w_{\bar\zeta}(\zeta)}}{1 - z\bar\zeta}\right)d\xi\,d\eta\ . \end{aligned} \tag{2}$$

Subtracting $i \operatorname{Im} w(0)$ proves

$$\begin{aligned} w(z) = {} & \frac{1}{2\pi i}\int_{|\zeta|=1} \operatorname{Re} w(\zeta)\,\frac{\zeta + z}{\zeta - z}\,\frac{d\zeta}{\zeta} + i\ \operatorname{Im}\ w(0) \\ & - \frac{1}{2\pi}\int_{|\zeta|<1}\left(\frac{w_{\bar\zeta}(\zeta)}{\zeta}\ \frac{\zeta + z}{\zeta - z} + \frac{\overline{w_{\bar\zeta}(\zeta)}}{\bar\zeta}\ \frac{1 + z\bar\zeta}{1 - z\bar\zeta}\right)d\xi\,d\eta\ . \end{aligned} \tag{3}$$

This is the Cauchy-Schwarz-Pompeiu representation. For analytic functions, where $w_{\bar z} = 0$, this formula reduces to the Cauchy-Schwarz formula

$$w(z) = \frac{1}{2\pi i}\int_{|\zeta|=1} \operatorname{Re} w(\zeta)\,\frac{\zeta + z}{\zeta - z}\,\frac{d\zeta}{\zeta} + i \operatorname{Im} w(0)\ .$$

Schwarz [12] has proved that for $\varphi \in C(\partial\mathbb{D};\mathbb{R}), c \in \mathbb{R}$, the function

$$w(z) = \frac{1}{2\pi i}\int_{|\zeta|=1} \varphi(\zeta)\,\frac{\zeta + z}{\zeta - z}\,\frac{d\zeta}{\zeta} + ic \tag{4}$$

is an analytic function in $\mathbb{D}$ satisfying $\operatorname{Re} w = \varphi$ on $\partial\mathbb{D}$, $\operatorname{Im} w(0) = c$. From (4)

$$\operatorname{Re} w(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \varphi(\zeta) \left(\frac{\zeta}{\zeta - z} + \frac{\bar{\zeta}}{\overline{\zeta - z}} - 1 \right) \frac{d\zeta}{\zeta} \tag{5}$$

is seen. The kernel function appearing here is the Poisson kernel, and (5) reflects the Poisson representation for harmonic functions. The representation (3) suggests the following result, see [1, 4, 5].

Theorem 2.1. *The Schwarz problem for the inhomogeneous Cauchy-Riemann equation in the unit disc*

$$w_{\bar{z}} = f \quad in\ \mathbb{D}, \qquad \operatorname{Re} w = \gamma \quad on\ \partial\mathbb{D}, \qquad \operatorname{Im}\ w(0) = c$$

is uniquely solvable in distributional sense for $f \in L_1(\mathbb{D}; \mathbb{C}), \gamma \in C(\partial\mathbb{D}; \mathbb{R})$, $c \in \mathbb{R}$. *The solution is given as*

$$\begin{aligned} w(z) = ic + \frac{1}{2\pi i} \int_{|\zeta|=1} \gamma(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} \\ - \frac{1}{2\pi} \int_{|\zeta|<1} \left(\frac{f(\zeta)}{\zeta} \frac{\zeta + z}{\zeta - z} + \frac{\overline{f(\zeta)}}{\bar{\zeta}} \frac{1 + z\bar{\zeta}}{1 - z\bar{\zeta}} \right) d\xi\, d\eta \ . \end{aligned} \tag{6}$$

Proof. Rewriting the Schwarz kernel as

$$\frac{\zeta + z}{\zeta - z} = \frac{2\zeta}{\zeta - z} - 1$$

it is seen that on the right-hand side of (6) all terms are analytic up to

$$Tf(z) = -\frac{1}{\pi} \int_{|\zeta|<1} f(\zeta) \frac{d\xi\, d\eta}{\zeta - z} \ .$$

This operator is the Pompeiu operator. The weak derivative of Tf with respect to $\bar{z}$ is f, see [13]. Hence (6) provides a distributional solution to the inhomogeneous Cauchy-Riemann equation $w_{\bar{z}} = f$. Moreover, the area integral for $|z| = 1$ is purely imaginary as there

$$\frac{1 + z\bar{\zeta}}{1 - z\bar{\zeta}} = \frac{\overline{z + \zeta}}{\overline{z - \zeta}} \ .$$

Thus the boundary condition is satisfied and $\operatorname{Im} w(0) = c$ is obvious. ∎

