

# Hardy-Littlewood Inequalities and $Q_p$ -Spaces

Guangbin Ren and Uwe Kähler

**Abstract.** We establish Hardy-Littlewood inequalities for monogenic and harmonic functions and consider their applications to the definition of  $Q_p$ -spaces in Clifford analysis.

**Keywords:** *Hardy-Littlewood inequalities, monogenic functions,  $Q_p$ -spaces*

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## 1. Introduction

In recent years a new scale of function spaces emerged from the field of complex analysis, the so-called  $Q_p$ -spaces. These spaces are defined in the following way [1]: Let  $B^2 = \{z : |z| < 1\}$  be the unit disk in  $\mathbb{C}$ ,  $\varphi_a(z) = (a - z)(1 - \bar{a}z)^{-1}$  the automorphisms which map the unit disk onto itself. Then we can define the semi-norm

$$|f|_{Q_p} = \sup_{a \in B^2} \int_{B^2} |f'(z)|^2 (1 - |\varphi_a(z)|^2)^p dx dy < \infty,$$

and we have  $Q_p = \{f \in \mathcal{H}(B^2) : |f|_{Q_p} < \infty\}$ , where  $\mathcal{H}(B^2)$  denotes the set of holomorphic functions over the unit disk. These  $Q_p$ -spaces form a scale of function spaces with the following properties:

$$\mathcal{D} \subset Q_p \subset Q_q \subset BMOA, \quad 0 < p < q < 1,$$

where  $\mathcal{D}$  is the Dirichlet space and  $BMOA$  denotes the space of all analytic  $BMO$ -functions, i.e., functions of Bounded Mean Oscillation. Moreover,

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Guangbin Ren: Department of Mathematics, University of Science and Technology of China, Hefei 230026, P. R. China and Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal; e-mail [rengb@ustc.edu.cn](mailto:rengb@ustc.edu.cn)

Uwe Kähler: Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal; e-mail [uwek@mat.ua.pt](mailto:uwek@mat.ua.pt)

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$Q_1 = BMOA$  and  $Q_p = \mathcal{B}$  for all  $p > 1$ , where  $\mathcal{B}$  denotes the complex Bloch space, i.e.,

$$\mathcal{B} = \{f \in \mathcal{H}(B^2) : \sup_{z \in B^2} (1 - |z|^2)|f'(z)| < \infty\}.$$

This scale of spaces was also generalized in different ways to higher dimensions, e.g., in [9] and [5]. Let us remark that these spaces are different to the usual weighted Bergman spaces, but several ideas and methods from the theory of weighted Bergman spaces over the unit ball can be carried over to this context.

Moreover, one of the main tools for the investigation of weighted Bergman spaces in the complex unit ball consists in the Hardy-Littlewood inequalities. Let  $f \in \mathcal{H}(B^2)$ ,  $0 < q < \infty$ . Furthermore, let  $0 < s < \infty$ ,  $-1 < p < \infty$  then we have for the integral mean

$$M_q(r, f) = \left\{ \int_{S^1} |f(r\zeta)|^q d\sigma(\zeta) \right\}^{\frac{1}{q}},$$

where  $S^1$  denotes the unit sphere, the following property:

$$f(0) + \int_0^1 M_q^s\left(\frac{\partial f}{\partial z}, r\right)(1-r)^{p+s} dr \simeq \int_0^1 M_q^s(f, r)(1-r)^p dr,$$

where  $X \simeq Y$  means that there exist constants  $C_1$  and  $C_2$ , such that  $C_1X \leq Y \leq C_2X$ . These inequalities can be generalized to higher dimensions using different canonical differential operators, mainly the gradient and the Euler operator, as a replacement for the complex derivative.

The main goal of this paper is to extend Hardy-Littlewood inequalities into Clifford analysis, not only using the above mentioned operators, but also the tangential derivative and the hypercomplex derivative. Additionally, in the cases were it is possible we would like to apply them to take a deeper look into the definition of  $Q_p$ -spaces in Clifford analysis.

## 2. Preliminaries

We shall denote by  $\mathcal{Cl}_n$  the universal real Clifford algebra generated by the vector space  $\mathbb{R}^n$  together with its orthonormal basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  endowed with the multiplication rules  $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}$ , for all  $i, j = 1, \dots, n$ . It will be a  $2^n$ -dimensional real associative algebra with basis given by  $\{1, \mathbf{e}_A = \mathbf{e}_{h_1} \cdots \mathbf{e}_{h_k}, A \subset \mathbb{N}\}$ , where  $A = \{h_1, \dots, h_k\} \subset N = \{1, \dots, n\}$  for  $1 \leq h_1 < \dots < h_k \leq n$ . Hence, each element  $x \in \mathcal{Cl}_n$  can be written as a linear combination of the elements of the basis. The particular linear combination of basic elements with equal length  $k$  is called a  $k$ -vector and we shall denote by  $[x]_k$  the  $k$ -vector part of  $x \in \mathcal{Cl}_n$ . We introduce an *involution automorphism* in the algebra

$\mathcal{Cl}_n$ , denoted *conjugation*  $x \rightarrow \bar{x}$ , defined by its action on the basis elements as  $\bar{1} = 1$  and  $\overline{(\mathbf{e}_{h_1} \cdots \mathbf{e}_{h_s})} = (-1)^{s+[s/2]} \mathbf{e}_{h_1} \cdots \mathbf{e}_{h_s}$  for all  $h_1, \dots, h_s$  satisfying  $1 \leq h_1 < \cdots < h_s \leq n$ . We define an arbitrary vector by the paravector  $z \in \mathcal{Cl}_n$  as the element  $z = x_0 + x_1 \mathbf{e}_1 + \cdots + x_n \mathbf{e}_n$ , where all  $x_i$  are real.