Corollary 2.2. *The Schwarz problem for the inhomogeneous anti-Cauchy-Riemann equation in the unit disc*

$$w_z = f \quad in\ \mathbb{D}, \qquad \operatorname{Re} w = \gamma \quad on\ \partial\mathbb{D}, \qquad \operatorname{Im}\ w(0) = c$$

for $f \in L_1(\mathbb{D};\mathbb{C}), \gamma \in C(\partial\mathbb{D};\mathbb{R}), c \in \mathbb{R}$, *is uniquely solvable in distributional sense through*

$$
\begin{aligned}
w(z) = ic + \frac{1}{2\pi i}\int_{|\zeta|=1} \gamma(\zeta)\,\overline{\frac{\zeta+z}{\zeta-z}}\,\frac{d\zeta}{\zeta} \\
- \frac{1}{2\pi}\int_{|\zeta|<1}\left(\frac{f(\zeta)}{\bar\zeta}\,\overline{\frac{\zeta+z}{\zeta-z}} + \frac{\overline{f(\zeta)}}{\zeta}\,\frac{1+\bar z\zeta}{1-\bar z\zeta}\right) d\xi\, d\eta\ .
\end{aligned} \tag{7}
$$

This follows from Theorem 2.1 applied to the function $\bar w$ and taking the complex conjugate of (6).

3. Poisson and inhomogeneous Bitsadze equation

There are two principally different second order differential operators available in the complex plane, the Laplace and the Bitsadze operator. As it is known, boundary value problems well-posed with respect to the Laplace operator can be ill-posed for the Bitsadze operator as, e.g., the Dirichlet problem is where the solution is prescribed at the boundary, see [8]. The Schwarz problem, however, is well-posed for both operators.

Theorem 3.1. *The Schwarz problem for the Poisson equation in the unit disc*

$$w_{z\bar z} = f \text{ in } \mathbb{D},\quad \operatorname{Re} w = \gamma_0,\quad \operatorname{Re} w_z = \gamma_1 \text{ on } \partial\mathbb{D},\quad \operatorname{Im}\, w(0) = c_0,\quad \operatorname{Im}\, w_z(0) = c_1$$

is uniquely solvable in distributional sense for $f \in L_1(\mathbb{D};\mathbb{C}), \gamma_0, \gamma_1 \in C(\partial\mathbb{D};\mathbb{R})$, $c_0, c_1 \in \mathbb{R}$. *The solution is given by*

$$
\begin{aligned}
w(z) = {} & ic_0 + ic_1(z+\bar z) - \frac{1}{2\pi i}\int_{|\zeta|=1}\gamma_0(\zeta)\overline{\frac{\zeta+z}{\zeta-z}}\,\frac{d\bar\zeta}{\bar\zeta} \\
& - \frac{1}{2\pi i}\int_{|\zeta|=1}\gamma_1(\zeta)\Big[\zeta\log(1-z\bar\zeta)^2 - \bar\zeta\log(1-\bar z\zeta)^2 + z - \bar z\Big]\frac{d\zeta}{\zeta} \\
& + \frac{1}{\pi}\int_{|\zeta|<1}\Big\{f(\zeta)\Big(\log|\zeta-z|^2 - \log(1-\bar z\zeta)\Big) - \overline{f(\zeta)}\log(1-\bar z\zeta)\Big\}\, d\xi\, d\eta \\
& - \frac{1}{\pi}\int_{|\zeta|<1}\Big\{f(\zeta)\Big(\frac{\log(1-\bar z\zeta)}{\zeta^2} + \log|\zeta|\Big) \\
& - \overline{f(\zeta)}\Big(\frac{\log(1-z\bar\zeta)}{\bar\zeta^2} + \log|\zeta|\Big)\Big\}\, d\xi\, d\eta \\
& + \frac{1}{\pi}\int_{|\zeta|<1}\Big(\frac{f(\zeta)}{\zeta} + \frac{\overline{f(\zeta)}}{\bar\zeta}\Big)\frac{z-\bar z}{2}\, d\xi\, d\eta\ .
\end{aligned} \tag{8}
$$

Proof. Decomposing the problem into the system

$$
\begin{aligned}
w_z &= \omega \text{ in } \mathbb{D}, \quad \operatorname{Re} w = \gamma_0 \text{ on } \partial\mathbb{D}, \quad \operatorname{Im} w(0) = c_0 \\
\omega_{\bar z} &= f \text{ in } \mathbb{D}, \quad \operatorname{Re} \omega = \gamma_1 \text{ on } \partial\mathbb{D}, \quad \operatorname{Im} \omega(0) = c_1
\end{aligned}
$$

and combining the solutions

$$
\begin{aligned}
w(z) = ic_0 &- \frac{1}{2\pi i}\int_{|\zeta|=1} \gamma_0 \overline{\frac{\zeta+z}{\zeta-z}}\,\frac{d\bar\zeta}{\bar\zeta} \\
&- \frac{1}{2\pi}\int_{|\zeta|<1}\left(\frac{\omega(\zeta)}{\bar\zeta}\,\overline{\frac{\zeta+z}{\zeta-z}} + \frac{\overline{\omega(\zeta)}}{\zeta}\,\frac{1+\bar z\zeta}{1-\bar z\zeta}\right) d\xi\, d\eta \\
\omega(z) = ic_1 &+ \frac{1}{2\pi i}\int_{|\zeta|=1} \gamma_1(\zeta)\frac{\zeta+z}{\zeta-z}\,\frac{d\zeta}{\zeta} \\
&- \frac{1}{2\pi}\int_{|\zeta|<1}\left(\frac{f(\zeta)}{\zeta}\,\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\bar\zeta}\,\frac{1+z\bar\zeta}{1-z\bar\zeta}\right) d\xi\, d\eta\,,
\end{aligned}
$$

the result follows. Here the relations

$$
\begin{aligned}
\frac{1}{2\pi}\int_{|\zeta|<1}\left(\frac{1}{\bar\zeta}\,\overline{\frac{\zeta+z}{\zeta-z}} - \frac{1}{\zeta}\,\frac{1+\bar z\zeta}{1-\bar z\zeta}\right) d\xi\, d\eta &= -z-\bar z \\
\frac{1}{2\pi}\int_{|\zeta|<1}\frac{\tilde\zeta+\zeta}{\tilde\zeta-\zeta}\,\overline{\frac{\zeta+z}{\zeta-z}}\,\frac{d\xi\, d\eta}{\bar\zeta} &= 2\tilde\zeta\log|\tilde\zeta-z|^2 - 2\tilde\zeta\log(1-\bar z\tilde\zeta) \\
&\quad - \tilde\zeta\log|\tilde\zeta|^2 + z \\
\frac{1}{2\pi}\int_{|\zeta|<1}\frac{\tilde\zeta+\zeta}{\overline{\tilde\zeta-\zeta}}\,\frac{1+\bar z\zeta}{1-\bar z\zeta}\,\frac{d\xi\, d\eta}{\zeta} &= -2\bar{\tilde\zeta}\log(1-\bar z\tilde\zeta) + \bar{\tilde\zeta}\log|\tilde\zeta|^2 - \bar z \\
\frac{1}{2\pi}\int_{|\zeta|<1}\frac{1+\zeta\bar{\tilde\zeta}}{1-\zeta\bar{\tilde\zeta}}\,\overline{\frac{\zeta+z}{\zeta-z}}\,\frac{d\xi\, d\eta}{\bar\zeta} &= \frac{2}{\bar{\tilde\zeta}}\log(1-z\bar{\tilde\zeta}) + z \\
\frac{1}{2\pi}\int_{|\zeta|<1}\frac{1+\bar\zeta\tilde\zeta}{1-\bar\zeta\tilde\zeta}\,\frac{1+\bar z\zeta}{1-\bar z\zeta}\,\frac{d\xi\, d\eta}{\zeta} &= -\frac{2}{\tilde\zeta}\log(1-\bar z\tilde\zeta) - \bar z
\end{aligned}
$$

are used. Instead of iterating these two solutions, verification of (8) providing a solution is simpler and done by direct calculations. However, then the unique solvability has to be shown. For this aim, let w_1 and w_2 be two solutions. Then $\omega = w_1 - w_2$ would be a harmonic function with homogeneous Schwarz data $\omega_{z\bar z} = 0$ in $\mathbb{D}$, $\operatorname{Re}\omega = 0$, $\operatorname{Re}\omega_z = 0$ on $\partial\mathbb{D}$, $\operatorname{Im}\omega(0) = 0$ and $\operatorname{Im}\omega_z(0) = 0$. As ω_z is an analytic function, say φ' in $\mathbb{D}$, then integrating $\omega_z = \varphi'$ means $\omega = \varphi + \bar\psi$, where also ψ is an analytic function in $\mathbb{D}$. Then $\operatorname{Re}\omega_z = 0$ on $\partial\mathbb{D}, \operatorname{Im}\omega_z(0) = 0$ means $\operatorname{Re}\varphi' = 0$ on $\partial\mathbb{D}$, $\operatorname{Im}\varphi'(0) = 0$. From Theorem 2.1 φ' is seen to be identically zero, i.e., φ is a constant, say a. Then from $\operatorname{Re}\omega = 0$ on