As in the complex case, we denote by  $\text{Sc}(z) = x_0$  the *scalar part* of the paravector  $z$  and by  $x = \text{Vec}(z) = x_1 \mathbf{e}_1 + \cdots + x_n \mathbf{e}_n$  its *vectorial part*. The *conjugate element* of  $z$  is given by  $\bar{z} = \text{Sc}(z) - \text{Vec}(z) = x_0 - x_1 \mathbf{e}_1 - \cdots - x_n \mathbf{e}_n$  and satisfies  $z\bar{z} = \bar{z}z = x_0^2 + x_1^2 + \cdots + x_n^2$ , the Euclidean norm of  $z$  considered as an element of the vectorial space  $\mathbb{R}^{n+1}$ . Moreover, each non-zero paravector  $z$  has a unique inverse  $z^{-1} = \frac{\bar{z}}{|z|^2}$ .

Furthermore, a Clifford-valued function  $w = w(z)$  has a representation  $w(z) = \sum_{A \subset N} w_A(z) \mathbf{e}_A$  with real-valued components  $w_A(z)$ . Properties such as continuity, differentiability, integrability, and so on, which are ascribed to the  $\mathcal{Cl}_n$ -valued function  $w$  have to be fulfilled by all real-valued components  $w_A$ .

For this setting we can introduce two differential operators depending on if we consider functions defined over vectors or paravectors. In the case of vectors, i.e., functions  $w : \Omega \subset \mathbb{R}^n \mapsto \mathcal{Cl}_{0,n}$  we have the usual Dirac operator

$$\partial w = \sum_{i=1}^n \mathbf{e}_i \frac{\partial w}{\partial x_i}$$

which factorizes the  $n$ -dimensional Laplace operator  $\Delta_n = -\partial\partial$ . In the case were it will be necessary we will write  $\partial_x$  to emphasize that we mean the Dirac operator with respect to the vector variable  $x$ .

The *generalized Cauchy-Riemann operator*  $D = \partial_{x_0} + \sum_{i=1}^n \mathbf{e}_i \partial_{x_i}$  factorizes the  $n + 1$ -dimensional Laplace operator  $\Delta$  in the sense that  $\Delta = \bar{D}D = D\bar{D}$ , where  $\bar{D} = \partial_{x_0} - \sum_{i=1}^n \mathbf{e}_i \partial_{x_i}$  is the *conjugate* of the generalized Cauchy-Riemann operator. Between  $D$  and  $\bar{D}$  exist the following relations:

$$Df + \bar{D}f = 2 \frac{\partial f}{\partial x_0}, \quad Df - \bar{D}f = 2 \sum_{i=1}^n \mathbf{e}_i \frac{\partial f}{\partial x_i}$$

so that we have  $\bar{D}f = 2\partial_0 f$  for monogenic functions.

A Clifford-valued function  $w(z) = \sum_{A \subset N} w_A(z) \mathbf{e}_A$  is said to be (*left-*) *monogenic* in  $\Omega$  if either  $\partial w(z) = 0$  or  $Dw(z) = 0$  for all  $z \in \Omega$  depending on the context. In the case of the generalized Cauchy-Riemann operator the term  $\frac{1}{2} \bar{D}w(z)$ , where  $w(z)$  is a given monogenic function, can be considered as the hypercomplex derivative, i.e., as generalization of the complex derivative from complex analysis, a result shown in [10].

Let us finally remark that there is a natural embedding of the Clifford algebra  $\mathcal{Cl}_n$  into the even subalgebra  $\mathcal{Cl}_{n+1}^+$ . Indeed, let  $\epsilon_0, \epsilon_1, \dots, \epsilon_n$  be the orthogonal basis for the vectorial space  $\mathbb{R}^{n+1}$  associated to the universal Clifford

algebra  $\mathcal{Cl}_{n+1}$  with multiplication rules  $\epsilon_i\epsilon_j + \epsilon_j\epsilon_i = -2\delta_{ij}$ ,  $i, j = 1, \dots, n + 1$ . Then the embedding is given by

$$\mathbf{e}_j = -\epsilon_0\epsilon_j, \quad j = 1, \dots, n; \tag{1}$$

for details we refer to [6, page 63]. Using the above embedding the Dirac operator  $\partial = \sum_{i=0}^n \epsilon_i \partial_{x_i}$  relates itself with the Cauchy-Riemann operator  $D$  by  $-\epsilon_0\partial = D$ . Based on this embedding, results regarding the analysis of the Dirac operator can be carried over to the case of the Cauchy-Riemann operator easily. For more information about these topics and general Clifford analysis we refer to [2] and [11].