$\partial\mathbb{D}$, Im $\omega(0) = 0$ it follows Re $\psi = -$ Re a on $\partial\mathbb{D}$ and Im $\psi(0) =$ Im a. Again applying Theorem 2.1 shows $\psi(z) = -\bar{a}$ identically in $\mathbb{D}$. Thus ω is identically zero in $\mathbb{D}$. ∎

Corollary 3.2. *The Schwarz problem for the Poisson equation in the unit disc*

$$w_{z\bar{z}} = f \text{ in } \mathbb{D}, \quad \text{Re } w = \gamma_0, \quad \text{Re } w_{\bar{z}} = \gamma_1 \text{ on } \partial\mathbb{D}, \quad \text{Im } w(0) = c_0, \quad \text{Im } w_{\bar{z}}(0) = c_1$$

for $f \in L_1(\mathbb{D};\mathbb{C}), \gamma_0, \gamma_1 \in C(\partial\mathbb{D};\mathbb{R}), c_0, c_1 \in \mathbb{R}$ *is uniquely solvable in distributional sense for* $f \in L_1(\mathbb{D};\mathbb{C}), \gamma_0, \gamma_1 \in C(\partial\mathbb{D};\mathbb{R}), c_0, c_1 \in \mathbb{R}$. *The solution is given by*

$$\begin{aligned} w(z) = {} & ic_0 + ic_1(z+\bar{z}) + \frac{1}{2\pi i}\int_{|\zeta|=1} \gamma_0(\zeta)\,\frac{\zeta+z}{\zeta-z}\,\frac{d\zeta}{\zeta} \\ & + \frac{1}{2\pi i}\int_{|\zeta|=1} \gamma_1(\zeta)\left[\zeta\log(1-z\bar{\zeta})^2 - \bar{\zeta}\log(1-\bar{z}\zeta)^2 + z - \bar{z}\right]\frac{d\zeta}{\zeta} \\ & + \frac{1}{\pi}\int_{|\zeta|<1} \left\{ f(\zeta)(\log|\zeta-z|^2 - \log(1-z\bar{\zeta})) - \overline{f(\zeta)}\log(1-z\bar{\zeta})\right\} d\xi\, d\eta \\ & - \frac{1}{\pi}\int_{|\zeta|<1} \left\{ f(\zeta)\left(\frac{\log(1-z\bar{\zeta})}{\bar{\zeta}^2} + \log|\zeta|\right)\right. \\ & \left. - \overline{f[\zeta)}\left(\frac{\log(1-\bar{z}\zeta)}{\zeta^2} + \log|\zeta|\right)\right\} d\xi\, d\eta \\ & - \frac{1}{\pi}\int_{|\zeta|<1} \left(\frac{f(\zeta)}{\bar{\zeta}} + \frac{\overline{f(\zeta)}}{\zeta}\right)\frac{z-\bar{z}}{2}\, d\xi\, d\eta\,. \end{aligned} \tag{8'}$$

This dual result to Theorem 3.1 follows by interchanging the roles of z and $\bar{z}$. It can be obtained by applying (8) to $\bar{w}$ and complex conjugation. In a similar manner the Bitsadze equation can be treated.

Theorem 3.3. *The Schwarz problem for the inhomogeneous Bitsadze equation in the unit disc*

$$w_{\bar{z}\bar{z}} = f \text{ in } \mathbb{D}, \quad \text{Re } w = \gamma_0, \quad \text{Re } w_{\bar{z}} = \gamma_1 \text{ on } \partial\mathbb{D}, \quad \text{Im } w(0) = c_0, \quad \text{Im } w_{\bar{z}}(0) = c_1$$

for $f \in L_1(\mathbb{D};\mathbb{C}), \gamma_0, \gamma_1 \in C(\partial\mathbb{D};\mathbb{R}), c_0, c_1 \in \mathbb{R}$ *is uniquely solvable in distributional sense through*

$$\begin{aligned} w(z) = {} & ic_0 + ic_1(z+\bar{z}) + \frac{1}{2\pi i}\int_{|\zeta|=1} \gamma_0(\zeta)\,\frac{\zeta+z}{\zeta-z}\,\frac{d\zeta}{\zeta} \\ & - \frac{1}{2\pi i}\int_{|\zeta|=1} \gamma_1(\zeta)\,\frac{\zeta+z}{\zeta-z}\left(\zeta - z + \overline{\zeta - z}\right)\frac{d\zeta}{\zeta} \\ & + \frac{1}{2\pi}\int_{|\zeta|<1} \left(\frac{f(\zeta)}{\zeta}\,\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\bar{\zeta}}\,\frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}\right)\left(\zeta - z + \overline{\zeta - z}\right) d\xi\, d\eta\,. \end{aligned} \tag{9}$$

Proof. Rewriting the problem as the system

$$
\begin{aligned}
&w_{\bar z} = \omega \text{ in } \mathbb{D}, \quad \text{Re } w = \gamma_0 \text{ on } \partial\mathbb{D}, \quad \text{Im } w(0) = c_0 \\
&\omega_{\bar z} = f \text{ in } \mathbb{D}, \quad \text{Re } \omega = \gamma_1 \text{ on } \partial\mathbb{D}, \quad \text{Im } \omega(0) = c_1
\end{aligned}
$$

and combining the components of its solution

$$
\begin{aligned}
w(z) &= ic_0 + \frac{1}{2\pi i}\int_{|\zeta|=1} \gamma_0(\zeta)\frac{\zeta+z}{\zeta-z}\,\frac{d\zeta}{\zeta} \\
&\quad - \frac{1}{2\pi}\int_{|\zeta|<1}\left(\frac{\omega(\zeta)}{\zeta}\,\frac{\zeta+z}{\zeta-z} + \frac{\overline{\omega(\zeta)}}{\bar\zeta}\,\frac{1+z\bar\zeta}{1-z\bar\zeta}\right) d\xi\, d\eta \\
\omega(z) &= ic_1 + \frac{1}{2\pi i}\int_{|\zeta|=1} \gamma_1(\zeta)\frac{\zeta+z}{\zeta-z}\,\frac{d\zeta}{\zeta} \\
&\quad - \frac{1}{2\pi}\int_{|\zeta|<1}\left(\frac{f(\zeta)}{\zeta}\,\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\bar\zeta}\,\frac{1+z\bar\zeta}{1-z\bar\zeta}\right) d\xi\, d\eta\ ,
\end{aligned}
$$

the solution (9) is obtained. Here the relations

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{|\zeta|<1}\left(\frac{1}{\bar\zeta}\,\frac{\overline{\zeta+z}}{\overline{\zeta-z}} - \frac{1}{\zeta}\,\frac{1+\bar z\zeta}{1-\bar z\zeta}\right) d\xi\, d\eta = -z-\bar z \\
&\frac{1}{2\pi}\int_{|\zeta|<1}\frac{\tilde\zeta+\zeta}{\tilde\zeta-\zeta}\,\frac{\zeta+z}{\zeta-z}\,\frac{d\xi\, d\eta}{\zeta} = \frac{\tilde\zeta+z}{\tilde\zeta-z}\,\overline{(\tilde\zeta-z)} \\
&\frac{1}{2\pi}\int_{|\zeta|<1}\frac{\overline{\tilde\zeta+\zeta}}{\overline{\tilde\zeta-\zeta}}\,\frac{1+z\bar\zeta}{1-z\bar\zeta}\,\frac{d\xi\, d\eta}{\bar\zeta} = \frac{1+z\bar{\tilde\zeta}}{1-z\bar{\tilde\zeta}}\,(\tilde\zeta-z) \\
&\frac{1}{2\pi}\int_{|\zeta|<1}\frac{1+\zeta\bar{\tilde\zeta}}{1-\zeta\bar{\tilde\zeta}}\,\frac{\zeta+z}{\zeta-z}\,\frac{d\xi\, d\eta}{\zeta} = \frac{1+z\bar{\tilde\zeta}}{1-z\bar{\tilde\zeta}}\,\overline{(\tilde\zeta-z)} \\
&\frac{1}{2\pi}\int_{|\zeta|<1}\frac{1+\bar\zeta\tilde\zeta}{1-\bar\zeta\tilde\zeta}\,\frac{1+z\bar\zeta}{1-z\bar\zeta}\,\frac{d\xi\, d\eta}{\bar\zeta} = \tilde\zeta+z
\end{aligned}
$$

are used. The uniqueness of the solution follows from the unique solvability of the Schwarz problem for the inhomogeneous Cauchy-Riemann equation, Theorem 2.1.