Let us also fix some additional notations we need for the Hardy-Littlewood inequalities. Let  $0 < p, q \leq \infty, 0 < \alpha < \infty$ . Furthermore, let  $B = B^n$  denote the unit ball in  $\mathbb{R}^n$ ,  $S = S^{n-1}$  its surface,  $d\sigma$  the normalized surface measure on  $S^{n-1}$ . For real-valued or clifford-valued functions  $f : B^n \mapsto \mathbb{R}$  we can introduce the integral mean of  $f$  by means of spherical coordinates  $x = r\xi$ ,  $r = |x|$ ,  $\xi \in S$ ,

$$M_q(r, f) = \left\{ \int_S |f(r\xi)|^q d\sigma(\xi) \right\}^{\frac{1}{q}}, \quad 0 < q < \infty,$$

and  $M_\infty(r, f) = \sup_{\xi \in S} |f(r\xi)|$ . This integral mean gives rise to the norm

$$\|f\|_{p,q,\alpha} = \left\{ \int_B M_q^p(r, f)(1 - r^2)^{p\alpha-1} dr \right\}^{\frac{1}{p}}$$

and, in case of  $p = \infty$ ,  $\|f\|_{\infty,q,\alpha} = \sup_{0 < r < 1} (1 - r^2)^\alpha M_q(r, f)$ .

Furthermore, in addition to the gradient  $\nabla f$  we will consider the following canonical differential operators:

$$Ef = \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}, \quad T_{ij}f = \left( x_i \frac{\partial f}{\partial x_j} - x_j \frac{\partial f}{\partial x_i} \right).$$

Hereby,  $E$  represents the Euler operator and  $T = \{T_{ij}\}_{i < j}$  the tangential derivative. Additionally, we will consider the operator

$$E_s = sI + \sum_{j=1}^n x_j \frac{\partial}{\partial x_j},$$

where  $I$  denotes the identity operator. Obviously, we have  $E_0 = E$ . The operator  $E_s, s > 0$ , is invertible in the space  $C^\infty(B)$ . We will denote the inverse operator by  $I_s$ , i.e.,  $E_s I_s f = I_s E_s f = f$ . Please, note that the operator  $I_s$  has the integral representation

$$I_s f(x) = \int_0^1 f(tx) t^{s-1} dt, \quad x \in B.$$

Let us finally remark that  $C$  will always denote a constant (different from case to case). For non-negative quantities  $X$  and  $Y$  we will often write  $X \lesssim Y$  if  $X \leq CY$  for some constant  $C$ . Also, we write  $X \simeq Y$  if  $X \lesssim Y$  and  $Y \lesssim X$ .

### 3. Hardy-Littlewood inequalities for harmonic and monogenic functions

Let us denote by  $H = \{f \in \mathcal{C}^2(B) : \Delta f = 0\}$  the space of all harmonic functions over the unit ball. In the first place we have the following lemmas due to Pavlovič [12].

**Lemma 3.1.** *Let  $0 < p, q \leq \infty$ ,  $-\infty < \alpha < \infty$ . If  $f \in H$ , then*

$$\|\nabla f\|_{p,q,\alpha+1} \lesssim \|f\|_{p,q,\alpha}.$$

**Lemma 3.2.** *Let  $0 < p, q \leq \infty$ ,  $-\infty < \alpha < \infty$ , and  $s > 0$ . If  $f \in H$ , then*

$$\|I_s f\|_{p,q,\alpha} \lesssim \|f\|_{p,q,\alpha+1}.$$

**Lemma 3.3.** *Let  $0 < p, q \leq \infty$ ,  $-\infty < \alpha < \infty$ ,  $k \in \mathbb{N}$ . To each  $\varepsilon \in (0, 1)$  there corresponds a constant  $C(\varepsilon, k)$  independent of  $f \in H$  such that*

$$\sup_{B_\varepsilon(0)} \sum_{j=0}^{k-1} |\nabla^j f(x)| \leq C(\varepsilon, k) \|f\|_{p,q,\alpha},$$

where  $B_\varepsilon(0)$  denotes the ball with center 0 and radius  $\varepsilon$ .

We would like to remark that combining Lemma 3.1 and Lemma 3.3 we get

$$\sum_{j=0}^{k-1} |\nabla^j f(0)| + \|\nabla^k f\|_{p,q,\alpha+k} \lesssim \|f\|_{p,q,\alpha}.$$

As we stated in the end of the previous section it holds

$$E_s I_s f = I_s E_s f = f \quad (s > 0) \quad (2)$$

for a  $\mathcal{C}^\infty$ -function  $f$  on  $B$ . Therefore, by replacing  $f$  with  $E_s f$  in Lemma 3.2 we get

$$\|f\|_{p,q,\alpha} \lesssim \|E_s f\|_{p,q,\alpha+1}.$$

Unfortunately, at the moment this is only valid for  $s > 0$ . The problem is that the approach of Pavlovič relies heavily on the property (2) which fails for  $s = 0$ . But when  $s = 0$ , we can use the following substitution of (2): For a  $\mathcal{C}^\infty$ -function  $f$  on  $B$  with  $f(0) = 0$  we have  $E I_0 f = I_0 E f = f$ , where  $I_0 f(x) = \int_0^1 f(tx) t^{-1} dt$ .