The solutions to the homogeneous Bitsadze equation $\partial_{\bar z}^2 w = 0$ are called bianalytic functions. The reason is that its solutions are first order polynomials in $\bar z$ with analytic coefficients. The solution of the equation $\partial_{\bar z}^n w = 0$ are called polyanalytic, in particular n-analytic. Formula (9) suggests a generalization. ∎

Theorem 3.4. *The Schwarz problem for the inhomogeneous polyanalytic equation in the unit disc*

$$
\partial_{\bar z}^n w = f \ \textit{in}\ \mathbb{D}, \quad \text{Re } \partial_{\bar z}^\nu w = \gamma_\nu \ \textit{on}\ \partial\mathbb{D}, \quad \text{Im } \partial_{\bar z}^\nu w(0) = c_\nu \quad (0 \le \nu \le n-1)
$$

is uniquely solvable in distributional sense for $f \in L_1(\mathbb{D};\mathbb{C}), \gamma_\nu \in C(\partial\mathbb{D};\mathbb{R})$, $c_\nu \in \mathbb{R}, 0 \le \nu \le n-1$. *The solution is given by*

$$\begin{aligned} w(z) = {} & i\sum_{\nu=0}^{n-1} \frac{c_\nu}{\nu!}(z+\bar z)^\nu + \sum_{\nu=0}^{n-1} \frac{(-1)^\nu}{2\pi i \nu!} \int_{|\zeta|=1} \gamma_\nu(\zeta) \frac{\zeta+z}{\zeta-z}(\zeta - z + \overline{\zeta - z})^\nu \frac{d\zeta}{\zeta} \\ & + \frac{(-1)^n}{2\pi(n-1)!} \\ & \times \int_{|\zeta|<1} \left(\frac{f(\zeta)}{\zeta} \frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\bar\zeta} \frac{1+z\bar\zeta}{1-z\bar\zeta} \right) (\zeta - z + \overline{\zeta - z})^{n-1} \, d\xi \, d\eta \ . \end{aligned} \tag{10}$$

Proof. For $n = 1$ formula (10) is just (6). Assuming (10) holds for $(n-1)$ rather than for n the problem is rewritten as the system

$$\begin{aligned} &\partial_{\bar z}^{n-1} w = \omega \text{ in } \mathbb{D}, \quad \operatorname{Re} \partial_{\bar z}^\nu w = \gamma_\nu \text{ on } \partial\mathbb{D}, \qquad \operatorname{Im} \partial_{\bar z}^\nu w(0) = c_\nu \quad (0 \le \nu \le n-2) \\ &\omega_{\bar z} = f \text{ in } \mathbb{D}, \qquad \operatorname{Re} \omega = \gamma_{n-1} \text{ on } \partial\mathbb{D}, \qquad \operatorname{Im} \omega(0) = c_{n-1} \end{aligned}$$

with the solution

$$\begin{aligned} \omega(z) = {} & i c_{n-1} + \frac{1}{2\pi i} \int_{|\zeta|=1} \gamma_{n-1}(\zeta) \frac{\zeta+z}{\zeta-z} \frac{d\zeta}{\zeta} \\ & - \frac{1}{2\pi} \int_{|\zeta|<1} \left(\frac{f(\zeta)}{\zeta} \frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\bar\zeta} \frac{1+z\bar\zeta}{1-z\bar\zeta} \right) d\xi \, d\eta \end{aligned}$$

and

$$\begin{aligned} w(z) = {} & i\sum_{\nu=0}^{n-2} \frac{c_\nu}{\nu!}(z+\bar z)^\nu + \sum_{\nu=0}^{n-2} \frac{(-1)^\nu}{2\pi i \nu!} \int_{|\zeta|=1} \gamma_\nu(\zeta) \frac{\zeta+z}{\zeta-z} (\zeta - z + \overline{\zeta - z})^\nu \frac{d\zeta}{\zeta} \\ & + \frac{(-1)^{n-1}}{2\pi(n-2)!} \int_{|\zeta|<1} \left(\frac{\omega(\zeta)}{\zeta} \frac{\zeta+z}{\zeta-z} + \frac{\overline{\omega(\zeta)}}{\bar\zeta} \frac{1+z\bar\zeta}{1-z\bar\zeta} \right) \\ & \times (\zeta - z + \overline{\zeta - z})^{n-2} \, d\xi \, d\eta \ . \end{aligned}$$

Combining these results relation (10) follows on the basis of

$$\begin{aligned} I_1 &= \frac{1}{2\pi} \int_{|\zeta|<1} \left[\frac{\tilde\zeta + \zeta}{\tilde\zeta - \zeta} \frac{1}{\zeta} \frac{\zeta+z}{\zeta-z} + \frac{1+\tilde\zeta\bar\zeta}{1-\tilde\zeta\bar\zeta} \frac{1}{\bar\zeta} \frac{1+z\bar\zeta}{1-z\bar\zeta} \right] (\zeta - z + \overline{\zeta - z})^{n-2} \, d\xi \, d\eta \\ &= \frac{1}{n-1} \frac{\tilde\zeta + z}{\tilde\zeta - z} (\tilde\zeta - z + \overline{\tilde\zeta - z})^{n-1} \\ I_2 &= \frac{1}{2\pi} \int_{|\zeta|<1} \left[\frac{1+\zeta\bar{\tilde\zeta}}{1-\zeta\bar{\tilde\zeta}} \frac{1}{\zeta} \frac{\zeta+z}{\zeta-z} + \frac{\overline{\tilde\zeta + \zeta}}{\overline{\tilde\zeta - \zeta}} \frac{1}{\bar\zeta} \frac{1+z\bar\zeta}{1-z\bar\zeta} \right] (\zeta - z + \overline{\zeta - z})^{n-2} \, d\xi \, d\eta \\ &= \frac{1}{n-1} \frac{1+z\bar{\tilde\zeta}}{1-z\bar{\tilde\zeta}} (\tilde\zeta - z + \overline{\tilde\zeta - z})^{n-1} \ . \end{aligned}$$

These two relations follow from

$$I_1 = -\frac{1}{\pi}\int_{|\zeta|<1}\left\{\frac{\tilde\zeta + z}{\tilde\zeta - z}\left[\frac{1}{\zeta-\tilde\zeta} - \frac{\tilde\zeta}{1-\tilde\zeta\bar{\zeta}} - \frac{1}{\zeta - z} + \frac{z}{1-z\bar{\bar{\zeta}}}\right] + \frac{1}{2}\left(\frac{1}{\zeta} - \frac{1}{\bar{\bar{\zeta}}}\right)\right\}$$

$$\times\,(\zeta - z + \overline{\zeta - z})^{n-2}\,d\xi\,d\eta$$

$$= -\frac{1}{2\pi}\,\frac{\tilde\zeta + z}{\tilde\zeta - z}\int_{|\zeta|<1}\left[\frac{1}{\zeta}\,\frac{\zeta+\tilde\zeta}{\zeta-\tilde\zeta} - \frac{1}{\bar{\bar{\zeta}}}\,\frac{1+\tilde\zeta\bar{\zeta}}{1-\tilde\zeta\bar{\zeta}} - \frac{1}{\zeta}\,\frac{\zeta+z}{\zeta-z} + \frac{1}{\bar{\bar{\zeta}}}\,\frac{1+z\bar{\zeta}}{1-z\bar{\bar{\zeta}}}\right]$$

$$\times\left(\zeta - z + \overline{\zeta - z}\right)^{n-2}d\xi\,d\eta - \frac{1}{2\pi}\int_{|\zeta|<1}\left(\frac{1}{\zeta} - \frac{1}{\bar{\bar{\zeta}}}\right)\left(\zeta - z + \overline{\zeta - z}\right)^{n-2}d\xi\,d\eta$$

$$I_2 = -\frac{1}{\pi}\int_{|\zeta|<1}\left\{\frac{1+z\bar{\bar{\tilde\zeta}}}{1-z\bar{\bar{\tilde\zeta}}}\left[\frac{1}{\overline{\overline{\zeta-\tilde\zeta}}} - \frac{\bar{\bar{\tilde\zeta}}}{1-\zeta\bar{\bar{\tilde\zeta}}} - \frac{1}{\zeta - z} + \frac{z}{1-z\bar{\bar{\zeta}}}\right] - \frac{1}{2}\left(\frac{1}{\zeta} - \frac{1}{\bar{\bar{\zeta}}}\right)\right\}$$

$$\times\left(\zeta - z + \overline{\zeta - z}\right)^{n-2}d\xi\,d\eta$$

$$= -\frac{1}{2\pi}\,\frac{1+z\bar{\bar{\tilde\zeta}}}{1-z\bar{\bar{\tilde\zeta}}}\int_{|\zeta|<1}\left[\frac{1}{\bar{\bar{\zeta}}}\,\frac{\overline{\overline{\zeta+\tilde\zeta}}}{\overline{\overline{\zeta-\tilde\zeta}}} - \frac{1}{\zeta}\,\frac{1+\bar{\bar{\tilde\zeta}}\zeta}{1-\bar{\bar{\tilde\zeta}}\zeta} - \frac{1}{\zeta}\,\frac{\zeta+z}{\zeta-z} + \frac{1}{\bar{\bar{\zeta}}}\,\frac{1+z\bar{\zeta}}{1-z\bar{\bar{\zeta}}}\right]$$