Now, using the integral representation of  $I_0$  we get the estimate

$$|I_0 f(r\xi)| \leq C_1 \sup_{B_\varepsilon(0)} |f| + C_2 \int_0^r |f(t\xi)| dt,$$

and following the same reasoning as in Lemma 3.2 in [12] we obtain

**Proposition 3.4.** *Let  $0 < p, q \leq \infty, 0 < \alpha < \infty$ . If  $f \in H$  and  $f(0) = 0$ , then*

$$\|I_0 f\|_{p,q,\alpha} \leq C \|f\|_{p,q,\alpha+1}. \tag{3}$$

Now, let us observe that if  $f \in H$ , then  $Ef \in H$  and  $Ef(0) = 0$ . Therefore, by replacing  $f$  in (3) with  $Ef$  we obtain  $\|f\|_{p,q,\alpha} \leq C \|Ef\|_{p,q,\alpha+1}$  for  $Ef(0) = 0$ . If we substitute  $f$  by  $f - f(0)$  then we get the following result:

$$\|f\|_{p,q,\alpha} \lesssim |f(0)| + \|Ef\|_{p,q,\alpha+1}.$$

From  $|Ef(x)| \leq |\nabla f(x)|$  it follows

$$\|f\|_{p,q,\alpha} \lesssim |f(0)| + \|Ef\|_{p,q,\alpha+1} \leq |f(0)| + \|\nabla f\|_{p,q,\alpha+1}.$$

By induction we obtain that for any  $f \in H$

$$\|f\|_{p,q,\alpha} \lesssim |f(0)| + \|E^k f\|_{p,q,\alpha+k} \lesssim \sum_{j=0}^{k-1} |\nabla^j f(0)| + \|\nabla^k f\|_{p,q,\alpha+k}.$$

Combining this with Lemma 3.1 we get the following theorem.

**Theorem 3.5.** *Let  $0 < p, q \leq \infty, 0 < s < \infty, k \in \mathbb{N}$ . If  $f \in H_m$ , then*

$$\begin{aligned} \|f\|_{p,q,s} &\simeq |f(0)| + \|E^k f\|_{p,q,s+k} \\ \|f\|_{p,q,s} &\simeq \sum_{|\alpha| < k} |\partial^\alpha f(0)| + \sum_{|\alpha|=k} \|\partial^\alpha f\|_{p,q,s+k}. \end{aligned}$$

Let us remark that this result is essentially due to Pavlovič. Now, we consider the case of the tangential derivative.

**Theorem 3.6.** *Let  $0 < p, q \leq \infty, 0 < \alpha < \infty, k \in \mathbb{N}$ . If  $f \in H$ , then*

$$\|f\|_{p,q,\alpha} \simeq |f(0)| + \|T^k f\|_{p,q,\alpha+k}.$$

We would like to remark that Choe-Koo-Yi [4] proved this result with the restriction that  $p = q \in [1, \infty], \alpha = 0$ . Our proof is based on the proof of the following lemma.

**Lemma 3.7.** *Let  $0 < p, q \leq \infty, 0 < \alpha < \infty$  and  $k \in \mathbb{N}$  even. If  $f \in H$ , then*

$$\|f\|_{p,q,\alpha} \simeq |f(0)| + \|T^k f\|_{p,q,\alpha+k}.$$

**Proof.** By definition, it is easy to show that  $T^2 f = -2EE_s f$  with  $s = n - 2$  for any harmonic function  $f$ . Therefore, the following items are all equivalent:

$$\|f\|_{p,q,\alpha}, \|E_s f\|_{p,q,\alpha+1}, \|EE_s f\|_{p,q,\alpha+2}, \|T^2\|_{p,q,\alpha+2}$$

for any  $f$  with  $f(0) = 0$ . This proves Lemma 3.7 for  $k = 2$  and for even  $k$  by induction. ■

To apply the same idea to the proof of our Theorem 3.6, i.e., estimate the term  $Tf = -\sqrt{2}E^{\frac{1}{2}}E_s^{\frac{1}{2}}f$  with  $s = n - 2$ , we need to take a look at fractional Euler operators  $E_s^\beta f$ , which are defined for  $\beta > 0$  and  $s \geq 0$  by acting on the expansion of  $f = \sum_k \sum_{|\nu|=k} H_\nu^k(x)a_{k,\nu}$  into spherical harmonics

$$E_s^\beta f = \sum_k \sum_{|\nu|=k} H_k^\nu(z)a_{k,\nu}(k + s)^\beta.$$

The inverse operator  $I_s^\beta$  is defined by

$$I_s^\beta f = \sum_k \sum_{|\nu|=k} H_k^\nu(z)a_{k,\nu}(k + s)^{-\beta}.$$

To do this we start with the following propositions.