$$\times\left(\zeta - z + \overline{\zeta - z}\right)^{n-2}d\xi\,d\eta - \frac{1}{2\pi}\int_{|\zeta|<1}\left(\frac{1}{\zeta} - \frac{1}{\bar{\bar{\zeta}}}\right)\left(\zeta - z + \overline{\zeta - z}\right)^{n-2}d\xi\,d\eta$$

and the next result. ∎

Lemma 3.5. *For $|z| < 1$, $|\tilde\zeta| < 1$ and $k \in \mathbb{N}_0$, it holds*

$$\frac{1}{k+1}\left(\tilde\zeta - z + \overline{\tilde\zeta - z}\right)^{k+1}$$
$$= \frac{(-1)^{k+1}}{k+1}(z + \bar z)^{k+1} \tag{11}$$
$$- \frac{1}{2\pi}\int_{|\zeta|<1}\left(\frac{1}{\zeta}\,\frac{\zeta+\tilde\zeta}{\zeta-\tilde\zeta} - \frac{1}{\bar{\bar{\zeta}}}\,\frac{1+\tilde\zeta\bar{\zeta}}{1-\tilde\zeta\bar{\zeta}}\right)\left(\zeta - z + \overline{\zeta - z}\right)^{k}d\xi\,d\eta\ .$$

Corollary 3.6. *For $|z| < 1$ and $k \in \mathbb{N}_0$, it holds*

$$\frac{1}{2\pi}\int_{|\zeta|<1}\left(\frac{1}{\zeta} - \frac{1}{\bar{\bar{\zeta}}}\right)\left(\zeta - z + \overline{\zeta - z}\right)^{k}d\xi\,d\eta = 0 \tag{12}$$

and

$$\frac{(-1)^{k+1}}{k+1}(z + \bar z)^{k+1}$$
$$= \frac{1}{2\pi}\int_{|\zeta|<1}\left(\frac{1}{\zeta}\,\frac{\zeta+z}{\zeta-z} - \frac{1}{\bar{\bar{\zeta}}}\,\frac{1+z\bar{\zeta}}{1-z\bar{\bar{\zeta}}}\right)\left(\zeta - z + \overline{\zeta - z}\right)^{k}d\xi\,d\eta\ . \tag{13}$$

Proof. For fixed $z \in \mathbb{D}$ the function $w(\tilde{\zeta}) = i(\tilde{\zeta} - z + \overline{\tilde{\zeta} - z})^{k+1}/(k+1)$ satisfies $w_{\bar{\tilde{\zeta}}}(\tilde{\zeta}) = i(\tilde{\zeta} - z + \overline{\tilde{\zeta} - z})^k$ in $\mathbb{D}$, $\mathrm{Re}\, w = 0$ on $\partial\mathbb{D}$ and $\mathrm{Im}\, w(0) = \frac{(-1)^{k+1}}{k+1}(z+\bar{z})^{k+1}$. According to Theorem 2.1 the solution to this Schwarz problem is given by (11). Inserting here $\tilde{\zeta} = 0$ gives (12), and putting $\tilde{\zeta} = z$ shows (13). ∎

Remark 3.7. The relation (13) not used in the proof of (10) can be proved independently from (11). Consider for this the function $w(z) = i(z + \bar{z})^{k+1}/(k+1)!$ satisfying the higher order Schwarz problem

$$\partial_{\bar{z}}^{k+1} w(z) = i \text{ in } \mathbb{D}$$

$$\mathrm{Re}\, \partial_{\bar{z}}^{\nu} w(z) = \mathrm{Re}\, \frac{i}{(k+1-\nu)!)}(z+\bar{z})^{k+1-\nu} = 0 \text{ on } \partial\mathbb{D},\ \mathrm{Im}\, \partial_{\bar{z}}^{\nu} w(0) = 0$$

with $0 \le \nu \le k$. Applying (10) for $n = k+1$ gives (13).

Remark 3.8. Theorem 3.4 can easily be proved by verifying (10) to be a solution to the problem. Differentiating the right-hand side of (10) with respect to $\bar{z}$ reduces it to the same expression with $n-1$ rather than n. Uniqueness of the solution to the Schwarz problem considered follows from the fact that all analytic coefficients of the polyanalytic function of order n with vanishing Schwarz data vanish identically. Formula (10) was given in the case of homogeneous Schwarz data in [4]. For a different proof of formula (10) see [5].

Besides the Schwarz problem the two other basic boundary value problems in complex analysis, the Dirichlet and the Neumann problems although ill-posed can be considered as well for polyanalytic as for polyharmonic functions. This is done in some different papers [2, 3, 6, 7].

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