**Proposition 3.8.** *Suppose  $f \in H$ ,  $\beta > 0$ ,  $0 < q \leq \infty$ ,  $s \geq 0$ ,  $0 < r < 1$ , and set  $u = \min(1, q)$ . Then there exists a constant  $C = C(q, \beta, n)$  such that*

$$M_q^u(r, I_s^\beta f) \leq C \int_0^1 \left(\ln \frac{1}{\rho}\right)^{\beta u - 1} \rho^{us - 1} M_q^u(r\rho, f) d\rho. \tag{4}$$

**Proposition 3.9.** *If  $f \in H$ ,  $\beta > 0$ ,  $0 < q \leq \infty$ , and  $s \geq 0$ , then for any  $0 \leq r < 1$*

$$M_q(r, E_s^\beta f) \leq C(1 - r)^{-\beta} M_q(r, f).$$

For the proofs we refer to [13, Propositions 3.1 and 3.2]. The proofs are exactly the same in our case if we use spherical harmonics. These propositions allow us to prove the following theorem.

**Theorem 3.10.** *Let  $0 < p, q \leq \infty, 0 < \alpha < \infty, \beta > 0$ . If  $f \in H$ , then*

$$\|f\|_{p,q,\alpha} \simeq |f(0)| + \|E_s^\beta f\|_{p,q,\alpha+k}.$$

**Proof.** For the sake of simplicity we will consider the case  $s = 0$ . The general case follows from the obvious modification of the proof.

The inequality “ $\gtrsim$ ” is a direct corollary of Proposition 3.9, the fact that  $f(0) = M_q(0, f)$ , and the monotonicity of  $M_q(r, f)$ . To prove the inverse one, we first consider the case of  $0 < p < \infty$ . Denote  $u = \min\{q, 1\}$ . Let  $r = t^{p\beta+1}$ , then for any  $f \in H(\Omega)$  we have

$$\int_0^1 (1 - r^2)^{p\alpha-1} M_q^p(r, I^\beta f) dr \leq C \int_0^1 t^{p\beta} (1 - t^2)^{p\alpha-1} M_q^p(t, I^\beta f) dt.$$

Denote  $h(r) = r^{-1}M_q(r, f)$ . Since  $1 \leq r^{-u}$  for any  $r \in (0, 1)$  and  $u > 0$ , Proposition 3.8 implies

$$\begin{aligned} & \int_0^1 (1-r^2)^{p\alpha-1} M_q^p(r, I^\beta f) dr \\ & \leq C \int_0^1 r^{p\beta} (1-r^2)^{p\alpha-1} \left\{ \int_0^1 (\log 1/\rho)^{\beta u-1} \rho^{u-1} h(r\rho)^u d\rho \right\}^{\frac{p}{u}} dr. \end{aligned}$$

Because  $r^{-1}M_q(r, f)$  is a nondecreasing continuous function, this can be controlled by  $C \int_0^1 r^{-p} (1-r)^{p\beta} (1-r^2)^{p\alpha-1} M_q^p(r, f) dr$ , that is,

$$\int_0^1 (1-r^2)^{p\alpha-1} M_q^p(r, I^\beta f) dr \leq C \int_0^1 r^{-p} (1-r)^{p\beta} (1-r^2)^{p\alpha-1} M_q^p(r, f) dr.$$

Assume that  $f(0) = 0$ , then  $f = I^\beta(E^\beta f)$ . By replacing  $f$  with  $E^\beta f$  in the above inequality, we get

$$\begin{aligned} \int_0^1 (1-r^2)^{p\alpha-1} M_q^p(r, f) dr & \lesssim \int_0^1 (1-r^2)^{p\alpha-1} M_q^p(r, I^\beta(E^\beta f)) dr \\ & \lesssim \int_0^1 (1-r)^{p\beta} (1-r^2)^{p\alpha-1} M_q^p(r, E^\beta f) dr. \end{aligned}$$

Now, replacing  $f$  by  $f - f(0)$ , due to  $E^\beta(f - f(0)) = E^\beta f$ , we get

$$\int_0^1 (1-r^2)^{p\alpha-1} M_q^p(r, f) dr \lesssim |f(0)| + \int_0^1 (1-r)^{p\beta} (1-r^2)^{p\alpha-1} M_q^p(r, E^\beta f) dr.$$

When  $p = \infty$ , we have that for any  $t, u > 0$

$$\int_0^1 \frac{(\log 1/\rho)^{\beta u-1} \rho^{tu-1} (1-r\rho)^{-1}}{(1-r\rho)^{\beta u} (1-(r\rho)^2)^{p\alpha-1}} d\rho \leq C \frac{(1-r)^{-1}}{(1-r^2)^{u\alpha-1}}. \tag{5}$$

Now,  $M_q(r, E^\beta f) = O(\frac{1}{(1-r)^{\beta-1}(1-r^2)^{p\alpha-1}})$  as  $r \rightarrow 1^-$ , then it follows from Proposition 3.8

$$M_q(r, f - f(0)) \leq C \left\{ \int_0^1 (\log 1/\rho)^{\beta u-1} \rho^{-1} M_q^u(r\rho, E^\beta f) \right\}^{1/u} \leq C \frac{(1-r)^{-1}}{(1-r^2)^{p\alpha-1}}.$$

The ‘‘only if’’ part follows from Proposition 3.9. ■

Now, we can proceed with the proof of Theorem 3.6.

**Proof of Theorem 3.6.** From  $Tf = -\sqrt{2}E^{\frac{1}{2}}E_s^{\frac{1}{2}}f$  and our above estimates we get that the following items are all equivalent:

$$\|f\|_{p,q,\alpha}, \|E_s^{\frac{1}{2}}f\|_{p,q,\alpha+\frac{1}{2}}, \|E^{\frac{1}{2}}E_s^{\frac{1}{2}}f\|_{p,q,\alpha+1}, \|T\|_{p,q,\alpha+1}$$

for any  $f$  with  $f(0) = 0$ . The general case follows by induction. ■



Based on the fact that monogenic functions are also harmonic functions the above Hardy-Littlewood inequalities for the Euler operator and the tangential derivative are also valid for monogenic functions.

Let us now take a look at the case of the hypercomplex derivative. This concept of a derivative works only in the case of monogenic functions as null-solutions of the generalized Cauchy-Riemann operator. Therefore, our above used approach does not work in this case. Nevertheless, we can state the following theorems.

**Theorem 3.11.** *Let  $f$  be monogenic,  $1 < p < \infty$ ,  $-1 < b < \infty$  and  $0 < a < \infty$ . Then it holds*

$$\int_0^1 (1-r)^b M_p^a(r, f) dr \lesssim |f(0)| + \int_0^1 (1-r)^b M_p^a(r, |g|) dr + \int_0^1 (1-r)^{a+b} M_p^a(r, |\overline{D}f|) dr,$$

where  $|g(x')| = \sum_{k=0}^{\infty} \frac{(1-|x'|^2)^{k/2}}{k!} (|\sum_{i=1}^n (\partial_{x'}^k) \frac{\partial f}{\partial x_i}(x')|)$ .

**Theorem 3.12.** *Let  $f$  be monogenic,  $1 < p < \infty$ ,  $-1 < b < \infty$  and  $0 < a < \infty$ . Then it holds*

$$\int_0^1 (1-r)^{a+b} M_p^a(r, |\overline{D}f|) dr \lesssim \int_0^1 (1-r)^b M_p^a(r, f) dr.$$

The proof of Theorem 3.12 follows from  $|\overline{D}f| \leq C|\nabla f|$  and Lemma 3.1. For the proof of Theorem 3.11 we need the following two lemmas.

**Lemma 3.13.** *Let  $1 \leq k < \infty, \mu > 0, \delta > 0, h : (0, 1) \mapsto [0, \infty)$  measurable, then*

$$\int_0^1 (1-r)^{k\mu-1} \left( \int_0^r (r-t)^{\delta-1} h(t) dt \right)^k dr \leq C \int_0^1 (1-r)^{k\mu+k\delta-1} h^k(r) dr.$$

For the proof we refer to [7, p.758].

**Lemma 3.14.** *If  $h(r)$  is a positive continuous nondecreasing function of  $r$ , and  $\beta > 0, 0 < c < q < \infty$ , then for  $0 < s < 1$*

$$\left( \int_0^1 (1-r)^{\beta q-1} h^q(rs) dr \right)^{\frac{1}{q}} \leq C \left( \int_0^1 (1-r)^{\beta c-1} h^c(rs) dr \right)^{\frac{1}{c}}.$$

For the proof we refer to the proof of Lemma 5 in [14].

**Proof of Theorem 3.11.** Let  $f$  be a monogenic function in the unit ball. Then we have for  $x, w \in B_{n+1}, x = x_0 + x', w = w_0 + w'$  ( $x' = \text{Vec } x, w' = \text{Vec } w$ )

$$f(x) = f(w) + \int_0^1 \sum_{k=0}^n (x_k - w_k) \frac{\partial f}{\partial x_k} (t(x - w) + w) dt.$$

Setting  $w = 0$  we get

$$f(x) = f(0) + \int_0^1 x_0 \frac{\partial f}{\partial x_0} (tx_0 + tx') dt + \int_0^1 \sum_{k=1}^n \frac{\partial f}{\partial x_k} (tx_0 + tx') dt.$$

Because of the fact that the function  $f$  is monogenic we have  $\bar{D}f = 2 \frac{\partial f}{\partial x_0}$ , and therefore

$$\begin{aligned} f(x) &= f(0) + \int_0^1 \frac{x_0}{2} \bar{D}f(tx_0 + tx') dt \\ &\quad + \int_0^1 \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \sum_{k=0}^{\infty} \frac{1}{k!} (-tx_0)^k [(\partial_{x'})^k f(0, tx')] \right) dt, \end{aligned}$$

where we used the Cauchy-Kovalevskaja extension of  $f$ . This follows from [6, p. 151] and the natural imbedding of the Clifford algebra  $\mathcal{Cl}_n$  into the even subalgebra  $\mathcal{Cl}_{n+1}^+$  mentioned in the preliminaries. Now, writing

$$|g(tx')| = \sum_{k=0}^{\infty} \frac{(1 - |tx'|^2)^{k/2}}{k!} \left( \left| \sum_{i=1}^n (\partial_{x'})^k \frac{\partial f}{\partial x_i} (tx') \right| \right)$$

we have

$$|f(x)|^p \leq C \left( |f(0)|^p + \left( \int_0^1 |g(tx')| dt \right)^p + \left( \int_0^1 |\bar{D}f(tx)| dt \right)^p \right), \quad p \geq 1,$$

due to  $|x_0 \frac{\partial f}{\partial x_0}(tx)| \leq \frac{1}{2} |\bar{D}f(tx)|$  and the elementary inequality  $(a + b)^p \leq 2^{p-1}(a^p + b^p), p \geq 1, a > 0, b > 0$ . Now, Minkowski's inequality leads to

$$\begin{aligned} M_p(r, f) &\leq C \left( |f(0)| + \int_0^1 M_p(tr, |g|) + \int_0^1 M_p(tr, |\bar{D}f|) dt \right) \\ &\leq C \left( |f(0)| + r^{-1} \int_0^r M_p(s, |g|) + r^{-1} \int_0^r M_p(s, |\bar{D}f|) ds \right). \end{aligned} \tag{6}$$

Moreover, let  $a \geq 1$ . Setting  $r = \rho^{a+1}$ , using the monotonicity of means

and the inequality  $(1 - \rho^{a+1})^b \leq K(1 - \rho)^b, -1 < b < \infty$ , we get

$$\begin{aligned} \int_0^1 (1 - r)^b M_p^a(r, f) dr &\leq C \int_0^1 (1 - \rho)^b M_p^a(\rho, f) \rho^a d\rho \\ &\leq C \left( |f(0)|^a + \int_0^1 (1 - \rho)^b \left( \int_0^r M_p(t, |g|) dt \right)^a d\rho \right. \\ &\quad \left. + \int_0^1 (1 - \rho)^b \left( \int_0^r M_p(t, |\overline{D}f|) dt \right)^a d\rho \right). \end{aligned}$$

The application of Lemma 3.13 with  $k = a, \mu = (1 + b)/a, \delta = 1, h(t) = M_p(t, |\overline{D}f|)$  as well as  $h(t) = M_p(t, |g|)$  results in

$$\begin{aligned} \int_0^1 (1 - r)^b M_p^a(r, f) dr &\leq C \left( |f(0)|^a + \int_0^1 (1 - r)^{a+b} M_p^a(r, |g|) dr \right. \\ &\quad \left. + \int_0^1 (1 - r)^{a+b} M_p^a(r, |\overline{D}f|) dr \right), \end{aligned}$$

which proves our theorem for  $a \geq 1$ .

In case of  $0 < a < 1$  we can apply Minkowski's inequality to (6) and by setting  $c = a, q = \beta = 1$  and  $h(r) = M_p(r, |\overline{D}f|)$  as well as  $h(r) = M_p(r, |g|)$  in Lemma 3.14 we obtain

$$\begin{aligned} M_p(r, f) &\leq C \left( |f(0)|^a + \int_0^1 M_p(tr, |g|) dt + \int_0^1 M_p(tr, |\overline{D}f|) dt \right) \\ &\leq C \left( |f(0)|^a + \left( \int_0^1 (1 - t)^{a-1} M_p^a(tr, |g|) dt \right)^{\frac{1}{a}} \right. \\ &\quad \left. + \left( \int_0^1 (1 - t)^{a-1} M_p^a(tr, |\overline{D}f|) dt \right)^{\frac{1}{a}} \right). \end{aligned}$$

This leads to

$$\begin{aligned} r^a M_p^a(r, f) &\leq C \left( |f(0)|^a + \int_0^r (r - s)^{a-1} M_p^a(s, |g|) dt \right. \\ &\quad \left. + \int_0^r (r - s)^{a-1} M_p^a(s, |\overline{D}f|) dt \right). \end{aligned}$$

Like in the first case we have

$$\begin{aligned} \int_0^1 (1 - r)^b M_p^a(r, f) dr &\leq C (|f(0)|^a \\ &\quad + \int_0^1 (1 - \rho)^b \left( \int_0^\rho (\rho - s)^{a-1} M_p^a(s, |g|) ds \right) d\rho \\ &\quad + \int_0^1 (1 - \rho)^b \left( \int_0^\rho (\rho - s)^{a-1} M_p^a(s, |\overline{D}f|) ds \right) d\rho). \end{aligned}$$

The use of Lemma 3.13 with  $k = 1, \mu = 1 + b, \delta = a$  and  $h(s) = M_p^a(s, |\overline{D}f|)$  results in

$$\int_0^1 (1-r)^b M_p^a(r, f) dr \leq C \left( |f(0)|^a + \int_0^1 (1-r)^{a+b} M_p^a(r, |g|) dr + \int_0^1 (1-r)^{a+b} M_p^a(r, |\overline{D}f|) dr \right),$$

which proves our theorem for  $0 < a < 1$ . ■

The following propositions follow directly from these theorems.

**Proposition 3.15.** *Let  $f$  be monogenic in the unit ball,  $b > -1, 1 < p < \infty$ , then*

$$\int_{B_{n+1}} |f(x)|^p (1 - |x|^2)^b dB_{n+1} \lesssim |f(0)|^p + \int_{B_{n+1}} |g(x)|(1 - |x|^2)^b dB_{n+1} + \int_{B_{n+1}} |\overline{D}f|^p (1 - |x|^2)^{b+p} dB_{n+1}.$$

**Proposition 3.16.** *Let  $f$  be monogenic in the unit ball,  $b > -1, 1 < p < \infty$ , then*

$$\int_{B_{n+1}} |\overline{D}f|^p (1 - |x|^2)^{b+p} dB_{n+1} \lesssim \int_{B_{n+1}} |f(x)|^p (1 - |x|^2)^b dB_{n+1}.$$

We can sum up the above results in the following theorem.

**Theorem 3.17.** *Let  $f \in \ker D, 1 < p < \infty, -1 < b < \infty$  and  $0 < a < \infty$ . Then it holds*

$$\int_0^1 (1-r)^b M_p^a(r, f) dr \simeq |f(0)| + \int_0^1 (1-r)^b M_p^a(r, |g|) dr + \int_0^1 (1-r)^{a+b} M_p^a(r, |\overline{D}f|) dr,$$

with  $|g(x')| = \sum_{k=0}^{\infty} \frac{(1-|x'|^2)^{k/2}}{k!} (|\sum_{i=1}^n ((\partial_{x'})^k \frac{\partial f}{\partial x_i}(x'))|)$ .

### 4. $Q_p$ -spaces in Clifford analysis

Essentially, there are two different approaches to define  $Q_p$ -spaces in Clifford analysis, based on the possibility to define monogenic functions via the Dirac operator or the Cauchy-Riemann operator.

Using the Dirac operator, i.e., considering vectors, and the Möbius transformations  $\varphi_a(x) = (a - x)(1 - \bar{a}x)^{-1}, |a| < 1$ , which map the unit disk in  $\mathbb{R}^n$  onto itself, we can use the following definition by J. Cnops and R. Delanghe [5]:

**Definition 4.1.** Let  $f : B \subset \mathbb{R}^n \mapsto \mathcal{C}l_{0,n}$  a function defined over the unit ball in  $\mathbb{R}^n$ , then the  $Q_p$ -space is the space of all monogenic functions, such that the semi-norm

$$|f|_{Q_p} = \sup_{a \in B} \int_B \sum_{k=1}^n \left| \frac{\partial f}{\partial x_k} \right|^2 (1 - |\varphi_a(z)|^2)^p dB,$$

is finite, i.e.,  $Q_p = \{f \in \ker \partial : |f|_{Q_p} < \infty\}$ .

In this definition they use the gradient  $|\nabla f|^2 = \left| \frac{\partial f}{\partial x_k} \right|^2$  as a replacement of the complex derivative. As we will see we can also use other canonical differential operators.

To this end we can apply our Hardy-Littlewood inequalities from the previous section. Based on the fact that this scale of  $Q_p$ -spaces is conformally invariant [5] we can conclude that the gradient can be replaced by any of our other canonical differential operators, a result which is of major importance in defining  $Q_p$ -spaces in a hyperbolic setting [3].

**Theorem 4.2.** *Let  $f \in Q_p$  then the following conditions are equivalent:*

1.  $\sup_{a \in B} \int_B |\nabla f|^2 (1 - |\varphi_a(z)|^2)^p dB < \infty$
2.  $\sup_{a \in B} \int_B |Ef|^2 (1 - |\varphi_a(z)|^2)^p dB < \infty$
3.  $\sup_{a \in B} \int_B |Tf|^2 (1 - |\varphi_a(z)|^2)^p dB < \infty$

For the proof we only remark that making a change of variable  $x = \varphi_a(z)$  (conformal invariance!) we get

$$\begin{aligned} |f(0)| + \int_B |\nabla f(z)|^2 (1 - |\varphi_a(z)|^2)^p dB \\ &= |f(\varphi_a(a))| + \int_B |\nabla f(\varphi_a(x))|^2 (1 - |x|^2)^p dB \\ &\approx |f(\varphi_a(a))| + \int_B |Ef(\varphi_a(x))|^2 (1 - |x|^2)^p dB \\ &= |f(0)| + \int_B |Ef(z)|^2 (1 - |\varphi_a(z)|^2)^p dB. \end{aligned}$$

Therefore, (i) implies (ii). The proofs of the other implications follow the same lines.

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